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ISOMORPHISMS OF ALGEBRAS OF GENERALIZED FUNCTIONS

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Abstract. We show that for smooth manifolds X and Y, any isomorphism between the algebras of generalized functions (in the sense of Colombeau) on X, resp. Y is given by composition with a unique generalized function from Y to X. We also characterize the multiplicative linear functionals from the Colombeau algebra on X to the ring of generalized numbers. Up to multiplication with an idempotent generalized number, they are given by an evaluation map at a compactly supported generalized point on X.

1. Introduction

It is a well-known theorem in commutative Banach algebra theory that the isomorphisms between algebras of ($\mathbb{C}$-valued) continuous functions on compact Hausdorff topological spaces X, resp. Y are given by composition with a unique homeomorphism from Y to X [15]. When X, Y are smooth Hausdorff manifolds, isomorphisms between algebras of smooth functions on them are similarly given by composition with a unique diffeomorphism from Y to X. Classically, this is proven by interpreting points of the manifold as multiplicative linear functionals on the corresponding algebras of smooth functions (this result is sometimes referred to as 'Milnor's exercise' [16, 21]) and it may be used as a basis for an axiomatic algebraic derivation of classical tensor analysis. In its full generality, the theorem was only recently established [10, 22]. A measure-theoretic analogue has been given in [20].

From the point of view of analysis, it is natural to look whether a similar theorem can hold for generalized functions such as Schwartz distributions [29]. By the nature of the problem, it is then necessary to consider algebras of generalized functions containing the vector space of distributions. The most widely used theory of such algebras is due to J.F. Colombeau [3, 4, 7, 8, 24]. It was introduced primarily as a tool for studying nonlinear partial differential equations and has been particularly successful in understanding solutions of equations with non-smooth coefficients and strongly singular data [5, 14, 23, 26]. It is an extension of the theory of distributions providing maximal consistency with respect to classical algebraic operations [11] in view of L. Schwartz's impossibility result [28]. Under the influence

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of applications of a primarily geometric nature (e.g. in Lie group analysis of differential equations and in general relativity, cf. [30] for a recent survey), a geometric theory of Colombeau generalized functions arose [6, 9, 11, 12, 18]. In particular, for X, Y smooth paracompact Hausdorff manifolds, Colombeau generalized functions from X to Y can be defined [17]. Recently, a definition of distributions from X to Y was proposed as a quotient of a subspace of the space $\mathcal{G}[X,Y]$ of so-called special Colombeau generalized functions from X to Y [19].

Denoting the algebra of (complex-valued) Colombeau generalized functions on a smooth paracompact Hausdorff manifold X (resp. Y) by $\mathcal{G}(X)$ (resp. $\mathcal{G}(Y)$), we show more generally (theorem 5.1 and its corollary) that algebra homomorphisms $\mathcal{G}(X) \to \mathcal{G}(Y)$ are characterized as compositions with locally defined Colombeau generalized functions from X to Y, up to multiplication with an idempotent element of $\mathcal{G}(Y)$ (which is necessarily locally constant on Y). When the homomorphism is an isomorphism, the idempotent element necessarily equals 1 and the generalized function from X into Y is uniquely determined.

Our technique is based on a characterization of the multiplicative $\widetilde{\mathbb{C}}$-linear functionals on $\mathcal{G}(X)$, where $\widetilde{\mathbb{C}}$ denotes the ring of Colombeau generalized complex numbers. Up to multiplication with an idempotent element of $\widetilde{\mathbb{C}}$, these functionals coincide with the evaluation maps at generalized points [11, §3.2] in $\mathcal{G}(X)$.

2. Preliminaries

The ring $\widetilde{\mathbb{C}}$ of (complex) Colombeau generalized numbers is defined as $\mathcal{M}/\mathcal{N}$, where

$$\begin{aligned}\mathcal{M} &= \{(z_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1)} : (\exists b \in \mathbb{R})(|z_\varepsilon| = \mathrm{O}(\varepsilon^b),\ \text{as } \varepsilon \to 0)\}\\ \mathcal{N} &= \{(z_\varepsilon)_\varepsilon \in \mathcal{M} : (\forall b \in \mathbb{R})(|z_\varepsilon| = \mathrm{O}(\varepsilon^b),\ \text{as } \varepsilon \to 0)\}.\end{aligned}$$

Colombeau generalized numbers arise naturally as evaluations of a Colombeau generalized function at a point in its domain. The subring of $\widetilde{\mathbb{C}}$ consisting of those elements that have a net of real numbers as a representative, is denoted by $\widetilde{\mathbb{R}}$. Nets in $\mathcal{M}$ are called moderate, nets in $\mathcal{N}$ negligible. The element $\tilde{z} \in \widetilde{\mathbb{C}}$ with representative $(z_\varepsilon)_\varepsilon$ is denoted by $[(z_\varepsilon)_\varepsilon]$.
$\widetilde{\mathbb{C}}$ is a complete topological ring with zero divisors for the so-called sharp topology [1, 27]. For $(z_\varepsilon)_\varepsilon \in \mathcal{M}$, let $\mathrm{v}((z_\varepsilon)_\varepsilon) := \sup\{a \in \mathbb{R} : |z_\varepsilon| = \mathrm{O}(\varepsilon^a)$, as $\varepsilon \to 0\}$. Then for $\tilde{z} = [(z_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{C}}$, $|\tilde{z}|_\mathrm{e} := e^{-\mathrm{v}((z_\varepsilon)_\varepsilon)}$ is an ultra-pseudonorm on $\widetilde{\mathbb{C}}$. The sharp topology on $\widetilde{\mathbb{C}}$ is the corresponding ultrametric topology.

Let $S \subseteq (0,1)$. Let $e_S = [(\chi_S(\varepsilon))_\varepsilon] \in \widetilde{\mathbb{R}}$, where χ_S is the characteristic function of S. Then $e_S \neq 0$ iff $0 \in \overline{S}$, the topological closure of S. Further, as is proven in [2], every idempotent element in $\widetilde{\mathbb{C}}$ is of the form e_S. Further algebraic properties of $\widetilde{\mathbb{C}}$ are described in [1, 2].

By a smooth manifold, we will mean a second countable Hausdorff $\mathcal{C}^\infty$ manifold of finite dimension (without boundary).

