Isomorphic Operator Algebras and Conjugate Inner Functions

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I. Introduction

Let D denote the open unit disk in the complex plane, $D=\{z: |z|<1\}$, and let m be normalized arclength measure on the boundary ∂D of D. If ϕ is a nonconstant inner function on D, then $C=C_\phi$ denotes the composition operator on $H^2=H^2(D)$ determined by ϕ—$C_\phi(f)=f\circ\phi$. Here $\circ$ denotes function composition. That C_ϕ is bounded is proven in [7; 8]. The operator C_ϕ does not tell everything about the analytic function ϕ. Indeed, if e_n is the function $e_n(z)=z^n$, then $C_{e_n}(e_m)=e_{nm}$ so that, for $n>1$, C_{e_n} is the direct sum of a 1-dimensional identity operator and a pure isometry of infinite multiplicity. As such, they are all unitarily equivalent to each other. On the other hand, e_n covers the disk n times so that these functions are not the same.

Each f in H^∞ defines the analytic Toeplitz operator T_f on H^2 by $T_f(h)=fh$. Let $\mathbf{A}=\mathbf{A}_\phi$ denote the norm closed algebra generated by C_ϕ and all the analytic Toeplitz operators. Note that $C_\phi T_f=T_{f\circ\phi}C_\phi$, so that $\mathbf{A}$ is commutative just in case ϕ is the identity function $\phi(z)=z$. From here on, the same notation will be used to denote the H^∞ function, its boundary function, its Toeplitz operator, and even its Gelfand transform. This convention is convenient and will cause no confusion.

Two inner functions ϕ and ψ are conjugate if there is an analytic homeomorphism τ of D satisfying $\tau\circ\psi=\phi\circ\tau$. We prove the following:

THEOREM 1. *If ϕ and ψ are nonconstant, nonperiodic inner functions, then they are conjugate if and only if the algebras $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$ are isomorphic.*

Here, ϕ is periodic if $\phi^{(n)}(z)=z$, where $\phi^{(n)}$ denotes the n-fold iterate of ϕ. The analytic homeomorphisms of D are the Möbius transformations

$$\tau(z)=c\frac{z-a}{1-\bar{a}z},$$

where $|a|<1$ and $|c|=1$. Theorem 1 is just the analytic version of what is done in [1; 2; 4; 5] for composition operators on L^2 spaces.

If τ is a homeomorphism as in the theorem, then $C_\tau C_\phi C_\tau^{-1}=C_\psi$ and $C_\tau fC_\tau^{-1}=f\circ\tau$, so that the map $\Gamma(a)=C_\tau aC_\tau^{-1}$ is an isomorphism of $\mathbf{A}_\phi$

Received July 23, 1990. Revision received July 3, 1991.
Michigan Math. J. 39 (1992).

onto $\mathbf{A}_\psi$. The interesting part of the theorem, then, is in the other direction. We must first study the algebra $\mathbf{A}_\phi$ more carefully.

II. The Algebra A

The measure $m\circ\phi^{-1}$ given by $m\circ\phi^{-1}(E)=m(\phi^{-1}(E))$ is absolutely continuous with respect to m, and

$$\frac{dm\circ\phi^{-1}}{dm}=P_{\phi(0)}(z)=\operatorname{Re}\left(\frac{z+\phi(0)}{z-\phi(0)}\right)$$

is the Poisson kernel for evaluation at $\phi(0)$ [7]. Let $h(z)$ be the reciprocal of the normalized Cauchy kernel:

$$h(z)=\frac{1-\overline{\phi(0)}z}{(1-|\phi(0)|^2)^{1/2}}.$$

Then h and h^{-1} are in H^∞, and $|h|^2=(P_{\phi(0)})^{-1}$ almost everywhere on ∂D. For any measurable function f on ∂D, $\int|h|^2\circ\phi f\circ\phi\, dm=\int f\, dm$ and $U=U_\phi=T_{h\circ\phi}C_\phi$ is an isometry. If f is in H^∞ then so is $C_\phi(f)$. Since $m\circ\phi^{-1}$ is not only absolutely continuous but also equivalent to m, C_ϕ is an isometry on H^∞. Note that the set of operators of the form $\sum_{n=0}^N f_n U^n$, with f_n in H^∞, is dense in $\mathbf{A}$. These operators will be called polynomials. Clearly $(C_\phi)^n=C_{\phi^{(n)}}$. On the other hand,

$$(U_\phi)^n=\left(\prod_{k=1}^n T_{h\circ\phi^{(k)}}\right)C_{\phi^{(n)}},$$

which is generally not the same as $U_{\phi^{(n)}}=T_{h_n\circ\phi^{(n)}}C_{\phi^{(n)}}$ where h_n is the outer function satisfying

$$|h_n|=\left[\frac{dm\circ\phi^{-(n)}}{dm}\right]^{-1/2}=[P_{\phi^{(n)}(0)}]^{-1/2}$$

almost everywhere on ∂D.

