Available online at http://scik.org

J. Math. Comput. Sci. 3 (2013), No. 5, 1335-1352

ISSN: 1927-5307

ISOMORPHISMS ON STRONGLY CONNECTED GROUP AUTOMATON

S. JEYA BHARATHI[1], A. JEYANTHI[2,*]

[1]Department of Mathematics, Thiagarajar College of Engineering, Madurai, India

[2]Department of Mathematics, Anna University Regional Center, Madurai, India

Abstract: Direct product of a permutation strongly connected automaton and a synchronizing strongly connected Aleshin Type automaton is also a strongly connected automaton. An automaton is called quasi-ideal automaton if and only if all the following conditions are satisfied; (i) It is strongly connected (ii) the minimal ideal of its transition semi group is a right group (iii) the ranges of the idempotent elements of the minimal ideal of its transition group form a merging of a partition on its set of states. An automaton is isomorphic to the direct product of a permutation strongly connected automaton and synchronizing strongly connected Aleshin type automaton if and only if it is a quasi ideal automaton.

Keywords: Aleshin Type automata, strongly connected, Permutation automata, direct product, quotient, minimal ideal, right group, simple group.

2000 AMS Subject Classification: 68Q70.

1. Introduction

When considering the direct product of automata, one would intuitively think that the direct product of two strongly connected automata would be strongly connected. Fleck [5] showed that for any non trivial (more than one state) automata A and B, if A is homomorphic to B (or vice versa) then A $\times$ B is not strongly connected. This is because an automaton homomorphism is a transition preserving function. Any input string which sends a state s to s itself in A must send every state t to t itself in B if there is a homomorphism from A to B. Therefore there is no transition between the states (p,a) and (q,b) in A $\times$ B when a $\neq$b. Thus, if the direct product of two strongly connected automata is strongly connected then there is no homomorphism between them. When the transition monoid of an automaton is a group, we say the automaton is a permutation automaton.

Corresponding author

Received June 9, 2013

When an automaton has at least one reset input function, i.e. an input function which maps every state into a single fixed state, we say the automaton is a synchronizing (reset) automaton. Hong [7] showed that the direct product of a permutation strongly connected automaton and a reset strongly connected automaton is strongly connected. If a strongly connected automaton is decomposable, then the quotient automata obtained by decomposition must be strongly connected because of the canonical homomorphisms. If an automaton can be decomposed into two quotient automata, it is isomorphic to the direct product of its quotient automata. Specifically, if a strongly connected automaton can be decomposed into a permutation strongly connected quotient automaton and a synchronizing strongly connected quotient automaton, then the quasi ideal automaton is isomorphic to the direct product of a permutation strongly connected automaton and a synchronizing strongly connected automaton.

2. Preliminaries

The definition of an automaton in this paper is from [3]. For a non-empty finite set X, we denote the free monoid over X by X* and the empty input string by ε. An automaton 1 is a triple $A = (Q, X, \delta)$ where Q is a set of states; X is a non empty set called the input alphabet; $\delta : Q \times X^* \to Q$ is the transition function satisfying $\forall\ p \in Q$, $\forall x,y \in X^*$, $\delta(p, xy) = \delta(\delta(p,x),y)$ and $\delta(p,\varepsilon) = p$. We often denote $\delta(p,x)$ by px when there is no chance of ambiguity. An automaton is finite if and only if its set of states is finite. All automata are finite in this paper.

For the sake of brevity, $A = (Q, X, \delta)$ and $B = (R, X, \gamma)$ are finite non-trivial automata throughout this paper. An automaton A is strongly connected (transitive) if for every $p,a, \in S$, there exists $x \in \Sigma^*$ such that $\delta(p,x) = a$. A mapping $\alpha : Q \to R$ such that $\alpha(\delta(p,x)) = \gamma(\alpha(p),x)$ for every $p \in Q$, $x \in X^*$ is a homomorphism from A to B. An isomorphism is a bijective homomorphism. If there is an isomorphism between A and B, we say A is isomorphic to B, denoted by $A \cong B$.

Define a relation $\equiv_A$ on X* by $\forall x,y \in X^*$, $x \equiv_A y$ if and only if $\delta(p, x) = \delta(p,y)$ for every $p \in Q$. The relation in $\equiv_A$ is an equivalence relation. Let $x \in X$, we denote the equivalence class $\{y \in X^* : x \equiv_A y\}$ by $|x|_A$. We denote $\{|x|_A : x \in X^*\}$ by M(A). We define an operation on M(A) by $|x|_A|y|_A = |xy|_A$ and $|\varepsilon|_A|x|_A = |x|_A|\varepsilon|_A = |x|_A|\varepsilon|_A = |x|_A$ for every $|x|_A.|y|_A \in M(A)$. M(A) with this operation is a monoid with the identity $|\varepsilon|_A$ and is called the transition monoid of A. We denote $\{|x|_A : x \in X^+\}$ by S(A) where $X^+ = X^* - \{\varepsilon\}$. We define an operation on S(A) by $|x|_A|y|_A = |xy|_A$ for every $|x|_A, |y|_A \in S(A)$. S(A) with this operation is a semigroup and is called the transition semigroup of A.

