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Isomorphisms of abelian group algebras

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Let G and H be locally compact groups with group algebras $\boldsymbol{L}(G)$ and $\boldsymbol{L}(H)$ respectively. If G and H are isomorphic groups, the correspondence between points of G and H given by the isomorphism induces an isomorphism of the group algebras. The purpose of this paper is to make a contribution to the converse question: assuming that $\boldsymbol{L}(G)$ and $\boldsymbol{L}(H)$ are isomorphic algebras, under what conditions can it be asserted that the underlying groups are isomorphic?

As a case of the problem, one can ask when an automorphism of a single group algebra is induced by an automorphism of the underlying group. A discussion appeared in the author's thesis (Harvard, 1950)[1], where the group was assumed to be commutative. The principal result there stated was Theorem 3 of this paper, except that only automorphisms were considered. Further information is hard to obtain even for simple groups; for example, it is not known whether there are any automorphisms of the algebra on the line, except a few obvious and trivial ones.

At a late stage of this work I learned that J. WENDEL has obtained Theorem 3 (for an operator assumed to be isometric) even for non-abelian group algebras. More recently he has established all of Theorem 3. While his methods and mine undoubtedly are related, it is difficult to make direct comparisons because of the complexity of the general case. I am grateful to Dr. WENDEL for correspondence about the problem, and for a summary of [7] before it appeared in print.

We shall consider only abelian groups, where the Fourier transform is a convenient tool. Our main result, Theorem 4, asserts the following: If T is an operator mapping $\boldsymbol{L}(G)$ isomorphically onto $\boldsymbol{L}(H)$ *with bound less than two*, where G and H are locally compact abelian groups, and if the dual group of G or of H is connected, then G and H are isomorphic groups, and T is the natural isomorphism of algebras induced by the group isomorphism. Since Theorem 3 is a corollary of the more complicated methods used here, we are not giving its original proof.

The argument used to prove Theorem 4 is a modification of a proof of A. BEURLING for the following theorem: if for each real λ

$$e^{i\lambda\varphi(x)}$$

[1] Most of the content of §§ 1 and 2 and Theorem 3 appeared in my thesis. I am pleased to record my indebtedness to Professor L. H. LOOMIS, who directed the thesis. The other theorems are generalizations of unpublished results of Professor A. BEURLING, to whom I am obliged for much advice, and for permission to publish these theorems depending essentially on his work.

is the Fourier-Stieltjes transform of a measure on the line having total variation at most two, then φ is a linear function. A corollary of BEURLING's theorem and the preliminary sections of this paper is that any automorphism of the group algebra of the line has bound greater than two (as a linear operator), unless it is of a trivial type.

In the first section a characterization is given of functions which are Fourier-Stieltjes transforms of bounded measures. This theorem is used in the second section to extend an isomorphism of group algebras to an isomorphism of the corresponding algebras of bounded measures. In the third section more analysis is performed on group algebras, extending theorems known on the line. The results of the paper are proved in the fourth section, and some examples and comments are collected in the final section.

§ 1. Fourier-Stieltjes transforms

Let G be a locally compact group (always taken abelian) with dual group $\hat{G}$, and let dx and $d\hat{x}$ be the Haar measures on G and $\hat{G}$ respectively. If μ is a measure of Radon on G (hereafter called simply a bounded measure), its Fourier-Stieltjes transform is given by the formula

$$\hat{\mu}(\hat{x}) = \int_G \overline{(x, \hat{x})}\, d\mu(x).$$

Then $\hat{\mu}$ is a bounded continuous function defined on $\hat{G}$. The transform is a linear operation, and the convolution of two measures is carried onto the ordinary product of their transforms. The set of Fourier-Stieltjes transforms is thus an algebra under the pointwise operations and multiplication by complex scalars. Call this algebra $\boldsymbol{B}(\hat{G})$. A norm is introduced into the algebra of bounded measures, or their transforms, by defining

$$\|\hat{\mu}\| = \|\mu\| = \sup \left|\int_G \varphi(x)\, d\mu(x)\right|$$

over measurable functions φ with essential bound one. The total mass of μ in a set E is defined by the same formula, where φ is required to vanish outside E.