Let Σ denote the σ-algebra of Borel subsets of ∂D and $\Sigma_n=\phi^{-(n)}(\Sigma)=\{\phi^{-(n)}(S): S\in\Sigma\}$, and let E_n denote the conditional expectation given Σ_n. So if f is any positive or integrable function on ∂D, then $E_n(f)$ is Σ_n measurable and $\int_S E_n(f)\, dm=\int_S f\, dm$ for each S in Σ_n. If f is Σ_n measurable then $E_n(fg)=fE_n(g)$ for any function g. Also, $E_n(f)$ is positive whenever f is positive, and $\|E_n(f)\|_\infty\le\|f\|_\infty$ if f is bounded. There is a function g satisfying $f=g\circ\phi^{(n)}$ if and only if $E_n(f)=f$, and in this case g is unique up to a set of measure 0. We shall write $g=f\circ\phi^{-(n)}$.

PROPOSITION 2.

1. $E_n\left(\prod_{k=1}^n|h\circ\phi^{(k)}|^2\right)=\left(\frac{dm\circ\phi^{-(n)}}{dm}\circ\phi^{(n)}\right)^{-1}.$

2. *If f is in H^2, then fU^n defines a bounded operator if and only if $E_n(|f|^2)$ is bounded and*

$$\|fU^n\|^2 = \left\| E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right) \frac{dm\circ\phi^{-(n)}}{dm} \circ \phi^{(n)} \right\|_\infty .$$

Proof. 1. Both U^n and $U_{\phi^{(n)}}$ are isometries, so if $S \in \Sigma$ then $\|U^n(\chi_S)\|^2 = \|U_{\phi^{(n)}}(\chi_S)\|^2 = m(S)$. That is,

$$\int \left(\prod_{k=1}^n |h\circ\phi^{(k)}|^2\right) \chi_{\phi^{-(n)}(S)}\, dm = \int \left(\frac{dm\circ\phi^{-(n)}}{dm}\circ\phi^{(n)}\right)^{-1} \chi_{\phi^{-(n)}(S)}\, dm$$

as desired.

2. Suppose f is in H^2. Since $|h|^2$ and $dm\circ\phi^{-(n)}/dm$ are just Poisson kernels, there are constants K_1 and K_2 such that

$$K_1|f|^2 \le |f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2 \frac{dm\circ\phi^{-(n)}}{dm}\circ\phi^{(n)} \le K_2|f|^2.$$

This and the fact that E_n preserves inequalities yields that $E_n(|f|^2)$ is bounded if and only if

$$E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right) \frac{dm\circ\phi^{-(n)}}{dm}\circ\phi^{(n)}$$

is. If $g \in H^2$ then

$$\begin{aligned}
\|fU^n(g)\|^2 &= \int |f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2 |g\circ\phi^{(n)}|^2\, dm \\
&= \int E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right) |g\circ\phi^{(n)}|^2\, dm \\
&= \int E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right)\circ\phi^{-(n)} \left(\frac{dm\circ\phi^{-(n)}}{dm}\right) |g|^2\, dm \\
&\le \left\| E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right)\circ\phi^{-(n)} \left(\frac{dm\circ\phi^{-(n)}}{dm}\right) \right\|_\infty \|g\|_2^2 \\
&= \left\| E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right) \left(\frac{dm\circ\phi^{-(n)}}{dm}\circ\phi^{(n)}\right) \right\|_\infty \|g\|_2^2.
\end{aligned}$$

The last equality follows because C_ϕ is an isometry on H^∞. This shows that

$$\|fU^n\|^2 \le \left\| E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right) \frac{dm\circ\phi^{-(n)}}{dm} \circ \phi^{(n)} \right\|_\infty .$$

