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Isoperimetric characterization of upper curvature bounds

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1. Introduction

1.1. Main result

We say that a metric space X satisfies the *Euclidean isoperimetric inequality for curves* if any closed Lipschitz curve $\gamma: S^1 \to X$ bounds a Lipschitz map of the unit disc $v: \overline{D} \to X$ whose parameterized Hausdorff area is at most $\ell_X^2(\gamma)/4\pi$. Here, $\ell_X(\gamma)$ denotes the length of γ in X. We refer to the first two sections below for the notion of parameterized Hausdorff area and other basic notions of metric geometry involved in the following main theorem of the present paper.

Theorem 1.1. *Let X be a proper metric space in which any pair of points is connected by a curve of finite length. Let X^i denote the set X with the induced length metric. The space X^i is* CAT(0) *if and only if X satisfies the Euclidean isoperimetric inequality for curves.*

This result provides an analytic access to upper curvature bounds and can be used to recognize upper curvature bounds without being able to identify geodesics or angles, a situation often appearing in metric constructions; cf. [1]. For instance, it is used in [25] to control upper curvature bounds under conformal changes of the distance, a result inaccessible by purely geometric means. Theorem 1.1 admits a natural generalization to non-zero curvature bounds; see Theorem 1.4 below.

The "only if" part of our theorem is folklore and follows as an easy consequence of Reshetnyak's majorization theorem, [37]. With a different definition of area, the "only if" part already appears in [3] at the very origin of the theory of spaces with upper curvature bounds.

Results closest to the much subtler "if" part of our theorem have been proven in [7] and [35] in the case of surfaces. Beckenbach and Rado proved in [7] our Theorem 1.1 for smooth 2-dimensional Riemannian manifolds, finding a connection between log-subharmonicity, isoperimetric inequalities and curvature bounds. In [35] the result of [7] was extended to some singular surfaces and non-zero curvature bounds.

Our theorem is motivated by Gromov's characterization [14] of Gromov hyperbolic spaces by subquadratic or small quadratic isoperimetric inequalities for large curves. Theorem 1.1 can be viewed as the borderline case and the non-rough version of Gromov's theorem and its optimal improvement in [42]. In fact, Theorem 1.1 admits a large scale version discussed in [43]. On the technical side, our result and its proof has some similarities with the work by Petrunin and Stadler in [33] and [31] on the curvature of discs satisfying some minimality properties.

1.2. Main idea

Essentially, the strategy of our proof of the "if" part is to reduce the problem in a general metric space to the situation considered in [35]. Recall the following simple consequence of the Gauss equation in Riemannian geometry: a minimal surface has curvature no larger than the ambient space. We reverse this idea and find a curvature bound for the total space by proving that all minimal discs have the corresponding curvature bound.

Theorem 1.2. *Let X be a proper metric space which satisfies the Euclidean isoperimetric inequality for curves. Let Γ be a Jordan curve of finite length in X and let $u: \overline{D} \to X$ be a solution of the Plateau problem in the space X for the boundary curve Γ. Then, the intrinsic minimal disc Z associated with u is a* CAT(0) *space.*

We refer to [26], [27] and §6 below for the notion of a solution of the Plateau problem and the associated intrinsic minimal disc. By definition of the intrinsic minimal disc Z, the solution of the Plateau problem u in Theorem 1.2 factorizes as $u=\bar{u}\circ P$ for a surjective map $P: \overline{D} \to Z$ and a 1-Lipschitz map $\bar{u}: Z \to X$. Moreover, $\bar{u}$ sends the boundary circle ∂Z of Z in an arclength-preserving way onto Γ.

It is not difficult to see that Theorem 1.2 implies Theorem 1.1. Assume for simplicity that the proper space X with the Euclidean isoperimetric inequality for curves is a length space. The existence of a solution u of the Plateau problem for any rectifiable Jordan curve Γ in X is proved in [26], generalizing [30] to the setting of metric spaces. In order to prove that X is CAT(0), one needs to prove that any Jordan triangle in X is thin; cf. §3 below. However, Theorem 1.2 implies that Γ is thin in the intrinsic minimal disc Z. Using the map $\bar{u}$ (which majorizes Γ in the sense of [37] and [2, §9.8]), this easily implies that Γ is thin in X as well.

1.3. Main steps

The proof of Theorem 1.2 involves several steps. First, a special case of the Blaschke–Santalo inequality implies that, among normed planes, only the Euclidean plane satisfies the Euclidean isoperimetric inequality, [41]. The quasi-convexity of the Hausdorff area proved in [10] and a natural blow-up argument imply that X has only "Euclidean tangent spaces", at least as far as infinitesimal properties of Sobolev maps with values in X are concerned. This is the property (ET) introduced in [26], which greatly simplifies the description of Sobolev maps and solutions of the Plateau problem.

In particular, the solution of the Plateau problem u in Theorem 1.2 is a conformal map; see §4. Thus, there exists a non-negative Borel function $f \in L^2(D)$, *the conformal factor of* u, such that, for *almost all* curves γ in D, the length of the image of γ under u is controlled by f:

$$\ell_X(u \circ \gamma) = \int_\gamma f. \tag{1.1}$$

The next step goes back to [7] and shows that the isoperimetric inequality forces f to be log-subharmonic.

The subsequent step, contained in [35], relates log-subharmonicity of conformal factors to non-positive curvature in the sense of Alexandrov. More precisely, the length metric defined on D by setting the length of *every* rectifiable curve $\gamma \subset D$ to be $\int_\gamma f$ is locally CAT(0). The metric space Y defined in this way is intimately related to the intrinsic minimal disc Z. The only difference is that (1.1) holds in Y for *all*, and in Z for *almost all*, rectifiable curves γ. In particular, we have a 1-Lipschitz map $I: Y \to Z$ which preserves the length of almost all curves.

The final, rather subtle step is devoted to the proof that the spaces Z and (the completion of) Y are identical. While the analytically defined conformal factor f controls the lengths of *almost all* curves in Z, it cannot control the lengths of all curves: contracting one interval in D to a point does not change the conformal factor. In particular, f does not a priori control the most important boundary curve. Applying some cutting and pasting tricks, we reduce the final step to the question whether the length of the boundary curve "is controlled by the conformal factor". Using general structural results about the intrinsic minimal discs obtained in [27], the final step reduces to the following.

THEOREM 1.3. *Let Z be a geodesic metric space homeomorphic to the closed disc $\overline{D}$. Denote by ∂Z the boundary circle and assume that $Z \setminus \partial Z$ is locally* CAT(0). *Then, the following are equivalent*:

(1) *Z is* CAT(0);

(2) *$Z \setminus \partial Z$ with the metric induced from Z is a length space*;

(3) *for any Jordan curve* $\eta \subset Z$ *the open disc* J_η *enclosed by* η *in* Z *satisfies*

$$\mathcal{H}^2(J_\eta) \leqslant \frac{\ell_Z^2(\eta)}{4\pi}.$$

Throughout the text, we denote by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure. Thus, condition (3) in Theorem 1.3 is the geometric (unparameterized) version of the Euclidean isoperimetric inequality. This theorem does not sound very surprising. The implication from (1) to (3) is an easy consequence of Reshetnyak's majorization theorem. The equivalence of (2) and (1) is not very difficult either. In contrast, the proof of the main implication from (3) to (1) is rather long and technical, and comprises one half of this paper. We refer the reader to §10, where the principal steps in the proof of Theorem 1.3 are explained in more detail. One might have the following example in mind in order to grasp the problem one faces when trying to prove this implication.

Start with a complicated Jordan curve Γ in $\mathbb{R}^2$, for example Koch's snowflake. Define Z as the closure of the Jordan domain of Γ, and consider a new metric on Z associated with a new length structure ([9]) given as follows. We let all curves in the interior $Z\setminus\Gamma$ of Z have Euclidean lengths, but we give the boundary curve $\Gamma=\partial Z$ some finite length by using an artificial parametrization of Γ by a large circle. The arising metric space keeps the Euclidean topology of the disc and is flat outside of ∂Z. The artificial shortening of Γ turns Γ into a local geodesic in Z, singular in the following sense. Some small parts of Γ cannot be approximated by curves in $Z\setminus\partial Z$ with almost the same length. Thus $Z\setminus\partial Z$ is not a length space. The (not very deep) implication (2) to (1) shows that Z cannot be CAT(0). Thus, the proof of the implication (3) to (1) must detect in this space Z a Jordan curve η violating the isoperimetric inequality. This Jordan curve η must consist of a Euclidean arc attached at both endpoints to an arc contained in the boundary ∂Z. The boundary ∂Z, a local geodesic in Z, is a very complicated curve, when considered from the controllable part $Z\setminus\partial Z$. This makes the finding of the desired curve η technically involved. (To make the example more complicated, change first the metric inside the Jordan domain of Γ by a smooth conformal factor in such a way that the curvature is everywhere non-positive and tends to $-\infty$ in the neighborhood of Γ).

1.4. Generalization to non-zero curvature bounds

Theorem 1.1 generalizes to other curvature bounds. The extension is achieved along the same route and involves only minor difficulties of notational and technical nature. In order to formulate the statement, we introduce the notion of a *Dehn function.* Let X be a metric space. Let $\delta\colon(0,\infty)\to[0,\infty]$ be a function. We say that X satisfies the *δ-isoperimetric inequality*, if, for any $r>0$, any Lipschitz curve $\gamma: S^1 \to X$ of length $\leqslant r$ bounds a Lipschitz disc $u: \overline{D} \to X$ of parameterized Hausdorff area $\leqslant \delta(r)$. The Dehn function δ_X of X (with respect to Lipschitz discs) is the infimum of all functions $\delta: (0, \infty) \to [0, \infty]$ for which X satisfies the δ-isoperimetric inequality.

For any real number $\varkappa$, we consider the simply connected space form $M^2_\varkappa$ of curvature $\varkappa$, and denote by $R_\varkappa \in (0, \infty]$ twice the diameter of $M^2_\varkappa$. Let $\delta_\varkappa$ be the Dehn function of $M^2_\varkappa$. We can now state the generalization of Theorem 1.1 to non-zero curvature bounds.

THEOREM 1.4. *Let X be a proper metric space in which any pair of points is connected by a curve of finite length. Let X^i be the set X with the induced length metric. The space X^i is* $\mathrm{CAT}(\varkappa)$ *if and only if the Dehn function δ_X of X satisfies $\delta_X \leqslant \delta_\varkappa$ on the interval* $(0, R_\varkappa)$.

1.5. Structure of paper and final comments

The paper consists of two parts and one appendix. The first part, which relies heavily on the existence and regularity theory of solutions of the Plateau problem, reduces Theorem 1.1 and Theorem 1.2 to Theorem 1.3. We closely follow the plan sketched above. The second part is devoted to the proof of Theorem 1.3. It consists of purely 2-dimensional metric geometry. The structure of this part is explained in §10. In the appendix, we explain the minor additional difficulties arising in the proof of Theorem 1.4, and sketch the solutions of these problems.

Remark 1.1. Once Theorem 1.1 has been proven, the statements of Theorem 1.2 and Theorem 1.3 can be strengthened; see [28], [31], [33] and [40].

Remark 1.2. In convex and metric geometry, there are many natural ways to measure area of 2-rectifiable sets and Lipschitz discs besides the Hausdorff area. The most famous among such definitions of area are Gromov's mass* and the Holmes–Thompson definition of area. We refer to [41] and [27] for lengthy discussions on definitions of area. As explained above, the first step of the proof of our main theorem uses an inequality from convex geometry to exclude all non-Euclidean tangent planes. The argument applies to all quasi-convex definitions of area μ with the following property: among all normed planes, only the Euclidean plane satisfies the Euclidean isoperimetric inequality with respect to μ. For the Holmes–Thompson definition of area μ^{HT}, the Euclidean isoperimetric inequality holds sharply for all normed planes. Thus, Theorem 1.1 is valid for any quasi-convex definition of area μ which satisfies $\mu \geqslant \mu^{\mathrm{HT}}$ on all normed planes with equality only on the Euclidean plane. In particular, Theorem 1.1 remains valid for Gromov's mass* and for Ivanov's "inscribed Riemannian" definition of area.

The validity of the Euclidean isoperimetric inequality for curves with respect to the Holmes–Thompson definition of area might be related to other forms of convexity beyond CAT(0).

Remark 1.3. If the constant $1/4\pi$ in the formulation of the Euclidean isoperimetric inequality is replaced by any smaller constant, then the space X^i in Theorem 1.1 turns out to be a tree [29], [42].

Remark 1.4. As a consequence of Theorem 1.1, a proper geodesic metric space with Euclidean isoperimetric inequality for curves must be contractible. It would be interesting to know whether any topological conclusions can be drawn, if the constant $1/4\pi$ is replaced by a slightly larger constant $1/4\pi < C < 1/2\pi$.

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Part I. Structure of minimal discs

2. Basics on metric spaces

2.1. Notation

The Euclidean norm of a vector $v \in \mathbb{R}^2$ is denoted by $|v|$. We denote the open unit disc in $\mathbb{R}^2$ by D. Connected open subsets of $\mathbb{R}^2$ will be called *domains*. A metric space is called *proper* if its closed bounded subsets are compact. We will denote distances in a metric space X by d or d_X. Let $X = (X, d)$ be a metric space. The open ball in X of radius r and center $x_0 \in X$ is denoted by

$$B(x_0, r) = B_X(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

A *Jordan curve* in X is a subset $\Gamma \subset X$ which is homeomorphic to S^1. Given a Jordan curve $\Gamma \subset X$, a continuous map $\gamma: S^1 \to X$ is called a *weakly monotone parametrization* of

Γ if γ is a uniform limit of homeomorphisms $\gamma_i: S^1 \to \Gamma$. For $m \geqslant 0$, the m-dimensional Hausdorff measure on X is denoted by $\mathcal{H}^m = \mathcal{H}^m_X$. The normalizing constant is chosen in such a way that on Euclidean $\mathbb{R}^m$ the Hausdorff measure $\mathcal{H}^m$ equals the Lebesgue measure $\mathcal{L}^m$.

If no confusion is possible, we will identify parameterized curves and their unparameterized images, and denote them by the same symbol. The length of a curve γ in a metric space X will be denoted by $\ell_X(\gamma)$, or simply by $\ell(\gamma)$. A continuous curve of finite length is called *rectifiable*. A (local) *geodesic* in a space X is a (locally) isometric map from an interval to X. A space X is called *a geodesic space* if any pair of points in X is connected by a geodesic. A space X is a *length space* if, for all $x, y \in X$, the distance $d(x, y)$ equals $\inf_\gamma \{\ell_X(\gamma)\}$, where γ runs over the set of all curves connecting x and y.