Let X be a smooth manifold. By $K \Subset X$, we denote a compact subset K of X. Let $\xi \in \mathfrak{X}(X)$ denote a vector field on X and L_ξ its Lie derivative. Then the (so-called special) algebra $\mathcal{G}(X)$ of Colombeau generalized functions on X is defined as $\mathcal{E}_M(X)/\mathcal{N}(X)$, where

$$\begin{aligned}\mathcal{E}_M(X) = \big\{&(u_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty(X))^{(0,1)} : (\forall K \Subset X)(\forall k \in \mathbb{N})(\exists b \in \mathbb{R})\\ &(\forall \xi_1, \dots, \xi_k \in \mathfrak{X}(X))\big(\sup_{p\in K} |L_{\xi_1} \cdots L_{\xi_k} u_\varepsilon(p)| = \mathrm{O}(\varepsilon^b),\ \text{as } \varepsilon \to 0\big)\big\}\\ \mathcal{N}(X) = \big\{&(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X) : (\forall K \Subset X)\\ &(\forall b \in \mathbb{R})\big(\sup_{p\in K} |u_\varepsilon(p)| = \mathrm{O}(\varepsilon^b),\ \text{as } \varepsilon \to 0\big)\big\}.\end{aligned}$$

See also [11, §3.2] for several equivalent definitions.

A net $(p_\varepsilon)_\varepsilon \in X^{(0,1)}$ is called compactly supported [11, §3.2] if there exists $K \Subset X$ and $\varepsilon_0 > 0$ such that $p_\varepsilon \in K$, for $\varepsilon < \varepsilon_0$. Denoting by d_h the Riemannian distance induced by a Riemannian metric h on X, two nets $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ are called equivalent if the net $(d_h(p_\varepsilon, q_\varepsilon))_\varepsilon$ is negligible (this does not depend on the choice of h). The equivalence classes w.r.t. this relation are called compactly supported generalized points on X. The set of compactly supported generalized points on X will be denoted by $\widetilde{X}_c$. If $u \in \mathcal{G}(X)$ and $p \in \widetilde{X}_c$, the point value $u(p) \in \widetilde{\mathbb{C}}$ is the generalized number with representative $(u_\varepsilon(p_\varepsilon))_\varepsilon$ (this does not depend on the representatives).

Let X, Y be smooth manifolds. The space $\mathcal{G}[X, Y]$ of c-bounded Colombeau generalized functions from X to Y is similarly defined as a quotient of the set $\mathcal{E}_M[X, Y]$ of moderate, c-bounded nets of smooth maps $X \to Y$ ([11, Def. 3.2.44]) by a certain equivalence relation $\sim$ ([11, Def. 3.2.46]).

We will use a slightly modified version of the space $\mathcal{G}[X, Y]$ where we do not require the nets to be globally defined. Let $\mathcal{E}_{M,\mathrm{ld}}[X, Y]$ be the set of all nets $(u_\varepsilon)_\varepsilon$ of smooth maps defined on $X_\varepsilon \subseteq X \to Y$ with the property that $(\forall K \Subset X)(\exists \varepsilon_0 > 0)(\forall \varepsilon < \varepsilon_0)(K \subseteq X_\varepsilon)$ and satisfying the c-boundedness and moderateness conditions for elements of $\mathcal{E}_M[X, Y]$, i.e. (cf. [11, Def. 3.2.44]),

(1) $(\forall K \Subset X)(\exists \varepsilon_0 > 0)(\exists K' \Subset Y)(\forall \varepsilon < \varepsilon_0)(u_\varepsilon(K) \subseteq K')$
(2) for each $k \in \mathbb{N}$, each chart (V, ϕ) in X, each chart (W, ψ) in Y, each $L \Subset X$ and each $L' \Subset Y$, there exists $N \in \mathbb{N}$ such that

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L')} \left\| D^{(k)}(\psi \circ u_\varepsilon \circ \phi^{-1})(\phi(p)) \right\| = \mathrm{O}(\varepsilon^{-N}).$$

Let $\sim$ be defined on $\mathcal{E}_{M,\mathrm{ld}}[X, Y]$ as in [11, Def. 3.2.46], i.e., $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$ iff

(1) for each $K \Subset X$, $\sup_{p\in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0$ as $\varepsilon \to 0$ for some (and hence every) Riemannian metric h on Y

(2) for each $k \in \mathbb{N}$, each $m \in \mathbb{N}$, each chart (V, ϕ) in X, each chart (W, ψ) in Y, each $L \Subset X$ and each $L' \Subset Y$,

$$\sup_{p \in L \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')} \left\| D^{(k)}(\psi \circ u_\varepsilon \circ \phi^{-1} - \psi \circ v_\varepsilon \circ \phi^{-1})(\phi(p)) \right\| = \mathrm{O}(\varepsilon^m).$$

Then $\mathcal{G}_{\mathrm{ld}}[X, Y] := \mathcal{E}_{M,\mathrm{ld}}[X, Y]/\sim$ is the space of locally defined c-bounded Colombeau generalized functions $X \to Y$. By definition, $\mathcal{G}[X, Y]$ is a subset of $\mathcal{G}_{\mathrm{ld}}[X, Y]$.

Remark. Under mild topological restrictions on X, $\mathcal{G}_{\mathrm{ld}}[X, Y] = \mathcal{G}[X, Y]$. E.g., it is sufficient that $(\forall K \Subset X)(\exists f \in \mathcal{C}^\infty(X, X))(\overline{f(X)} \Subset X \,\&\, f_{|K} = \mathrm{id}_K)$. This appears to be fulfilled in almost all practical cases. It is unknown if $\mathcal{G}_{\mathrm{ld}}[X, Y] \neq \mathcal{G}[X, Y]$ for some smooth manifolds X, Y.

3. Surjectivity of multiplicative $\widetilde{\mathbb{C}}$-linear maps

Throughout this paper, $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$,... are faithful, commutative $\widetilde{\mathbb{C}}$-algebras with 1. Faithfulness of $\mathcal{A}$ means that for each $\lambda \in \widetilde{\mathbb{C}} \setminus \{0\}$, $\lambda 1 \neq 0$ in $\mathcal{A}$; hence we can identify the subring $\{\lambda 1 : \lambda \in \widetilde{\mathbb{C}}\}$ with $\widetilde{\mathbb{C}}$. By a linear map $\mathcal{A}_1 \to \mathcal{A}_2$, a $\widetilde{\mathbb{C}}$-linear map is meant. In particular, a multiplicative linear functional on $\mathcal{A}$ is meant to be a multiplicative $\widetilde{\mathbb{C}}$-linear map $\mathcal{A} \to \widetilde{\mathbb{C}}$.

Lemma 3.1.

