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Isomorphism Theorems for Variants of Semigroups of Linear Transformations

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Abstract

If S is a semigroup and $a \in S$, the semigroup $(S, \circ)$ defined by $x \circ y = xay$ for all $x, y \in S$ is called a *variant* of S and $(S, \circ)$ is denoted by (S, a). In 2003-2004, Tsyaputa characterized when two variants of the following transformation semigroups are isomorphic : the symmetric inverse semigroup, the full transformation semigroup and the partial transformation semigroup on a finite nonempty set. In this paper, we consider the semigroups under composition of all linear transformations of a finite-dimensional vector space over a finite field. We determine when its variants are isomorphic. We also obtain as a consequence in the same matter for the full $n \times n$ matrix semigroup over a finite field.

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1 Introduction and Preliminaries

The cardinality of a set X is denoted by $|X|$. The value of the mapping α at x in the domain of α shall be written as $x\alpha$. The range (image) of α is denoted by ran α.

If S is a semigroup and $a \in S$, the semigroup $(S, \circ)$ defined by $x \circ y = xay$ for all $x, y \in S$ is called a *variant* of S and it is denoted by (S, a). Variants of abstract semigroups were first studied by Hickey [2] in 1983. In fact, variants of concrete semigroups of relations were earlier considered by Magill [8] in 1967. Hickey [2, 1, 3, 4, 5] introduced various results relating to variants of semigroups. Khan and Lawson [7] determined an element a in a regular semigroup with identity and an inverse semigroup such that (S, a) is a regular semigroup.

It is interesting to know when two variants of a certain semigroup are isomorphic. It is clear that if S is a semigroup with identity and a, b are units of S, $(S, a) \cong (S, b)$ through the mapping $x \mapsto axb^{-1}$, in particular, $(S, a) \cong S$ through $x \mapsto ax$. In this case, a^{-1} is the identity of (S, a). Moreover, if S has a zero 0, then 0 is clearly the zero of the variant (S, a) of S for every $a \in S$.

For a nonempty set X, let $T(X), P(X)$ and $I(X)$ denote the full transformation semigroup, the partial transformation semigroup and the symmetric inverse semigroup. Notice that $T(X)$ and $I(X)$ are subsemigroups of $P(X)$. If X is a finite set containing n elements, let T_n, P_n and I_n stand for $T(X), P(X)$ and $I(X)$, respectively. For $\alpha \in P_n$ and $k \in \{1, \ldots, n\}$, let

$$\alpha_k = |\{y \in \text{ran } \alpha \mid |y\alpha^{-1}| = k\}|.$$

The n-tuple $(\alpha_1, \ldots, \alpha_n)$ is called the *type of* α. In 2003-2004, Tsyaputa [9, 10] provided the remarkable results on variants of I_n, T_n and P_n as follows : for $\alpha, \beta \in I_n, (I_n, \alpha) \cong (I_n, \beta)$ if and only if $|\text{ran } \alpha| = |\text{ran } \beta|$, for $\alpha, \beta \in T_n, (T_n, \alpha) \cong (T_n, \beta)$ if and only if α and β have the same type and this is true for variants of P_n.

We recall some notations and basic knowledge in linear algebra. Let V and W be vector spaces over a field F. Let $L_F(V, W)$ be the set of all linear transformations $\alpha : V \to W$ and let $L_F(V)$ stand for $L_F(V, V)$. Then $L_F(V)$ is a semigroup under composition. Note that 1_V, the identity mapping on V and 0_V, the zero mapping on V are the identity and the zero of the semigroup $L_F(V)$, respectively. If $\theta \in L_F(V, W)$, then $\dim_F V = \dim_F \ker\theta + \dim_F \text{ran } \theta$. We call $\dim_F \ker\theta$ and $\dim_F \text{ran } \theta$ the *nullity* and the *rank* of θ, respectively and they are denoted respectively by nullity θ and rank θ. If B is a basis of V, B' is a basis of W and $\theta \in L_F(V, W)$ is such that $\theta|_B$ is a bijection from B onto B', then θ is an isomorphism from V onto W. If B_1 is a basis of $\ker\theta$ and B a basis of V containing B_1, then $(B \smallsetminus B_1)\theta$ is a basis of ran θ and for distinct $u, v \in B \smallsetminus B_1, u\theta \neq v\theta$. In particular, if W is a subspace of V, B_1 a basis of W and B a basis of V containing B_1, then $\{v + W \mid v \in B \smallsetminus B_1\}$ is a basis of the quotient space V/W and for distinct $u, v \in B \smallsetminus B_1, u + W \neq v + W$.