When viewing $x \in X^*$ as a function from Q to Q, i.e. $x : Q \to Q$, $x(p) = \delta(p,x)$ for every $p \in Q$ we call x an input function. We denote the range of an input function x by $I_{mA}(x) = \{\delta(p,x) : p \in Q\}$ and the rank of x by $|I_{mA}(x)|$. A right ideal $\mathfrak{R}$ of a semigroup S is a nonempty subset of Q such that $r \in \mathfrak{R}$ and $p \in Q$ imply $rp \in \mathfrak{R}$. This is equivalent to $\mathfrak{R}Q \subseteq \mathfrak{R}$. Similarly a left ideal of Q is a nonempty subset **L** satisfying Q $\mathbf{L} \subseteq \mathbf{L}$. An ideal I of a semigroup S is a subset which is both a left and a right ideal J of S, $J \subseteq I$ implies J = I.

An idempotent element of a semigroup S is an element $e \in S$ such that ee = e. Any finite semigroup has at least one idempotent element.

Let $I(A) = \{ |x|_A \in S(A) : |I_{mA}(x)| \text{ is minimal}\}$ be the minimal ideal of the transition semigroup S(A) (by [8] Chapter 2 Preposition 1.3). I(A) is called the minimal transition ideal of A. I(A) is a finite semigroup. Thus it has at least one idempotent element. Denoted the set of all the idempotent elements in I(A) by E(A). Note that $E(A) \subseteq I(A)$. A is a permutation automaton if and only if its transition monoid M(A) is a group if and only if px =qx,implies p = q for all$p,q \in Q, x \in X^*$.It is possible that the transition semigroup S(A) of an automaton is a group even when A is not a permutation automaton. So if an automaton A is strongly connected and its transition semigroup S(A) is a group then its transition monoid M(A) is a group,i.e. A is a permutation automaton. Let $x \in X^*$, x is a reset input function of A if there exists $q \in Q$ such that $\delta(p,x) = q \ \forall p \in Q$.If x is a reset input function of A then $[x]_A$ is a right zero element of the transition semigroup S(A). Thus $z\,x \equiv_A x$ and xz is a reset input function of A $\forall \ x \in X^*$, If x is a reset input function of A. An automaton A is called a synchronizing (reset) automaton, if A has atleast one reset input function. The minimal ideal I(A) of a synchronizing automaton, A consists of all reset input functions of A, hence a right zero group[4].A semigroup S(A) is right simple if S itself is the only right ideal of S.A semigroup S is right simple if and only if aS = S for every $a \in S$ if and only if $\forall a,b \in S$,there exists $x \in S$ such that ax = b [4]. A semigroup is a right group if and only if it is a right simple group and has an idempotent element [4].Thus the finite semigroup is a right group if and only if it is right simple.

3. Strongly Connected Automaton

Definition1

An automaton A is triplet A =(Q,X,δ) where Q is a nonempty finite set of states, X is a nonempty finite set of binary alphabets X and δ is the next state function called state transition function such that δ(a, xy) =δ(δ(a,x),y) and δ(a,ε)=a ∀ a∈Q and all x,y∈X^*.Here X^* is the free semigroup generated by the elements of X and ε is its identity.

Definition 2.

Let A=(Q,X,δ) be an automaton, A permutation ρ on Q is called an automorphism of the automaton A if ρ(δ(a,x))=δ(ρ(a),x) for all a∈Q, and x∈X^*.Then set of all automorphisms of A forms a group, denoted G(A) and it is called automarphism group of A. Here the product gh∈G(A) of g, h∈G(A) means gh(a)=g(h(a))∀a∈Q.

Definition 3.

An automation A= (Q,X,δ) is called a permutation automaton if δ(a,σ) is a permutation on Q, ∀σ∈X (ie) every input function is bijective.

δ	0	1
p	p	q
q	r	p
r	q	r

Table(i): Transition Table for A.

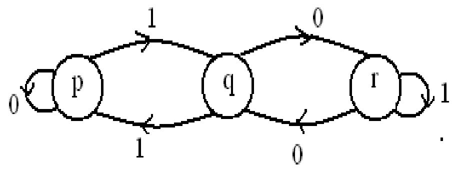


Fig.1.permutation automaton

Definition 4. Synchronizing Automaton

In the theory of DFA, a synchronizing word (or) reset sequence is a word in the input alphabet of the DFA which sends any state of the DFA to one and the same state.(ie) An Automaton admitting reset word is called Synchronizing Automaton.

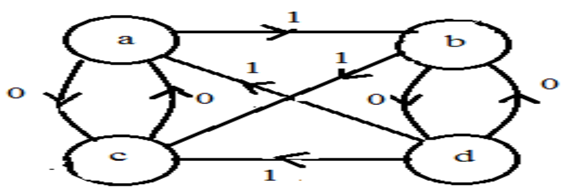


Fig.2. Synchronizing Automaton

If there is word x∈X* such that a,b,c ∈Q where δ(a,x)=δ(b,x) (or equivalently|δ(Q,w)|=1). Then we say that w synchronizes A then A is synchronizing Automaton.

Definition 5. Strongly connected Automaton

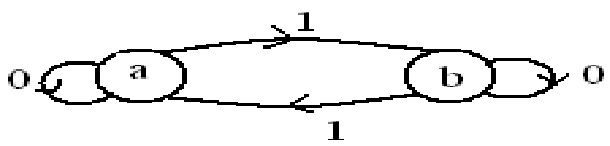


Fig.3 Strongly Connected Automata

An Automaton A= (Q,X,δ) is strongly connected if for every a,b∈Q, there exists x∈X* such that δ(a,x) =b.(ie) for every input, there exists path between the two states.