(1) *If a multiplicative linear map ϕ: $\mathcal{A}_1 \to \mathcal{A}_2$ is surjective, then $\phi(1) = 1$.*
(2) *A multiplicative linear functional m on $\mathcal{A}$ is surjective iff $m(1) = 1$.*

Proof. (1) Let $u \in \mathcal{A}_1$ such that $\phi(u) = 1$. Then $\phi(1) = \phi(1)\phi(u) = \phi(u) = 1$.
(2) If $m(1) = 1$, then $m(\lambda 1) = \lambda$, $\forall \lambda \in \widetilde{\mathbb{C}}$, so m is surjective. The converse holds by part 1. □

Proposition 3.2. *Suppose that there exists a multiplicative linear map ϕ_0: $\mathcal{A}_1 \to \mathcal{A}_2$ with $\phi_0(1) = 1$. Let ϕ be any multiplicative linear map $\mathcal{A}_1 \to \mathcal{A}_2$. Then there exists a multiplicative linear map ψ: $\mathcal{A}_1 \to \mathcal{A}_2$ with $\psi(1) = 1$ such that $\phi = \phi(1) \cdot \psi$.*
If $\mathcal{A}$ is a topological algebra and ϕ_0, ϕ are continuous, then ψ is also continuous.

Proof. Let $\psi = \phi + (1 - \phi(1))\phi_0$. □

E.g., if $\mathcal{A}_1 = \mathcal{G}(X)$, X a manifold, then for any $p \in X$, δ_p: $\mathcal{A}_1 \to \mathcal{A}_2$: $\delta_p(u) = u(p)1$ is a multiplicative linear map $\mathcal{A}_1 \to \mathcal{A}_2$ with $\delta_p(1) = 1$. In particular, the study of multiplicative linear functionals on $\mathcal{G}(X)$ is reduced to the surjective ones.

4. Multiplicative $\widetilde{\mathbb{C}}$-linear functionals on $\mathcal{G}(X)$

For a (non-zero) multiplicative $\mathbb{C}$-linear functional m on a $\mathbb{C}$-algebra A, $A/\operatorname{Ker} m \cong \mathbb{C}$ is a field, so $\operatorname{Ker} m$ is a maximal ideal. If A is a Banach algebra, the converse also holds: for a maximal ideal $M \lhd A$, $A/M \cong \mathbb{C}$ by the Gelfand-Mazur theorem [15, 3.2.4], and the canonical surjection $A \to A/M$ determines a multiplicative $\mathbb{C}$-linear functional. Since $\widetilde{\mathbb{C}}$ is not a field, the kernel of a multiplicative $\widetilde{\mathbb{C}}$-linear functional on a $\widetilde{\mathbb{C}}$-algebra $\mathcal{A}$ will not be a maximal ideal. This motivates the following definition.

Definition. An ideal $I \lhd \mathcal{A}$ is maximal with respect to the property $I \cap \widetilde{\mathbb{C}}1 = \{0\}$ iff $J \lhd \mathcal{A}$, $I \subseteq J$ and $J \cap \widetilde{\mathbb{C}}1 = \{0\}$ imply that $I = J$.

It is easy to see that for a surjective multiplicative $\widetilde{\mathbb{C}}$-linear functional m on $\mathcal{A}$, $\operatorname{Ker} m$ is an ideal maximal with respect to $\operatorname{Ker} m \cap \widetilde{\mathbb{C}}1 = \{0\}$.

Definition. Let $u \in \mathcal{A}$ and $S \subseteq (0,1)$ with $0 \in \overline{S}$. Then u is called invertible w.r.t. S iff there exists $v \in \mathcal{A}$ such that $uv = e_S$.

Lemma 4.1. *Let $I \lhd \mathcal{A}$. The following are equivalent:*

1. $I \cap \widetilde{\mathbb{C}}1 = \{0\}$
2. *for each $S \subseteq (0,1)$ with $0 \in \overline{S}$, if $u \in \mathcal{A}$ and u is invertible with respect to S, then $u \notin I$.*

Proof. (1) $\Rightarrow$ (2): let u be invertible w.r.t. S. Should $u \in I$, then also $0 \neq e_S \in I$, so $I \cap \widetilde{\mathbb{C}}1 \neq \{0\}$.
(2) $\Rightarrow$ (1): let $\lambda \in \widetilde{\mathbb{C}} \setminus \{0\}$. Then there exists $S \subseteq (0,1)$ with $0 \in \overline{S}$ such that λ is invertible w.r.t. S, so $\lambda \notin I$. □

We denote the complement of $S \subseteq (0,1)$ by S^c.

Lemma 4.2. *Let $u \in \mathcal{A}$ and $S \subseteq (0,1)$ with $0 \in \overline{S}$. Then u is invertible w.r.t. S iff $ue_S + e_{S^c}$ is invertible.*

Proof. If $uv = e_S$, for some $v \in \mathcal{A}$, then $(ue_S + e_{S^c})(ve_S + e_{S^c}) = uve_S + e_{S^c} = 1$.
Conversely, if $(ue_S + e_{S^c})v = 1$, for some $v \in \mathcal{A}$, then multiplying by e_S shows that $u(ve_S) = e_S$. □

Corollary 4.3. *Let X be a smooth submanifold of $\mathbb{R}^d$. Let $S \subseteq (0,1)$ with $0 \in \overline{S}$. Let $u \in \mathcal{G}(X)$. Then the following are equivalent:*

1. *u is invertible w.r.t. S (as an element of $\mathcal{G}(X)$)*
2. *$u(\tilde{x})$ is invertible w.r.t. S (as an element of $\widetilde{\mathbb{C}}$), for each $\tilde{x} \in \widetilde{X}_c$.*

Proof. This is a combination of the previous lemma with proposition A.3. □

Proposition 4.4. *Let X be a smooth submanifold of $\mathbb{R}^d$. Let $I \lhd \mathcal{G}(X)$. If $(\forall p \in \widetilde{X}_c)(\exists u_p \in I)(u_p(p) \neq 0)$, then I is not the kernel of a surjective multiplicative linear functional on $\mathcal{G}(X)$.*