It is clear that for a basis B of V,

$$|L_F(V, W)| = |\{\alpha \mid \alpha : B \to W\}| = |W|^{|B|}.$$

In particular, if V and W are finite-dimensional, F is finite, $\dim_F V = n$ and $\dim_F W = k$, then $W \cong F^k$ as vector spaces and hence

$$|L_F(V, W)| = |F^k|^n = |F|^{(\dim_F V)(\dim_F W)} < \infty.$$

For a positive integer n and a field F, let $M_n(F)$ be the multiplicative semigroup of all $n \times n$ matrices over a field F. If V is finite-dimensional and $\dim_F V = n$, then there exists a semigroup isomorphism $\varphi : L_F(V) \to M_n(F)$ which preserves ranks ([6], p. 330 and 336-337).

In this paper, we shall prove that if V is a finite-dimensional vector space over a finite field F and $\theta_1, \theta_2 \in L_F(V)$, then $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ if and only if rank $\theta_1 =$ rank θ_2. As a consequence, we have that if F is a finite field and $P_1, P_2 \in M_n(F)$, then $(M_n(F), P_1) \cong (M_n(F), P_2)$ if and only if rank $P_1 =$ rank P_2.

2 Main Result

To prove the main result, the following series of lemmas is needed.

Lemma 2.1. *Let S be a semigroup with identity and $a, b \in S$. If there exist units u, v in S such that $uav = b$, then $(S, a) \cong (S, b)$.*

Proof. Define $\varphi : S \to S$ by $x\varphi = v^{-1}xu^{-1}$ for all $x \in S$. It is evident that φ is a bijection. If $x, y \in S$, then

$$(xay)\varphi = v^{-1}xayu^{-1} = v^{-1}xu^{-1}uavv^{-1}yu^{-1} = v^{-1}xu^{-1}bv^{-1}yu^{-1} = (x\varphi)b(y\varphi).$$

Thus φ is an isomorphism from (S, a) onto (S, b). □

Lemma 2.2. *Let V be a vector space over a field F and $\theta_1, \theta_2 \in L_F(V)$. If* rank $\theta_1 =$ rank θ_2, nullity $\theta_1 =$ nullity θ_2 *and* $\dim_F(V/\text{ran } \theta_1) = \dim_F(V/\text{ran } \theta_2)$, *then there exist isomorphisms $\varphi, \psi \in L_F(V)$ such that $\varphi\theta_1\psi = \theta_2$.*

Proof. Let B_1 and B_2 be bases of $\ker\theta_1$ and $\ker\theta_2$, respectively and let $\bar{B}_1$ be a basis of V containing B_1 and $\bar{B}_2$ a basis of V containing B_2. It follows that $(\bar{B}_1 \smallsetminus B_1)\theta_1$ and $(\bar{B}_2 \smallsetminus B_2)\theta_2$ are bases of ran θ_1 and ran θ_2, respectively. We also have $|(\bar{B}_1 \smallsetminus B_1)\theta_1| = |\bar{B}_1 \smallsetminus B_1|$ and $|(\bar{B}_2 \smallsetminus B_2)\theta_2| = |\bar{B}_2 \smallsetminus B_2|$. Next let $\bar{\bar{B}}_1$ be a basis of V containing $(\bar{B}_1 \smallsetminus B_1)\theta_1$ and $\bar{\bar{B}}_2$ a basis of V containing $(\bar{B}_2 \smallsetminus B_2)\theta_2$. By assumption, $|B_1| = |B_2|$ and $|(\bar{B}_1 \smallsetminus B_1)\theta_1| = |(\bar{B}_2 \smallsetminus B_2)\theta_2|$. Therefore there exists an isomorphism $\varphi \in L_F(V)$ such that $B_2\varphi = B_1$ and $(\bar{B}_2 \smallsetminus B_2)\varphi = \bar{B}_1 \smallsetminus B_1$. Since $\dim_F(V/\text{ran } \theta_1) = \dim_F(V/\text{ran } \theta_2)$, it follows that $|\bar{\bar{B}}_1 \smallsetminus (\bar{B}_1 \smallsetminus B_1)\theta_1| = \dim_F(V/\text{ran } \theta_1) = \dim_F(V/\text{ran } \theta_2) = |\bar{\bar{B}}_2 \smallsetminus (\bar{B}_2 \smallsetminus B_2)\theta_2|$. Let $\pi : \bar{\bar{B}}_1 \smallsetminus (\bar{B}_1 \smallsetminus B_1)\theta_1 \to \bar{\bar{B}}_2 \smallsetminus (\bar{B}_2 \smallsetminus B_2)\theta_2$ be a bijection. Note that $\bar{\bar{B}}_1 = (\bar{B}_1 \smallsetminus B_1)\theta_1 \cup (\bar{\bar{B}}_1 \smallsetminus (\bar{B}_1 \smallsetminus B_1)\theta_1) = ((\bar{B}_2 \smallsetminus B_2)\varphi\theta_1) \cup (\bar{\bar{B}}_1 \smallsetminus (\bar{B}_1 \smallsetminus B_1)\theta_1)$.