To show equality, pick g in H^2 so that $|g|$ approximates, in the L^2 sense, the characteristic function of the set on which

$$E_n\left(|f|^2 \prod_{k=1}^n |h\circ\phi^{(k)}|^2\right)\circ\phi^{-(n)} \left(\frac{dm\circ\phi^{-(n)}}{dm}\right)$$

is almost its maximum. □

We next define coordinate maps Π_n on $\mathbf{A}$ such that $\Pi_n(\sum_{k=0}^N f_k U^k) = f_n$. Since fU^n may be a bounded operator even if f is not a bounded function, it is necessary to define coordinate spaces that may be larger than H^∞. Let $K_n = \{f \in H^2 : E_n(|f|^2) \in L^\infty\}$ and $|||f|||_n = \|fU^n\|$.

PROPOSITION 3. *For each n, K_n is a Banach space, $H^\infty \subseteq K_n \subseteq K_{n+1} \subseteq H^2$, and the inclusion operators are bounded.*

We let K_0 be H^∞, E_0 the identity operator, and $|||\ |||_0 = \|\ \|_\infty$.

PROPOSITION 4. *If ϕ is not periodic, then for each $n = 0, 1, 2, 3, \ldots$ there is a map Π_n from $\mathbf{A}$ to K_n such that $|||\Pi_n(a)|||_n \leq \|a\|$ and $\Pi_n(\sum_{k=0}^N f_k U^k) = f_n$.*

Proof. It suffices to show that $|||f_n|||_n \leq \|\sum_{k=0}^N f_k U^k\|$. Let $a = \sum_{k=0}^N f_k U^k$. If, for some n, $\{z : |z| = 1, \phi^{(n)}(z) = z\}$ has positive measure, then $\phi^{(n)}(z) = z$ for almost all z on ∂D. This case has been excluded. Consequently, the set of fixed points of $\phi^{(n)}$ on ∂D has measure zero; that is, ϕ is aperiodic. As in Halmos [6], if $E \subseteq \partial D$ with $m(E) > 0$ and if k is a natural number, then there is a subset F of E with $m(F) > 0$ such that the sets $\phi^{-(n)}(F)$, $n = 1, 2, \ldots, k$, are pairwise disjoint.

Let

$$E = \left\{z : |z| = 1, \left(E_n\left(|f|^2 \prod_{i=1}^n |h \circ \phi^{(i)}|^2\right)\right)^{1/2} \circ \phi^{(-n)} \left(\frac{dm \circ \phi^{-(n)}}{dm}\right)^{1/2} > |||f_n|||_n - \epsilon\right\}$$

Now pick $F \subset E$ and $m(F) > 0$ such that the sets $\phi^{(-k)}(F)$ are pairwise disjoint for $1 \leq k \leq N$. The operator a can act on $L^2(m)$ as well as H^2. We temporarily use $\|a\|_L$ and $\|a\|_H$ to denote the norms of these two operators. Pick g in $L^2(m)$ such that $\|g - \chi_F\| \leq \epsilon / \|a\|_L$, $\|g\| \leq \|\chi_F\|$, and $z^s g \in H^2$ for some s. Then

$$\begin{aligned}
(|||f_n|||_n - \epsilon)^2 \|\chi_F\|^2 &\leq \int \left(E_n\left(|f|^2 \prod_{i=1}^n |g \circ \phi^{(i)}|^2\right)\right) \circ \phi^{(-n)} \left(\frac{dm \circ \phi^{-(n)}}{dm}\right) \chi_F \, dm \\
&= \int E_n\left(|f_n|^2 \prod_{i=1}^n |g \circ \phi^{(i)}|^2\right) \chi_F \circ \phi^{(n)} \, dm \\
&= \int |f_n|^2 \prod_{i=1}^n |g \circ \phi^{(i)}|^2 \chi_F \circ \phi^{(n)} \, dm \\
&\leq \int \sum_{k=0}^N |f_k|^2 \prod_{i=1}^k |g \circ \phi^{(i)}|^2 (\chi_F \circ \phi^{(k)}) |(\phi^{(k)})^s|^2 \, dm \\
&= \int \left| \sum_{k=0}^N f_k \prod_{i=1}^k g \circ \phi^{(i)} (\chi_F \circ \phi^{(k)}) (\phi^{(k)})^s \right|^2 dm \\
&= \int |a(z^s \chi_F)|^2 \, dm = \|a(z^s \chi_F)\|^2 \leq (\|a(z^s g)\| + \epsilon)^2 \\
&\leq (\|a\|_H \|z^s h\| + \epsilon)^2 = (\|a\|_H \|g\| + \epsilon)^2 \\
&\leq (\|a\|_H \|\chi_F\| + \epsilon)^2.
\end{aligned}$$