Proof. Suppose that I is the kernel of a surjective multiplicative linear functional. Then $I \cap \widetilde{\mathbb{C}}1 = \{0\}$ and $I + \widetilde{\mathbb{C}}1 = \mathcal{G}(X)$, so each of the coordinate functions $x_i \in I + \widetilde{\mathbb{C}}1$ ($i \in \{1, \dots, d\}$), i.e., for each i, there exists $\lambda_i \in \widetilde{\mathbb{C}}$ such that $x_i - \lambda_i 1 \in I$. Write $\lambda = (\lambda_1, \dots, \lambda_d) \in \widetilde{\mathbb{C}}^d$ and consider $|x - \lambda|^2 = \sum_i (x_i - \overline{\lambda}_i 1)(x_i - \lambda_i 1) \in I$.
We distinguish 3 cases.
(1) $\lambda \in \widetilde{X}_c$. Notice that by corollary A.2, this property is well-defined. By hypothesis, there exists $u_\lambda \in I$ with $u_\lambda(\lambda) \neq 0$. Then also $|x - \lambda|^2 + |u_\lambda|^2 \in I$, and there exists $S \subseteq (0,1)$, $0 \in \overline{S}$, such that $u_\lambda(\lambda) \in \widetilde{\mathbb{C}}$ is invertible w.r.t. S. Let $\tilde{x} \in \widetilde{X}_c$ with representative $(x_\varepsilon)_\varepsilon$. By proposition A.4, there exist m, $k \in \mathbb{N}$ such that

$$(\exists \varepsilon_0 > 0)(\forall \varepsilon \in S \cap (0, \varepsilon_0))(|x_\varepsilon - \lambda_\varepsilon| \leq \varepsilon^m \Rightarrow |u_{\lambda,\varepsilon}(x_\varepsilon)| \geq \varepsilon^k).$$

Thus $(\exists \varepsilon_0 > 0)$ $(\forall \varepsilon \in S \cap (0, \varepsilon_0))$

$$\left(|x_\varepsilon - \lambda_\varepsilon|^2 + |u_{\lambda,\varepsilon}(x_\varepsilon)|^2 \geq \begin{cases} \varepsilon^{2k}, & |x_\varepsilon - \lambda_\varepsilon| \leq \varepsilon^m \\ \varepsilon^{2m}, & |x_\varepsilon - \lambda_\varepsilon| \geq \varepsilon^m \end{cases} \right),$$

and we conclude by corollary 4.3 that $|x - \lambda|^2 + |u_\lambda|^2 \in I$ is invertible w.r.t. S, a contradiction.
(2) $\lambda \in \widetilde{X} \setminus \widetilde{X}_c$, where $\widetilde{X} = \{\tilde{x} \in \widetilde{\mathbb{R}}^d : (\exists \text{ repr. } (x_\varepsilon)_\varepsilon \text{ of } \tilde{x})(\forall \varepsilon)(x_\varepsilon \in X)\}$. Let $(K_n)_{n\in\mathbb{N}}$ be a compact exhaustion of X with $K_n \subseteq (K_{n+1})^\circ$, $\forall n \in \mathbb{N}$ (where the interior is taken in the relative topology on X). Consider a representative $(\lambda_\varepsilon)_\varepsilon$ of λ such that $\lambda_\varepsilon \in X$, $\forall \varepsilon$. As $\lambda \notin \widetilde{X}_c$, there exists a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ such that $\lambda_{\varepsilon_n} \in X \setminus K_n$ for each n. As the Euclidean distance $d(X \setminus K_n, K_{n-1}) > 0$ for each n, $v(x) = |x - \lambda_\varepsilon|^2 \in I$ evaluated in any compactly supported point of X is invertible w.r.t. $S = \{\varepsilon_n : n \in \mathbb{N}\}$, a contradiction.
(3) If $\lambda \in \widetilde{\mathbb{C}}^d \setminus \widetilde{X}$, then for any representative $(\lambda_\varepsilon)_\varepsilon$ of λ, $(d(\lambda_\varepsilon, X)_\varepsilon)_\varepsilon$ is not a negligible net. This means that there exists $S \subseteq (0,1)$ with $0 \in \overline{S}$ and $m \in \mathbb{N}$ such that $d(\lambda_\varepsilon, X) \geq \varepsilon^m$, for each $\varepsilon \in S$. This also means that $v(x) = |x - \lambda_\varepsilon|^2 \in I$ evaluated in any compactly supported point of X is invertible w.r.t. S, a contradiction. □

Theorem 4.5. *Let X be a smooth manifold.*

(1) *The surjective multiplicative linear functionals on $\mathcal{G}(X)$ are*

$$\delta_p : \mathcal{G}(X) \to \widetilde{\mathbb{C}} : \delta_p(u) = u(p),$$

where $p \in \widetilde{X}_c$.

(2) *The multiplicative linear functionals on $\mathcal{G}(X)$ are*

$$e\delta_p : \mathcal{G}(X) \to \widetilde{\mathbb{C}} : e\delta_p(u) = eu(p),$$

where $p \in \widetilde{X}_c$ and $e \in \widetilde{\mathbb{R}}$ idempotent.

Proof. First, let X be a smooth submanifold of $\mathbb{R}^d$.
(1) Let m be a surjective multiplicative linear functional on $\mathcal{G}(X)$. Then by proposition 4.4, there exists $p \in \widetilde{X}_c$ such that $u(p) = 0$, $\forall u \in \operatorname{Ker} m$. I.e., $\operatorname{Ker} m \subseteq \operatorname{Ker} \delta_p$. But $\operatorname{Ker} m$ is maximal w.r.t. $\operatorname{Ker} m \cap \mathbb{C}1 = \{0\}$ and $\operatorname{Ker} \delta_p \cap \mathbb{C}1 = \{0\}$, so $\operatorname{Ker} m = \operatorname{Ker} \delta_p$. So for each $u \in \mathcal{G}(X)$, as $u - u(p) \in \operatorname{Ker} m = \operatorname{Ker} \delta_p$, $m(u) = m(u - u(p) + u(p)) = m(u(p)) = u(p)$, so $m = \delta_p$.
(2) This follows from part 1 and proposition 3.2.
Now let X be any smooth manifold. It follows from Whitney's embedding theorem [13] that there exists a smooth embedding f: $X \to \mathbb{R}^d$, for some $d \in \mathbb{N}$. Let m: $\mathcal{G}(X) \to \widetilde{\mathbb{C}}$ be a surjective multiplicative linear functional. For $u \in \mathcal{G}(f(X))$, $u \circ f \in \mathcal{G}(X)$ (corollary A.7). Then μ: $\mathcal{G}(f(X)) \to \widetilde{\mathbb{C}}$: $\mu(u) = m(u \circ f)$ is a surjective multiplicative linear functional, so there exists $p \in \widetilde{f(X)}_c$ such that $\mu(u) = u(p)$, $\forall u \in \mathcal{G}(f(X))$. For each $v \in \mathcal{G}(X)$, $v \circ f^{-1} \in \mathcal{G}(f(X))$, so $m(v) = \mu(v \circ f^{-1}) = v(f^{-1}(p))$, where $f^{-1}(p) \in \widetilde{X}_c$ [11, 3.2.55]. □

5. Algebra homomorphisms $\mathcal{G}(X) \to \mathcal{G}(Y)$

Theorem 5.1. *Let $X \subseteq \mathbb{R}^{d_1}$, $Y \subseteq \mathbb{R}^{d_2}$ be smooth submanifolds.*