Definition 6. Synchronizing strongly connected Aleshin type Automaton

γ	0	1
a	c	b
b	b	c
c	a	a

Table (ii): Transition Table for

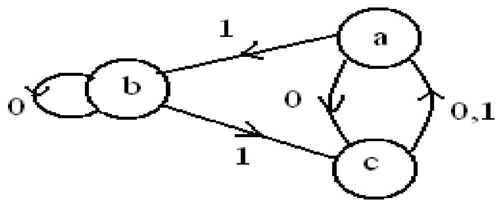


Fig.4. Aleshin Type Automata

Here B=(R,X,γ) is said to be the synchronizing strongly connected Aleshin type automaton, if the automaton is strongly connected, synchronized Aleshin type automaton[1]

Lemma 1.

A synchronizing strongly connected Aleshin type automaton has a property that the cardinality of the minimal transition ideal equals the number of [10] states of the automaton. Here B is the synchronizing strongly connected automata.

Lemma 2.

Let A be a strongly connected automaton. If A has at least one reset input function then there exists a unique (up to the equivalence relation $\equiv_A$) reset input function x_p , for every $p \in Q$, such that $\delta(q, x_p) = p$ for all $q \in Q$

Definition 7.

The direct product of an automaton A and an automaton B is defined as the automaton $AxB=(QxR, X, \delta_{AxB})$ where $\delta_{AxB}((p,a),x)=\delta(p,x),\gamma(a,x)$ for every $p \in Q$, $a \in R$, $x \in X^*$. Here A&B have the same binary input X.

4. Direct product of an Automaton A x B

Theorem 1.

The [7] direct product of a permutation strongly connected automaton and a synchronizing strongly connected Aleshin type automaton is also strongly connected.

Proof.

Let A be a permutation strongly connected automaton and B is synchronizing strongly connected Aleshin [2] type automaton. Let $(p,a),(q,b) \in QxR$. Since A is strongly connected, there exists $x \in X^*$ such that $\delta(p,x)=q$. since B is synchronous automaton, there exists a reset input function $y \in X^*$ such that $\gamma(a,y)=b \ \forall \ a \in R$. In particular $\gamma(a,y)=b$, M(A) is a group. Since A is a permutation automaton Thus there exists $z \in X^*$ such that $zy \equiv_A \varepsilon$. Consider the input string $xzy \in X^*$. $xzy=x(zy) \equiv_A x\varepsilon \equiv_A x$. On the other hand, $xzy \equiv_B y$ since y is a reset input function of B. Now $\delta_{AxB}((p,a), xzy) = (\delta(p,xzy), \gamma(q,xzy) = (\delta(p,x),\gamma(a,y)) = (q,b)$. Hence AxB is strongly connected.

δ_{AxB}	0	1
(p,a)	(p,c)	(q,b)
(p,b)	(p,b)	(q,c)
(p,c)	(p,a)	(q,a)
(q,a)	(r,c)	(p,b)
(q,b)	(r,b)	(p,c)
(q,c)	(r,a)	(p,a)
(r,a)	(q,c)	(r,b)
(r,b)	(q,b)	(r,c)
(r,c)	(q,b)	(r,a)

Table (iii) Transition Table for AxB

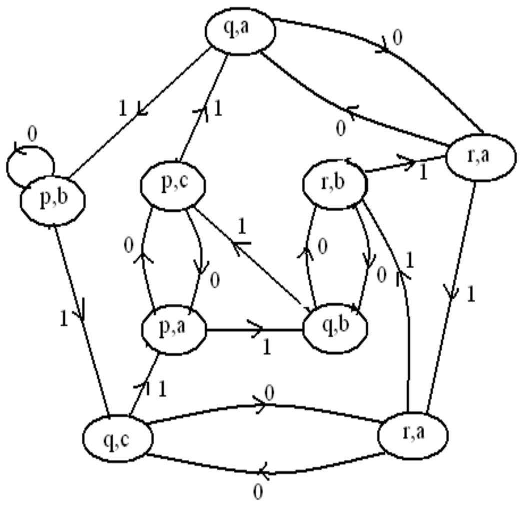


Fig.5. Synchronizing strongly connected Aleshin type

The direct product of A and B, AxB = (QxR, X, δ_{AxB}) where δ_{AxB} is defined by the transition table.

{(p,a)x:x∈X*} = {(p,b)x: x∈X*} {(p,c)x:x∈X*}

{(q,a)x:x∈X*} = {(q,b)x: x∈X*} {(q,c)x:x∈X*}

{(r,a)x:x∈X*} = {(r,b)x: x∈X*} {(r,c)x:x∈X*}

Therefore A x B is strongly connected.

Lemma 3

Let A and B be automata. Let x ∈ X* then

(i) $I_{mAxB} \in I_{(A)} = I_{mA}(x) \times I_{mB}(x)$

(ii) $[x]|_{AxB} \in I(AxB)$ if and only if $|x|_A \in I(A)$ and $|x|_B \in I(B)$

Proof

(i) It is obviously true

By (i), $| I_{mAxB}(x)|$ is minimal if and only if both $|I_{mA}(x)|$ and $| I_{mB}(x)|$ are minimal

If $\mathfrak{R}$ is a right group and E is the set of all idempotent in $\mathfrak{R}$, then $\mathfrak{R} = \cup_{e\in E}{}^{Re}$. By [8], thus $I(A) = \cup_{[e]A\in E(A)} I(A)[e]_A$ if I(A) is a right group.

Lemma 4.
Let A be a strongly connected automaton. If I(A) is a right group, then $Q = \cup_{e \in E(A)} I_{mA}(e)$.

Proof.