2.2. Length metric associated with a map

We refer the reader to [9], [32], [27] for discussions of the following construction and related topics. Let X' and X be metric spaces. Let $u: X' \to X$ be a continuous map. Assume that, for any $y_1, y_2 \in X'$, there exists a continuous curve $\gamma: I \to X'$ connecting y_1 and y_2 such that the curve $u \circ \gamma$ has finite length. Then, we let $d_u(y_1, y_2) \in [0, \infty)$ be the infimum of lengths of all such curves $u \circ \gamma$. The so-defined function $d_u: X' \times X' \to [0, \infty)$ is a pseudo-distance on the set X'. The corresponding metric space Z_u, which arises from X' by identifying pairs of points with d_u-distance zero, is a length space. We will call it the *length metric space associated with the map u.*

By construction, the space Z_u associated with the map u comes with a canonical, possibly non-continuous, surjective projection $P: X' \to Z_u$ and a 1-Lipschitz map $\bar{u}: Z_u \to X$ such that $u = \bar{u} \circ P$.

The most prominent example of this construction is given as follows. Let X be a metric space in which any pair of points is connected by a curve of finite length. Then, the *length space* X^i *associated with* X is the special case $X^i = Z_u$ of the above construction for the identity map $u = \mathrm{Id}: X \to X$. If X is proper, then, due to the theorem of Arzela–Ascoli, any pair of points in X is connected by a curve of shortest length. Therefore, the space X^i is a geodesic space. The completeness of X implies that X^i is complete as well. The 1-Lipschitz map $\bar{u}: X^i \to X$ from above is the identity, in this case. The map $P = \bar{u}^{-1}: X \to X^i$ need not be continuous, but it sends curves of finite length in X to continuous curves of the same length in X^i.

2.3. Polygons and triangles

A polygon in a metric space X is a closed curve $\gamma\colon[a,b]\to X$ such that, for some $a=t_1\leqslant...\leqslant t_n=b$, all restrictions $\gamma\colon[t_i,t_{i+1}]\to X$ are geodesics. A Jordan curve Γ which can be parameterized as a polygon will be called a *Jordan polygon*. If $n=3$, we obtain the notions of a *triangle* and *Jordan triangle*, respectively.

2.4. Parameterized area of Lipschitz maps

Let K be a Borel subset of $\mathbb{R}^2$ and let $u\colon K\to X$ be a Lipschitz map into some metric space X. The set K can be decomposed as a disjoint union

$$K=A\cup\bigcup_{i=1}^{\infty}K_i,$$

in such a way that all K_i are compact and A has measure zero, and such that $u\colon K_i\to X$ is either injective or $\mathcal{H}^2(u(K_i))=0$; see [19]. The parameterized area of u, which generalizes the classical parameterized area of smooth maps, is given by (see [27, §2.4])

$$\operatorname{Area}(u):=\sum_{i=1}^{\infty}\mathcal{H}^2(u(K_i))=\int_{u(K)}N(x)\,d\mathcal{H}^2(x),$$

where $N(x)$ is the cardinality of $u^{-1}(x)$. The parameterized area can be computed by a metric transformation formula, cf. [26, p. 3].

Note that, for any biLipschitz homeomorphism $F\colon K_0\to K\subset\mathbb{R}^2$, the parameterized areas of $u\colon K\to X$ and $u_0=u\circ F\colon K_0\to X$ coincide.

3. Upper curvature bounds

3.1. Definition

For a triangle Γ in a metric space X we consider the (unique up to Euclidean motions) comparison triangle $\Gamma_0\subset\mathbb{R}^2$ with the same side-lengths as Γ. The triangle Γ is called *thick* (more precisely 0-thick) if there are points on Γ_0 which have smaller distance than the corresponding points on Γ; cf. [6]. Otherwise, the triangle is called *thin* (or CAT(0)-*triangle* in the terminology of [6]).

A complete geodesic metric space X is CAT(0) if there are no thick triangles in X. The following observation allows the restriction to Jordan triangles.

Lemma 3.1. *Let X be a complete geodesic metric space. If X is not* CAT(0), *then there exists a thick Jordan triangle in X.*

Proof. If there are two different geodesics between a pair of points, then we find parts of these geodesics that build a Jordan curve. This Jordan curve is a geodesic bigon, a degenerate case of a triangle, which is automatically thick.

Otherwise, geodesics are uniquely determined by their endpoints. Given a thick triangle with vertices A_1, A_2 and A_3, we find a uniquely determined Jordan triangle with vertices A_1', A_2' and A_3' in the union of the sides, by taking A_i' to be the last common point of the sides A_iA_j and A_iA_k. If the triangle $A_1'A_2'A_3'$ is thin, then so is the triangle $A_1A_2A_3$, by Alexandrov's lemma; cf. [6, Lemma 3.5]. Thus, we have found a thick Jordan triangle $A_1'A_2'A_3'$ in X. $\square$

3.2. Majorization theorem

Let X be a CAT(0) space. Due to the majorization theorem of Reshetnyak ([37]), any closed curve $\gamma\colon[0,l]\to X$ parameterized by arclength is *majorized* by a closed convex set $\overline{\Omega}\subset\mathbb{R}^2$ in the following sense; cf. [2] and [6]. There exists a simple closed parametrization by arclength $\eta\colon[0,l]\to\mathbb{R}^2$ of the boundary $\partial\Omega$ and a 1-Lipschitz map $M\colon\overline{\Omega}\to X$ such that $M\circ\eta=\gamma$. Then, for any biLipschitz parametrization $F\colon\overline{D}\to\overline{\Omega}$, the area of the Lipschitz disc $M\circ F$ is bounded by $\mathrm{Area}(M\circ F)\leqslant\mathcal{H}^2(\Omega)$. The isoperimetric inequality in $\mathbb{R}^2$ yields $\mathrm{Area}(M\circ F)\leqslant l^2/4\pi$. Now, it is easy to deduce the following result.

Lemma 3.2. *Let X be a* CAT(0) *space. Then, any Lipschitz curve $\gamma\colon S^1\to X$ of length l is the boundary of a Lipschitz map $u\colon\overline{D}\to X$ with* $\mathrm{Area}(u)\leqslant l^2/4\pi$.

Proof. Let $\gamma_0\colon S^1\to X$ be a parametrization of γ proportional to arclength. The existence of a Lipschitz map $u_0\colon\overline{D}\to X$ extending γ_0 with the right bound on the area follows from the paragraph preceding the lemma. We attach to u_0 a Lipschitz annulus of zero area connecting γ_0 and γ by a linear reparametrization; cf. [29, Lemma 3.6]. The arising Lipschitz disc u has the same area as u_0, and provides the required filling of γ. $\square$

3.3. Curvature bounds via majorization

The majorization theorem is closely related to the following observation.

Lemma 3.3. *Let X be a complete geodesic metric space. The space X is* CAT(0) *if and only if, for any Jordan triangle $\Gamma\subset X$, there exists a* CAT(0) *space Z and a* 1-*Lipschitz map $F\colon Z\to X$ which sends some closed rectifiable curve $\Gamma'\subset Z$ in an arclength-preserving way onto Γ.*

Proof. If X is CAT(0) then, for any triangle $\Gamma\subset X$, we can take $Z=X$ and $F=\mathrm{Id}\colon Z\to X$. Now, assume that any Jordan triangle in X is majorized by a CAT(0) space

Z as in the formulation of the lemma. In order to prove that X is CAT(0), we only need to prove that any Jordan triangle Γ is thin. Consider a majorization $F: Z \to X$ of the triangle Γ. Then, the preimage in Γ' of any geodesic contained in Γ is a geodesic in Z of the same length, cf. [2, p. 88]. Hence, Γ' is a geodesic triangle in Z with the same side-lengths as Γ. Thus, Γ and Γ' have the same comparison triangle Γ_0 in $\mathbb{R}^2$. Since Z is CAT(0), the triangle Γ' is thin. Since $F: \Gamma' \to \Gamma$ is 1-Lipschitz, we deduce that Γ is thin as well. □

3.4. Local curvature bounds

A metric space X has non-positive curvature if any point in X has a CAT(0) neighborhood. A complete geodesic metric space X of non-positive curvature is CAT(0) if and only if X is simply connected, by a version of the theorem of Cartan–Hadamard [6, §6].

3.5. Reshetnyak's gluing theorem

Let $X^{\pm}$ be CAT(0) spaces with closed convex subsets $A^{\pm} \subset X^{\pm}$. If $G: A^+ \to A^-$ is an isometry, then the space X arising from gluing X^+ and X^- along the isometry G is CAT(0); cf. [9, Theorem 9.1.21]. Localizing the statement, we see that a gluing of two spaces of non-positive curvature along isometric locally convex subsets is again a space of non-positive curvature.

4. Generalities on Sobolev maps

We assume some knowledge of Sobolev maps with values in a metric space and refer to [16], [21] [26], [27], [39] and references therein for explanations.

Let Ω be a bounded domain in $\mathbb{R}^2$ and X be a complete metric space. A map $u \in L^2(\Omega, X)$ is contained in the (Newton–)Sobolev space $N^{1,2}(\Omega, X)$ if there exists a Borel function $\varrho \in L^2(\Omega)$ such that, for 2-*almost all Lipschitz curves* $\gamma: [a,b] = I \to \Omega$, the composition $u \circ \gamma$ is continuous and

$$\ell_X(u \circ \gamma) \leqslant \int_\gamma \varrho := \int_a^b \varrho(\gamma(t)) |\gamma'(t)| \, dt. \tag{4.1}$$

We refer to [16] for a thorough discussion of the notion of 2-almost all curves. For the present paper it is sufficient to know that, for any biLipschitz embedding $F: I \times I \to \Omega$ and almost all $t \in I$, inequality (4.1) holds true for the curve $\gamma_t(s) = F(t,s)$. There exists a minimal function $\varrho = \varrho_u$ satisfying the condition above, uniquely defined up to sets of measure zero. It will be called the *generalized gradient* of u. The integral $\int_\Omega \varrho_u^2(z)\,dz$ coincides with the Reshetnyak energy (see [39], [26]), which we denote by $E_+^2(u)$.

Let $u\in N^{1,2}(\Omega,X)$ be arbitrary. For almost all $z\in\Omega$ there exists a semi-norm $\operatorname{ap}\operatorname{md} u_z$ on $\mathbb{R}^2$ called the approximate metric differential, such that the following conditions hold true, [18], [26, §4], [27, Lemma 3.1]. The map $z\mapsto \operatorname{ap}\operatorname{md} u_z$ into the space of semi-norms has a Borel measurable representative. For 2-almost all curves $\gamma\colon I\to\Omega$ we have

$$\ell_X(u\circ\gamma)=\int_I \operatorname{ap}\operatorname{md} u_{\gamma(t)}(\gamma'(t))\,dt. \tag{4.2}$$

Moreover, for almost any $z\in\Omega$, we have

$$\varrho_u(z)=\sup_{v\in S^1} \operatorname{ap}\operatorname{md} u_z(v).$$

There is a countable, disjoint decomposition $\Omega=S\cup\bigcup_{1\leqslant i<\infty} K_i$ into a set S of measure zero and compact subsets K_i such that the restriction of u to any K_i is Lipschitz continuous. The (parameterized Hausdorff) area of the Sobolev map u is defined to be

$$\operatorname{Area}(u):=\sum_{i=1}^{\infty}\operatorname{Area}(u_i),$$

where u_i denotes the Lipschitz continuous restriction of u to K_i. This number $\operatorname{Area}(u)$ is finite, independent of the decomposition and generalizes the area of Lipschitz discs; cf. [27, §3.6].

A map $u\in N^{1,2}(\Omega,X)$ is called *conformal* if, at almost all $z\in\Omega$, the semi-norm $\operatorname{ap}\operatorname{md} u_z$ is a multiple $f(z)\cdot s_0$ of the standard Euclidean norm s_0 on $\mathbb{R}^2$. In this case, $f\in L^2(\Omega)$ will be called the *conformal factor* of u. The conformal factor f of a conformal map $u\in N^{1,2}(\Omega,X)$ coincides with the generalized gradient ϱ_u. In the conformal case, equation (4.2) therefore simplifies to

$$\ell_X(u\circ\gamma)=\int_\gamma f, \tag{4.3}$$

valid for 2-almost all curves γ in Ω. Moreover, the restriction of u to any subdomain $O\subset\Omega$ satisfies

$$\operatorname{Area}(u|_O)=\int_O f^2. \tag{4.4}$$

Any map $u\in N^{1,2}(D,X)$ has a well-defined trace $\operatorname{tr}(u)\in L^2(S^1,X)$. If $u\in N^{1,2}(D,X)$ has a representative with a continuous extension to $\overline{D}$, then $\operatorname{tr}(u)$ is the restriction of this extension to the boundary circle.

5. Excluding non-Euclidean norms in tangent spaces

5.1. Isoperimetric sets in normed planes

Let V be a 2-dimensional normed space. There exists a convex subset $\mathbb{I}_V \subset V$ with the largest possible area among all convex sets with the same length of the boundary $\partial\mathbb{I}_V$. This subset is unique up to translations and dilations, and is called the *isoperimetric set*; [41]. The following reformulation of the Blaschke–Santalo inequality shows that the Euclidean isoperimetric inequality never holds in V, unless V is Euclidean; cf. Remark 1.2.

Lemma 5.1. *In the notations above*,

$$\mathcal{H}^2(\mathbb{I}_V) \geqslant \frac{\ell_V^2(\partial\mathbb{I}_V)}{4\pi}, \tag{5.1}$$

with equality if and only if V is Euclidean.

Proof. After rescaling (cf. [41, equation (4.10)]), we may assume $2\mathcal{H}^2(\mathbb{I}_V)=\ell_V(\partial\mathbb{I}_V)$. Then, (5.1) is equivalent to $\mathcal{H}^2(\mathbb{I}_V)\leqslant\pi$, with equality if and only if V is Euclidean. However, due to [41, equation (4.14)], this is exactly the statement of the 2-dimensional Blaschke–Santalo inequality [41, Theorem 2.3.3]. □

5.2. Formulation of the claim

A complete metric space X has property (ET) if, for any map $u\in N^{1,2}(D,X)$, almost all approximate metric differentials $\operatorname{ap}\operatorname{md} u_z$ are Euclidean norms or degenerate semi-norms, [26, §11]. Examples of spaces with property (ET) are spaces with 1-sided curvature bounds and sub-Riemannian manifolds. We refer to [26] for a thorough discussion of this property. The aim of this section is to prove the following.

Theorem 5.2. *Let X be a proper metric space with Euclidean isoperimetric inequality for curves. Then*, *X has property* (ET).