(1) *Let ϕ: $\mathcal{G}(X) \to \mathcal{G}(Y)$ be a morphism of algebras (i.e., a multiplicative $\widetilde{\mathbb{C}}$-linear map). Then there exists $f \in (\mathcal{G}(Y))^{d_1}$, c-bounded into X and $e \in \mathcal{G}(Y)$ idempotent such that*
$$\phi(u) = e \cdot (u \circ f), \quad \forall u \in \mathcal{G}(X).$$
If $\phi(1) = 1$, then $e = 1$ and f is uniquely determined.
(2) *If ϕ: $\mathcal{G}(X) \to \mathcal{G}(Y)$ is an isomorphism of algebras (i.e., additionally, ϕ is bijective), then the map f has an inverse $f^{-1} \in (\mathcal{G}(X))^{d_2}$, c-bounded into Y such that ϕ^{-1} is given by composition with f^{-1}. As a map $\widetilde{Y}_c \to \widetilde{X}_c$, f is bijective. In this case,* $\dim X = \dim Y$.

Proof. (1) First, let $\phi(1) = 1$. Let $\tilde{y} \in \widetilde{Y}_c$ arbitrary. Then the map $\delta_{\tilde{y}} \circ \phi$ is a multiplicative linear functional on $\mathcal{G}(X)$. It is also surjective, as $\delta_{\tilde{y}}(\phi(1)) = 1$. So by theorem 4.5, there exists $f(\tilde{y}) \in \widetilde{X}_c$ such that $\delta_{\tilde{y}} \circ \phi = \delta_{f(\tilde{y})}$. So
$$(\forall u \in \mathcal{G}(X))(\forall \tilde{y} \in \widetilde{Y}_c)((\phi(u))(\tilde{y}) = u(f(\tilde{y}))). \tag{1}$$
In particular, for $u_i(x) = x_i \in \mathcal{G}(X)$, $i = 1, \dots, d_1$, we see that
$$(\phi(u_1), \dots, \phi(u_{d_1})) \in (\mathcal{G}(Y))^{d_1} \tag{2}$$
is the unique generalized function which coincides with f when evaluated at generalized points in $\widetilde{Y}_c$ (because an element of $\mathcal{G}(Y)$ is completely determined by its values in $\widetilde{Y}_c$ [11, Thm. 3.2.8]). With a slight abuse of notation, we will therefore also denote it by f. By proposition A.6, f is c-bounded into X. So by proposition A.5, for each $u \in \mathcal{G}(X)$, the componentwise composition $u \circ f$ defines an element of $\mathcal{G}(Y)$. By eqn. (1), it coincides with

$\phi(u)$ on each compactly supported point in $\widetilde{Y}_c$, so $u \circ f = \phi(u)$ in $\mathcal{G}(Y)$. Clearly, f is completely determined by $f_i = u_i \circ f = \phi(u_i)$ ($i = 1, \dots, d_1$). For general ϕ, this follows by proposition 3.2 and the fact that $\phi(1)$ is idempotent.

(2) Applying part 1 to ϕ^{-1}, we find $g \in (\mathcal{G}(X))^{d_2}$, c-bounded into Y such that ϕ^{-1} is given by composition with g. To see that $g = f^{-1}$, we show that $f \circ g = \mathrm{id}_{\mathcal{G}(X)} \in (\mathcal{G}(X))^{d_1}$, where $\mathrm{id}_{\mathcal{G}(X)}$ is the generalized function with representative $(\mathrm{id}_X)_\varepsilon$.

By eqn. (2) and because ϕ^{-1} is given by composition with g,

$$\begin{aligned} f \circ g = (f_1 \circ g, \dots, f_{d_1} \circ g) &= (\phi^{-1}(f_1), \dots, \phi^{-1}(f_{d_1}))\\ &= (\phi^{-1}(\phi(u_1)), \dots, \phi^{-1}(\phi(u_{d_1}))) = (u_1, \dots, u_{d_1}). \end{aligned}$$

Similarly, $g \circ f = \mathrm{id}_{\mathcal{G}(Y)} \in (\mathcal{G}(Y))^{d_2}$. From these equalities, it follows also that f^{-1} is the inverse of f as a pointwise map on compactly supported generalized points.

In order to prove that $\dim X = \dim Y$, we consider the equality $f \circ g = \mathrm{id}_{\mathcal{G}(X)}$ on representatives. By proposition A.5, there exist representatives $(f_\varepsilon)_\varepsilon$ of f and $(g_\varepsilon)_\varepsilon$ of g such that $f_\varepsilon \circ g_\varepsilon = \mathrm{id}_X + n_\varepsilon$ holds locally and for sufficiently small ε, where $(n_\varepsilon)_\varepsilon \in (\mathcal{N}(X))^{d_1}$. Let $x \in X$ be fixed. Denoting the differential of a map f at x by $df(x)$, we have $df_\varepsilon(g_\varepsilon(x)) \circ dg_\varepsilon(x) = d(\mathrm{id}_X + n_\varepsilon)(x)$ by the chain rule. As elements in $\mathcal{N}(X)$ also satisfy the negligibility-estimates for the derivatives [11, p. 278], $\mathrm{rank}(d(\mathrm{id}_X + n_\varepsilon)(x)) = \mathrm{rank}(\mathrm{id}_{T_xX}) = \dim X$ for sufficiently small ε, whereas

$$\mathrm{rank}(df_\varepsilon(g_\varepsilon(x)) \circ dg_\varepsilon(x)) \le \mathrm{rank}(df_\varepsilon(g_\varepsilon(x))) \le \dim Y,$$

since $f_\varepsilon \in \mathcal{C}^\infty(Y, \mathbb{R}^{d_1})$. Hence $\dim X \le \dim Y$. By $g \circ f = \mathrm{id}_{\mathcal{G}(Y)}$, we similarly obtain $\dim Y \le \dim X$. ☐

Corollary 5.2. *Let X, Y be smooth manifolds.*

1. *Let ϕ: $\mathcal{G}(X) \to \mathcal{G}(Y)$ be a morphism of algebras (i.e., a multiplicative $\widetilde{\mathbb{C}}$-linear map). Then there exists $f \in \mathcal{G}_{\mathrm{ld}}[Y, X]$ and $e \in \mathcal{G}(Y)$ idempotent such that*
$$\phi(u) = e \cdot (u \circ f), \quad \forall u \in \mathcal{G}(X).$$
If $\phi(1) = 1$, then $e = 1$ and f is uniquely determined.
2. *If ϕ: $\mathcal{G}(X) \to \mathcal{G}(Y)$ is an isomorphism of algebras (i.e., additionally, ϕ is bijective), then the map f has an inverse $f^{-1} \in \mathcal{G}_{\mathrm{ld}}[X, Y]$ such that ϕ^{-1} is given by composition with f^{-1}. As a map $\widetilde{X}_c \to \widetilde{Y}_c$, f is bijective. In this case, $\dim X = \dim Y$.*

Proof. It follows from Whitney's embedding theorem [13] that there exist smooth embeddings ι_1: $X \to \mathbb{R}^{d_1}$ and ι_2: $Y \to \mathbb{R}^{d_2}$, for some $d_1, d_2 \in \mathbb{N}$.