The functions summable on G for Haar measure are an algebra $\boldsymbol{L}(G)$ under addition and convolution. Since the convolution of a function and a measure is again a function, $\boldsymbol{L}(G)$ is an ideal in the algebra of bounded measures. We denote the family of Fourier transforms of summable functions by $\boldsymbol{F}(\hat{G})$; then $\boldsymbol{F}(\hat{G})$ is a Banach algebra in the norm inherited from $\boldsymbol{L}(G)$, and is an ideal in $\boldsymbol{B}(\hat{G})$ by the previous remark. We are going to show that $\boldsymbol{F}(\hat{G})$ is not an ideal in any larger ring of functions. In other words, if a function $\hat{\varphi}$ defined on $\hat{G}$ has the property that $\hat{f}\cdot\hat{\varphi} \in \boldsymbol{F}(\hat{G})$ for every $\hat{f} \in \boldsymbol{F}(\hat{G})$, then $\hat{\varphi}$ is itself an element of $\boldsymbol{B}(\hat{G})$.

Theorem 1. Let $\hat{\varphi}$ be a function defined on $\hat{G}$. A necessary and sufficient condition for $\hat{\varphi}$ to be the Fourier-Stieltjes transform of a bounded measure on G is that $\hat{f}\cdot\hat{\varphi} \in \boldsymbol{F}(\hat{G})$ for every $\hat{f} \in \boldsymbol{F}(\hat{G})$.

We only have to consider the sufficiency of the condition. Define an operator U in $\boldsymbol{L}(G)$ by setting

$$\widehat{Uf} = \hat{f} \cdot \hat{\varphi};$$

that is, Uf is the unique summable function whose transform is $\hat{f} \cdot \hat{\varphi}$. U is evidently linear. To show it is continuous, suppose that f_n converges to f, and Uf_n converges to g in $\boldsymbol{L}(G)$. According to the closed graph theorem ([1], p. 41), it is enough to show that $Uf = g$. Now the convergence of summable functions carries with it the uniform convergence of the Fourier transforms, so we have:

$$\widehat{Uf}_n = \hat{f}_n \cdot \hat{\varphi} \text{ converges to } \hat{g},$$

$$\hat{f}_n \text{ converges to } \hat{f}.$$

The second assertion implies that

$$\hat{f}_n \cdot \hat{\varphi} \text{ converges to } \hat{f} \cdot \hat{\varphi}.$$

Hence $\hat{f} \cdot \hat{\varphi} = \hat{g}$, or $Uf = g$.

Let $\{e_a\}$ be a directed system of functions running through an approximate identity for $\boldsymbol{L}(G)$, and set

$$\hat{\varphi}_a = \hat{e}_a \cdot \hat{\varphi} = \widehat{Ue}_a.$$

Then by hypothesis each $\hat{\varphi}_a \in \boldsymbol{F}(\hat{G})$, and the directed system converges to $\hat{\varphi}$ uniformly on compact sets. Define a linear functional F in $\boldsymbol{L}(\hat{G})$, the algebra of functions summable on $\hat{G}$:

$$F(\hat{h}) = \int_{\hat{G}} \hat{\varphi}(\hat{x}) \hat{h}(\hat{x}^{-1}) \, d\hat{x}$$

for any summable $\hat{h}$ with Fourier transform h. Then we have

$$\begin{aligned} F(\hat{h}) &= \lim_a \int_{\hat{G}} \hat{\varphi}_a(\hat{x}) \hat{h}(\hat{x}^{-1}) \, d\hat{x} \\ &= \lim_a \int_{\hat{G}} \int_G \overline{(x, \hat{x})} \, U e_a(x) \hat{h}(\hat{x}^{-1}) \, dx \, d\hat{x} \\ &= \lim_a \int_G h(x) \, U e_a(x) \, dx. \end{aligned}$$

Hence

$$|F(\hat{h})| \leq \|U e_a\| \cdot \|h\|_\infty \leq \|U\| \cdot \|h\|_\infty,$$

since U is bounded and each function e_a has norm one.

Of course F is a continuous functional on $\boldsymbol{L}(\hat{G})$ for the norm of that space. We have just proved the stronger fact that F is continuous with respect to the uniform norm of Fourier transforms. A theorem of BOCHNER [2] and SCHOENBERG [6] asserts that this is sufficient to assure the existence of a bounded measure of norm not exceeding $\|U\|$ whose transform is $\hat{\varphi}$, and this completes the proof of the theorem. We remark that Bochner and Schoenberg state their theorem for the real line, but Schoenberg's formulation and proof apply to arbitrary groups.

Theorem 1 has been proved frequently in special cases ([3], [4], and [9]). Dr. E. Akutowicz suggested to me the desirability of finding a proof on arbitrary abelian groups.