5.3. Sobolev–Dehn function

For the limiting arguments used in the proof of Theorem 5.2, it is better to use a variant of the Dehn function with Sobolev, instead of Lipschitz discs, due to better stability properties. For a complete metric space X, we let the *Sobolev–Dehn function* of X be the minimal function $\delta_X^{\mathrm{Sob}}\colon(0,\infty)\to[0,\infty]$ for which the following holds true. For any Lipschitz curve $\gamma\colon S^1\to X$ of length at most r and any $\varepsilon>0$ there exists a Sobolev map $u\in N^{1,2}(D,X)$ with $\operatorname{tr}(u)=\gamma$ and $\operatorname{Area}(u)\leqslant\delta_X^{\mathrm{Sob}}(r)+\varepsilon$.

Since any Lipschitz disc is contained in $N^{1,2}(D,X)$, the Sobolev–Dehn function δ_X^{Sob} is bounded from above by the (Lipschitz–)Dehn function δ_X of the space X. If the space X is Lipschitz 1-connected, for instance a Banach or a CAT(0) space, then $\delta_X = \delta_X^{\mathrm{Sob}}$, [29, Propostion 3.1]. For any space X which satisfies the Euclidean isoperimetric inequality for curves, we have $\delta_X^{\mathrm{Sob}}(r) \leqslant r^2/4\pi$.

5.4. Limiting arguments

Property (ET) can be thought of as an infinitesimal property of the space, informally expressed by the condition that tangent spaces do not contain non-Euclidean normed planes. This idea can be made precise using blow-ups of metric spaces as a special case of ultralimits of the rescaled original space. We refer to [26, §11] for details and just recall the following fact.

Lemma 5.3. *Let X be a complete metric space and ω a non-principal ultrafilter on $\mathbb{N}$. Assume that, for all $x \in X$ and all sequences t_i of positive real numbers converging to zero, the following holds true: any normed plane V contained in the blow-up*

$$B = \lim_\omega \left(\frac{1}{t_n} X, x \right)$$

is Euclidean. Then, X has property (ET).

The only property of blow-ups needed for the proof of Theorem 5.2 is the following stability of quadratic isoperimetric inequalities from [29, Theorem 1.8]. Here it is crucial to work with Sobolev–Dehn functions and properness is used in an essential way.

Lemma 5.4. *Let X be a proper geodesic metric space with $\delta_X^{\mathrm{Sob}}(r) \leqslant r^2/4\pi$. Then, for any blow-up B of X as in Lemma 5.3, we have $\delta_B^{\mathrm{Sob}}(r) \leqslant r^2/4\pi$ for all $r \geqslant 0$.*

5.5. Quasi-convexity of the Hausdorff area

Using Lemma 5.1 and [10], we readily obtain the following result.

Proposition 5.5. *Assume that a complete metric space B contains a non-Euclidean normed plane V. Then, the Sobolev–Dehn function of B satisfies $\delta_B^{\mathrm{Sob}}(r) > r^2/4\pi$ for all $R > 0$.*

Proof. Let $\mathbb{I}_V \subset V$ be an isoperimetric set of V whose boundary $\partial \mathbb{I}_V$ has length r. The quasi-convexity of the Hausdorff area proved in [10], together with Lemma 5.1, implies that

$$\delta_B^{\mathrm{Sob}}(r) \geqslant \mathcal{H}^2(\mathbb{I}_V) > \frac{r^2}{4\pi};$$

see also [26, §2.4] and [27, §10.2]. This finishes the proof. □

Combining Lemmas 5.3–5.5, we finish the proof of Theorem 5.2.

Remark 5.1. A more direct but slightly more technical proof of Theorem 5.2 can be provided along the lines of [42, Theorem 5.1], also including the case of non-proper target spaces X.

6. Solutions of the Plateau problem

6.1. Solution of the Plateau problem

Let X be a proper metric space with the Euclidean isoperimetric inequality for curves. Due to Theorem 5.2, the space X satisfies property (ET). Let Γ be a Jordan curve in X of finite length. Consider the non-empty set $\Lambda(\Gamma, X)$ of all maps $v \in N^{1,2}(D, X)$ whose trace is a weakly monotone parametrization of Γ. A solution of the *Plateau problem* for the boundary curve Γ is a conformal map $u \in \Lambda(\Gamma, X)$ which has smallest area among all maps in $\Lambda(\Gamma, X)$. Equivalently, u is a map with minimal Reshetnyak energy $E_+^2(u)$ among all maps in $\Lambda(\Gamma, X)$, [26, Theorem 11.4]. Due to [26, Corollary 11.5], a solution of the Plateau problem exists for every Jordan curve Γ of finite length in X. Any such solution of the Plateau problem has the following property, [26, Theorem 1.4], [27, Proposition 1.8].

Theorem 6.1. *Let Γ be a Jordan curve of finite length in X, and let u be a solution of the Plateau problem for the curve Γ. Then, u has a representative, again denoted by u, which continuously extends to $\overline{D}$. For any Jordan curve $\eta \subset \overline{D}$ with Jordan domain $J \subset D$,*

$$\mathrm{Area}(u|_J) \leqslant \frac{\ell_X^2(u \circ \eta)}{4\pi}.$$

In fact, from [26, Theorem 1.4], one can conclude that u is locally Lipschitz on D.

6.2. Intrinsic minimal disc

Let X, Γ and u be as in Theorem 6.1. In [27] it was shown that the intrinsic pseudo-metric d_u on $\overline{D}$ described in §2.2 is well defined, finite-valued and continuous with respect to the Euclidean metric. As in §2.2, denote by $Z = Z_u$ the associated metric space. Then, the following holds true; see [27, Theorems 1.1, 1.2, 1.4 and 1.5].

THEOREM 6.2. *Let Γ be a Jordan curve in X of finite length, and let $u\colon \overline{D}\to X$ be a continuous solution of the Plateau problem with boundary Γ. Let $Z=Z_u$ be the associated length metric space, $P\colon \overline{D}\to Z$ be the canonical projection and $\bar{u}\colon Z\to X$ be the unique map with $u=\bar{u}\circ P$. Then, the following properties hold*:

(1) *Z is a geodesic space homeomorphic to $\overline{D}$ and P is continuous*; *the preimage $P^{-1}(Z\setminus\partial Z)$ is homeomorphic to D.*

(2) *the map $\bar{u}\colon Z\to X$ is* 1*-Lipschitz and sends ∂Z in an arclength-preserving way onto* Γ;

(3) *for any curve $\gamma\subset\overline{D}$ we have $\ell_X(u\circ\gamma)=\ell_Z(P\circ\gamma)$*;

(4) *for any open $V\subset D$ we have* $\operatorname{Area}(u|_V)=\mathcal{H}^2_Z(P(V))$;

(5) *given any Jordan curve $\eta\subset Z$ and the corresponding Jordan domain $O\subset Z$, we have*

$$\mathcal{H}^2(O)\leqslant\frac{\ell_Z^2(\eta)}{4\pi}.$$

The space Z in Theorem 6.2 will be called the *intrinsic minimal disc* associated with u.

6.3. Reduction to Theorem 1.2

We can now prove the following result.

PROPOSITION 6.3. *Theorem* 1.2 *implies Theorem* 1.1.

Proof. The "only if" part of Theorem 1.1 has already been verified in Lemma 3.2, since the identity map $\mathrm{Id}\colon X^i\to X$ is 1-Lipschitz and lengths of curves in X and in X^i coincide. Let now X be a proper metric space which satisfies the Euclidean isoperimetric inequality for curves. Assume in addition that any pair of points in X is connected by a curve of finite length. Consider the induced length space X^i. Since X is proper, the space X^i is a complete geodesic metric space. We are going to prove that X^i is CAT(0). We take an arbitrary Jordan triangle $\Gamma\subset X^i$ and need to majorize it by some CAT(0) space, in the sense of Lemma 3.3. Now, Γ has the same length when viewed as a curve in X. We find a solution u of the Plateau problem for the curve $\Gamma\subset X$ and apply Theorems 6.1 and 6.2 to $\Gamma\subset X$. As in Theorem 6.2, we denote by Z the intrinsic minimal disc associated with u. Thus, Z is a compact geodesic metric space homeomorphic to $\overline{D}$ and there exists a 1-Lipschitz map $\bar{u}\colon Z\to X$ which maps the boundary ∂Z in an arclength-preserving way to Γ. Since Z is a geodesic space, the map $\bar{u}$ considered as a map to X^i is still 1-Lipschitz and arclength preserving on ∂Z. Assuming that Theorem 1.2 holds true, the space Z is CAT(0). Thus, Lemma 3.3 implies that X^i is CAT(0) as well. $\square$

7. The conformal factor

7.1. An integral inequality

Let X be a complete metric space which satisfies the Euclidean isoperimetric inequality for curves. Let Γ be a Jordan curve of finite length in X, and let $u\colon \overline{D}\to X$ be a solution of the Plateau problem as in Theorem 6.1. Let $f\in L^2(D)$ be the conformal factor of u. Applying Theorem 6.1 to concentric circles and using (4.4) and (4.3) we deduce the following result.

Lemma 7.1. *The conformal factor $f\in L^2(D)$ satisfies the inequality*

$$\int_{B(z,r)} f^2 \leqslant \frac{1}{4\pi}\left(\int_{\partial B(z,r)} f\right)^2 \tag{7.1}$$

for any $z\in D$ and almost any $0<r<1-|z|$.

7.2. Log-subharmonic functions

Recall that a function $f\colon U\to[-\infty,\infty)$ defined on a domain $U\subset\mathbb{R}^2$ is called *subharmonic* if f is upper semi-continuous, contained in L^1_{loc} and satisfies $f(z)\leqslant⨍_{B(z,s)} f$ for all $z\in U$ and all $s>0$ with $B(z,s)\subset U$. Here and below, we denote by $⨍_T g=⨍_T g\,d\mu$ the integral mean value $\frac{1}{\mu(T)}\int_T g\,d\mu$ of a function integrable with respect to a measure μ. A function $f\in L^1_{\mathrm{loc}}(U)$ has a subharmonic representative if and only if $\Delta f\geqslant 0$ in the distributional sense. This representative is uniquely defined at each point by $f(z)=\lim_{s\to 0}⨍_{B(z,s)} f$.

A function $f\colon U\to[0,\infty)$ is called *log-subharmonic* if $\log(f)$ is a subharmonic function. Any log-subharmonic function is locally bounded. Log-subharmonic functions are intimately related to non-positive curvature.

Inequality (7.1) turns out to imply log-subharmonicity (the other implication is true as well, cf. [7], but will not be needed here).

Proposition 7.2. *Any non-negative function $f\in L^2(D)\setminus\{0\}$ which satisfies (7.1) has a log-subharmonic representative.*

Proof. We can rewrite (7.1) in terms of the integral averages as

$$⨍_{B(z,r)} f^2 \leqslant \left(⨍_{\partial B(z,r)} f\right)^2. \tag{7.2}$$

For continuous positive functions f satisfying (7.2), the log-subharmonicity is proved in [7, lemma on p. 665]. The general case reduces to the case of smooth positive functions as follows; cf. [13]. Applying Hölder's inequality to (7.2), we infer

$$⨍_{B(z,r)} f \leqslant ⨍_{\partial B(z,r)} f. \tag{7.3}$$

Thus, f has a subharmonic representative; see [13, Lemma 4.6 and Remark 4.8]. In particular, f is locally bounded.

A combination of (7.2) and (7.3) directly implies that, for any $\delta>0$, the function $f^\delta(z):=f(z)+\delta$ satisfies (7.2) as well. If f^δ has a log-subharmonic representative for all $\delta>0$ then (after changing to the subharmonic representative) we obtain f as a limit of locally uniformly bounded log-subharmonic functions. Then, by the classical convergence theorems for subharmonic functions (cf. [5, §3.7, Exercise 3.15]), the function f has a log-subharmonic representative. Therefore, it suffices to prove the proposition under the assumption that f is everywhere positive.

For this, we take any $\varepsilon>0$ and consider the usual mollified functions

$$f_\varepsilon\colon B(0,1-\varepsilon)\longrightarrow[0,\infty)$$

obtained by convolutions with a standard (Friedrichs) mollifier. We use the observation of [13, Lemma 4.5] (going back to the proof in [7]) that the smooth function f_ε still satisfies (7.2) for all balls contained in $B(0,1-\varepsilon)$. Due to [7], f_ε is log-subharmonic. By the limiting argument as above f has a log-subharmonic representative as well. This finishes the proof. □

7.3. Conclusion

Taking Proposition 7.2 and Lemma 7.1 together, we have shown that the conformal factor $f\in L^2(D)$ of our minimal disc u has a log-subharmonic representative $\bar f$. From now on, we will replace f by $\bar f$ and assume that f is log-subharmonic.

8. Metric defined by a conformal factor

We refer to [38] and the references therein for a detailed description of the theory, a special case of which is sketched here. Let U be a domain in $\mathbb{R}^2$, and let $f\colon U\to[0,\infty)$ be a log-subharmonic function.

For a Lipschitz curve $\gamma\colon[a,b]\to U$, define the f-length of γ to be

$$L_f(\gamma):=\int_\gamma f=\int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

The f-length does not change if γ is reparameterized. As f is locally bounded, the f-length of any Lipschitz curve is finite.

Define $d_f\colon U\times U\to[0,\infty)$ by

$$d_f(z,z'):=\inf\{L_f(\gamma):\gamma \text{ is a Lipschitz curve between } z \text{ and } z'\}. \tag{8.1}$$

This function d_f defines a metric on U, and the identity map $i: U \to (U, d_f)$ from the Euclidean subset U to the new metric space is a homeomorphism, [38, Theorem 7.1.1]. Denote by Y the metric space (U, d_f).

The metric d_f does not change if, in (8.1), the infimum is taken over all injective curves of bounded variation of turn instead over all Lipschitz curves, [38, p. 101]. If injective curves γ_n of uniformly bounded variation of turn converge pointwise to the curve γ in U then, due to [38, Theorem 8.4.4],

$$\lim_{n\to\infty} L_f(\gamma_n) = L_f(\gamma). \tag{8.2}$$

Thus, the distance $d_f(z_1, z_2)$ can be defined by formula (8.1), where the infimum is taken over the set of all polygonal curves $\gamma \subset U$ between z_1 and z_2.