I(A) itself is its only right ideal, by [8] then $Q = \cup_{[x]_A \in I(A)} I_{mA}(x)$. Since $I_{mA}(xe) \subseteq I_{mA}(e)$ for $[x]_A \in I(A)$ and $[e]_A \in E(A)$. We have $Q = \cup_{[xe]_A \in I(A)E(A)} I_{mA}(xe) = \cup_{[e]_A \in E(A)} I_{mA}(e)$.

X	$\equiv_A$	p	q	r
ε		p	q	r
0		p	r	q
1		q	p	r
00	ε	p	q	r
01		q	r	p
10		r	p	q
11		p	q	r
010		r	q	p
011	0	p	r	q
100	1	q	p	r
101	010	r	q	p
0100	01	q	r	p
0101	10	r	p	q

Table(iv) : Input function of A

	ε	0	1	01	10	010	011	100
ε	ε	0	1	01	10	010	011	100
0	0	ε	01	1	010	10	11	100
1	1	10	11	101	110	1010	1011	11
01	01	010	011	ε	0110	10	1	011

Table(v): Cayley Table of I(A)

X	$\equiv_B$	a	b	c
ε		a	b	c
0		c	b	a
1		b	c	a
00	ε	a	b	c
01		a	c	b
10		b	a	c
11		c	a	b
010		c	a	b
011		b	a	c

	ε	0	1	01	10	010
ε	ε	0	1	01	10	010
0	0	ε	01	1	010	10
1	1	10	ε	010	0	01
01	01	010	0	10	ε	1
010	010	01	10	0	1	ε

Table(vii): Cayley Table of I(B)

100		b	c	a
101		c	b	a
0100		a	c	b
0101	ε	a	b	c

Table(vi) : Input function of B

Definition 8.

An automaton A is called a quasi ideal automaton if

(i) A is strongly connected

(ii) Its minimal transition ideal I(A) is a right group.

(iii) the ranges of the idempotent elements of its minimal transition ideal I(A) form a merging of a partition on Q that is, $\cup_{[e]_A \in E(A)} I_{mA}(e) = Q$ and $\forall [e]_A, [f]_A \in E(A)$, $I_{mA}(e) \cap I_{mA}(f) = \varphi$ or $I_{mA}(e) = I_{mA}(f)$.

Theorem 2.

The direct product of a permutation strongly connected automaton and synchronizing strongly connected Aleshin type Automaton is a quasi ideal automaton.

Proof.

Let A be a permutation strongly connected automaton and B be a synchronizing strongly connected Aleshin type automaton, By Theorem 1, A x B is strongly connected Automaton.Let $[x]_{AxB}$, $[y]_{AxB} \in I_{(AxB)}$, By Lemma 3(ii), $[x]_A$, $[y]_A \in I(A)$ and $[x]_B$, $[y]_B \in I(B)$ since A is permutation automaton, M(A) is a group. Therefore, there exists $x' \in X^*$ such that $xx' \equiv_A \varepsilon$. Then $xx'y \equiv_A y$. Since $[y]_B \in I(B)$, y is a reset input function of B. We have $xx'y \equiv_{AxB} y$, that is the minimal transition ideal $I_{(AxB)}$ of AxB is right simple hence a right group. By Lemma 4, $Q \times R = \cup_{[e]_{AxB} \in E(AxB)} I_{mAxB}(e)$, by the Table (viii) .

Suppose $I_{mAxB}(e) \cap I_{mAxB}(f) \neq \varphi$ for $[e]_{AxB}$, $[f]_{AxB} \in E(AxB)$. By Lemma 3(i), we have $I_{mA}(e) \cap I_{mA}(f) \neq \varphi$ and $I_{mB}(e) \cap I_{mB}(f) \neq \varphi$. Both e and f are reset input function of B, From Fig.(v), $|I_{mB}(e)| = |I_{mB}(f)| = 1$.

So, $I_{mB}(e) = I_{mB}(f)$. M(A) is group, $I_{mA}(e) = I_{mA}(f) = Q$. Hence $I_{m(AxB)}(e) = I_{mA}(e) \times I_{mB}(e) = I_{mA}(f) \times I_{mB}(f) = I_{m(AxB)}(f)$. Therefore $\{ I_{m(AxB)}(e) : [e]_A \in E(AxB)\}$ forms a partition of Q x R.

We have shown that we can compose a quasi-ideal automaton by taking the direct product of a permutation strongly connected automaton and a synchronizing strongly connected automaton. We shall decompose a quasi-ideal automaton by using automaton congruence relations.

An automaton congruence relation on an automaton A is an equivalence relation θ on Q compatible with the transition function, i.e. p θ q implies δ(p,z) θ δ(q,z) for all $z \in X^*$. We denote the θ equivalence class of $p \in Q$ by $[p]_\theta = \{q \in Q : q\ \theta\ p\}$ and $Q / \theta = \{[p]_\theta : p \in Q\}$. With an automaton congruence relation θ on A, we can construct θ-quotient automaton $A/\theta = (Q/\theta, X, \delta_{A/\theta})$ where , $\delta_{A/\theta} : Q/\theta \times X^* \to Q/\theta$ is defined by $\delta_{A/\theta}(|p|_\theta,z)=[\delta(p,z)]_\theta$ for all $z \in X^*$.

Theorem 3.

Let A be a strongly connected automaton. If the minimal ideal I(A) is a right group, then there exists anautomaton congruence relation π on Q defined by$\forall p,q \in Q, p\pi q$ if and only if δ(p,x) = δ(q,x) for every $[x]_A \in I(A)$.The π-quoitent automaton A/π is a permutation strongly connected automaton.