§ 2. Isomorphisms of group algebras

Let G and H be given groups and let T be a one-one mapping of $\boldsymbol{L}(G)$ onto $\boldsymbol{L}(H)$ such that

$$T(f+g) = Tf + Tg$$
$$T(f * g) = Tf * Tg$$
$$T(\lambda f) \quad = \lambda Tf$$

for any $f, g \in \boldsymbol{L}(G)$ and any complex scalar λ. T is called an *isomorphism* of $\boldsymbol{L}(G)$ onto $\boldsymbol{L}(H)$. If G and H are the same group, T is an *automorphism* of $\boldsymbol{L}(G)$. Since the topology of a commutative semi-simple Banach algebra is uniquely determined, T and its inverse must be continuous. Evidently the inverse of T is an isomorphism of $\boldsymbol{L}(H)$ onto $\boldsymbol{L}(G)$.

The operator T carries the algebraic structure of $\boldsymbol{L}(G)$ onto that of $\boldsymbol{L}(H)$, and in particular maps regular maximal ideals onto ideals of the same kind. Identifying the regular maximal ideals of an algebra with points of the dual group, we have defined a mapping τ of $\hat{G}$ onto $\hat{H}$.

Lemma 1. τ is a homeomorphism of $\hat{G}$ onto $\hat{H}$.

It is trivial to verify that τ is a one-one mapping of $\hat{G}$ onto $\hat{H}$. To establish its bicontinuity, recall that a set of regular maximal ideals is closed (that is, the corresponding set of points in the dual group is closed) just if the intersection of the ideals in the set is contained in no other regular maximal ideal. Since this property is preserved under τ and its inverse, τ is bicontinuous.

The lemma states in particular that if the algebras of two groups are isomorphic, then the dual groups are homeomorphic. The homeomorphisms which can arise in this way are not further described, and this is the crux of the problem, but one interesting deduction can be made immediately. Suppose the algebra $\boldsymbol{L}(R)$ of the line is isomorphic to $\boldsymbol{L}(H)$ for some group H. Then $\hat{H}$ is homeomorphic to $\hat{R}$, and so $\hat{H}$ is the line provided with another group operation. But the group operation in R is unique, in the sense that if $*$ is a continuous group operation in R, then there is a homeomorphism φ of R onto itself such that

$$x * y = \varphi^{-1}[\varphi(x) + \varphi(y)]$$

for all numbers x and y. So $\hat{R}$ and $\hat{H}$ are isomorphic groups, and R and H must be isomorphic too. Hence *the real line is characterized by its group algebra.* However, it does not follow from these considerations that an automorphism of $\boldsymbol{L}(R)$ is the natural one induced by a point mapping of R onto itself. This will not be the case unless the homeomorphism τ is multiplicative.

Lemma 2. For any $f \in \boldsymbol{L}(G)$ and all $\hat{x} \in \hat{G}$,

$$\widehat{Tf}(\tau \hat{x}) = \hat{f}(\hat{x}).$$

That is, if the operator T is looked at as a mapping of $\boldsymbol{F}(\hat{G})$ onto $\boldsymbol{F}(\hat{H})$, it is determined in the natural way by the point mapping τ. If $\hat{\varphi}$ is any function defined on $\hat{G}$, let $\hat{\varphi}^{\tau}$ be the function on $\hat{H}$ given by the formula

$$\hat{\varphi}^{\tau}(\tau \hat{x}) = \hat{\varphi}(\hat{x}).$$

Then the lemma states that

$$\widehat{Tf} = \hat{f}^{\tau}.$$

Since T is homogeneous we only have to show that

$$\hat{f}(\hat{x}) = 0 \text{ is equivalent to } \widehat{Tf}(\tau \hat{x}) = 0,$$

and

$$\hat{f}(\hat{x}) = 1 \text{ is equivalent to } \widehat{Tf}(\tau \hat{x}) = 1.$$

The first fact is contained in the definition of τ as a mapping of maximal ideals. The second fact is reduced to the first by the observation that $\hat{f}(\hat{x}) = 1$ just if the transform of $f * g - g$ vanishes at $\hat{x}$ for each summable g.

Lemma 3. A necessary and sufficient condition for a function $\hat{\psi}$ defined on $\hat{H}$ to belong to $\boldsymbol{B}(\hat{H})$ is that $\hat{\psi} = \hat{\varphi}^{\tau}$ for some $\hat{\varphi} \in \boldsymbol{B}(\hat{G})$.

In other words, the mapping of functions induced by τ, which carries $\boldsymbol{F}(\hat{G})$ onto $\boldsymbol{F}(\hat{H})$ by the last lemma, actually carries $\boldsymbol{B}(\hat{G})$ onto $\boldsymbol{B}(\hat{H})$ as well.