For any compactly contained subdomain $V \subset U$, we have that the restriction $i: V \to Y$ is Lipschitz continuous, as f is locally bounded. For any Lipschitz curve γ in U, we have $\ell_Y(i \circ \gamma) = L_f(\gamma)$, [36]. We deduce from (4.2) that $i: V \to Y$ is conformal with conformal factor f. Therefore,

$$\mathcal{H}^2(i(V)) = \int_V f^2. \tag{8.3}$$

The main results of Reshetnyak's analytic theory of Alexandrov surfaces of (integral) bounded curvature [38, Theorems 7.1 and 7.2] take in our case the following form.

THEOREM 8.1. *The space $Y = (U, d_f)$ constructed above has non-positive curvature. Conversely, for any space M of non-positive curvature which is homeomorphic to a surface without boundary the following is true. For any point $x \in M$, there exists a neighborhood of x isometric to some $Y = (U, d_f)$, where U is a domain in $\mathbb{R}^2$ and $f: U \to [0, \infty)$ is log-subharmonic.*

Proof. We merely explain why Theorem 8.1 is a special case of Reshetnyak's results, relying on the results presented in [38].

Recall that a locally compact length space X has non-positive curvature if and only if any point $x \in X$ has a neighborhood U such that any triangle Γ in U has a non-positive excess, [3, p. 36]. (Here, the excess of a triangle is the sum of its angles minus π.) Thus, a length space X homeomorphic to a surface without boundary has non-positive curvature if and only if it has bounded (integral) curvature in the sense of Aleksandrov, and the (signed) curvature measure of X is non-positive.

Now both claims of Theorem 8.1 are exactly Theorems 7.1 and 7.2 in [38]. □

9. Reduction to Theorem 1.3

9.1. Formulation

The aim of this section is to prove the following result.

PROPOSITION 9.1. *Theorem 1.3 implies Theorem 1.2.*

Thus, we assume that Theorem 1.3 is true. Let X be a proper metric space which satisfies the Euclidean isoperimetric inequality for curves. Let Γ be a Jordan curve of finite length in X. Let $u:\overline{D}\to X$ be a solution of the Plateau problem in X for the boundary curve Γ. Denote by Z the intrinsic minimal disc associated with u as in Theorem 6.2, and let $P:\overline{D}\to Z$ be the canonical surjective map. Let Z_0 be the open disc $Z\setminus\partial Z$, and denote by D_0 the preimage $P^{-1}(Z_0)$. Then, D_0 is homeomorphic to the open disc and, in particular, $D_0\subset D$.

Let f denote the conformal factor of u, which is log-subharmonic, due to §7. Denote by Y_0 the open disc D_0 equipped with the length metric d_f as introduced in the previous section. Theorem 8.1 implies that Y_0 has non-positive curvature. Let $i:D_0\to Y_0$ denote the canonical homeomorphism (identity map). Let $I:Y_0\to Z_0\subset Z$ denote the composition $P\circ i^{-1}$. We now easily conclude, using Theorem 1.3, the following result.

LEMMA 9.2. *If $I:Y_0\to Z_0$ is a local isometry, then Z is* CAT(0).

Proof. If I is a local isometry then it is locally injective. By the invariance of domains we see that I is an open map. Since I is surjective and Y_0 non-positively curved, the space Z_0 has non-positive curvature. Due to Theorem 6.2, the space Z satisfies the isoperimetric inequality for Jordan curves as required in Theorem 1.3 (3). From Theorem 1.3, we deduce that Z is CAT(0). □

Therefore, in order to prove Proposition 9.1, we only need to show that $I:Y_0\to Z_0$ is a local isometry.

9.2. Properties of the map I

We claim and prove the following.

LEMMA 9.3. *The map $I:Y_0\to Z_0$ is 1-Lipschitz.*

Proof. Since the metric in Y_0 can be defined using f-lengths of polygonal curves, we only need to prove that $L_f(\gamma)\geqslant\ell_Z(P\circ\gamma)=\ell_X(u\circ\gamma)$ for any straight segment $\gamma:[a,b]\to D$. Consider the variation γ_s, for $-\varepsilon<s<\varepsilon$, of $\gamma_0=\gamma$ through segments parallel to γ. For any s, we have $L_f(\gamma_s)=\int_{\gamma_s} f$. Moreover, by the definition of the conformal factor, we have $\ell_X(u\circ\gamma_s)=\int_{\gamma_s} f$ for *almost all* $s\in(-\varepsilon,\varepsilon)$. Thus, we find a sequence $s_n\to 0$ such that $L_f(\gamma_{s_n})=\ell_X(u\circ\gamma_{s_n})$. The result follows from

$$L_f(\gamma)=\lim_{n\to\infty} L_f(\gamma_{s_n})=\lim_{n\to\infty}\ell_X(u\circ\gamma_{s_n})\geqslant \ell_X(u\circ\gamma_0)=\ell_X(u\circ\gamma),$$

where we have used (8.2) for the first equality. □

By construction, (4.4), (8.3) and Theorem 6.2 (4), we obtain the following result.

Lemma 9.4. *The map I preserves the Hausdorff measure $\mathcal{H}^2$. More precisely, for any open subset $V\subset D_0$ we have*

$$\mathcal{H}^2(i(V))=\int_V f^2=\mathcal{H}^2(P(V)).$$

9.3. The conclusion

We can now finish the proof of the main result of this section.

Proof of Proposition 9.1. By the definition of the metrics on Y_0 and Z, it suffices to show that, for any simple curve $\eta\colon[a,b]\to Y_0$, we have $\ell_{Y_0}(\eta)=\ell_Z(I\circ\eta)$. Assume the contrary, and consider a curve η with

$$\ell_{Y_0}(\eta)>\ell_Z(I\circ\eta).$$

In order to obtain a contradiction, we will roughly proceed as follows. We will first complement a subcurve of η to a Jordan curve and equip the closure of the corresponding Jordan domain with a new metric, to which Theorem 1.3 will be applied. Inside the domain, the new metric will come from that of Y_0, while the length of the boundary will come from that of its image in Z.

To be more concrete, we first replace the curve η by a subcurve, and may assume that either $I\circ\eta$ is a point, or that no subarc of η is mapped by I to a point. Further replacing η by a slightly smaller subcurve, we may assume that η is part of a Jordan curve T such that the closure of $T\setminus\eta$ is a rectifiable arc η' in Y_0. Let $J\subset Y_0$ denote the Jordan domain of T whose closure is $\bar{J}=J\cup T$.

We call admissible any curve in $\bar{J}$ that is a finite concatenation of simple curves either completely contained in η, or intersecting η at most in a finite set of points. Define the new length $\mathcal{L}^+(\gamma)$ of such an admissible curve γ to be the sum of the Y_0-lengths of arcs outside η and the lengths of the images in Z of the subarcs contained in η. The pseudo-distance $d^+\colon\bar{J}\times\bar{J}\to[0,\infty]$ associated with the length functional $\mathcal{L}^+$ is finite-valued and continuous with respect to the Euclidean topology on $\bar{J}$, since η' and $I\circ\eta$ have finite length. Denote the corresponding metric space by S, and let $Q\colon \bar{J}\to S$ be the canonical projection.

Then, S is a compact length space, hence it is a geodesic space. The map Q is a local isometry outside T and a homeomorphism outside η (here and below, we consider $\bar{J}$ with the metric restricted from Y_0, not with a length metric!). If I does not send η to a point (hence is bijective by assumption), then Q is bijective, hence a homeomorphism. If I sends η to a point, then Q collapses η to a point in S. In both cases, S is homeomorphic to $\bar{D}$. The restriction $I\colon \bar{J}\to Z$ factorizes through Q and, for any curve $k\subset\bar{J}$, we have $\ell_S(Q\circ k)\geqslant\ell_Z(I\circ k)$ by Lemma 9.3.

We claim that S is CAT(0). Indeed, $S\setminus\partial S$ is locally isometric to J. Thus $S\setminus\partial S$ has non-positive curvature. Due to Theorem 1.3, we only need to verify the isoperimetric inequality for Jordan curves $c\subset S$. For any Jordan curve c in S, there is a unique Jordan curve $\hat{c}$ in $\bar{J}$ which is mapped by Q to c. Let $G\subset J$ denote the Jordan domain of $\hat{c}$. Then, $Q(G)\subset S$ is the Jordan domain of c and has the same Hausdorff measure as $G\subset Y_0$, since Q is a local isometry outside T.

We have

$$\ell_S(c)=\ell_S(Q\circ\hat{c})\geqslant\ell_Z(I\circ\hat{c})=\ell_X(u\circ i^{-1}\circ\hat{c}).$$

Moreover,

$$\mathcal{H}^2_S(Q(G))=\mathcal{H}^2_{Y_0}(G)=\mathcal{H}^2_Z(I(G))=\mathrm{Area}(u|_{i^{-1}(G)}).$$

Since $i^{-1}(G)\subset D$ is the Jordan domain of $i^{-1}(\hat{c})$, the desired inequality

$$\mathcal{H}^2_S(Q(G))\leqslant\frac{\ell^2_S(c)}{4\pi}$$

follows from Theorem 6.1. Therefore, an application of Theorem 1.3 finishes the proof of the claim that S is CAT(0).

Another application of Theorem 1.3 implies that the metric on $S\setminus\partial S$ is a length metric. Therefore, the length-preserving map $Q^{-1}\colon S\setminus\partial S\to J$ is 1-Lipschitz. Thus, it extends to a 1-Lipschitz map $\widehat{Q}^{-1}\colon S\to\bar{J}$. By continuity, the composition $\widehat{Q}^{-1}\circ Q$ must be the identity on $\bar{J}$.

By assumption, $\ell_S(Q\circ\eta)<\ell_{Y_0}(\eta)$. We obtain a contradiction to $\eta=\widehat{Q}^{-1}\circ Q\circ\eta$, since the 1-Lipschitz map $\widehat{Q}^{-1}$ cannot increase lengths. □

Thus, Theorems 1.1 and 1.2 are reduced to Theorem 1.3.

Part II. Geometry of strange metric discs

10. Plan of the second part of the paper

The second part of the paper is devoted to the proof of Theorem 1.3. Before we start with the actual proof of the theorem, we recollect in §11 some basic observations about the local geometry of non-positively curved surfaces. In §12 we prove that the completion of a non-positively curved open disc is CAT(0) and homeomorphic to a closed disc whenever compact. All but the main implication (3) to (1) of Theorem 1.3 turn out to be relatively simple. The proofs of these simple implications are provided in §13.

In §14 we embark on the proof of the implication (3) to (1) of Theorem 1.3, thus assuming the isoperimetric inequality and trying to prove that the disc Z is CAT(0). We use subdomains of Z in order to reduce the situation to the case where the "problematic" part of ∂Z consists of a geodesic $c\subset\partial Z\subset Z$. We consider the complement

$$Y_0 = Z \setminus \partial Z$$

equipped with the induced length metric and the completion Y of Y_0, which is CAT(0) and compact. The space Y comes along with a canonical 1-Lipschitz map

$$I: Y \longrightarrow Z,$$

which is a local isometry in Y_0. This reduces the question to the situation described in the introduction (the example of Koch's snowflake): the disc Z arises from a CAT(0) disc Y by possibly shortening or collapsing a part of the boundary curve $\eta\subset Y$.

If I is an isometry, then we are done. Otherwise, some part η of the boundary curve ∂Y is sent by I to a curve in ∂Z of smaller length. In order to obtain a contradiction to the isoperimetric inequality, it suffices to extend small parts of η to Jordan curves T in Y which bound almost optimal isoperimetric regions in Y. The map I then shortens the length of T, but leaves the area of the enclosed Jordan domain unchanged, and thus $I(T)\subset Z$ violates (3) of Theorem 1.3.

If η is rectifiable, we approximate η by geodesics and use the approximation of Y by flat cones in order to reduce the problem to the situation where η is a line in $\mathbb{R}^2$. In that case, one can take as suitable Jordan curves T large parts of sufficiently large circles complemented by a short chord contained in η. This is carried out in §15. The more difficult case of a non-rectifiable curve η is carried out in §16. Here, the non-rectifiability of η provides parts which are contracted by an arbitrary large amount. This allows sufficient flexibility in the choice of critical Jordan curves in Y.

11. Geometry of non-positively curved surfaces

11.1. Basic geometric features

Let M be a metric space of non-positive curvature homeomorphic to D. We refer to [12], [24] and [38] for the investigations of such and related more general spaces. Let $x \in M$ be arbitrary. We find a small open metric ball $U = B(x, \varepsilon)$ around x whose closure is a compact CAT(0) space. Any geodesic starting at x can be extended to a geodesic in $\bar{U}$ of length 2ε, [24, §4]. The space of directions Σ_x is a circle of some length $\alpha \geqslant 2\pi$. By definition, the tangent cone $T_x M$ at the point x is the Euclidean cone over Σ_x.

11.2. Hinges

Let M and $x \in U = B(x, \varepsilon) \subset M$ be as above, and let γ_1 and γ_2 be geodesics of length $\geqslant \varepsilon$ starting in x and having only the point x in common. Then, γ_1 and γ_2 intersect the boundary circle of $B(x, \varepsilon)$ at one point each. Denote by Γ the union of (the images of) γ_1 and γ_2 inside $B(x, \varepsilon)$. Then, Γ divides $B(x, \varepsilon)$ into two closed subsets $H_\pm$ homeomorphic to closed half-planes intersecting in their common boundary Γ. We call $H_\pm$, equipped with the induced length metric, the *hinges* (of size ε) defined by γ_1 and γ_2.

We claim that both hinges have non-positive curvature. If γ_1 and γ_2 concatenate to a geodesic, then $H_\pm$ are convex in $B(x, \varepsilon)$ and the claim follows. Otherwise, we extend γ_1 by a geodesic γ_1^+ of length ε starting in x to a geodesic γ of length 2ε. Then, γ divides $B(x, \varepsilon)$ into two convex subsets $A_\pm$ with common boundary γ. Without loss of generality, we may assume that H_+ is contained in A_+. Then, $\gamma \cup \gamma_2$ divide $B(x, \varepsilon)$ into three convex hinges H_+, A_- and a third hinge H' (between γ_2 and γ_1^+). The hinge H_+ has non-positive curvature by convexity, and H_- is the result of gluing H' and A_- along the geodesic γ_1^+, hence it has non-positive curvature by Reshetnyak's gluing theorem. This finishes the proof of the claim.

Lemma 11.1. *Let M be a space of non-positive curvature homeomorphic to the open disc. Let Γ be a Jordan polygon in M. Then, the closed Jordan domain $\bar{J}$ of Γ with its intrinsic metric is* CAT(0).

Proof. The space $\bar{J}$ is compact and simply connected. At any point $x \in \bar{J}$, a small ball around x is either open in M or isometric to a hinge described above. Therefore, $\bar{J}$ is non-positively curved. The lemma follows from the theorem of Cartan–Hadamard. $\square$

11.3. Approximation by flat cones

For the next result, we will use Theorem 8.1 from above and a theorem about the approximation of Alexandrov surfaces by their tangent cones.