(1) Let ϕ: $\mathcal{G}(X) \to \mathcal{G}(Y)$ be a multiplicative $\widetilde{\mathbb{C}}$-linear map with $\phi(1) = 1$. Then $\widetilde{\phi}$: $\mathcal{G}(\iota_1(X)) \to \mathcal{G}(\iota_2(Y))$: $\widetilde{\phi}(u) = \phi(u \circ \iota_1) \circ \iota_2^{-1}$ is a multiplicative $\widetilde{\mathbb{C}}$-linear map with $\widetilde{\phi}(1) = 1$. By the previous theorem and by corollary A.7,

there exists $\widetilde{f} \in \mathcal{G}_{\mathrm{ld}}[\iota_2(Y), \iota_1(X)]$ such that $\widetilde{\phi}$ is given by composition with $\widetilde{f}$. So for each $u \in \mathcal{G}(X)$, $\phi(u) = \widetilde{\phi}(u \circ \iota_1^{-1}) \circ \iota_2 = u \circ (\iota_1^{-1} \circ \widetilde{f} \circ \iota_2)$. By the analogue of [11, Cor. 3.2.59] for $\mathcal{G}_{\mathrm{ld}}[X,Y]$, $f = \iota_1^{-1} \circ \widetilde{f} \circ \iota_2 \in \mathcal{G}_{\mathrm{ld}}[Y,X]$. Unicity of f follows from unicity of $\widetilde{f}$.
The result for general ϕ follows again from proposition 3.2.
(2) We similarly find $\widetilde{g} \in \mathcal{G}_{\mathrm{ld}}[\iota_1(X), \iota_2(Y)]$ with $g = \iota_2^{-1} \circ \widetilde{g} \circ \iota_1 \in \mathcal{G}_{\mathrm{ld}}[X,Y]$. By the previous theorem, $\widetilde{f} \circ \widetilde{g}$ is the identity in $\mathcal{G}_{\mathrm{ld}}[\iota_1(X), \iota_1(X)]$. So $f \circ g = \iota_1^{-1} \circ \mathrm{id}_{\mathcal{G}_{\mathrm{ld}}[\iota_1(X), \iota_1(X)]} \circ \iota_1 = \mathrm{id}_{\mathcal{G}_{\mathrm{ld}}[X,X]}$, and similarly, $g \circ f = \mathrm{id}_{\mathcal{G}_{\mathrm{ld}}[Y,Y]}$. It follows again that $g = f^{-1}$ as pointwise maps on compactly supported generalized points. □

Concerning idempotent elements in $\mathcal{G}(X)$, we can be more explicit:

Proposition 5.3. *Let X be a smooth manifold. Let $e \in \mathcal{G}(X)$ be idempotent. Then on every connected component of X, e is an idempotent constant.*

Proof. If X is an open subset of $\mathbb{R}^d$, this is proven in [2]. Let X be an arbitrary manifold. Consider a chart (V, ψ) of X and $x \in V$. Then the local representation $e \circ \psi^{-1} \in \mathcal{G}(\psi(V))$ is an idempotent, and therefore equal to some constant $c \in \widetilde{\mathbb{C}}$ in a connected, open neighbourhood W of $\psi(x)$. So $e = c$ in the open neighbourhood $\psi^{-1}(W)$ of x. Therefore, for every $c \in \widetilde{\mathbb{C}}$, $\{x \in X : (\exists U \text{ open neighbourhood of } x)(e_{|U} = c)\}$ is open and closed in X. Consequently, on every connected component C of X, each $x \in C$ has an open neighbourhood U such that $e_{|U} = c$, for some constant $c \in \widetilde{\mathbb{C}}$ independent of $x \in C$. The proposition follows by the fact that $\mathcal{G}(C)$ is a sheaf of differential algebras on C ([11, Prop. 3.2.3]). □

Appendix A. Colombeau generalized functions on a manifold embedded in $\mathbb{R}^d$

In this appendix, we extend some results that are well-known in the special case where X is an open subset of $\mathbb{R}^d$ to the case of a submanifold of $\mathbb{R}^d$.

Lemma A.1. *Let X be a connected smooth submanifold of $\mathbb{R}^d$. Let h be the Riemannian metric on X induced by the Euclidean metric in $\mathbb{R}^d$. Let $K \Subset X$. Then there exists $C \in \mathbb{R}^+$ such that for each p, $q \in K$, $|p - q| \le d_h(p,q) \le C\,|p - q|$.*

Proof. $d_h(p,q)$ is the infimum of the distances between p, q along paths on X, and therefore at least equal to the Euclidean distance between p and q. For the other inequality, suppose first that p, q lie in a sufficiently small neighbourhood of a given point $p_0 \in K$. It is an exercise in elementary differential geometry that in this case, $d_h(p,q) \le C\,|p-q|$ (with $C \to 1$ as p, $q \to p_0$).
If the inequality would not hold globally on K, one could construct sequences $(p_m)_m$, $(q_m)_m$ of points in K such that $d_h(p_m, q_m) \ge m\,|p_m - q_m|$. Because K is compact, there is a subsequence $(m_k)_k$ such that $p_{m_k} \to p \in K$,

$q_{m_k} \to q \in K$. By continuity, $d_h(p,q) \geq m\,|p-q|$, for each $m \in \mathbb{N}$, so $|p-q| = 0$ and $p = q$. This contradicts the inequality in a sufficiently small neighbourhood of p. ☐

Corollary A.2. *Let X be a smooth submanifold of $\mathbb{R}^d$. The compactly supported generalized points in $\widetilde{X}_c$ are in 1-1 correspondence with the elements of $\widetilde{\mathbb{R}}^d$ which have a representative that consists of elements of K, for some $K \Subset X$. More specifically, the injection is given by the (well-defined) map $\widetilde{X}_c \to \widetilde{\mathbb{R}}^d$ which is the identity-map on representatives.*