Here, the two sums are equal because the function $\chi_F \circ \phi^{(k)}$ is supported on $\phi^{-(k)}(F)$, and these sets are disjoint. □

PROPOSITION 5. *If ϕ is not periodic, then the coordinate maps satisfy the product rule*

$$\Pi_k(ab) = \sum_{i=0}^{k} (\Pi_i(a))(\Pi_{k-i}(b)\circ\phi^{(i)}).$$

Proof. This identity is true if a and b are polynomials. That it is true for all a and b follows by continuity. □

PROPOSITION 6. *The closed ideal of* $\mathbf{A}$ *generated by* U^{n+1} *is* $\bigcap_{i=0}^{n} \ker \Pi_i$.

Proof. That the intersection of these kernels is a closed ideal that contains U^{n+1} follows from the product rule and the continuity of the Π_i. Conversely, suppose $\Pi_i(a)=0$ for $i \le n$. Let p_k be a sequence of polynomials in $\mathbf{A}$ that converges to a; then $p_k - \sum_{i=0}^{n} \Pi_i(p_k)U^i$ is in the ideal generated by U^{n+1} and converges to a. Thus a is in the ideal generated by U^{n+1}. □

It is necessary to examine the multiplicative linear functionals on $\mathbf{A}$. Let M denote the space of nonzero multiplicative functionals on $\mathbf{A}$, and let Δ be the maximal ideal space of H^∞. We will think of D as a subset of Δ and write $f(z)$ instead of $z(f)$. For $z \in \Delta$, let $M_z = \{\alpha \in M : \alpha(f) = f(z)$ for all f in $H^\infty\}$. If ϕ is not periodic then no M_z is empty. Indeed, if α_z is defined by $\alpha_z(a) = \Pi_0(a)(z)$, then α_z is in M_z.

The inner function ϕ can be extended from D to a transformation of Δ as follows. If f is in H^∞, let $T(f) = f\circ\phi$, and let T^* be the adjoint transformation of T. Then, if $z \in \Delta$ and f and g are in H^∞, it is easily verified that $T^*(z)(fg) = T^*(z)(f)T^*(z)(g)$ so that T^* maps Δ to Δ, and if z is in D then $T^*(z)(f) = z(f\circ\phi) = f(\phi(z))$ or $T^*(z) = \phi(z)$. The restriction of T^* to Δ is the desired extension of ϕ and will be called ϕ as well.

Let A_r denote the disk algebra on the closed disk $\bar{D}_r = \{|z| \le r\}$; A_r is thus the uniform closure of the polynomials in the algebra of all continuous functions on $\bar{D}_r$. We write A for A_1, and A_0 is just the field of complex numbers.

PROPOSITION 7. *Suppose that ϕ is not periodic.*

(a) *If $z \in \Delta$ and $\phi(z) \neq z$, then $M_z = \{\alpha_z\}$.*

(b) *If $z \in \Delta$ and $\phi(z) = z$, then there is an r $(0 \le r \le 1)$ and a bounded algebra homomorphism ρ_z of* $\mathbf{A}$ *to A_r such that $\alpha \in M_z$ if and only if there is a ξ in $\bar{D}_r$ such that $\alpha(a) = \rho_z(a)(\xi)$ for all a in* $\mathbf{A}$.

(c) *If $z \in D$ and $\phi(z) = z$, then $r = 1$ and ρ_z maps onto A.*

Proof. (a) If z is not a fixed point of ϕ, then pick f in H^∞ with $f(z) \neq f(\phi(z))$. Then, for $\alpha \in M_z$, $\alpha(U)f(z) = \alpha(U)\alpha(f) = \alpha(Uf) = \alpha((f\circ\phi)U) = \alpha((f\circ\phi)\alpha(U)) = f(\phi(z))\alpha(U)$. Thus $\alpha(U) = 0$. So α agrees with α_z on sums of the form $\sum_{i=0}^{n} f_i U^i$. But such sums are dense in $\mathbf{A}$, so $\alpha = \alpha_z$.