LEMMA 11.2. *Let M be a space of non-positive curvature which is homeomorphic to a surface* (*possibly with boundary*). *Let $x \in M$ be a point. Let γ_1 and γ_2 be geodesics starting at x and enclosing a positive angle. Then*, *there exist a closed interval $T \subset \mathbb{R}$, a ball O around the vertex o of the Euclidean cone CT over T*, *and a biLipschitz map $E\colon O \to E(O) \subset M$ with the following properties*:

(1) *the map E sends the vertex o of CT to x*;

(2) *E sends initial parts of the boundary rays of the flat hinge CT isometrically onto the initial parts of γ_1 and γ_2*;

(3) *we have*

$$\lim_{v,w \to o} \frac{d(E(v), E(w))}{d(v, w)} = 1.$$

Thus, the biLipschitz constant of the restriction of E to small balls around o goes to 1 with the radius of the balls tending to zero. Clearly, the length of the interval T, and hence the total angle of the Euclidean hinge CT, is not less than the angle between γ_1 and γ_2.

Proof of Lemma 11.2. We first assume that x is not on the boundary ∂M. The existence of a biLipschitz map $E'\colon O \to M$ from a ball O around the origin $0 \in T_x M$, such that conditions (1) and (3) of Lemma 11.2 hold true, is the content of a theorem of Burago, [11], stated in [38, Theorem 9.10]. Note that this theorem is applicable, due to Theorem 8.1 above. Composing E' with a self-isometry of $T_x M$, we may assume that the initial part of a given ray η starting at the origin zero is sent by E' to a curve $E' \circ \eta$ whose starting direction coincides with the starting direction of η. Consider the rays η_1 and η_2 in $T_x M$ whose starting directions are $\gamma_1'(0), \gamma_2'(0) \in \Sigma_x \subset T_x M$. We now find a biLipschitz map of $T_x M$ to itself, which fixes zero, has at zero the identity as its differential, and which sends the initial part of η_i to $(E')^{-1}(\gamma_i)$. Composition of E' with this biLipschitz map provides the required map E upon restriction to the smaller hinge in O between the rays η_i.

Let us now assume that x is contained in the boundary ∂M. We choose any simple arc γ connecting a point on γ_1 with a point on γ_2, and such that γ and the corresponding parts of γ_1 and γ_2 constitute together a Jordan curve Γ. Consider the union V of the corresponding Jordan domain and the curve $\Gamma \setminus \gamma$. Since γ_1 and γ_2 are geodesics and $x \in \partial M$, the set V is locally convex in M, by topological reasons. Thus, in order to find a biLipschitz embedding required in Lemma 11.2, we may replace M by V, and therefore assume that ∂M is the union of the geodesics γ_1 and γ_2. But (a small ball around x in) such a space is isometric to a hinge in some manifold N *without boundary* which still has non-positive curvature, as we see by applying Reshetnyak's gluing theorem twice. (First we glue to V along γ_1 a hinge of large angle from $\mathbb{R}^2$. In the so arising space the boundary is a geodesic and we may double the arising space to obtain the required manifold N.) By construction, V is the smaller hinge of the two hinges determined by $\gamma_1, \gamma_2 \subset N$. Thus, the map E constructed above (for the space N) has its image in V. This finishes the proof of Lemma 11.2. $\square$

Using the notation in Lemma 11.2, we will call *hinge* between two geodesics γ_1 and γ_2 the intersection of $E(O)$ with a small metric ball $B(x, r)$. This provides an extension of the definition of a hinge to the case of points at the boundary.

12. Completions of 2-dimensional open discs

Before embarking on the proof of Theorem 1.3, we will study the interiors of discs appearing in that theorem and their completions. The following basic result generalizes [8].

PROPOSITION 12.1. *Let Y_0 be a length space homeomorphic to the open disc. If Y_0 is non-positively curved then the completion Y of Y_0 is* CAT(0).

Proof. Choose Jordan curves Γ'_n with increasing Jordan domains whose union is Y_0. Approximate Γ'_n by Jordan polygons Γ_n. Let J_n be the corresponding Jordan domains and denote by Y_n the closure of these Jordan domains, equipped with their intrinsic metric.

By Lemma 11.1, every Y_n is CAT(0). The completion Y isometrically (and canonically) embeds into the CAT(0) space Y', obtained as an ultralimit of the Y_n (choosing the same fixed point lying in $Y_1 \subset Y_n$ as the base point of the spaces Y_n). Hence, Y is isometric to a subset of the CAT(0) space Y'. Due to completeness, Y is a closed subset of Y'. Since Y_0 is a length space, its completion Y is a length space as well. Therefore, Y must be convex in Y'. Thus Y is CAT(0). $\square$

The reader should consult [4, p. 1270] for references on homology manifolds used in the proof of the next lemma.

LEMMA 12.2. *Let Y_0 be a non-positively curved length metric space homeomorphic to D. If the completion Y of Y_0 is compact, then Y is homeomorphic to $\overline{D}$.*

Proof. By construction, Y_0 is dense in Y. Since Y_0 is locally complete, Y_0 is open in Y. The space Y is a separable CAT(0) space, hence it is contractible and locally contractible. The topological dimension of Y coincides with its geometric dimension;

see [20]. But Y embeds isometrically into an ultralimit of 2-dimensional CAT(0)-spaces, therefore the geometric dimension of Y is at most 2; see [20] and [23, Lemma 11.1]. Thus, Y has topological dimension 2.

Set $\partial Y := Y \setminus Y_0$. We claim that, for any $z \in \partial Y$, the local homology with integer coefficients $H_*(Y, Y \setminus \{z\})$ vanishes. In order to see this, note that $Y \setminus \{z\}$ is connected since Y_0 is connected. By dimensional reasons, the contractibility of Y, and the long exact sequence of the pair $(Y, Y \setminus \{z\})$, we only need to prove that $H_1(Y \setminus \{z\}) = 0$. Since Y is locally contractible and Y_0 is contractible, it is sufficient to prove that any closed curve $\gamma: S^1 \to Y \setminus \{z\}$ can be approximated by closed curves with images in Y_0. Covering γ by small metric balls $B(x, r)$, we observe that it is sufficient to prove that $B(x, r) \cap Y_0$ is connected for any $x \in Y$ and $r > 0$. But this follows from the fact that Y_0 is a length space and Y is the completion of Y_0. This finishes the proof of the claim.

Thus, Y is a homology 2-manifold with boundary ∂Y. Therefore, ∂Y is a homology 1-manifold and, due to [34], the doubling Y^+ of Y along the boundary ∂Y is a homology 2-manifold without boundary. Due to [44, §IX.5.9 and §IX.5.10], for $n = 1, 2$ any homology n-manifold without boundary is a manifold without boundary. Thus, Y^+ is a 2-manifold and ∂Y is a closed 1-submanifold. We deduce that Y is a manifold with boundary. Since Y_0 is homeomorphic to D, the space Y must be homeomorphic to $\overline{D}$. □

13. Simple implications

We now embark on the proof of Theorem 1.3. Thus, from now on, let Z be a geodesic metric space homeomorphic to $\overline{D}$ and such that the space $Z \setminus \partial Z$ is non-positively curved.

13.1. (1) implies (3)

Suppose that the space Z is CAT(0), and consider a Jordan curve $\eta \subset Z$ of finite length ℓ. Then, Reshetnyak's majorization theorem (Lemma 3.2) provides a Lipschitz disc $u: \overline{D} \to Z$ filling η of area at most $\ell^2/4\pi$. For topological reasons, the image of u must contain the Jordan domain J of η. Therefore, $\mathcal{H}^2(J) \leqslant \mathrm{Area}(u) \leqslant \ell^2/4\pi$.

13.2. (2) implies (1)

Note that Z is the completion of $Z \setminus \partial Z$. Thus, if $Z \setminus \partial Z$ is a length space, then Z is CAT(0) by Proposition 12.1.

13.3. (1) implies (2)

Thus, we assume that Z is CAT(0), and claim that $Z\setminus\partial Z$ is a length space. The proof of this implication (probably well known to experts) is slightly more involved.

Consider arbitrary points $y_+, y_- \in Z\setminus\partial Z$. Let $\gamma: [a,b]\to Z$ be a geodesic between y_+ and y_-. Fix a positive $\varepsilon>0$. We need to find a curve γ_ε in $Z\setminus\partial Z$ which connects y_+ and y_- and has length at most $(1+2\varepsilon)\ell(\gamma)$.

For any $y=\gamma(t)$, we claim the existence of an open ball W^t around y with the following property. For any $z_1, z_2 \in W^t\cap\gamma$, there exists a curve $\eta\subset W^t$ connecting z_1 and z_2 with $\eta\cap\partial Z\subset\{z_1, z_2\}$ and such that

$$(1+\varepsilon)d_Z(z_1, z_2) \geqslant \ell(\eta).$$

Indeed, for $y\notin\partial Z$, in particular for $y=y_\pm$, the claim is evident. For any $y\in\partial Z$ we apply Lemma 11.2 to both parts of γ emanating from y and deduce the claim from the corresponding result in the flat hinge CT, where the claim is clear as well.

We can cover γ by a finite number of open balls $U_i=W^{t_i}$. By choosing an appropriate subsequence and rearrangement, we may assume that any two consecutive balls in the sequence U_i intersect. Choose arbitrary points y_i on γ in the intersection of U_i and U_{i+1}, with the only requirement that $y_i\notin\partial Z$ whenever $U_i\cap U_{i+1}\cap\gamma$ is not completely contained in ∂Z. We connect y_i and y_{i+1} by a curve $\eta_{i+1}\subset U_{i+1}$ provided by the definition of W^{t_i}. We may assume the first y_0 and the last y_m to be the ends $y_\pm$ of γ. Denote by η the concatenation of all η_i. Then, η is a curve in Z between y_+ and y_- which has length at most $(1+\varepsilon)\ell(\gamma)$. Moreover, η intersects ∂Z at most at finitely many points $y'_j=\eta(s_j)$.

For any such y'_j, a neighborhood of y'_j in ∂Z is contained in γ. We again apply Lemma 11.2 to both parts of γ emanating from y'_j, and note that, in this case, the image of the map E must be *open* in Z by the invariance of domains. Using the corresponding property in the flat cone CT, we find an arbitrarily short curve k_j with the following property. The curve k_j connects two points $\eta(a_j)$ and $\eta(b_j)$, with some $a_j<s_j<b_j$, and does not intersect ∂Z. Replacing $\eta|_{[a_j,b_j]}$ by k_j, we obtain the desired short connection between y_+ and y_-.

14. Some simplifications

The rest of the paper is devoted to the proof that (3) implies (1) in Theorem 1.3. Thus, from now on, we assume that Z is a geodesic metric space homeomorphic to $\overline{D}$, such that $Z\setminus\partial Z$ is non-positively curved and such that Z satisfies the isoperimetric inequality as stated in Theorem 1.3 (3). We need to prove that Z is CAT(0).

14.1. Subdomains

Let T be a Jordan curve of finite length in Z with Jordan domain J. Consider the closure $\bar{J}=J\cup T\subset Z$ with the induced length metric. Since T has finite length, the topologies induced by the length metric and by the induced metric coincide. Thus, $\bar{J}$ is a compact length metric space homeomorphic to $\bar{D}$. The compactness implies that $\bar{J}$ is a geodesic space. The identity map $F: \bar{J}\to Z$ preserves the lengths of all curves and the $\mathcal{H}^2$-area of all domains. Moreover, the restriction of F to $J=\bar{J}\setminus T$ is a local isometry. Therefore, the assumptions of Theorem 1.3 (3) are valid for the space $\bar{J}$ as well. As a consequence, we may reduce the problematic part of Z to a single geodesic.

LEMMA 14.1. *We may assume, in addition, the following conditions*:

(1) $\mathcal{H}^2(Z)$ *is finite*;

(2) *there exist a geodesic* $c: [a_0, b_0]\to\partial Z\subset Z$ *and some* a *and* b, *with* $a_0<a<b<b_0$, *such that* $Z\setminus c([a,b])$ *has non-positive curvature.*

Proof. Assume that some Z satisfies the assumption of Theorem 1.3,(3) but is not CAT(0). We are going to find a Jordan domain Z^- in Z which satisfies both assumptions of the lemma, but which is not CAT(0) either.

Due to §13.2, $Z\setminus\partial Z$ cannot be a length space. Thus, we find points $x, y\in Z\setminus\partial Z$ and $\varepsilon>0$ such that, for any curve $\gamma\subset Z\setminus\partial Z$ connecting x and y, we have $\ell(\gamma)>d_Z(x,y)+\varepsilon$.

Consider a geodesic $c: [a_0, b_0]\to Z$ in Z between x and y. Connect further x and y by some simple piecewise geodesic curve $\hat{c}$ in $Z\setminus\partial Z$ disjoint from c outside the endpoints. Consider the arising Jordan curve $T=c\cup\hat{c}$ and the corresponding closed Jordan domain Z^- with its induced length metric. Since T has finite length, we have $\mathcal{H}^2(Z^-)<\infty$ by the isoperimetric inequality. Outside the intersection of T with ∂Z, the space Z^- has non-positive curvature, by Lemma 11.1. Thus Z^- satisfies (1) and (2) required in the lemma, where $a<b$ can be chosen arbitrary in (a_0, b_0), sufficiently close to a_0 and b_0, respectively.

Assume that Z^- is CAT(0). Then, $Z^-\setminus T$, with the metric induced from Z^-, is a length space due to §13.3. Therefore, we find a curve γ in Z^- connecting x and y, such that γ does not intersect $T\cap\partial Z$ and such that the length of γ is arbitrary close to $\ell(c)=d_Z(x,y)$. This contradicts our assumption on x and y, shows that Z^- cannot be CAT(0), and finishes the proof of the lemma. □

14.2. Simple setting

We may and will assume from now on that Z satisfies the assumptions of Lemma 14.1. Thus, Z has non-positive curvature outside a geodesic c contained in ∂Z. We consider the complement $Z\setminus\partial Z$ and call the associated length space Y_0. Note that the identity map $I\colon Y_0\to Z\setminus\partial Z$ is a 1-Lipschitz homeomorphism. The length space Y_0 has non-positive curvature, thus, by Proposition 12.1, the completion Y of Y_0 is CAT(0). Moreover, $I\colon Y_0\to Z$ extends to a 1-Lipschitz map $I\colon Y\to Z$. Under these assumptions, we obtain the following uniform area estimate for balls in Y_0 near c.