Define $\psi \in L_F(V)$ on $\bar{\bar{B}}_1$ by

$$\psi = \begin{pmatrix} (u\varphi)\theta_1 & v \\ u\theta_2 & v\pi \end{pmatrix}_{\substack{u\in \bar{B}_2 \smallsetminus B_2, \\ v \in \bar{\bar{B}}_1 \smallsetminus (\bar{B}_1 \smallsetminus B_1)\theta_1}}.$$

Since $u\theta_2 \neq v\theta_2$ for distinct $u, v \in \bar{B}_2 \smallsetminus B_2$, it follows that $\psi|_{\bar{\bar{B}}_1}$ is a bijection from $\bar{\bar{B}}_1$ onto $\bar{\bar{B}}_2$. Hence ψ is an isomorphism of V. If $u \in B_2$, then $u\varphi \in B_1$, so $u\varphi\theta_1\psi = 0 = u\theta_2$. If $u \in \bar{B}_2 \smallsetminus B_2$, then by the definition of ψ, $u\varphi\theta_1\psi = u\theta_2$. Therefore we have $\varphi\theta_1\psi = \theta_2$, as desired. □

Lemma 2.3. *Let V be a finite-dimensional vector space over a field F and $\theta_1, \theta_2 \in L_F(V)$. If* rank θ_1 = rank θ_2*, then there exist isomorphisms $\varphi, \psi \in L_F(V)$ such that $\varphi\theta_1\psi = \theta_2$.*

Proof. Since $\dim_F V = \text{nullity } \theta_1 + \text{rank } \theta_1 = \text{nullity } \theta_2 + \text{rank } \theta_2$, $\dim_F V$ is finite and rank θ_1 = rank θ_2, it follows that nullity θ_1 = nullity θ_2. Also, we have $\dim_F(V/\text{ran } \theta_1) = \dim_F V - \text{rank } \theta_1 = \dim_F V - \text{rank } \theta_2 = \dim_F(V/\text{ran } \theta_2)$. Hence by Lemma 2.2, the desired result follows. □

Notice that the converse of Lemma 2.3 is clearly true.

The following lemma follows directly from Lemma 2.1 and Lemma 2.3.

Lemma 2.4. *Let V be a finite-dimensional vector space over a field F and $\theta_1, \theta_2 \in L_F(V)$. If* rank θ_1 = rank θ_2*, then $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$.*

Theorem 2.5. *Let V be a finite-dimensional vector space over a finite field F and $\theta_1, \theta_2 \in L_F(V)$. Then $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ if and only if* rank θ_1 = rank θ_2.