(b) Suppose z is a fixed point of ϕ. Let $r=\sup\{|\alpha(U)|: \alpha\in M_z\}$. Since M_z is compact, there is a β in M_z such that $|\beta(U)|=r$. If f is in H^∞, then $fU^n\in \mathbf{A}$ and $|f(z)|r^n=|\beta(fU^n)|\leq \|fU^n\|=|||f|||_n$. Suppose $|\xi_0|<r$. If $a=\sum_{n=0}^N f_nU^n$, then $|||f_n|||_n\leq \|a\|$ and

$$\left|\sum_{n=0}^N f_n(z)\xi_0^n\right|\leq \sum_{n=0}^N |f_n(z)||\xi_0|^n\leq \sum_{n=0}^N \|a\|\left(\frac{|\xi_0|}{r}\right)^n\leq \frac{\|a\|}{1-|\xi_0|/r}.$$

Hence the map $\alpha(\sum_{n=0}^N f_nU^n)=\sum_{n=0}^N f_n(z)\xi_0^n$ extends to all of $\mathbf{A}$. Clearly $\alpha\in M_z$ and so $\|\alpha\|=1$. If $\rho_z(\sum_{n=0}^N f_nU^n)$ is the polynomial in ξ $\sum_{n=0}^N f_n(z)\xi^n$, then $\rho_z(\sum_{n=0}^N f_nU^n)(\xi_0)$ is just $\alpha(\sum_{n=0}^N f_nU^n)$. Consequently

$$\left\|\rho_z\left(\sum_{n=0}^N f_nU^n\right)\right\|\leq \left\|\sum_{n=0}^N f_nU^n\right\|,$$

where the former norm is the supremum on D_r; thus ρ_z extends to be continuous on all of $\mathbf{A}$. Note that the set $\{\alpha(U): \alpha\in M_z\}$ is closed and so must be $\bar{D}_r$.

(c) If $z\in D$ is a fixed point then $r=1$. Indeed, if $|z|<1$ then evaluation at z has norm $(1-|z|^2)^{-1/2}$ as a linear functional on H^2. Thus, if $a=\sum_{n=0}^N f_nU^n$ is in $\mathbf{A}$, then

$$\begin{aligned}\sum_{n=0}^N |f_n(z)||\xi|^n &\leq \sum_{n=0}^N \|f_n\|_2(1-|z|^2)^{-1/2}|\xi|^n\\ &\leq \sum_{n=0}^N |||f_n|||_n(1-|z|^2)^{-1/2}|\xi|^n\leq \|a\|(1-|z|^2)^{-1/2}\sum_{n=0}^N|\xi|^n.\end{aligned}$$

Hence the map $a\to\sum_{n=0}^N f_n(z)\xi^n$ extends to $\mathbf{A}$ as long as $|\xi|<1$. Therefore $r=1$. It remains to show that ρ_z maps $\mathbf{A}$ onto A. But if g is in A then $g(U)$ is defined by the functional calculus, $g(U)$ is in $\mathbf{A}$, and $\rho_z(g(U))=g$. □

The map $z\to\alpha_z$ naturally imbeds Δ as a subset of M. Similarly, if z is a fixed point of ϕ such that the "radius" r of M_z is positive, then part (b) of Proposition 7 identifies M_z with the disk $\bar{D}_r$. So M is the union of disks, one of which looks like Δ and the others like true disks $\bar{D}$. Furthermore, $\mathbf{A}$ acts as an algebra of analytic functions of these disks, like H^∞ on Δ and like A on the others.

DEFINITION. A subset C of M is an analytic disk for $\mathbf{A}$ if

(a) C is the closure of its interior, and
(b) if $a\in\mathbf{A}$ and $\alpha(a)=0$ for all α in some nonempty open subset of C, then $\alpha(a)=0$ for all α in C.