Proof.
I(A) itself is its only right ideal. By [6], A/π is a permutation automaton. A/π is strongly connected since A is strongly connected.From Table(vi) and Table(vii). Since I(A) is a right group, so is I(A) a right simple group and has an idempotent element (00) and (11).

Theorem 4.
Let A be a quasi ideal automaton. There exists an automaton congruence relation ρ on Q defined by $\forall$ $p,q \in Q$, p ρ q if and only if $p,q \in I_{mA}(e)$ for some $[e]_A \in E(A)$. The [10] ρ - quotient automaton A/ρ is a automaton A/ρ is a synchronizing strongly connected automaton SSCA .

Proof.

From Table(viii), $\{ I_{mA}(e): [e]_A \in E(A)\}$ forms a partition on Q. Let ρ be the equivalence relation on Q induced by the partition that is for every $p,q \in Q$, p ρ q if and only if $p,q \in I_{mA}(e)$ for some $\{ I_{mA}(e): [e]_A \in E(A)\}$ forms a partition on Q. Let ρ be the equivalence relation on Q induced by the partition that is for every p,q∈Q, p ρ q if and only if p,q ∈ I_{mA}(e) for some $[e]_A$∈E(A) by Table(viii). To show ρ is an automaton congruence relation let p,q ∈Q, Z∈X*. Suppose p ρ q, then there exists $[e]_A$∈E(A) and p'q'∈Q such that p=p'e and q=q'e. Since I(A) is an ideal. $[ez]_A$∈I(A) I(A) is right group. I(A)= a? $a_{|f|A}a$? ?E(A)I(A)$|f|_{A.}$ Then there exists $[x]_A$∈I(A) and $|f|_A$∈E(A) such that ez $\equiv_A$ xf. Thus pz = p'ez = p'xf ∈ I_{mA}(f) and similarly qz ∈ I_{mA}(f). Therefore pz ρ qz. A/ρ is strongly connected since A is strongly connected. .From Table(ix),Let $[e]_A$∈E(A), p∈ I_{mA}(e). Consider $[p]_ρ$. Let $[q]_ρ$∈Q/p. Since te∈ I_{mA}(e), we have p ρ qe, that is $[p]_ρ$=$[qe]_ρ$ =$\delta_{A/\rho}$($[q]_{\rho,}$e) since $[q]_ρ$ was arbitrarily choosen, e is a reset input function of A/ρ. Thus A/ρ is a synchronizing automaton.

We have shown that we can decompose a quasi-ideal automaton into two strongly connected quotient automata, one is permuting and other is synchronizing.
Define a binary operation on relations π and ρ on a set Q by $\pi \circ \rho$ = {(p,q) ∈ Q × Q / ∃u∈Q} such that {(p,u)∈ π and (u,q) ∈ ρ} we denote the equality identity relation {(p,p) : p ∈ Q by 1_Q}.

Theorem 5.

Let A be an automaton.[6],[9], If there exists automaton congruence relations π and ρ on A such that $\pi \cap \rho = 1_Q$ and $\pi \circ \rho = Q \times Q$ then $A \cong A/\pi \times A/\rho$

Proof.

Define α:Q→Q/π x Q/ρ by α(p) = ($[p]_π$,$[p]_ρ$) for all p∈Q. α is well defined and a homomorphism. Let p,q∈Q and α(p) = α(q) that is ($[p]_π$,$[p]_ρ$) = ($[q]_π$,$[q]_ρ$). Then p π q and p ρ q that is (p,q) ∈ $\pi \cap \rho = 1_Q$. We have p = q. Hence α is injective. To show that α is surjective. Let ($[p]_π$,$[q]_ρ$)∈ Q/π × Q/ρ. Since (p,q)∈Q×Q = $\pi \circ \rho$. There exists u∈Q such that (p,u)∈π and(u, q)∈ρ. Thus $[p]_π$ = $[u]_π$ and $[u]_ρ$ = $[q]_ρ$. We have α(u) = ($[u]_π$,$[u]_ρ$) = ($[p]_π$.$[q]_ρ$) By Theorem 2, proved the if part. Let A be a Quasi-ideal Automaton. By Theorem 3,

A/π is a permutation strongly connected automaton where π is the automaton congruence relation defined on Q by p,q∈Q, p π q if and only if δ(p,x) = δ(q,x) for every $[x]_A$ ∈ I(A). By Theorem 4, A/ρ is a synchronizing strongly connected automaton where ρ is the automaton congruence relation defined on Q by ∀ p,q ∈ Q, p ρ q if and only if p,q ∈ I_{mA}(e) for some $[e]_A$ ∈ E(A). Let p,q ∈ Q.