Suppose $\hat{\varphi} \in \boldsymbol{B}(\hat{G})$, and let $\hat{f}$ be an arbitrary function of $\boldsymbol{F}(\hat{G})$. Then

$$\hat{f} \cdot \hat{\varphi} \in \boldsymbol{F}(\hat{G}),$$

so that

$$\hat{f}^{\tau} \cdot \hat{\varphi}^{\tau} = (\hat{f} \cdot \hat{\varphi})^{\tau} \in \boldsymbol{F}(\hat{H}).$$

Since $\hat{f}^{\tau}$ represents an arbitrary function of $\boldsymbol{F}(\hat{H})$ by Lemma 2, $\hat{\varphi}^{\tau} \in \boldsymbol{B}(\hat{H})$ by Theorem 1. The same argument applied to the inverse of T shows that all of $\boldsymbol{B}(\hat{H})$ is so covered, completing the proof.

The mapping thus defined of $\boldsymbol{B}(\hat{G})$ onto $\boldsymbol{B}(\hat{H})$ is an isomorphism of these algebras, which can be interpreted as an isomorphism of the algebra of measures on G onto the measures on H. This operator extends T, and we shall use the same letter to denote it. We want to show that the norm of T is not increased by the extension.

According to the theorem of Bochner and Schoenberg used in the proof of Theorem 1, $\|\hat{\mu}^{\tau}\|$ is the smallest constant M satisfying the inequality

$$\left| \int_{\hat{H}} \hat{\mu}^{\tau}(\hat{x}) \hat{h}(\hat{x}^{-1}) \, d\hat{x} \right| \leq M \| h \|_{\infty}$$

for every summable function $\hat{h}$ on $\hat{H}$, with transform h. If $\{e_\alpha\}$ is an approximate identity for $\boldsymbol{L}(G)$, $\hat{e}_\alpha = \widehat{Te^\tau_\alpha}$ converges to one uniformly on compact sets in $\hat{H}$ (using the fact that τ is a homeomorphism), and so the left side of the inequality is the limit in α of

$$\left|\int_{\hat{H}} \hat{e}^\tau_\alpha(\hat{x})\,\hat{\mu}^\tau(\hat{x})\,\hat{h}(\hat{x}^{-1})\,d\hat{x}\right|.$$

For any α this is not greater than $\|h\|_\infty$ multiplied by

$$\|\hat{e}^\tau_\alpha\cdot\hat{\mu}^\tau\| = \|T(e_\alpha * \mu)\| \leq \|T\|\cdot\|e_\alpha * \mu\| \leq \|T\|\cdot\|\mu\|.$$

So the constant M can be taken as $\|T\|\cdot\|\mu\|$, and consequently

$$\|\hat{\mu}^\tau\| = \|T\mu\| \leq \|T\|\cdot\|\mu\|.$$

Here $\|T\|$ refers to the norm of T defined in $\boldsymbol{L}(G)$, and so this inequality proves that T has the same bound in the wider space of bounded measures.

We digress to show how T can be extended in another way. By its definition, τ maps the characters of G onto those of H. Define an operator T^* mapping trigonometric polynomials on H into polynomials on G by setting

$$T^*\left(\sum_{i=1}^{n} c_i\,\tau\hat{x}_i\right) = \sum_{i=1}^{n} c_i\,\hat{x}_i\,.$$

Lemma 2 expresses the fact that for any trigonometric polynomial P on H and $h\in\boldsymbol{L}(G)$,

$$\int_H P(y)\,Th(y^{-1})\,dy = \int_G T^*P(x)\,h(x^{-1})\,dx.$$

Hence T^*, where it is defined, is the operator adjoint to T, mapping the conjugate space of $\boldsymbol{L}(H)$ onto the conjugate space of $\boldsymbol{L}(G)$. It is well-known (and easily verified) that the norm of the adjoint of an operator is equal to the norm of the operator itself. So if a sequence of polynomials P_n converges uniformly on H, the sequence T^*P_n converges uniformly on G, and in both cases the limit function is almost-periodic. We obtain in this way an operator T^* mapping the almost-periodic functions of H onto those of G. T^* is linear with bound $\|T\|$, and furthermore distributes over the convolution operation defined for almost-periodic functions. This isomorphism of the spaces of almost-periodic functions on the groups seems to embody the essential features of the situation, but I have not been able to deduce anything from it.

§ 3. Mean values

The isomorphism of $\boldsymbol{B}(\hat{G})$ onto $\boldsymbol{B}(\hat{H})$ which exists by Lemma 3 carries a point mass on G into some measure on H, whose Fourier-Stieltjes transform must have absolute value one at every point. The object of this section is to prove Theorem 2 below, which provides information about measures of this type.

If μ is a bounded measure on G, at most a denumerable number of points have positive mass. So μ has a unique decomposition

$$\mu = \sum_{i=1}^{\infty} a_i\,\mu_{p_i} + \nu,$$

where the a_i are complex constants, μ_{p_i} is the measure having unit mass at the point p_i, and ν is a measure vanishing on single points; the sum of the absolute values of the coefficients converges since μ is finite. We call a sum of point masses a discrete measure, and ν a continuous measure.