Proof. First assume that $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ through an isomorphism φ. Since 0_V is the zero of both $(L_F(V), \theta_1)$ and $(L_F(V), \theta_2)$, we have that $0_V\varphi = 0_V$. We claim that $\alpha\theta_1 = \beta\theta_1$ if and only if $(\alpha\varphi)\theta_2 = (\beta\varphi)\theta_2$ for all $\alpha, \beta \in L_F(V)$. Let $\alpha, \beta \in L_F(V)$ and assume that $\alpha\theta_1 = \beta\theta_1$. Then $\alpha\theta_1\lambda = \beta\theta_1\lambda$ for all $\lambda \in L_F(V)$, it follows that $(\alpha\varphi)\theta_2(\lambda\varphi) = (\beta\varphi)\theta_2(\lambda\varphi)$ for all $\lambda \in L_F(V)$. Since $(L_F(V))\varphi = L_F(V)$, we have $(\alpha\varphi)\theta_2 = (\alpha\varphi)\theta_2 1_V = (\beta\varphi)\theta_2 1_V = (\beta\varphi)\theta_2$. But since φ^{-1} is an isomorphism from $(L_F(V), \theta_2)$ onto $(L_F(V), \theta_1)$, if $(\alpha\varphi)\theta_2 = (\beta\varphi)\theta_2$, then from the above proof we have similarly that $(\alpha\varphi)\varphi^{-1}\theta_1 = (\beta\varphi)\varphi^{-1}\theta_1$, i.e., $\alpha\theta_1 = \beta\theta_1$. Therefore we prove that $\alpha\theta_1 = \beta\theta_1$ if and only if $(\alpha\varphi)\theta_2 = (\beta\varphi)\theta_2$. In particular, if $\beta = 0_V$, then $\alpha\theta_1 = 0_V$ if and only if $(\alpha\varphi)\theta_2 = 0_V$. This proves that for every $\alpha \in L_F(V), \alpha\theta_1 = 0_V$ if and only if $(\alpha\varphi)\theta_2 = 0_V$. It follows that ran $\alpha \subseteq \ker\theta_1$ if and only if ran $\alpha\varphi \subseteq \ker\theta_2$ for all $\alpha \in L_F(V)$. This proves that $(L_F(V, \ker\theta_1))\varphi = L_F(V, \ker\theta_2)$. Consequently, $|L_F(V, \ker\theta_1)| = |L_F(V, \ker\theta_2)|$. As mentioned in Section 1, $|L_F(V, \ker\theta_1)| = |F|^{(\dim_F V)(\text{nullity } \theta_1)}$ and $|L_F(V, \ker\theta_2)| = |F|^{(\dim_F V)(\text{nullity } \theta_2)}$. It follows that nullity θ_1 = nullity θ_2. Hence rank $\theta_1 = \dim_F V -$ nullity $\theta_1 = \dim_F V -$ nullity θ_2 = rank θ_2.

The converse follows directly from Lemma 2.4.

The proof is thereby completed. □

Corollary 2.6. *Let F be a finite field, n a positive integer and $P_1, P_2 \in M_n(F)$. Then $(M_n(F), P_1) \cong (M_n(F), P_2)$ if and only if* rank P_1 = rank P_2.

Proof. Let V be a vector space over F of dimension n. Then there exists a semigroup isomorphism $\varphi : L_F(V) \to M_n(F)$ which preserves ranks. Let $\theta_1, \theta_2 \in L_F(V)$ be such that $\theta_1\varphi = P_1$ and $\theta_2\varphi = P_2$. Then for all $\alpha, \beta \in L_F(V)$,

$$(\alpha\theta_1\beta)\varphi = (\alpha\varphi)P_1(\beta\varphi) \text{ and } (\alpha\theta_2\beta)\varphi = (\alpha\varphi)P_2(\beta\varphi).$$

Since $\varphi : L_F(V) \to M_n(F)$ is a bijection, it follows from the above equalities that φ is an isomorphism from $(L_F(V), \theta_1)$ onto $(M_n(F), P_1)$ and an isomorphism from $(L_F(V), \theta_2)$ onto $(M_n(F), P_2)$, i.e., $(L_F(V), \theta_1) \cong (M_n(F), P_1)$ and $(L_F(V), \theta_2) \cong (M_n(F), P_2)$.

First assume that $(M_n(F), P_1) \cong (M_n(F), P_2)$. This implies that $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$. By Theorem 2.5, rank θ_1 = rank θ_2. Since φ preserves ranks, it follows that rank P_1 = rank P_2.

Conversely, assume that rank P_1 = rank P_2. Then rank θ_1 = rank θ_2 since φ preserves ranks. By Theorem 2.5, $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$. Consequently, $(M_n(F), P_1) \cong (M_n(F), P_2)$. □

Remark 2.7. From Lemma 2.4 and the proof of Corollary 2.6 we can see that the following result holds : if F is a field (need not be finite), n a positive integer and $P_1, P_2 \in M_n(F)$ are such that rank P_1 = rank P_2, then $(M_n(F), P_1) \cong (M_n(F), P_2)$. In fact, the following result can be referred from Lemma 2.1 and the fact that if $P_1, P_2 \in M_n(F)$ are such that rank P_1 = rank P_2, then P_1 is equivalent to P_2, i.e., $P_1 = Q_1P_2Q_2$ for some invertible matrices Q_1, Q_2 in $M_n(F)$ ([6], p. 338).

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