X	$\equiv_{AxB}$	(p,a)	(p,b)	(p,c)	(q,a)	(q,b)	(q,c)	(r,a)	(r,b)	(r,c)
ε		(p,a)	(p,b)	(p,c)	(q,a)	(q,b)	(q,c)	(r,a)	(r,b)	(r,c)
0		(p,c)	(p,b)	(p,a)	(r,c)	(r,b)	(r,a)	(q,c)	(q,b)	(q,a)
1		(q,b)	(q,c)	(q,a)	(p,b)	(p,c)	(p,a)	(r,b)	(r,c)	(r,a)
00	ε	(p,a)	(p,b)	(p,c)	(q,a)	(q,b)	(q,c)	(r,a)	(r,b)	(r,c)
01		(q,a)	(q,c)	(q,b)	(r,a)	(r,c)	(r,b)	(p,a)	(p,c)	(p,b)
11	ε	(p,c)	(p,a)	(p,b)	(q,c)	(q,a)	(q,b)	(r,c)	(r,a)	(r,b)
10		(r,b)	(r,a)	(r,c)	(p,b)	(p,a)	(p,c)	(q,b)	(q,a)	(q,c)
000		(p,c)	(p,b)	(p,a)	(r,c)	(r,b)	(r,a)	(q,c)	(q,b)	(q,a)
001		(q,b)	(q,c)	(q,a)	(p,b)	(p,c)	(p,a)	(r,b)	(r,c)	(r,a)
010		(r,c)	(r,a)	(r,b)	(q,c)	(q,a)	(q,b)	(p,c)	(p,a)	(p,b)
011		(p,b)	(p,a)	(p,c)	(r,b)	(r,a)	(r,c)	(q,b)	(q,a)	(q,c)
100		(q,b)	(q,c)	(q,a)	(p,b)	(p,c)	(p,a)	(r,b)	(r,c)	(r,a)
101		(r,c)	(r,b)	(r,a)	(q,c)	(q,b)	(q,a)	(p,c)	(p,b)	(p,a)
110		(p,a)	(p,c)	(p,b)	(r,a)	(r,c)	(r,b)	(q,a)	(q,c)	(q,b)
111		(q,a)	(q,b)	(q,c)	(p,a)	(p,b)	(p,c)	(r,a)	(r,b)	(r,c)

Table(viii) : Input function of AxB

X	$\equiv_{AxB}$	**p**	**q**	**r**
ε		**p**	**q**	**r**
00		**p**	**q**	**r**
11		**p**	**q**	**r**

Table (ix): Idempotent of AxB

X	**A/π**	**p**	**q**	**r**
ε		**p**	**q**	**r**
0		**p**	**r**	**q**
1		**q**	**p**	**r**

Table (x) merging of p,q,r

Proof.

By Theorem 2, proved the if part. Let A be a Quasi-ideal Automaton. By Theorem 3, A/π is a permutation strongly connected automaton where π is the automaton congruence relation defined on Q by p,q∈Q, p π q, if and only if δ(p,x) = δ(q,x) for every $[x]_A$ ∈ I(A). By Theorem 4, A/ρ is a synchronizing strongly connected automaton where ρ is the automaton congruence relation defined on Q by ∀ p,q ∈ Q, p ρ q if and only if p,q ∈ $I_{mA}(e)$ for some $[e]_A$∈E(A). Let p,q ∈ Q.

ε	a	b	c	a	b	c	a	b	c
0	c	b	a	c	b	a	c	b	a
1	b	c	a	b	c	a	b	c	a
00	a	b	c	a	b	c	a	b	c
01	a	c	b	a	c	b	a	c	b
11	c	a	b	c	a	b	c	a	b
10	b	a	c	b	a	c	b	a	c
000	c	b	a	c	b	a	c	b	a
001	b	c	a	b	c	a	b	c	a
010	c	a	b	c	a	b	c	a	b
011	b	a	c	b	a	c	b	a	c
100	b	c	a	b	c	a	b	c	a
101	c	b	a	c	b	a	c	b	a
110	a	c	b	a	c	b	a	c	b

Table(xi): Cayley Table of I(A×B) of merging a,b,c

X	$\equiv_{AxB}$	**a**	**b**	**c**
ε		**a**	**b**	**c**
00		**a**	**b**	**c**
0101		**a**	**b**	**c**

Table (xii): Idempotent of AxB

X	**A/ρ**	**a**	**b**	**c**
ε		**a**	**b**	**c**
0		**c**	**b**	**a**
1		**b**	**c**	**a**

Table (xiii) merging of a,b,c

Suppose (p,q) ∈ $\pi \cap \rho$. Then p,q ∈ $I_{mA}(e)$. For some e∈E(A) since p ρ q. There exist p'q'∈Q such that p = p'e and q = q'e. We have pe = qe since p π Q. Now p = p'e = p'ee = pe = qe = q'ee = q'e = q. Hence (p,q)∈I_Q that is $\pi\cap\rho = I_Q$. Let (p,q)∈ Q×Q. By Lemma 2, q∈$I_{mA}(e)$ for some $[e]_A$∈E(A). Let $[x]_A$∈I(A). Since I(A) is rightsimple, ez$\equiv_A$x forz∈X*.By[9] px=pez=peez=(pe)x. So, pπpe. Since pe∈$I_{mA}(e)$, peρq. Hence (p,q)∈ $\pi\circ\rho$,we have $\pi\circ\rho$=Q×Q. By then [5] A ≅A/π × A/ρ

Theorem 6.

An automaton is isomorphic to the direct product of a permutation strongly connected automaton and synchronizing strongly connected Aleshin type [8] automaton if and only if it is a quasi ideal automaton.

Proof.

By Theorem 2, proved the if part. Let A be a Quasi-ideal Automaton. By Theorem 3, A/π is a permutation strongly connected automaton where π is the automaton congruence relation defined on Q by p,q∈Q, p π q if and only if δ(p,x) = δ(q,x) for every $[x]_A$ ∈ I(A). By Theorem 4, A/ρ is a synchronizing strongly connected automaton where ρ is the automaton congruence relation defined on Q by ∀ p,q ∈ Q, p ρ q if and only if p,q ∈ $I_{mA}(e)$ for some $[e]_A$∈E(A). Let p,q ∈ Q.