PROPOSITION 8. *The maximal analytic disks in M are Δ and those M_z with positive radius.*

Proof. That Δ and the nontrivial M_z are analytic disks is clear. Since U vanishes only on Δ and $z-z_0$ vanishes only on M_{z_0}, these are maximal analytic disks. If C is any analytic disk, then the interior of C must have nontrivial intersection with the interior of either Δ or one of the M_{z_0}. In the first case U must vanish on C, in the latter case $z-z_0$ does. In either case, C must be contained in one of the indicated disks. □

III. Proof of Theorem 1

Suppose ϕ and ψ are two aperiodic inner functions and Γ is an algebra isomorphism from $\mathbf{A}_\phi$ to $\mathbf{A}_\psi$. Let M and N denote the spaces of multiplicative functionals on $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$. Then Γ induces a map γ from N to M given by $\alpha(\Gamma(a))=\gamma(\alpha)(a)$. The map γ is a homeomorphism since the topologies on M and N are determined by their corresponding algebras. The defining equation for γ shows that it maps analytic disks to analytic disks. In particular, $\gamma(\Delta)$ is a maximal analytic disk in M. But since no $M_z=\bar{D}_r$ is homeomorphic to Δ, it must be that $\gamma(\Delta)=\Delta$. That is, if Π_n denotes the nth coordinate map for both $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$, and if (for w in Δ) β_w is the functional in N given by $\beta_w(b)=\Pi_0(b)(w)$, then there is a $z=\tau(w)$ in Δ such that $\gamma(\beta_w)=\alpha_z$. So for a in $\mathbf{A}_\phi$, $\beta_w(\Gamma(a))=\alpha_{\tau(w)}(a)$; in particular, if f is in H^∞ then $f(\tau(w))=\Pi_0(\Gamma(f))(w)$.

LEMMA 9. $\Pi_0(\Gamma(U_\phi))=0$.

Proof. If $w\in\Delta$, then

$$\Pi_0(\Gamma(U_\phi))(w)=\beta_w(\Gamma(U_\phi))=\alpha_{\tau(w)}(U_\phi)=\Pi_0(U_\phi)(\tau(w))=0. \qquad \square$$

LEMMA 10. $\Pi_1(\Gamma(U_\phi))\neq 0$.

Proof. If $\Pi_1(\Gamma(U_\phi))=0$, then $\Gamma(U_\phi)$ is in the closed ideal of $\mathbf{A}_\psi$ generated by $(U_\psi)^2$. Then the isomorphism Γ induces a homomorphism of the quotient Banach algebra $\mathbf{A}_\phi/\mathbf{A}_\phi U_\phi$ onto the algebra $\mathbf{A}_\psi/\mathbf{A}_\psi(U_\psi)^2$, where $\mathbf{A}_\phi U_\phi$ and $\mathbf{A}_\psi(U_\psi)^2$ denote (respectively) the closed ideals of $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$ generated by U_ϕ and $(U_\psi)^2$. But this is impossible because the former quotient is commutative and the latter is not. □

The homeomorphism τ of Δ has been constructed. It remains to show that $\phi\circ\tau=\tau\circ\psi$. If $f\in H^\infty$ then $U_\phi f=(f\circ\phi)U_\phi$. Hence,

$$\Pi_1(\Gamma(U_\phi)\Gamma(f))=\Pi_1(\Gamma(f\circ\phi)\Gamma(U_\phi)).$$

But

$$\begin{aligned}\Pi_1(\Gamma(U_\phi)\Gamma(f))&=\Pi_1(\Gamma(U_\phi))\Pi_0(\Gamma(f))\circ\psi+\Pi_0(\Gamma(U_\phi))\Pi_1(\Gamma(f))\\&=\Pi_1(\Gamma(U_\phi))\Pi_0(\Gamma(f))\circ\psi\end{aligned}$$

by the product rule and the fact that $\Pi_0(\Gamma(U_\phi))=0$. Furthermore,

$$\begin{aligned}\Pi_1(\Gamma(f\circ\phi)\Gamma(U_\phi))&=\Pi_1(\Gamma(f\circ\phi))\Pi_0(\Gamma(U_\phi))\circ\psi+\Pi_0(\Gamma(f\circ\phi))\Pi_1(\Gamma(U_\phi))\\&=\Pi_0(\Gamma(f\circ\phi))\Pi_1(\Gamma(U_\phi)).\end{aligned}$$

Therefore,

$$0=\Pi_1(\Gamma(U_\phi))\Pi_0(\Gamma(f))\circ\psi-\Pi_0(\Gamma(f\circ\phi))\Pi_1(\Gamma(U_\phi))$$
$$=\Pi_1(\Gamma(U_\phi))[\Pi_0(\Gamma(f))\circ\psi-\Pi_0(\Gamma(f\circ\phi))].$$

Here the product of two functions, analytic in D, is the zero function. One of the functions is not identically zero, so the other must be:

$$\Pi_0(\Gamma(f))\circ\psi-\Pi_0(\Gamma(f\circ\phi))=0.$$

Recalling that $\Pi_0(\Gamma(g))=g\circ\tau$ for g in H^∞, we have $f\circ\tau\circ\psi=f\circ\phi\circ\tau$. But $f\in H^\infty$ is arbitrary, so $\tau\circ\psi=\phi\circ\tau$. Finally, if e is the identity function of D, $e(z)=z$, then

$$\tau(z)=e(\tau(z))=\alpha_{\tau(z)}(e)=\alpha_z(\Gamma(z))=\Pi_0(\Gamma(e))(z),$$

so τ is analytic on D.

