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Isoperimetric problem for uniform enlargement

by

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Abstract. We consider an isoperimetric problem for product measures with respect to the uniform enlargement of sets. As an example, we find (asymptotically) extremal sets for the infinite product of the exponential measure.

1. Introduction. Let (X, μ) be a separable topological space equipped with a Borel probability measure. Assume that to each point $x \in X$ there corresponds an open neighborhood $D(x)$ with the following symmetry property: for any $x, y \in X$,

$$\text{if} \quad x \in D(y), \quad \text{then} \quad y \in D(x). \tag{1.1}$$

For every non-empty set $A \subset X$, we define its *enlargement* by

$$\operatorname{enl}(A) = \bigcup_{a \in A} D(a), \tag{1.2}$$

and consider the problem of finding the function

$$R_\mu(p) = \inf_{\mu(A) \ge p} \mu(\operatorname{enl}(A)), \tag{1.3}$$

where the sup is over all Borel sets $A \subset X$ of measure $\mu(A) \ge p$.

Usually the enlargement is built with the help of a metric (or pseudometric) in X, say d, by taking for $D(x)$ the open ball $D(x, h)$ with center x and radius $h > 0$. Then $\operatorname{enl}(A) = A^h$ is the open h-neighbourhood of A, and $R_\mu(p) = R_\mu(p, h)$ depends also on h. Next, in applications of (1.3) to distribution of Lipschitz functions f on X, one fixes p and varies h. For $A = \{x : f(x) \le m\}$, where m is the median of f, we have $A^h \subset \{x : f(x) <$

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$m+h\}$, and (1.2) immediately gives inequalities for deviations of the form

$$\mu\{f-m\ge h\}\le 1-R_\mu(1/2,h),$$
$$\mu\{f-m\le -h\}\le 1-R_\mu(1/2,h).$$

Here, the bound $1-R_\mu(1/2,h)$ is sharp in the class of all Lipschitz functions: when $f(x)=\mathrm{dist}(A,x)$, we come back to (1.3) for the case $p=1/2$. However, we will study R_μ as a function of p only, and the parameter h will not play any role in such abstract setting.

Consider the finite product (X^n,μ^n), the nth power of (X,μ). The *uniform enlargement* in X^n is defined by the family of cubes

$$D_n(x)=D(x_1)\times\ldots\times D(x_n),\qquad x=(x_1,\ldots,x_n)\in X^n,$$

so, again as in definition (1.2), for any $A\subset X^n$,

$$\mathrm{enl}(A)=\bigcup_{a\in A}D_n(a). \tag{1.4}$$

If the enlargement in X is defined via a metric d as above, then the uniform enlargement in X^n corresponds to the uniform metric (or supremum-distance) $d_n(x,y)=\sup_{1\le i\le n}d(x_i,y_i)$ with the same parameter h.

The isoperimetric problem in (X^n,μ^n) with respect to the uniform enlargement consists in finding the function R_{μ^n}. To solve the isoperimetric problem in the infinite product (X^∞,μ^∞) for the enlargement generated by the infinite-dimensional cubes $D_\infty(x)=D(x_1)\times D(x_2)\times\ldots$, $x=(x_i)_{i\ge1}\in X^\infty$, one needs to find the function $R_{\mu^\infty}(p)=\inf_n R_{\mu^n}(p)$, $0<p<1$, i.e.,

$$R_{\mu^\infty}(p)=\inf_n\inf_{\mu^n(A)\ge p}\mu^n(\mathrm{enl}(A)), \tag{1.5}$$

where the second infimum is over all Borel sets $A\subset X^n$ of measure $\mu^n(A)\ge p$. That is, R_{μ^∞} provides the optimal function for the dimension free inequalities $\mu^n(\mathrm{enl}(A))\ge R(\mu^n(A))$.

In this note, we prove that, in many "good" cases, R_{μ^∞} is completely determined by R_μ.

THEOREM 1.1. *If the function R_μ is concave on $(0,1)$ and $R_\mu\ne1$ identically, then R_{μ^∞} represents the maximal function among all increasing bijections $R\le R_\mu$ in $(0,1)$ such that, for any $p,q\in(0,1)$,*

$$R(pq)\le R(p)\,R(q), \tag{1.6}$$
$$S(pq)\ge S(p)\,S(q), \tag{1.7}$$

where $S(p)=1-R(1-p)$.