Theorem 2. If the Fourier-Stieltjes transform of μ has absolute value one at every point, then $\sum_{i=1}^{\infty} |a_i|^2 = 1$.

This remarkable fact was found by Professor BEURLING on the real line; it is related to a theorem of WIENER ([8], p. 146). The idea of the proof is that the transform of the continuous part of μ is small in mean value:

$$\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} |\hat{\nu}(\hat{x})|^2 d\hat{x} = 0,$$

and so the discrete part cannot be small. In generalizing to arbitrary groups we have to describe this entirely on G, for want of a suitable averaging process on $\hat{G}$. This is accomplished by the following lemma.

Lemma 4. Let $\{V_a\}$ be a fundamental system of neighborhoods of the identity in G. Let e_a be the function equal to $1/\sqrt{m V_a}$ on V_a, and zero elsewhere ($m V_a$ is the Haar measure of V_a). Then for any bounded measure ν, $e_a * \nu$ exists almost everywhere and belongs to $L^2(G)$. If ν is a continuous measure, then $e_a * \nu$ converges to zero in $L^2(G)$.

The last assertion contains the force of the lemma, and of course is trivial in the case of a discrete group, where every measure is discrete. We exclude this case. Then there are neighborhoods of the identity with Haar measure as small as we please; and if ν is continuous, there are neighborhoods of the identity containing as little of the total mass of ν as we please. For Haar measure and ν are regular measures.

The norm of e_a is constantly one in $L^2(G)$, but the system tends to zero in the norm of $L(G)$. Thus it is immediate that $e_a * \nu$ tends to zero in $L(G)$, whether or not ν is continuous. In $L^2(G)$ the situation is more delicate.

Each e_a is summable, so $e_a * \nu$ exists almost everywhere. By the Plancherel theorem for square-summable functions, the Fourier transform $\hat{e}_a$ of e_a belongs to $L^2(G)$, and so $\hat{e}_a \cdot \hat{\nu}$ is also square-summable. Since this function is the transform of $e_a * \nu$, the convolution belongs to $L^2(G)$, proving the first part of the lemma.

For any function of G, define

$$\tilde{f}(x) = \overline{f(x^{-1})},$$

with an analogous definition for measures. If $f \in L^2(G)$,

$$\|f\|_2^2 = f * \tilde{f}(0).$$

We have to show that

$$\|e_a * \nu\|_2^2 = e_a * \tilde{e}_a * \nu * \tilde{\nu}(0)$$

converges to zero in α.

Let us set

$$f_\alpha(x) = e_\alpha * \tilde{e}_\alpha(x) = \int_G e_\alpha(y)\, e_\alpha(x^{-1}y)\, dy\,;$$

then $f_\alpha(0) = 1$, $0 \leq f_\alpha(x) \leq 1$ everywhere, and $f_\alpha(x) = 0$ outside $V_\alpha^{-1} V_\alpha$. Let π be the measure $\nu * \tilde{\nu}$. The expression for the squared norm above becomes

$$f_\alpha * \pi(0) = \int_G f_\alpha(y^{-1})\, d\pi(y).$$

Taking account of the properties of f_α mentioned, the absolute value of the integral cannot exceed the total mass of π in $V_\alpha^{-1} V_\alpha$. Now π is a measure of Radon which, like ν, vanishes on points. So we can find neighborhoods of the identity containing as little mass as we please, and it follows that the absolute value of the integral converges to zero in α. This finishes the proof of the lemma.

We proceed to the theorem itself, retaining the notation already introduced. Having proved our lemma, we can use the idea of BEURLING's proof on the line. The transform $\hat{\mu}$ of μ is assumed to have absolute value one at every point. Hence for any $\hat{h} \in \boldsymbol{L}^2(\hat{G})$ with transform h,

$$\|\hat{\mu} \cdot \hat{h}\|_2 = \|\hat{h}\|_2,$$

which implies

$$\|\mu * h\|_2 = \|h\|_2.$$

If G is discrete, take h to be the function equal to one at the identity and zero elsewhere. Then $\mu * h$ represents the same distribution of mass as μ itself, and is a function of $\boldsymbol{L}^2(G)$ having norm one. Hence

$$\sum_{i=1}^{\infty} |a_i|^2 = 1,$$

and the theorem holds in this case. We assume now that G is not discrete.