$I_{mA}(0)$= $I_{mA}(1)$={p,q,r}=Q All inputs of A are permutations. M(A)=I(A) is a group. {p:x∈X^*}={q:x∈X^*}={r:x∈X^*}=Q. A is strongly connected. A is strongly connected permutation automaton.

Let B={R,X,γ) be an automaton where R = {a,b,c}, X= {0,1}and γ is defined by transition function. ImB(00)={a,b,c} and ImB(0101)={a,b,c}.From figure 4, and Table (viii) and (ix), Both 0 and 1 are the reset input functions of B. I(B) is a right zero semigroup. {ax:x∈ X*}={bx:x∈ X*}={cx:x∈ X*}. B is strongly connected. B is synchronizing strongly connected automaton. The Cayley table of I(AxB),∀ [x]AxB∈ I(AxB) , [x]AxB I(AxB)=I(AxB). I(AxB) is right simple. (00)(00)=(00) and (11)(11)=(11). Thus 00 and 11 are idempotent elements of I(AxB). I(AxB) is a right group. ImAxB(00)={ (p,c), (p,b), (p,a), (r,c) ,(r,b), (r,a) ,(q,c) (q,b), (q,a) }. From Table(iv), when we are merging all 'p' columns, 'q' columns and 'r' columns,we get.

ImAxB(00)={p,r,q}. ImAxB(11)= {(q,b) ,(q,c) ,(q,a) ,(p,b), (p,c), (p,a), ,(r,b), (r,c),(r,a)}. From Table(iv), when we are merging all ‘q’ columns, ‘p’ columns and ‘r’ columns, we get ImAxB(11)= {q,p,r}.They are disjoint and the union of them is equal to QxR.

Let C= (QxR,X, δ_c) be an automaton, where (QxR)={ (p,a), (p,b), (p,c) ,(q,a) (q,b), (q,c) ,(r,a) ,(r,b), (r,c) } and $\delta_{AxB} = \delta_C$ is defined by the transition Table(iii). Direct product of AxB is C ≌ AxB, so C is quasi-ideal automaton.

$\delta_{c/\pi}$	p	q	r
0	p	r	q
1	q	P	r

Table(xiv)

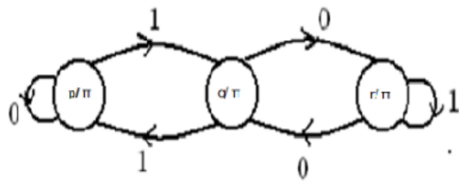
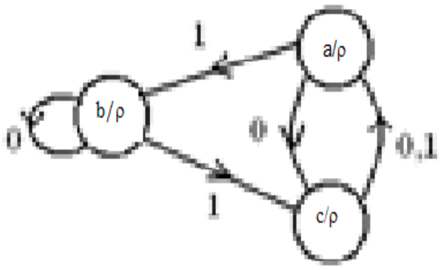


Fig.6

$\delta_{c/\rho}$	0	1
a	c	b
b	b	c
c	a	a



Table(xv)

Fig.7

I(AxB) is a right group, so is I(C). We define an automaton congruence relation [From Fig.6] π on C by ∀p,q∈Q,qπp if and only if δ_{AxB} (q,x)= δ_{AxB}(p,x) for every $[x]_C$=I(C). $[p]_\pi$={ (p,a), (p,b), (p,c)}; $[q]_\pi$={ q,a) (q,b), (q,c)} ;$[r]_\pi$={ (r,a) ,(r,b), (r,c)}.

We construct quotient automaton C/π = (Q/π, X, $\delta_{C/\pi}$) where Q/π={$[p]_\pi$, $[q]_\pi$, $[r]_\pi$} and [From Fig.6] $\delta_{C/\pi}$ is defined by the transition Table(x). E(C)={$[00]_C$,$[11]_C$}. I_{mC}(00)={p,r,q}, I_{mC}(11)={q,p,r}. They form the merging of partition on Q. We define on automaton congruence relation ρ on C by ∀ a,b ∈R, b ρ a if and only if a,b ∈ I_{mC}(e) for some $[e]_C$ ∈ E(C). From Fig.7 and Table(xi),$[a]_ρ$={(p,a),(q,a,(r,a)}, $[b]_ρ$={(p,b),(q,b),(r,b} and $[c]_ρ$={(p,c),(q,c),(r,c)}. we construct a quotient automaton C/ρ={R/ρ,X, $δ_{C/ρ}$}, Where R/ρ={$[a]_ρ$, $[b]_ρ$, $[c]_ρ$} and $δ_{C/ρ}$ is defined by the transition Table(xi).

Theorem 7.

In synchronized strongly connected aleshin type automata, reverse of inverse of aleshin type automaton is equal to inverse of reverse, i.e., **RI[A(S)]=IR[A(S)]**

Proof.