IV. The Periodic Case

What happens if ϕ is periodic? Which ϕ are periodic?

PROPOSITION 11. *If ϕ is an inner function of period n, then ϕ is conjugate to a rotation $\rho(z)=cz$, where $c^n=1$.*

Proof. According to [3], an analytic homeomorphism ϕ of D has either one fixed point in D, or one on the boundary of D, or two on the boundary of D. Furthermore, if ϕ has boundary fixed points, then one of them is attractive in the sense that $\phi^{(n)}(z)$ converges to this fixed point for each z in D. But this cannot happen if ϕ is periodic, so ϕ must have its fixed point α in D. Let $\tau(z)=(z-\alpha)/(\bar{\alpha}z-1)$. Then $\tau^{-1}\circ\phi\circ\tau$ has period n and fixes 0, and so must be a periodic rotation of D. □

A proposition similar to Proposition 4 is true for periodic ϕ.

PROPOSITION 12. *If ϕ has period n, then for each $k<n$ there is a bounded coordinate map Π_k of* **A** *onto H^∞ such that*

$$a=\sum_{k=0}^{n-1}\Pi_k(a)U^k$$

for each a in **A**.

Proof. It is first shown that if $a=\sum_{i=0}^{n-1}f_iU^i$ then $\|f_k\|\le\|a\|$. Let τ be an analytic homeomorphism of D such that $\tau^{-1}\circ\phi\circ\tau$ is a periodic rotation. If $E=\{z\in\partial D: |f_k(a)|\ge\|f_k\|-\epsilon\}$, then E intersects one of the sets $\{\tau(e^{i\theta}): (j-1)2\pi/n\le\theta<j2\pi/n\}$ in a set of positive measure. Let F be that set. Then the sets $\phi^{(-i)}(F)$, $0\le i<n$, are disjoint. The desired inequality now follows as in the proof of Proposition 4. Set $\Pi_k(\sum_{i=0}^{n-1}f_iU^i)=f_k$ and extend Π_k

continuously to all of **A**. Every a in **A** can be written as $\sum_{k=0}^{n-1} \Pi_k(a)U^k$, since this is true for polynomials and the set of polynomials is dense in **A**. □

In case ϕ has period n, $\mathbf{A}_\phi$ has only n complex homomorphisms.

PROPOSITION 13. *If ϕ has period n then $M = M_w$, where w is the fixed point of ϕ and $\{\alpha(U): \alpha \in M\}$ is just the set on* nth *roots of unity.*

Proof. If $\alpha \in M$ then $\alpha(U)^n = \alpha(U^n) = 1$, so $\alpha(U)$ is a root of unity. If e is the identity function in H^∞, $e(z) = z$, and if $\alpha \in M_w$, then $\alpha(U)w = \alpha(U)\alpha(e) = \alpha(Ue) = \alpha(e \circ \phi U) = \phi(w)\alpha(U)$. Thus w is the fixed point of ϕ. Conversely, if $d^n = 1$, then setting $\alpha(a) = (\sum_{i=0}^{n-1} \Pi_i(a)(w)d^i)$ defines a complex homomorphism with $\alpha(U) = d$. □

COROLLARY 14. *If $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$ are isomorphic and ϕ has period n, then so does ψ.*

So $\mathbf{A}_\phi$ is quite simple when ϕ is periodic, yet our theorem fails miserably in this case. If ϕ and ψ are rotations of period n, then for some i and j $(0 < i, j < n)$, $\phi^{(i)} = \psi$ and $\psi^{(j)} = \phi$, so that $\mathbf{A}_\phi$ and $\mathbf{A}_\psi$ are not only isomorphic but equal. Yet a simple computation shows that ϕ and ψ will be conjugate only if they are the same.

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