Write μ as the sum of its discrete and continuous parts:

$$\mu = \mu_0 + \nu.$$

If $\{e_\alpha\}$ is the directed system of the lemma,

$$\|e_\alpha * \mu_0 + e_\alpha * \nu\|_2 = \|e_\alpha * \mu\|_2 = \|e_\alpha\|_2 = 1;$$

at the same time

$$\|e_\alpha * \nu\|_2$$

converges to zero by the lemma. Hence

$$\|e_\alpha * \mu_0\|_2$$

approaches one.

Dropping the convergence index on e_α, let e_p be the translate of e by a group element p:

$$e_p(x) = e(p^{-1}x).$$

Then

$$e * \mu_0 = \sum_{i=1}^{\infty} a_i e_{p_i}.$$

Given an integer N, we can find α so high that the sets $p_i V_\alpha$ are disjoint for $i = 1, \ldots, N$. For such α we have

$$\| e * \mu_0 \|_2 \leq \left\| \sum_{i=1}^{N} a_i e_{p_i} \right\|_2 + \left\| \sum_{i=N+1}^{\infty} a_i e_{p_i} \right\|_2 =$$

$$= \left[\sum_{i=1}^{N} |a_i|^2 \right]^{1/2} + \left\| \sum_{i=N+1}^{\infty} a_i e_{p_i} \right\|_2 \leq \left[\sum_{i=1}^{N} |a_i|^2 \right]^{1/2} + \sum_{i=N+1}^{\infty} |a_i|.$$

Since the series

$$\sum_{i=1}^{\infty} |a_i|$$

converges, the last term above converges to

$$\left[\sum_{i=1}^{\infty} |a_i|^2 \right]^{1/2}.$$

At the same time the norm on the left converges to one, so that

$$1 \leq \left[\sum_{i=1}^{\infty} |a_i|^2 \right]^{1/2}.$$

The opposite inequality is obtained from the same calculations. In the chain of inequalities above we can reverse the sense of each sign, provided the second term is subtracted in each case instead of added. Then the limiting process is carried through as before. So the theorem is proved.

§ 4. Isomorphisms with prescribed bounds

T is an isomorphism of $\boldsymbol{L}(G)$ onto $\boldsymbol{L}(H)$. In this section we shall give two conditions on the bound of T which are sufficient to prove that G and H are isomorphic, and that T is the natural isomorphism induced by the isomorphism of the groups. The first condition is that T have bound one, and applies to any abelian algebras. Our second condition is that the bound of T be smaller than two, and applies to groups whose duals are connected. The chain of argument leading from Theorem 2 to this result is inspired by BEURLING's theorem for the line, although the generalization to groups requires some change in detail.

T has been extended to be an isomorphism of the algebra of bounded measures on G onto the corresponding algebra on H, and the norm of T is not increased by the extension. The unit mass at a point p of G is carried by T into a measure $\mu(p)$ on H, and for the Fourier-Stieltjes transforms we have the relation

$$\hat{\mu}(\tau \hat{x}) = \overline{(p, \hat{x})}.$$

So $\hat{\mu}$ has absolute value one at every point. Then by Theorem 2, μ has a unique decomposition

$$\mu = \mu_0 + \nu$$

where ν is a continuous measure, and μ_0 is discrete:

$$\mu_0 = \sum_{i=1}^{\infty} a_i \mu_{q_i}, \quad \sum_{i=1}^{\infty} |a_i|^2 = 1.$$

Let a be one of the coefficients having largest absolute value among all the a_i. Then we have

$$1 \leq |a| \sum_{i=1}^{\infty} |a_i| \leq |a| \cdot \|\mu\|.$$

Now μ is the image under T of a unit measure, and so has norm no larger than $\|T\|$. The last inequality then gives

$$1/\|T\| \leq |a|.$$

We shall use this inequality in the proofs of both theorems in this section.

Theorem 3. Suppose T is a norm-decreasing isomorphism of $\boldsymbol{L}(G)$ onto $\boldsymbol{L}(H)$. Then G and H are isomorphic groups, T is isometric, and T has the following representation: there is an isomorphism γ of G onto H, a fixed character $\hat{y}$ of H, and a constant k depending on the choice of Haar measure in H, such that

$$Tf = k \cdot \hat{y} \cdot f^{\gamma} \qquad (\text{all } f \in \boldsymbol{L}(G)).$$