Proof. By the fact that every $x \in X$ has a connected neighbourhood, $K \Subset X$ is contained in a finite number of connected components of X.
Two compactly supported nets $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ in $X^{(0,1)}$ represent the same generalized point in X iff $d_h(p_\varepsilon, q_\varepsilon) = \mathrm{O}(\varepsilon^m)$, $\forall m \in \mathbb{N}$. (By definition, this also implies that for a fixed sufficiently small ε, p_ε and q_ε lie in the same connected component.) By lemma A.1, this is equivalent with $|p_\varepsilon - q_\varepsilon| = \mathrm{O}(\varepsilon^m)$, $\forall m \in \mathbb{N}$ (this also implies that for a fixed sufficiently small ε, p_ε and q_ε lie in the same connected component, since any open cover of $K \Subset X$, in particular one that consists of connected sets, has a Lebesgue number), i.e., they represent the same element in $\widetilde{\mathbb{R}}^d$. ☐

Proposition A.3. *Let X be a smooth submanifold of $\mathbb{R}^d$. Let $u \in \mathcal{G}(X)$. Then the following are equivalent:*

(1) *u is invertible (as an element of $\mathcal{G}(X)$)*
(2) *$u(\tilde{x})$ is invertible (as an element of $\widetilde{\mathbb{C}}$), for each $\tilde{x} \in \widetilde{X}_c$.*

Proof. (1) $\Rightarrow$ (2) is analogous to [11, Thm. 1.2.5].
(2) $\Rightarrow$ (1): to show that a global inverse exists, it is enough to show that there exists an inverse in each local representation (w.r.t. charts), and that the compatibility-conditions between them are satisfied [11, Prop. 3.2.3]. As in [11, Thm. 3.2.8], part (2) is also satisfied for each local representation. So by [11, Thm. 1.2.5], local inverses exist. The compatibility-conditions for u^{-1} follow from the compatibility-conditions of u and the fact that inverses in $\mathcal{G}(X)$ are unique (for any manifold X). ☐

Proposition A.4 (Continuity in the sharp topology)**.** *Let X be a smooth submanifold of $\mathbb{R}^d$. Let $u \in \mathcal{G}(X)$ and let $K \Subset X$. Then for each $k \in \mathbb{N}$,*

$$(\exists m \in \mathbb{N})(\exists \varepsilon_0 > 0)(\forall \varepsilon \leq \varepsilon_0)$$
$$(\forall x, y \in K)(|x-y| \leq \varepsilon^m \Rightarrow |u_\varepsilon(x) - u_\varepsilon(y)| \leq \varepsilon^k).$$

Proof. If X is an open subset of $\mathbb{R}^d$, see e.g. [25, Prop. 3.1].
If X is a smooth manifold of $\mathbb{R}^d$, cover K by geodesically convex W_α with $\overline{W}_\alpha \Subset V_\alpha$ for charts (V_α, ψ_α) (as in [11, Thm. 3.2.8]). By compactness, a finite number $W_1, \dots, W_M$ is sufficient. Call the corresponding charts $(V_1, \psi_1), \dots, (V_M, \psi_M)$. By the existence of a Lebesgue number, we may suppose that x and y belong to the same W_i, if ε_0 is chosen sufficiently small

(and $m \geq 1$). So, let $k \in \mathbb{N}$. We apply the proposition to $u \circ \psi_i^{-1} \in \mathcal{G}(\psi_i(V_i))$, and we obtain $m_i \in \mathbb{N}$, $\varepsilon_i > 0$ such that

$$(\forall \varepsilon \leq \varepsilon_i)(\forall x, y \in W_i)(|\psi_i(x) - \psi_i(y)| \leq \varepsilon^{m_i} \Rightarrow |u_\varepsilon(x) - u_\varepsilon(y)| \leq \varepsilon^k).$$

Further, by [11, Lemma 3.2.6] and lemma A.1 (as W_i is connected),

$$|\psi_i(x) - \psi_i(y)| \leq C d_h(x, y) \leq C' |x - y|,$$

for some C, $C' \in \mathbb{R}^+$ (independent of x, $y \in W_i$). So, possibly after increasing m_i and decreasing ε_i,

$$(\forall \varepsilon \leq \varepsilon_i)(\forall x, y \in W_i)(|x - y| \leq \varepsilon^{m_i} \Rightarrow |u_\varepsilon(x) - u_\varepsilon(y)| \leq \varepsilon^k).$$

Choose $\varepsilon_0 \leq \varepsilon_1, \dots, \varepsilon_0 \leq \varepsilon_M$ and $m \geq m_1, \dots, m \geq m_M$. Then we obtain the statement of the proposition. □

Let $X \subseteq \mathbb{R}^{d_1}$, $Y \subseteq \mathbb{R}^{d_2}$ be smooth submanifolds. In analogy with the case where X, Y are open subsets of $\mathbb{R}^{d_1}$, resp. $\mathbb{R}^{d_2}$ ([11, 1.2.7]), $u \in (\mathcal{G}(X))^{d_2}$ is called c-bounded into Y if there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that

$$(\forall K \Subset X)(\exists K' \Subset Y)(\exists \varepsilon_0 > 0)(\forall \varepsilon \leq \varepsilon_0)(u_\varepsilon(K) \subseteq K'). \tag{3}$$

Proposition A.5. *Let $X \subseteq \mathbb{R}^{d_1}$, $Y \subseteq \mathbb{R}^{d_2}$ be smooth submanifolds. Let $u \in (\mathcal{G}(X))^{d_2}$ be c-bounded into Y and $v \in \mathcal{G}(Y)$. Then the composition $v \circ u$ defined on representatives by means of $(v \circ u)_\varepsilon = (v_\varepsilon \circ u_\varepsilon)$ is a well-defined generalized function in $\mathcal{G}(X)$.*

Proof. Notice that the net $(v_\varepsilon \circ u_\varepsilon)_\varepsilon$ is only locally defined; to find a globally defined representative, it can be multiplied by a net $(\chi_\varepsilon)_\varepsilon$ of smooth, compactly supported cut-off functions which is a representative of $1 \in \mathcal{G}(X)$. Well-definedness follows as in [11, Prop. 1.2.8]. □

Proposition A.6. *Let $X \subseteq \mathbb{R}^{d_1}$, $Y \subseteq \mathbb{R}^{d_2}$ be smooth submanifolds. Let $u \in (\mathcal{G}(X))^{d_2}$. Then the following are equivalent:*