LEMMA 14.2. *For any $y\in Y_0$ and all r such that $\overline{I(B_{Y_0}(y,r))}\cap\partial Z\subset c$, the area of the r-ball in Y_0 around y satisfies*

$$\mathcal{H}^2(B_{Y_0}(y,r))\geqslant \tfrac{1}{4}\pi r^2.$$

Proof. We proceed similarly to [27, §9.2], following the standard arguments leading to the boundary regularity of minimal surfaces under a chord-arc condition on the boundary.

Let $y\in Y_0$ and $r>0$ be as in the statement of the lemma. Since ∂Z is not completely contained in c, the assumption on r implies that there exists some "remote" point $x\in Y_0$ with the following properties: $d(x,y)>r$ and, for the connected component U of x in $Y_0\setminus B(y,r)$, the closure of $I(U)$ in Z contains a point in $\partial Z\setminus c$.

We now argue by contradiction and assume that $\mathcal{H}^2(B_{Y_0}(y,r))<\frac{1}{4}\pi r^2$. For any $t<r$, we consider the ball $B_{Y_0}(y,t)$ around y in Y_0, denote by $v(t)$ its area and by L_t its boundary in Y_0. Note that L_t separates x and y in Y_0. Set $w(t)=\mathcal{H}^1(L_t)$. By the co-area inequality, we have $\int_0^t w(s)\,ds\leqslant v(t)$ for all $t<r$. (See [27, Lemma 2.3], and note that Y_0 is countably 2-rectifiable, no non-Euclidean planes appear as tangent spaces of Y_0 and the distance function to any point is 1-Lipschitz). The contradiction to $v(r)<\frac{1}{4}\pi r^2$ follows by integration, once we have verified for almost all $t<r$ the inequality

$$v(t)\leqslant \frac{w^2(t)}{\pi}. \tag{14.1}$$

We claim that, for all t with finite $w(t)=\mathcal{H}^1(L_t)$, the set L_t contains a closed subset L_t' still separating x and y in Y_0, and such that L_t' is either a Jordan curve or homeomorphic to an open interval. Indeed, consider the topological sphere $S^2=Z/\partial Z$ obtained from Z by collapsing ∂Z to a point, equipped with the quotient metric. Let K be the closure of the image of $I(L_t)$ in S^2. Then, K separates the (images of the) points x and y in S^2. Since $w(t)<\infty$, it follows that K contains a Jordan curve K_0 which separates x and y; cf. [27, Corollary 7.5]. The preimage L_t' of K_0 in Y_0 is either a Jordan curve or an open interval, it is contained in L_t and separates x and y in Y_0.

Assume that L_t' is a Jordan curve. By the choice of x, the Jordan domain of L_t' contains y, and therefore the whole ball $B_{Y_0}(y,t)$. Thus,

$$v(t)\leqslant \frac{\ell_Z^2(I(L_t'))}{4\pi}\leqslant \frac{w^2(t)}{4\pi},$$

and hence (14.1) holds.

Assume now that L'_t is an open interval, which has finite length by assumption. Then, the closure of $I(L'_t)$ consists of $I(L'_t)$ and one or two points on ∂Z. By construction and assumption, these points are contained in $\overline{I(B_{Y_0}(y,r))}\cap\partial Z\subset c$. We consider the Jordan curve $T_t\subset Z$ given by $I(L'_t)$ and the part of c between the endpoints of $I(L'_t)$. Again, by the choice of x, the Jordan domain of T_t contains $I(y)$, and therefore the image $I(B_{Y_0}(y,t))$. On the other hand, the length of T_t is at most $2w(t)$. Indeed, c is a geodesic in Z, and thus the curve $I(L'_t)$ has at least half of the length of T_t. Therefore, the isoperimetric inequality gives us

$$v(t)\leqslant\frac{(2w(t))^2}{4\pi}=\frac{w^2(t)}{\pi},$$

finishing the proof of (14.1) and of the lemma. □

Under the assumptions of Lemma 14.1, we conclude the following result.

Proposition 14.3. *The space Y is compact and homeomorphic to $\overline{D}$. The restriction $I:\partial Y\to\partial Z$ is weakly monotone, and in particular the preimage $\eta=I^{-1}(c)$ is an arc. The restriction $I:Y\setminus\eta\to Z\setminus c$ is a bijective local isometry. The map $I:Y\to Z$ is an isometry if and only if I maps η to c in an arclength-preserving way.*

Proof. Assume that Y is not compact. Then, we find some $\varepsilon>0$ and an infinite sequence of points $y_n\in Y_0$ with pairwise distance at least 2ε. After choosing a subsequence, we may assume that $I(y_n)$ converges to a point $z\in Z$. If the point z has a CAT(0) neighborhood in Z, then, for a small ball B around z, the metric of $B\cap(Z\setminus\partial Z)$ is a length metric, due to §13.3. From this, we get $d_Y(y_n,y_m)=d_Z(I(y_n),I(y_m))$ for all n and m large enough. Therefore, the points y_n cannot be 2ε-separated.

Therefore, we may assume that z does not have a CAT(0) neighborhood. Hence, $z\in c([a,b])\subset c([a_0,b_0])$. Thus, we find some small $r<\varepsilon$ such that $B(z,3r)\cap\partial Z\subset c$. This allows us to apply Lemma 14.2 to y_n and deduce that, for all sufficiently large n, the area of the ball $B_{Y_0}(y_n,r)$ in Y_0 is at least $\frac{1}{4}\pi r^2$. But all these balls are disjoint. This contradicts the finiteness of the total area of Y_0 and finishes the proof that Y is compact. From Lemma 12.2, we deduce that Y is homeomorphic to $\overline{D}$.

Since $I(Y_0)$ is dense in Z and Y compact, we obtain $I(Y)=Z$. As $I:Y_0\to Z\setminus\partial Z$ is a homeomorphism, we infer that $I^{-1}(\partial Z)=\partial Y$. An open subset U of $\overline{D}$ is contractible if and only if $U\cap D$ is contractible. Since $I:Y\setminus\partial Y\to Z\setminus\partial Z$ is a homeomorphism, we deduce that preimages of open contractible sets are contractible. Therefore, the preimage of any point $z\in Z$ is a cell-like set (cf. [15, p. 97] or [27, §7]). For $z\in\partial Z$, the preimage is a cell-like subset of the circle ∂Y, and hence either a point or an arc. Therefore, the restriction $I:\partial Y\to\partial Z$ is weakly monotone.

In particular, the preimage η of the arc $c\subset\partial Z$ is an arc in ∂Y. For any point $z\in Z\setminus c$ there is a small ball O around z such that the metric on $O\setminus\partial Z$ is a length metric, as shown in §13.3. Thus, $I\colon I^{-1}(O)\to O$ is an isometry.

Assume now that I is arclength preserving on η. Then, I is bijective, and hence a homeomorphism. Moreover, for any geodesic γ in Z, the map $I\colon I^{-1}(\gamma)\to\gamma$ preserves the $\mathcal{H}^1$-measure. Thus, $I^{-1}(\gamma)$ has the same length as γ, and the map $I^{-1}\colon Z\to Y$ is 1-Lipschitz. Therefore, I is an isometry. □

In the sequel we will construct Jordan curves $T\subset Y$ whose intersection with η is an arc η_0. In this situation, the image $I\circ T$ is a Jordan curve in Z whose Jordan domain is the (locally isometric) image of the Jordan domain of T. The complementary part $k=T\setminus\eta_0$ is mapped by I in an arclength-preserving way. Since I is 1-Lipschitz and the image of η_0 is a geodesic, we get

$$\ell_Z(I\circ T)=\ell_Y(k)+\ell_Z(I\circ\eta_0)\leqslant 2\ell_Y(k).$$

Moreover, the number $\ell_Y(T)-\ell_Z(I\circ T)=\ell_Y(\eta_0)-\ell_Z(I\circ\eta_0)$ measures the deviation of I from being an isometry.

15. Rectifiable parts

15.1. Formulation

We continue to work under the standing assumptions of Lemma 14.1, and use the notation of Proposition 14.3. The aim of this section is to prove the following result.

PROPOSITION 15.1. *The map $I\colon Y\to Z$ preserves the length of any rectifiable subcurve of η.*

15.2. Euclidean domains of almost isoperimetric equality

We begin with a short Euclidean computation. For any sufficiently small $r>0$, let $T=T_r$ be a Jordan curve in $\mathbb{R}^2$ which consists of an arc of length $2\pi-2r$ on S^1 and a chord of length $2\sin r$. Then, $\ell(T)<2\pi$ and the Jordan domain J of T has area

$$\mathcal{H}^2(J)=\pi-r+\tfrac{1}{2}\sin(2r)>\pi-r^3.$$

Therefore, we deduce

$$\ell(T)-\sqrt{4\pi\mathcal{H}^2(J)}<2r^3.$$

We note that the curve T is contained in a hinge of angle $\pi-r$ enclosed between the chord and the tangent to S^1 at one of the endpoints of the chord. Rescaling the curve T suitably, we obtain the following lemma.

Lemma 15.2. *For any* $\varepsilon>0$ *there exist some* $L, \delta>0$ *with the following property. Let* $\gamma: [0,\infty)\to\mathbb{R}^2$ *be one of two rays bounding a hinge* H *of angle* $\geqslant\pi-\delta$ *in Euclidean* $\mathbb{R}^2$. *For any* $s>0$ *one can find a Jordan curve* $T_s\subset H$ *with Jordan domain* J_s *such that the following statements hold*:

(1) *the curve* T_s *contains the initial part of* γ *of length* s;

(2) $\ell(T_s)\leqslant Ls$;

(3) $\ell(T_s)-\sqrt{4\pi\mathcal{H}^2(J_s)}<\varepsilon s$.

15.3. Curved domains of almost isoperimetric equality

We apply the approximation of hinges by flat cones, provided by Lemma 11.2, and directly deduce the following from Lemma 15.2.

Lemma 15.3. *For any* $\varepsilon>0$ *there exist some* $L, \delta>0$ *with the following property. Let* M *be a metric space of non-positive curvature homeomorphic to a surface with boundary. Let* H *be a hinge in* M *of angle* $\geqslant\pi-\delta$, *and let* γ *be one of its bounding geodesics. Then, for all sufficiently small* $s>0$, *there exists a Jordan curve* $T_s\subset H\subset M$ *with Jordan domain* J_s *such that the conclusions* (1)–(3) *of Lemma* 15.2 *hold.*

The bound $s_0>0$, such that the conclusion of Lemma 15.3 holds for all $0<s<s_0$, depends on ε, the space M and the hinge H.

15.4. Differentials

In order to approach general rectifiable curves, we will use a Rademacher-type theorem. Let $\gamma: [p,q]\to X$ be a rectifiable curve parameterized by arclength in a CAT(0) space X. We say that γ is *differentiable* at the point $t\in(p,q)$ if the ingoing and outgoing directions of γ are "almost defined by almost opposite geodesic directions". More precisely, we require the following conditions to hold for the curves $\gamma^\pm(s):=\gamma(t\pm s)$. The angle between $\gamma^\pm$ is well defined (cf. [9, Definition 3.6.26]) and equal to π. Moreover, there are geodesics $\eta_n^\pm$ starting at $\gamma(t)$ such that the angles between $\eta_n^\pm$ and $\gamma^\pm$ are well defined and converge to zero as n tends to ∞.

In other words, the angle between η_n^+ and η_n^- converges to π and, for any $\delta>0$ and all sufficiently large n, there exists some $s_0>0$ such that, for all $0<s<s_0$, we have $d(\gamma(t\pm s),\eta_n^\pm(s))<\delta s$. The metric differentiability theorem implies the following result ([22, Theorem 1.6]).

Lemma 15.4. *Let* X *be a* CAT(0) *space and* $\gamma: [p,q]\to X$ *be a rectifiable curve parameterized by arclength. Then, for almost all* $t\in[p,q]$, *the curve* γ *is differentiable at* t.

Now, we are able to deduce that many small parts of any rectifiable curve can be complemented to Jordan curves almost violating the Euclidean isoperimetric inequality. In order to avoid minor difficulties, we restrict ourselves to boundary curves, the only case we will need.

PROPOSITION 15.5. *For any* $\varepsilon>0$ *there exists some* $L>0$ *with the following property. Let* M *be a metric space of non-positive curvature homeomorphic to a surface with boundary, and let* $\gamma:[p,q]\to\partial M$ *be a part of the boundary of* M *parameterized by arclength. Then, for a set* S *of full measure in* $[p,q]$ *and any* $t\in S$, *there exists some* $r_0=r_0(t)>0$ *such that the following statement holds. For any* $s<r_0$ *there exists a Jordan curve* T_s *in* M *with Jordan domain* J_s *such that*

(1) *the intersection of* T_s *with* γ *is an arc which contains* $\gamma|_{[t,t+s]}$;

(2) $\ell(T_s)\leqslant Ls$;

(3) $\ell(T_s)-\sqrt{4\pi\mathcal{H}^2(J_s)}<\varepsilon s$.

Proof. Choose some $L=L\left(\frac{1}{4}\varepsilon\right)$ provided by Lemma 15.3. Let $S\subset(p,q)$ be the set of points in which γ is differentiable. For any $t\in S$ and any $\delta>0$ we find geodesics $\gamma^{\pm}$, starting in $\gamma(t)$ at an angle not less than $\pi-\delta$, such that $d(\gamma(t\pm s),\gamma^{\pm}(s))<\delta s$ for all sufficiently small s.

Let H denote a small hinge in M enclosed by the geodesics $\gamma^{\pm}$. If δ and s are small enough, we apply Lemma 15.3 and find a Jordan curve $T'_s\subset H\subset M$ with Jordan domain J'_s and the following properties. The curve T'_s has length at most $2Ls$ and contains the initial part of γ^+ of length $2s$. Moreover,

$$\ell(T'_s)-\sqrt{4\pi\mathcal{H}^2(J'_s)}<\tfrac{1}{2}\varepsilon s.$$

We now connect the point $\gamma^+(2s)$ with a nearest point $\gamma(t+\hat{s})$ on $\gamma|_{[t,t+2s]}$ by a geodesic c_s. By the choice of γ^+, the length of c_s is at most $2\delta s$. Moreover,

$$|\hat{s}-2s|=d(\gamma^+(\hat{s}),\gamma^+(2s))\leqslant 2\delta s+\delta\hat{s},$$

by the triangle inequality. Therefore, for $\delta<\frac{1}{2}$, we deduce that $\hat{s}-2s<4\delta\hat{s}\leqslant 8\delta s$.