In Fig. 9 , **δ[(p,a), 1]= (p,c) ; δ[(p,b), 1] = (p,b); δ[(p,c), 1] = (p,a) ;**
δ[(q,a),1] = (r,c) ;

δ[(q,b),1] =(r,b), δ[(q,c),1] = (r,a); δ[(r,a) ,1] = (q,c), δ[(r,b),1] = (q,b) ; δ[(r,c),1] = (q,a)

δ[(p,a), 0]= (q,b) ; δ[(p,b), 0] = (q,c); δ[(p,c), 0] =(q,a); δ[(q,a),0] = (p,b) ;δ[(q,b),0] = (p,c),

δ[(q,c),0] = (p,a); δ[(r,a) ,0] = (r,b), δ[(r,b),0] = (r,c); δ[(r,c),0] = (r,a)

In Fig. 11, **δ[(p,a), 1]= (p,c) ; δ[(p,b), 1] = (p,b); δ[(p,c), 1] = (p,a) ;**

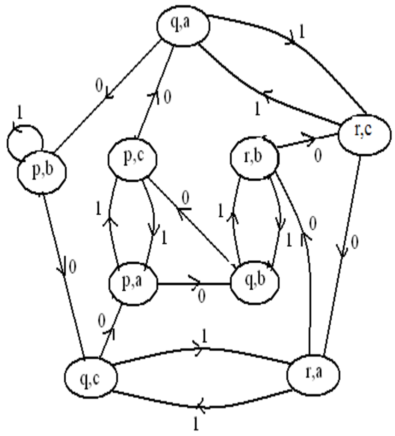
δ[(q,a),1] = (r,c) ;

δ[(q,b),1] =(r,b), δ[(q,c),1] = (r,a); δ[(r,a) ,1] = (q,c), δ[(r,b),1] = (q,b) ; δ[(r,c),1] = (q,a)

δ[(p,a), 0]= (q,b) ; δ[(p,b), 0] = (q,c); δ[(p,c), 0] =(q,a); δ[(q,a),0] = (p,b) ;δ[(q,b),0] =(p,c),

δ[(q,c),0] = (p,a); δ[(r,a) ,0] = (r,b), δ[(r,b),0] = (r,c) ; δ[(r,c),0] = (r,a)

Reverse Automata **Inverse Reverse Automata**



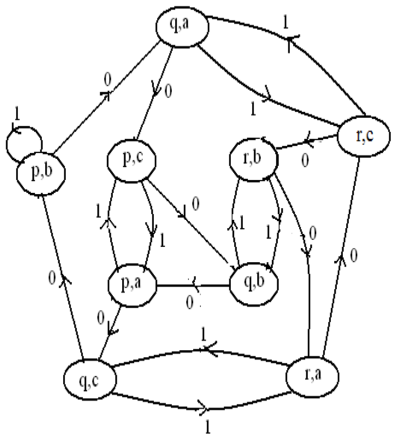
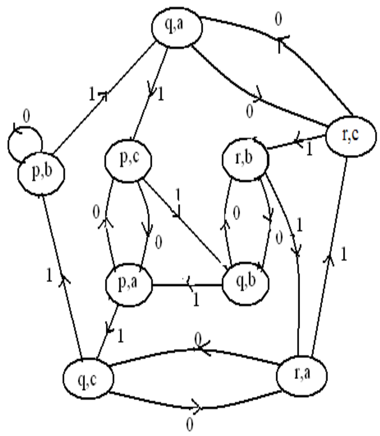


Fig.8 **Fig. 9**

Inverse Automata **Inverse Reverse Automata**



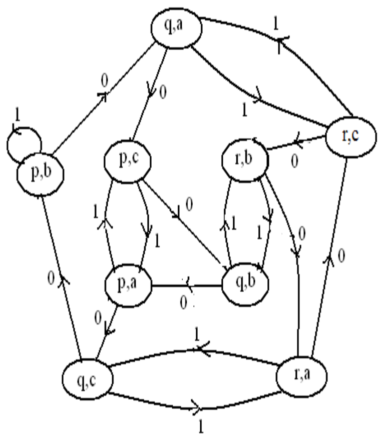


Fig. 10 **Fig.11**

Fig. 9 and Fig.11 R[I(A(S))] = I[R(A(s))]

Therefore, LHS = RHS, Hence the theorem.

Definition 9.

An automaton A is said to be persistent if the automaton is continuously, constantly working in their existing path with their respective input alphabet. Here the direct product of A x B is persistent

Conclusion

The strongly connected quasi Ideal automaton is isomorphic to the direct product of strongly connected permutation automaton and the synchronizing strongly connected Aleshin type automaton. The reverse of inverse of the strongly connected Aleshin type automaton is equal to the inverse of reverse of strongly connected Aleshin type automaton. So it is reversible and bireversible. Hence the direct product of strongly connected aleshin type automaton is invertible, also persistent.

REFERENCES

[1]Burger, M. and Mozes, S., Finitely presented simple groups and products of trees, C. R. Acad. Sci. Paris Sér. I Math., *324* (7) (1997), 747--752.

[2] Burger, M. and Mozes, S., Groups acting on trees: from local to global structure, Inst. Hautes Études Sci. Publ. Math. (92) (2000), 113--150 (2001).

[3] Z. Bavel, Introduction to the theory of automata (Reston Publishing Company, Reston, 1983).

[4] A. H. Cli_ord and G. B. Preston, The Algebraic Theory of Semigroups, Vol. 1 (American Mathematical Society, Providence, 1961).

[5] A. C. Fleck, On the strong connectedness of the direct product, IEEE Trans.Electronic Computers 16 (1967) 90.

[6] M. A. Harrison, Introduction to switching and automata theory (McGraw-Hill, New York, 1965).

[7] K. S. Hong, Strongly related automata and connectivity in the direct product of _nite automata, Dept. of Computer Science, University of Kansas, Master Research Report(1979 ?).

[8] G. Lallement, Semigroups and combinatorial applications (John Wiley & Sons, Inc, New York-Chichester-Brisbane-Toronto, 1979).

[9] Y. Masunaga, S. Noguchi and J. Oizumi, A Characterization of automata and a direct product decomposition, Journal of Computer and System Sciences 13 (1976) 74-89.

[10]On Direct product and Quotient of strongly connected Automata, Zino H.Hu Research Gate-2013