Since the bound of T is at most one, the inequality above implies that $|a| = 1$. Hence $\mu(p)$ is a point mass, with the continuous component ν vanishing altogether. Let the mass of $\mu(p)$ be concentrated at a point q of H, and let $\varrho(p)$ be the coefficient a, having absolute value one. In terms of Fourier-Stieltjes transforms, we have

$$\overline{(p, \hat{x})} = \varrho(p)\, \overline{(q, \tau\hat{x})}.$$

If τ maps the origin of $\hat{G}$ onto that of $\hat{H}$, then $\varrho(p)$ is one for all p. We assume for the present that this is the case. Then τ carries the characters on $\hat{G}$ into certain characters on $\hat{H}$; a character χ on $\hat{G}$ is carried into χ^{τ}. For arbitrary $\hat{x}$, $\hat{y} \in \hat{G}$ we have

$$\chi[\tau^{-1}(\tau\hat{x}\,\tau\hat{y})] = \chi^{\tau}(\tau\hat{x}\,\tau\hat{y}) = \chi^{\tau}(\tau\hat{x})\chi^{\tau}(\tau\hat{y}) = \chi(\hat{x})\chi(\hat{y}).$$

Since χ was an arbitrary character on $\hat{G}$,

$$\tau(\hat{x}\hat{y}) = \tau(\hat{x})\tau(\hat{y}).$$

τ is known to be a homeomorphism of $\hat{G}$ onto $\hat{H}$, and has just been shown multiplicative. Hence τ is an isomorphism of $\hat{G}$ onto $\hat{H}$. It follows that G and H are isomorphic groups, but there is still some computation to show how to describe T in terms of an isomorphism.

Define a mapping δ of H into G by the formula

$$(\delta y, \hat{x}) = (y, \tau\hat{x})$$

where $\hat{x}$ and y belong to $\hat{G}$ and to H respectively. We omit the verification that δ is one-one onto and bicontinuous, and so is an isomorphism of H onto G. Now according to Lemma 2, for any $f \in \boldsymbol{L}(G)$ and $\hat{x} \in \hat{G}$,

$$\int_G (x, \hat{x})\, f(x)\, dx = \int_H T f(y)\, (y, \tau \hat{x})\, dy.$$

Set $x = \delta y$, and let k be the uniquely determined constant such that

$$dx = k\, d\delta x.$$

Then the integral on the right becomes

$$\int_H T f^{\delta}(\delta y)\, (\delta y, \hat{x})\, dy = \frac{1}{k} \int_G T f^{\delta}(x)\, (x, \hat{x})\, dx.$$

The functions f and $T f^{\delta}/k$ thus have the same Fourier transforms, and must be identical. If γ is the inverse of δ, we have proved

$$T f = k f^{\gamma},$$

under the assumption that τ maps the identity of $\hat{G}$ onto that of $\hat{H}$.

In the general case, consider the isomorphism T_0 defined by

$$T_0 f = \hat{y}^{-1} \cdot T f$$

for an element $\hat{y}$ of $\hat{H}$. The Fourier transform of $T_0 f$ is just the transform of $T f$ translated by the fixed group element $\hat{y}$. So for a suitable choice of $\hat{y}$, the homeomorphism τ_0 associated with T_0 maps the identity of $\hat{G}$ onto that of $\hat{H}$. Applying the previous case,

$$T f = \hat{y} \cdot T_0 f = k \cdot \hat{y} \cdot f^{\gamma}.$$

The proof that G and H are isomorphic still applies, and T is isometric by the representation formula. This completes the proof of the theorem.

The original proof in the author's thesis did not appeal to Theorem 2, and so was more elementary than that presented here.

Theorem 4. Suppose that T is an isomorphism of $\boldsymbol{L}(G)$ onto $\boldsymbol{L}(H)$ with bound less than two; moreover assume at least one of $\hat{G}$ and $\hat{H}$ is connected. Then the conclusion of Theorem 3 holds.[1]

We begin again with the decomposition $\mu(p) = \mu_0 + \nu$; or taking transforms,

$$\hat{\mu}(\hat{x}) = \sum_{i=1}^{\infty} a_i \overline{q_i(\hat{x})} + \hat{\nu}(\hat{x}).$$

If q is that q_i belonging to the coefficient a,

$$|\hat{\mu}(\hat{x}) - a \overline{q(\hat{x})}| \leq \|\mu - a \mu_q\| \leq \|T\| - |a| \leq 3/2 - \varepsilon,$$

[1] Dr. WENDEL has informed me of two non-isomorphic finite groups, whose algebras are isomorphic under an operator of bound smaller than two. Thus some hypothesis such as connectedness is necessary for the truth of the theorem.

for some fixed number $\varepsilon > 0$ and all $\hat{x} \in \hat{H}$. For we know that

$$|a| \geq 1/\|T\| > 1/2.$$

Equivalently, there is a number a of modulus greater than one-half and a character q on $\hat{H}$ such that

$$|\hat{\mu}(\hat{x})\, q(\hat{x}) - a| \leq 3/2 - \varepsilon$$

for all $\hat{x} \in \hat{H}$. Since the values of the product function always lie on the unit circle, this inequality shows that $\hat{\mu}(\hat{x})\, q(\hat{x})$ never takes values in a certain arc of positive length on the circle. Of course μ, q, and a all depend on $p \in G$.