Since γ^+ is a geodesic, c_s does not intersect T'_s outside γ^+. Throwing away the common part of c_s and γ^+, we may assume that c_s intersects T'_s only at the initial point of c_s.

Let now the curve T_s arise from T'_s by replacing the initial γ^+-part of T'_s by c_s and the corresponding arc of γ between $\gamma(t)$ and $\gamma(t+\hat{s})$. By construction, the Jordan domain J_s of T_s contains the Jordan domain J'_s of T'_s, and hence $\mathcal{H}^2(J'_s)\leqslant\mathcal{H}^2(J_s)$. Moreover, the length of $\ell(T_s)$ is at most $\ell(T'_s)+2\delta s+8\delta s$.

Once δ and s have been chosen small enough, we see that the curve T_s satisfies all requirements of the lemma. □

15.5. Length preservation

Proof of Proposition 15.1. Let $\gamma\colon [p,q]\to\partial Y$ be a rectifiable subcurve contained in η. We may assume γ to be parameterized by arclength. Since the map I is 1-Lipschitz, we have $\ell_Z(I\circ\gamma)\leqslant\ell_Y(\gamma)$. If the inequality is strict, then we find some $\varepsilon>0$ and a set Q of positive measure in $[p,q]$ such that the following statement holds: for any $r\in Q$, there is some $\delta=\delta(r)>0$ such that, for all $h<\delta$, one has

$$\ell_Z(I\circ\gamma|_{[r,r+h]})\leqslant(1-2\varepsilon)h.$$

Applying Proposition 15.5, we find some $r\in Q$ and, for all sufficiently small $h>0$, we find a Jordan curve T_h as in Proposition 15.5 containing $\gamma_{[r,r+h]}$. Let J_h denote the Jordan domain of T_h. Then, $I(J_h)$ is the Jordan domain of the Jordan curve $I(T_h)$. Since on Y_0 the map I is a local isometry, we have $\mathcal{H}^2(J_h)=\mathcal{H}^2(I(J_h))$. By assumption and the 1-Lipschitz property of I, we see that $\ell_Z(I\circ T_h)\leqslant\ell_Y(T_h)-2\varepsilon h$.

We deduce $\ell_Z(I\circ T_h)-\sqrt{4\pi\mathcal{H}^2(I(J_h))}<-\varepsilon h$, which contradicts the isoperimetric assumption of Theorem 1.3 (3). This finishes the proof. □

16. Final steps

16.1. Formulation

We continue to use the notation from Proposition 14.3. The rest of the section is devoted to the proof of the following result.

PROPOSITION 16.1. *The map $I\colon Y\to Z$ is an isometry.*

Assume the contrary. Due to Proposition 14.3 and Proposition 15.1, the arc η is not rectifiable. We fix a parametrization of η as a simple curve $\eta\colon [p,q]\to Y$.

For the convenience of the reader, we first outline the main steps of the proof. In Lemma 16.3 we will deduce from the non-rectifiability of η the existence of points on η at which the "differential" of I is arbitrary small. We will then fix such a point y and search for a contradiction to the isoperimetric inequality in a small neighborhood of this point. In Lemma 16.4, we show that the area of small balls around y is almost Euclidean, and obtain in Corollary 16.5 a bound on the length of any curve surrounding such a ball. In Lemma 16.7 we connect y by a geodesic γ with a nearby point on η, and prove that γ and η are sufficiently close to each other, more precisely, they enclose (in a rather weak sense) an angle of at most $\arctan\left(\frac{1}{2}\right)$. In the final subsection, we take two geodesics connecting y with nearby points on η, lying on different sides of y. If the angle enclosed between these geodesics is at least π, then we obtain a contradiction in the same way as at the end of the proof of Proposition 15.1 above. If the angle is smaller than π, then we consider two points on these geodesics with small distance r from y. We connect these points by a "circular arc" inside the hinge (using Lemma 11.2), and we further connect these points to some points on η using Lemma 16.7. The arising curve is relatively short, but nevertheless surrounds a sufficiently large ball around y, leading to a contradiction with Corollary 16.5.

16.2. Bounding diameter by endpoints on η

We claim and prove the following.

LEMMA 16.2. *For any $p \leqslant t < r \leqslant q$ the diameter of $\eta|_{[t,r]}$ is at most $10d(\eta(t), \eta(r))$.*

Proof. Assume the contrary and set $l = d(\eta(t), \eta(r))$. Since $Y_0 = Y \setminus \partial Y$ is a length space, we find a simple curve k in Y of length less than $2l$ connecting $\eta(t)$ and $\eta(r)$ and building a Jordan curve T together with $\eta|_{[t,r]}$. The Jordan domain J of T has area at most $4l^2/\pi$, as the length of the image of T in Z is at most $2\ell_Y(k) = 4l$.

Since $\bar{J}$ (which contains $\eta|_{[t,r]}$) has diameter larger than $10l$, we find a point $x \in \bar{J}$ with distance at least $4l$ from the curve k. Thus, the ball $B_Y(x, 4l)$ is contained in $\bar{J}$. From Lemma 14.2 we see that $\mathcal{H}^2(J) \geqslant \frac{1}{4}\pi(2l)^2 = \pi l^2$. This contradicts $\mathcal{H}^2(J) \leqslant \frac{4}{\pi} l^2$ and finishes the proof. □

16.3. A consequence of non-rectifiability

Recall that $I \circ \eta$ is a weakly monotone parametrization of the geodesic $c \subset Z$. From now on, we will consider the curve c with this weakly monotone parametrization $c = I \circ \eta \colon [p, q] \to Z$, despite the fact that in the rest of the paper all geodesics are parameterized by arclength. We are going to find points at which the "differential" of I is arbitrary small, by using Lemma 16.2 and the fact that η is non-rectifiable.

LEMMA 16.3. *For any $\lambda > 0$ there exist some $t \in [p, q]$ and $\varepsilon > 0$ such that, for all $s \in [p, q]$ with $|s - t| < 2\varepsilon$, we have*

$$d(\eta(t), \eta(s)) \geqslant \lambda d(c(t), c(s)).$$

Proof. We assume the contrary and take some $\lambda > 0$ for which the claim is wrong. We are going to prove that η is rectifiable, in contradiction to our assumptions.

Consider an arbitrary $\varepsilon > 0$. For any $t \in [p, q]$, we find some $t^+ \neq t$ with $|t^+ - t| < 2\varepsilon$ and $d(\eta(t), \eta(t^+)) < \lambda d(c(t), c(t^+))$. If t is one of the endpoints p or q, then we set $t^- = t$.

If not then, by continuity of η and c, we find t^- arbitrarily close to t on the other side of t from t^+ such that $d(\eta(t^+),\eta(t^-))<\lambda d(c(t^+),c(t^-))$.

Denote by I_t the closed interval between t^- and t^+, which by our choice has length smaller than 2ε. Changing the order, if needed, we may assume that $t^-<t^+$ for any t. We find a finite covering of $[p,q]$, by some of these intervals $I_{t_1},...,I_{t_k}$, such that the intersection number of the covering is at most 2. We reorder the intervals and have $t_i^-\leqslant t_{i+1}^-\leqslant t_i^+$ for all i. Thus, all endpoints of all the intervals I_{t_i} define a 2ε-fine subdivision $p=s_1<s_1\leqslant...\leqslant s_{2k}=q$ of $[p,q]$. Each of the intervals $[s_i,s_{i+1}]$ is contained in exactly one or two of the intervals $[t_j^-,t_j^+]$. Due to Lemma 16.2, we have $d(\eta(s_i),\eta(s_{i+1}))\leqslant 10d(\eta(t_j^-),\eta(t_j^+))$ in this case. Summing up, we deduce that

$$\sum_{i=1}^{2k} d(\eta(s_i),\eta(s_{i+1}))\leqslant 20\biggl(\sum_{j=1}^{k} d(\eta(t_j^+),\eta(t_j^-))\biggr)<20\lambda\biggl(\sum_{j=1}^{k} d(c(t_j^-),c(t_j^+))\biggr)\leqslant 40\lambda\ell(c).$$

Since ε was arbitrary, we see that $40\lambda\ell(c)$ provides an upper bound for the length of η, in contradiction to the non-rectifiability of η. □

16.4. Setting

We choose some large $\lambda>0$, to be determined later. We find $t\in[p,q]$ and some $\varepsilon>0$ provided by Lemma 16.3. Since Z is non-positively curved in neighborhoods of the boundary points $c(a_0)=I(\eta(p))$ and $c(b_0)=I(\eta(q))$ (cf. Lemma 14.1), the map I is a local isometry in neighborhoods of $\eta(p)$ and $\eta(q)$. Hence, $t\in(p,q)$. In order to simplify the notation, we may and will assume $t=0$ and $[-\varepsilon,\varepsilon]\subset(p,q)$. Set $y=\eta(0)$ and note that $I(y)$ is not an endpoint of the geodesic c in Z. We choose some $r_0>0$ such that $B(y,2r_0)\cap\partial Y\subset\eta|_{[-\varepsilon,\varepsilon]}$.

We can now show that balls around y have almost Euclidean area. We emphasize that the point y and the radius r_0 depend on the choice of the constant λ.

Lemma 16.4. *For any $\alpha_0>0$ the following statement holds. If λ has been chosen large enough, then, for any $r<r_0$, the area of $O(y,r)=B(y,r)\cap Y_0$ can be estimated by*

$$\mathcal{H}^2(O(y,r))\geqslant(\pi-\alpha_0)r^2.$$

Proof. Approximating y by points in Y_0, we obtain from Lemma 14.2 the inequality $\mathcal{H}^2(O(y,r))\geqslant\frac14\pi r^2$ for all $r<r_0$. In order to improve the bound, we argue as in the proof of Lemma 14.2. We consider the distance function $f\colon Y_0\to\mathbb{R}$ defined by $f(z)=d(y,z)$. Then, $O(y,t)$ is the sublevel set $f^{-1}((0,t))$. Denote by $L_t\subset Y_0$ the level set $f^{-1}(t)$ and by $w(t)$ its length $\mathcal{H}^1(L_t)$. Set $v(t)=\mathcal{H}^2(O(y,t))$. By the co-area inequality, $w\in L^1([0,r_0])$

and $\int_0^t w(s)\,ds \leqslant v(t)$ for almost all $t \in (0, r_0)$. As in the proof of Lemma 14.2, for almost all $t \in (0, r_0)$, the set L_t contains an arc L'_t which connects two points on η and separates $O(y,t)$ from some fixed point $x \in Y_0$ at large distance from y.

Denote by $w_1(t) \leqslant w(t)$ the length $\mathcal{H}^1(L'_t)$. The statement of the lemma follows by integration, once we have verified for almost all $t \in (0, r_0)$ the inequality

$$v(t) \leqslant \frac{w_1^2(t)}{4\pi}(1+g_\lambda), \tag{16.1}$$

where the constant g_λ goes to zero as λ goes to ∞.

We already know $v(t) \geqslant \frac{1}{4}\pi t^2$, and we have seen in the proof of Lemma 14.2 that $w_1^2(t)/\pi \geqslant v(t)$. Therefore, $w_1(t) \geqslant \frac{1}{2}\pi t$.

The endpoints $e^{\pm}$ of L'_t must be contained in $\eta|_{[-\varepsilon,\varepsilon]}$ and lie on different sides of y. Moreover, by definition, $d(e^{\pm}, y) = t$. As in Lemma 14.2, we consider the Jordan curve T_t built by L'_t, and the part of η between $e^{\pm}$. From the choice of λ and ε, we deduce that the length of $I \circ (T_t \cap \eta)$ is at most $2t/\lambda$. Therefore,

$$\ell_Z(I \circ T_t) \leqslant w_1(t) + \frac{2}{\lambda} t \leqslant \left(1 + \frac{4}{\pi\lambda}\right) w_1(t).$$

Now, as in the proof of Lemma 14.2, the isoperimetric inequality in Z provides (16.1) with $1 + g_\lambda = (1 + 4/\pi\lambda)^2$. This finishes the proof. □

As a consequence, we get the following result.

COROLLARY 16.5. *If λ is large enough, then, for any $r < \frac{1}{3} r_0$, the following statement holds. Any curve k in $Y \setminus B_Y(y,r)$ which connects two points in $\eta([-\varepsilon,\varepsilon])$ on different sides of y satisfies the inequality $\ell_Y(k) > (\pi+3)r$.*

Proof. Assume the contrary. Fix a sufficiently small $\alpha_0 > 0$, choose $\lambda > 0$ such that Lemma 16.4 holds. Consider an arbitary $r < \frac{1}{3} r_0$ and a curve k violating the conclusion of the corollary. Then, we find a simple subcurve of k which still connects two points $e^{\pm}$ in $\eta([-\varepsilon,\varepsilon])$ on different sides of y, and does not intersect η between $e^{\pm}$. We replace k by this subcurve, and consider the Jordan curve T built by k and the part of η between $e^{\pm}$. The Jordan domain of T contains $O(y,r) = B(y,r) \cap Y_0$. Therefore,

$$\frac{\ell_Z^2(I \circ T)}{4\pi} \geqslant (\pi - \alpha_0) r^2, \tag{16.2}$$

due to Lemma 16.4 and the isoperimetric inequality in Z.

We have $\ell_Y(k) \geqslant \frac{1}{2} \ell_Z(I \circ T)$. If k does not intersect the $3r$-ball around y in Y, then we may replace the term r^2 by $(3r)^2$ on the right-hand side of (16.2). In this case, we arrive at a contradiction, once α_0 is small enough ($\alpha_0 = \frac{5}{9}\pi$ is sufficient here).

If k intersects the $3r$-ball around y in Y, then $e^{\pm}$ have distance at most $10r$ to y, since k has length at most $(\pi+3)r$. Therefore, $I\circ(T\cap\eta)$ has length at most $20r/\lambda$. Hence,

$$\frac{1}{4\pi}\left(\ell_Y(k)+\frac{20}{\lambda}r\right)^2 \geqslant (\pi-\alpha_0)r^2.$$

If λ has been chosen sufficiently large and α_0 sufficiently small, this contradicts the assumption $\ell_Y(k)\leqslant(\pi+3)r$. □

16.5. The boundary is close to a geodesic

Consider the geodesic $\gamma\colon[0,t_0]\to Y$ starting at y and ending at $\eta(\varepsilon)$. Let $P\colon Y\to\gamma$ be the nearest point projection, which is well defined and 1-Lipschitz, since Y is CAT(0). For any point $x\in Y$, denote by β_x the shortest geodesic from x to $P(x)$. Then, $P(\beta_x)=P(x)$. In particular, for $x,w\in Y$, the geodesics β_x and β_w are either disjoint or are sent by P to the same point, their common endpoint. Since any geodesic β_x encloses an angle of at least $\frac{1}{2}\pi$ with (the initial part of) γ at $P(x)$, we infer that $d(y,\beta_x)=d(y,P(x))$. In other words, any point on β_x has at least the same distance from $y=\gamma(0)$ as $P(x)$. For topological reasons, we have the following lemma.