1. *u is c-bounded into Y*
2. *for one, and thus for all representatives $(u_\varepsilon)_\varepsilon$ of u,*
$$(\forall K \Subset X)(\exists K' \Subset Y)(\forall m \in \mathbb{N})\big(\sup_{x \in K} d(u_\varepsilon(x), K') = \mathrm{O}(\varepsilon^m), \varepsilon \to 0\big)$$
(here d denotes the Euclidean distance in $\mathbb{R}^{d_2}$).
3. *as a pointwise function on compactly generalized points, $u(\widetilde{X}_c) \subseteq \widetilde{Y}_c$.*

Proof. (1) ⇒ (3): let $\tilde{x} \in \widetilde{X}_c$. For a representative $(x_\varepsilon)_\varepsilon$ of x, $x_\varepsilon \in K \Subset X$, for sufficiently small ε. If $(u_\varepsilon)_\varepsilon$ is a representative of u with $u_\varepsilon(K) \subseteq K' \Subset Y$ for sufficiently small ε, then $u_\varepsilon(x_\varepsilon) \in K'$ for sufficiently small ε, so $u(\widetilde{x}) \in \widetilde{Y}_c$.
(3) ⇒ (2): suppose that there exists $K \Subset X$ such that

$$(\forall K' \Subset Y)(\exists m \in \mathbb{N})(\forall \eta \in (0,1))(\exists \varepsilon < \eta)(\exists x \in K)(d(u_\varepsilon(x), K') \geq \varepsilon^m).$$

We distinguish 2 cases.
(a) $(\sup_{x \in K} d(u_\varepsilon(x), Y))_\varepsilon$ is not negligible, i.e., the previous formula also holds for Y itself instead of K'. Then we find a decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$, with $\varepsilon_n \to 0$ and $x_{\varepsilon_n} \in K$, with $d(u_{\varepsilon_n}(x_{\varepsilon_n}), Y) \geq \varepsilon^m$, for some m. Extend

$(x_{\varepsilon_n})_{n\in\mathbb{N}}$ to $(x_\varepsilon)_\varepsilon$, with $x_\varepsilon \in K$, $\forall\varepsilon$. Then it represents $\tilde{x} \in \widetilde{X}_c$ for which $u(\tilde{x}) \notin \widetilde{Y}_c$.
(b) $(\sup_{x\in K} d(u_\varepsilon(x), Y))_\varepsilon$ is negligible. Consider a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Y with $K_n \subseteq (K_{n-1})^\circ$, $\forall n \in \mathbb{N}$. Then we find a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, with $\varepsilon_n \to 0$, $m_n \in \mathbb{N}$ and $x_{\varepsilon_n} \in K$ such that $d(u_{\varepsilon_n}(x_{\varepsilon_n}), Y) < \varepsilon_n^{m_n+n} < \varepsilon_n^{m_n} \le d(u_{\varepsilon_n}(x_{\varepsilon_n}), K_n)$; in particular, there exists $y_n \in Y \setminus K_n$ such that $|y_n - u_{\varepsilon_n}(x_{\varepsilon_n})| \le \varepsilon_n^n$. Let $m < n$. As $K_m \subseteq (K_n)^\circ$, $d(y_n, K_m) \ge r \in \mathbb{R}^+$, so $u_{\varepsilon_n}(x_{\varepsilon_n}) \notin K_m$ as soon as n is large enough. Extend $(x_{\varepsilon_n})_{n\in\mathbb{N}}$ to $(x_\varepsilon)_\varepsilon$, with $x_\varepsilon \in K$, $\forall\varepsilon$. Then it represents $\tilde{x} \in \widetilde{X}_c$ for which $u(\tilde{x}) \notin \widetilde{Y}_c$.
(2) $\Rightarrow$ (1): let $(u_\varepsilon)_\varepsilon$ be a representative of u. Let W be a normal tubular neighbourhood of Y in $\mathbb{R}^{d_2}$ with associated smooth retraction q: $W \to Y$ (see [13]). By assumption, u is c-bounded into W, so the composition $q \circ u$ is a well-defined element of $(\mathcal{G}(X))^{d_2}$ and is c-bounded into Y. Let $K \Subset X$. By the fact that W is a normal tubular neighbourhood of Y, $q(x)$ is the unique element of Y that is closest to x, for each $x \in W$. So for sufficiently small ε, $\sup_{x\in K} |(q \circ u_\varepsilon)(x) - u_\varepsilon(x)| = \sup_{x\in K} d(u_\varepsilon(x), Y)$ which is negligible by assumption. It follows that $q \circ u = u$ as a generalized function in $(\mathcal{G}(X))^{d_2}$. □

Corollary A.7. (1) *Let $X \subseteq \mathbb{R}^{d_1}$, $Y \subseteq \mathbb{R}^{d_2}$ be smooth submanifolds. An element $u \in (\mathcal{G}(X))^{d_2}$ that is c-bounded into $Y \subseteq \mathbb{R}^{d_2}$ defines a unique element of $\mathcal{G}_{\mathrm{ld}}[X, Y]$ by restricting a representative satisfying eqn. (3) to (suitably chosen, depending on ε) compact subsets of X.*
(2) *Let X, Y be smooth manifolds. Let $u \in \mathcal{G}_{\mathrm{ld}}[X, Y]$ and $v \in \mathcal{G}(Y)$. Then the composition $v \circ u$ defined on representatives by means of $(v \circ u)_\varepsilon = v_\varepsilon \circ u_\varepsilon$ is a well-defined generalized function in $\mathcal{G}(X)$.*

Proof. (1) Let $(u_\varepsilon)_\varepsilon$ be a representative of u satisfying eqn. (3). Let $(K_n)_{n\in\mathbb{N}}$ be a compact exhaustion of X. Then for each $n \in \mathbb{N}$, there exists $K \Subset Y$ and $\varepsilon_n \in (0, 1)$ such that $u_\varepsilon(K_n) \subseteq K$, for each $\varepsilon \le \varepsilon_n$. We may suppose $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence. Let $v_\varepsilon = u_{\varepsilon|K_n}$, for each $\varepsilon_{n+1} < \varepsilon \le \varepsilon_n$. Then $(v_\varepsilon)_\varepsilon$ represents an element of $\mathcal{G}_{\mathrm{ld}}[X, Y]$. Well-definedness follows as in [11, Prop. 3.2.43].
(2) Analogous to [11, Prop. 3.2.58]. □

It follows that, in case $\mathcal{G}[X, Y] \subsetneq \mathcal{G}_{\mathrm{ld}}[X, Y]$, a characterization of algebra homomorphisms $\mathcal{G}(X) \to \mathcal{G}(Y)$ as compositions with generalized maps $\mathcal{G}[X, Y]$ is not possible ($\mathcal{G}_{\mathrm{ld}}[X, Y]$ has to be used instead).

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