One of $\hat{G}$ and $\hat{H}$ is assumed to be connected; since the spaces are homeomorphic, both are connected. A theorem of MACKEY [5] states that the union of the images of one-parameter subgroups in a connected group is dense. Applying this to $\hat{H}$, if r is a non-constant character on $\hat{H}$, there is a one-parameter subgroup Δ on which r is not constant. Define a function r^* on the line by setting

$$r^*(\lambda) = r(\Delta(\lambda)) \qquad (\lambda \in R).$$

Evidently r^* is a continuous non-constant character of the line, and so winds around the circle infinitely often as λ increases indefinitely.

In the same way define $\hat{\eta}^*$, a continuous function on the line into the circle. For each subgroup Δ, the product $\hat{\mu}^* q^*$ leaves an arc of the circle uncovered. Let r be any non-trivial character on $\hat{H}$, and choose a subgroup Δ on which r is not constant. Since $\hat{\mu}^* q^*$ is continuous, $\hat{\mu}^* q^* r^*$ winds around the circle infinitely often. This shows that q is uniquely determined by μ. In fact, for any character s, either $\hat{\mu} \cdot s$ fails to cover the circle (as is the case for $s = q$), or else there is a one-parameter subgroup such that $\hat{\mu}^* s^*$ winds around the circle infinitely often.

Since T is an isomorphism of measures, the unit mass at the point $p\,p'$ of G is carried into $\mu(p) * \mu(p')$. So $\hat{\mu}^2$ is the transform of the measure on H associated with the point p^2. We want to show that the corresponding character is just q^2. Indeed, since $\hat{\mu} \cdot q$ does not cover the circle, at least $(\hat{\mu}^* q^*)^2$ does not wind around the circle indefinitely for any subgroup Δ. By the remark above, q^2 must be the character determined by $\hat{\mu}^2$, and so actually $\hat{\mu}^2 q^2$ does not cover the circle.

Now the image of $\hat{\mu} \cdot q$ is an arc of the circle, since $\hat{H}$ is connected. Furthermore, by repeating the procedure above, none of the powers $(\hat{\mu} \cdot q)^{2^n}$ covers the circle. This is only possible if the image of $\hat{\mu} \cdot q$ is a single point, showing that $\hat{\mu}$ is a multiple of a character. The proof of Theorem 3 applies from the place where the mass of $\mu(p)$ was known to be concentrated at a single point for each $p \in G$, and so the theorem is shown.

There is no reason to suppose that two is the best constant in the statement of the theorem. The proof does not seem capable of extension. However, the method can probably be modified to cover the case $\|T\| = 2$, matching BEURLING'S result on the line.

§ 5. Conclusion

Examples are hard to find. Theorem 3 gives two classes of automorphisms of an algebra $\boldsymbol{L}(G)$ which always exist. The operation of multiplying summable functions by a fixed character of the group is actually an automorphism; among all bounded functions, only characters have this property. Then τ is simply translation in the dual group. Secondly, each group automorphism of G induces an automorphism of the algebra (introducing a factor k to compensate for expansion of measure). Among all point transformations of G onto itself, only the group automorphisms have this property. The homeomorphism τ is an automorphism of $\hat{G}$.

All these automorphisms are isometric. Theorems 3 and 4 state under appropriate conditions that only these types can arise. The question of finding and describing other automorphisms (or more generally, isomorphisms over non-isomorphic groups) is not touched in this paper, and seems to be too difficult to attack on general groups.

For compact groups, J. Riss and L. Schwartz have pointed out a class of non-trivial automorphisms. If G is compact, $\hat{G}$ is discrete. Let τ permute the points of some finite subset of $\hat{G}$, and leave other points fixed. It is easy to verify that the class $\boldsymbol{F}(\hat{G})$ is carried onto itself, and it is possible to write down the corresponding automorphism T. In particular, G can be taken to be the circle group and $\hat{G}$ the group of integers.

The problems which suggest themselves are of varied type. Theorem 4 needs to be elucidated for the most general abelian and even non-abelian groups, and the techniques of Banach algebras and group representations are pertinent. In another direction, Theorem 3 states that a group is determined by its group algebra. We have remarked that the line is determined by its algebra *as a topological ring*, without reference to the norm itself. Probably there are more general theorems of this type. On the other hand, the line and circle groups require the more powerful methods of concrete analysis. In particular the automorphism problem for the circle group lies in the classical domain of Fourier series.

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