Lemma 16.6. *The composition $P\circ\eta\colon[0,\varepsilon]\to\gamma$ is a weakly monotone parametrization of γ.*

Denote by Q the union of all geodesics β_x, where x runs over all points on $\eta|_{[0,\varepsilon]}$. By definition, Q contains $\eta|_{[0,\varepsilon]}$ and γ. Denote by Q_0 the intersection $Q\cap Y_0$, and consider the 1-Lipschitz continuous function $f\colon Q_0\to[0,t_0]$ which sends $x\in Q_0$ to the γ-parameter of $P(x)$, thus $f(x)=d(P(x),y)$. By definition, $f^{-1}(t)$ is exactly the (intersection with Y_0 of the) union of all geodesics β_w which start on $\eta|_{[0,\varepsilon]}$ and end in $\gamma(t)$.

For $t\in[0,t_0]$ we let $h(t)\geqslant 0$ be the infimum of lengths of all geodesics β_x which start at some point $x\in\eta|_{[0,\varepsilon]}$ and end at $\gamma(t)=P(x)$. By definition, $h(t)$ equals the minimum of the distance function to the geodesic γ on the compact set $P^{-1}(\gamma(t))\cap\eta|_{[0,\varepsilon]}$. By continuity, $P(\eta|_{[0,\varepsilon]})=\gamma$, and thus h is well defined. By compactness, for any $t\in[0,t_0]$ we find a geodesic $\beta^t=\beta_x$ of length $h(t)$ which starts on $\eta|_{[0,\varepsilon]}$ and ends at $\gamma(t)$. By minimality, the geodesic β^t intersects η only at the starting point. Again by compactness, the function $h(t)$ is lower semi-continuous.

We set $g(t)=\mathcal{H}^1(f^{-1}(t))$ for $t\in[0,t_0]$. By construction, we have $h(t)\leqslant g(t)$ for all t. By the co-area inequality, g is integrable and, for any $0\leqslant t<t'\leqslant t_0$, we have

$$\mathcal{H}^2(f^{-1}((t,t')))\geqslant\int_t^{t'} g(s)\,ds. \tag{16.3}$$

With these notation and preparations at hand, we can now show that γ and η do not diverge too fast from each other.

Lemma 16.7. *For all $t\in[0,t_0]$ there is some $t\leqslant t'\leqslant 2t$ with $h(t')\leqslant\frac{1}{2}t'$. Thus, there exists a geodesic $\beta^{t'}$ of length at most $\frac{1}{2}t'$ starting at $\gamma(t')$ orthogonally to γ and ending on $\eta|_{[0,\varepsilon]}$.*

Proof. Let $\mathcal{T}$ be the set of all $t\in[0,t_0]$ for which the claim is true. By definition, we have $h(t_0)=0$, and therefore $\left[\frac{1}{2}t_0,t_0\right]\subset\mathcal{T}$. By the semi-continuity of h, $\mathcal{T}$ is closed. Assume that $\mathcal{T}\neq[0,t_0]$ and let $t_3\in\left(0,\frac{1}{2}t_0\right]$ be the smallest number such $[t_3,t_0]\subset\mathcal{T}$.

Consider $t_2:=2t_3$. From the minimality of t_3, we infer that $h(t_2)\leqslant\frac{1}{2}t_2$ and that for any $t\in[t_3,t_2)$ the inequality $h(t)>\frac{1}{2}t$ holds true. Due to the semi-continuity of h and since $h(0)=0$, there exists a largest $t_1\in[0,t_3)$ with $h(t_1)\leqslant\frac{1}{2}t_1$. Summarizing, we have

$$h(t_1)\leqslant\tfrac{1}{2}t_1,\quad h(t_2)\leqslant\tfrac{1}{2}t_2,\quad t_2>2t_1\quad\text{and}\quad h(t)>\tfrac{1}{2}t\ \text{ for } t\in(t_1,t_2).$$

We are going to derive a contradiction to the isoperimetric inequality. Consider the geodesics β^{t_1} and β^{t_2}. By construction, these geodesics do not intersect η outside their endpoints. Moreover, $\gamma|_{(t_1,t_2)}$ does not intersect η, since otherwise h were equal to zero at the intersection point. Thus β^{t_1}, β^{t_2}, $\gamma|_{(t_1,t_2)}$ and the part of η between the endpoints of β^{t_1} and β^{t_2} constitute a Jordan curve T. Due to Lemma 16.6, the preimage $f^{-1}((t_1,t_2))$ is contained in the Jordan domain J of T. Since $g(t)\geqslant h(t)>\frac{1}{2}t$ for all $t\in(t_1,t_2)$, we deduce from (16.3) that

$$\mathcal{H}^2(J)>\int_{t_1}^{t_2}\frac{s}{2}\,ds=\frac{t_2^2-t_1^2}{4}. \tag{16.4}$$

We now estimate the length of $I\circ T$ in Z as follows. By assumption, the lengths of β^{t_1}, β^{t_2} and $\gamma|_{(t_1,t_2)}$ sum up to at most $\frac{1}{2}(t_1+t_2)+(t_2-t_1)$. Moreover, the distance of the starting point of β^{t_2} on η from y is at most $t_2+\frac{1}{2}t_2$. Thus, the η-part of $I\circ T$ is mapped to a part of the geodesic $c\subset Z$ which has length at most $3t_2/2\lambda$.

We set $q=t_1/t_2<\frac{1}{2}$. The isoperimetric inequality in Z gives us $\mathcal{H}^2(J)\leqslant\ell_Z^2(I\circ T)/4\pi$. Inserting the above estimates, we infer

$$\frac{1-q^2}{4}\leqslant\frac{1}{4\pi}\left(\frac{3}{2}-\frac{1}{2}q+\frac{3}{2\lambda}\right)^2. \tag{16.5}$$

The left-hand side is at least $\frac{3}{16}$ since $q<\frac{1}{2}$. The right-hand side is at most

$$\frac{9}{16\pi}\left(1+\frac{1}{\lambda}\right)^2.$$

Since $\pi>3$, we obtain a contradiction if λ is large enough. This finishes the proof. $\square$

16.6. Final conclusions

We look at the other side of y, and connect y with $\eta(-\varepsilon)$ by a geodesic γ_1. We apply to γ_1 the same considerations which we applied to γ above. We deduce that, for all sufficiently small t, there is some $t \leqslant t' \leqslant 2t$ and a geodesic $\alpha^{t'}$ from $\gamma_1(t')$ to a point on $\eta|_{[-\varepsilon,0]}$ such that $\ell(\alpha^{t'}) \leqslant \frac{1}{2}t' \leqslant t$ and $d(\alpha^{t'}, y) = t' \geqslant t$.

The contradiction is now achieved in two steps.

LEMMA 16.8. *The angle between γ and γ_1 must be at least π.*

Proof. Assume the contrary. We first claim that, for all sufficiently small t, there exists a curve k_t between $\gamma(t)$ and $\gamma_1(t)$ such that any point on k_t has distance at least t from y and such that $\ell(k_t) \leqslant \left(\pi + \frac{1}{2}\right)t$.

Indeed, if the angle between γ and γ_1 is not zero we apply Lemma 11.2 to the hinge between γ and γ_1. Thus, for any fixed $\delta > 0$ and all sufficiently small t, we find a curve k_t' of length at most $(1+\delta)\pi t$ which connects points $\gamma((1+\delta)\cdot t)$ and $\gamma_1((1+\delta)\cdot t)$ as the image of the corresponding circular arc in the flat hinge under the almost isometric map E provided by Lemma 11.2. Moreover, the distance of any point on k_t' to y is at least t. In order to obtain the required curve k_t, we just need to connect the endpoints of k_t' with $\gamma(t)$ and $\gamma_1(t)$ along γ and γ_1, respectively. On the other hand, if the angle between γ and γ_1 is zero (or just sufficiently small), we can obtain the required curve k_t for all sufficiently small t as follows: connect $\gamma(2t)$ with $\gamma_1(2t)$ by a geodesic, and then connect $\gamma(t)$ with $\gamma(2t)$ and $\gamma_1(2t)$ with $\gamma_1(t)$ along γ and γ_1, respectively.

Now, we consider a sufficiently small t such that the curve β^t has length at most $\frac{1}{2}t$. Such t exists by Lemma 16.7. Moreover, we apply Lemma 16.7 to the curve γ_1 instead of γ, and find some $t \leqslant t' \leqslant 2t$ and a geodesic $\alpha^{t'}$ with the properties provided by Lemma 16.7 and discussed prior to the present lemma.

Let the curve k be the concatenation of β^t, k_t, $\gamma_1|_{[t,t']}$ and $\alpha^{t'}$. By construction, the curve k lies completely outside the ball $B(y,t)$, it connects two points on η which lie on different sides of y, and the length of k is at most

$$\ell(k) \leqslant \tfrac{1}{2}t + \left(\pi + \tfrac{1}{2}\right)t + t + t \leqslant (\pi + 3)t.$$

This contradicts Corollary 16.5 and finishes the proof. □

The final lemma is proven similarly to the final step in the rectifiable case (Proposition 15.1).

LEMMA 16.9. *The angle between γ and γ_1 is strictly smaller than π.*

Proof. We assume the contrary and apply Lemma 15.3 to the hinge H between γ and γ_1. Thus, for all sufficiently small s, we find a Jordan curve T_s in the hinge H which contains the initial part of γ of length s, and such that, for the Jordan domain J_s of T_s, we have

$$\ell_Y(T_s)-\sqrt{4\pi\mathcal{H}^2(J_s)}<\tfrac{1}{3}s.$$

We now choose s to be such that β^s has length at most $\frac{1}{2}s$, and replace $\gamma|_{[0,s]}\subset T_s$ by the concatenation of β^s and the part of η between the starting point of β^s and y. The arising Jordan curve T_s' contains J_s in its Jordan domain. The image Jordan curve $I\circ T_s'$ has length at most

$$\ell_Z(I\circ T_s')\leqslant \ell_Y(T_s)-s+\frac{s}{2}+\frac{3s}{2\lambda}=\ell_Y(T_s)-\frac{s}{2}+\frac{3s}{2\lambda}.$$

If we have chosen $\lambda>9$, then the curve $I\circ T_s'$ does not satisfy the isoperimetric inequality in Z. □

The contradiction between Lemmas 16.9 and 16.8 finishes the proof of Proposition 16.1, and therefore the proof of Theorem 1.3.

Appendix A. Generalization to non-zero curvature bounds

We sketch the proof of Theorem 1.4, which generalizes Theorem 1.1 to the case of non-zero curvature bounds. We refer to [6] and [2] for basics on CAT($\varkappa$) spaces and recall that Reshetnyak's majorization theorem holds for all $\varkappa$. Thus, any closed curve Γ of length smaller than $R_\varkappa$ in any $CAT(\varkappa)$ space is majorized by a convex subset in $M^2_\varkappa$. Now, the proof of Lemma 3.2 shows the "only if" part of Theorem 1.4.

Starting the proof of the "if" part, let us assume that X satisfies the conditions of Theorem 1.4. Due to

$$\lim_{r\to 0}\frac{\delta_\varkappa(r)}{r^2}=\frac{1}{4\pi},$$

the arguments from §5 remain valid and prove that X satisfies property (ET). Arguing as in §6.3, we reduce the proof of the "if" part to the following claim: every intrinsic minimal disc Z in X corresponding to a solution of the Plateau problem $u\in\Lambda(\Gamma,X)$ is a CAT($\varkappa$) space. Here, Γ is any Jordan curve in X of length smaller than $R_\varkappa$.

The isoperimeric property of X implies the same isoperimetric property for all Jordan curves in Z. Thus, as in §7, we deduce that the conformal factor f of u satisfies the integral inequality (for all $z\in D$ and almost all $0<r<1-|z|$):

$$\int_{B(z,r)} f^2\leqslant \delta_\varkappa\biggl(\int_{\partial B(z,r)} f\biggr).$$

Arguing as in §7 and §8, and using [35] instead of [7], we obtain a metric of curvature $\leqslant\varkappa$ on the disc D which is defined on D by the canonical semi-continuous representative of the conformal factor f. As in §9, we reduce the proof to the following analogue of Theorem 1.3.

THEOREM A.1. *Let Z be a geodesic metric space homeomorphic to $\overline{D}$. Assume that, for any Jordan curve Γ in Z, the Jordan domain J enclosed by Γ satisfies the inequality $\mathcal{H}^2(J)\leqslant\delta_\varkappa(\ell(\Gamma))$. Assume further that $Z\setminus\partial Z$ has curvature $\leqslant\varkappa$. Then, Z is* $\mathrm{CAT}(\varkappa)$ *and $Z\setminus\partial Z$ is a length space.*

To prove Theorem A.1, we closely follow the second part of this paper. The approximation by flat cones Lemma 11.2 is valid without changes for all $\varkappa\neq 0$. As in §13.3, the $\mathrm{CAT}(\varkappa)$ property of Z implies that $Z\setminus\partial Z$ is a length space. In order to prove that Z is $\mathrm{CAT}(\varkappa)$, we need the following additional lemma which can be used instead of the theorem of Cartan–Hadamard.

LEMMA A.2. *If Z has curvature $\leqslant\varkappa$, then Z is* $\mathrm{CAT}(\varkappa)$.

Assuming that the lemma is wrong, we obtain an isometric embedding into Z of a circle Γ of length $2i<R_\varkappa$, where i is the injectivity radius of Z; cf. [6, §6]. Then, one can either obtain a contradiction by directly estimating the area of the Jordan domain J of Γ, which contains a rather large metric ball, or apply the fact that a round hemisphere is a minimal filling of a circle (cf. [17]) to deduce that

$$\mathcal{H}^2(J)\geqslant\frac{\ell^2(\Gamma)}{2\pi},$$

which contradicts the isoperimetric inequality.

Lemma A.2 shows that the closed Jordan domain of any Jordan polygon in $Z\setminus\partial Z$ is $\mathrm{CAT}(\varkappa)$ in its intrinsic metric. Thus, as in §12, we obtain that the completion Y of the space $Y_0=Z\setminus\partial Z$, equipped with the induced length metric, must be a $\mathrm{CAT}(\varkappa)$ space. From here, the rest of the proof goes without changes; we only need to restrict the attention to sufficiently small distances, where $\delta_\varkappa$ almost coincides with δ_0.

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