The case $R_\mu(p)=1$ for all p is possible, and then obviously $R_{\mu^\infty}=1$. Anyway, for concave R_μ, we thus get from Theorem 1.1:

COROLLARY 1.2. *$R_{\mu^\infty}=R_\mu$ if and only if $R_{\mu^2}=R_\mu$, and this holds if and only if the function $R=R_\mu$ satisfies (1.6) and (1.7).*

Geometrically the properties (1.6)–(1.7) mean that the extremal sets in the "one-dimensional" problem (1.3) remain extremal "on the plane" and for all higher dimensions. Or, in other words, if for all p (some of) the extremal sets "on the plane" are of the form $A\times X$, then for all p (some of) the extremal sets in X^n, $n>2$, are of the form $A\times X^{n-1}$. When $X=\mathbb{R}$ is the real line with the canonical enlargement $\mathrm{enl}(A)=A+(-h,h)$, and the measure μ is log-concave (that is, μ has a logarithmically concave density with respect to Lebesgue measure), then the extremal sets in (1.3) are of the form $(-\infty,a]$ or $[b,\infty)$, and thus the function R_μ is concave ([B2]):

$$R_\mu(p,h)=\min\{F(F^{-1}(p)+h),\ 1-F(F^{-1}(1-p)-h)\}, \tag{1.8}$$

where $F(x)=\mu((-\infty,x])$ denotes the distribution function of μ, and F^{-1} is the inverse function. It is further simplified for μ symmetric around a point: $R_\mu(p,h)=F(F^{-1}(p)+h)$.

In such a situation, one may wonder when the condition (1.6) is satisfied for all $h>0$ (the condition (1.7) becomes equivalent to (1.6) since $S=R^{-1}$), that is, when the standard half-spaces $\{x:x_1\le\mathrm{const}\}$ provide the minimum for $\mu^n(A+(-h,h)^n)$ with $\mu^n(A)\ge p$, whenever $p\in(0,1)$ and $h>0$ ($n\ge2$ is fixed). This turns out to be equivalent to the fact that the support of μ is the whole real line, and for all $p,q\in(0,1)$,

$$\frac{f(F^{-1}(pq))}{pq}\le\frac{f(F^{-1}(p))}{p}+\frac{f(F^{-1}(q))}{q}, \tag{1.9}$$

where f is the density of μ.

Of course, the Gaussian measure, with density $f(x)=\exp(-x^2)/\sqrt{2\pi}$, satisfies (1.9). Moreover, the standard half-spaces are extremal when the enlargement in $\mathbb{R}^n$ corresponds to the Euclidean metric ([ST], [Bor]), and as can easily be shown ([BH1]), this extremal property of standard half-spaces characterizes Gaussian measures.

As concerns the uniform metric, there is a relatively large family of symmetric log-concave probability distributions satisfying (1.9). For example, the measure with the distribution function $F(x)=1/(1+\exp(-x))$ is such a case. The related two-sided exponential distribution, with density $f(x)=\exp(-|x|)/2$, does not satisfy (1.9), and thus for $n\ge2$ the extremal sets are not half-spaces. In Section 7, we illustrate Theorem 1.1 on a non-symmetric example: we apply it to the one-sided exponential distribution ν, with density $f(x)=\exp(-x)$, $x>0$. In this case, according to (1.8), we have

$$R_\nu(p,h)=\min\{(1-\alpha)+\alpha p,\ p/\alpha\},\quad \alpha=\exp(-h). \tag{1.10}$$

THEOREM 1.3. *For all $p \in (0,1)$ and $h > 0$,*

$$R_{\nu^\infty}(p,h) = \min\{p^\alpha,\, 1-(1-p)^{1/\alpha}\}, \qquad \alpha = \exp(-h). \tag{1.11}$$

Note that

$$R_{\nu^\infty}(p,h) = \inf_{\nu^\infty(A)\ge p} \nu_*^\infty(A + h(-1,1)^\infty), \tag{1.12}$$

where ν_*^∞ stands for the inner measure, and the infimum is taken over all Borel sets $A \subset \mathbb{R}^\infty$ of measure $\nu^\infty(A) \ge p$. With the help of (1.11), one can indicate asymptotically extremal sets in (1.12). Namely, take for $A_n(p)$ the standard n-dimensional cube $(-\infty, a_n(p)]^n \times \mathbb{R} \times \ldots$ of measure p, hence $a_n(p) = -\log(1-p^{1/n})$, and for $B_n(p)$ the complement of $A_n(1-p)$. We observe that, for all $h > 0$,

$$\nu^\infty(A_n(p)^h) \to p^\alpha, \qquad \nu^\infty(B_n(p)^h) \to 1-(1-p)^{1/\alpha},$$

as $n \to \infty$, where A^h denotes the Minkowski sum $A + h(-1,1)^\infty$. Therefore, the infimum in (1.12) is attained asymptotically either at the standard n-dimensional cubes of measure p, or at the complements of such cubes of measure $1-p$. Note that these sets do not depend on the parameter h, but in order to choose the type (either $A_n(p)$ or $B_n(p)$), one should compare p^α and $1-(1-p)^{1/\alpha}$. A simple analysis shows that, for all $p \in [1/2, 1)$ and for all $\alpha \in (0,1)$, $p^\alpha < 1-(1-p)^{1/\alpha}$, and consequently, for all $h > 0$, we have

$$R_{\mu^\infty}(p,h) = p^\alpha,$$

and thus the cubes $A_n(p)$ are asymptotically extremal whenever $p \ge 1/2$. For $p < 1/2$, the type of extremal sets depends on h.

2. Existence of multiplicative moduli.

We need several elementary analytical propositions on so-called multiplicative moduli.

DEFINITION 2.1. A non-decreasing function R from $(0,1]$ into itself will be called a *multiplicative modulus* (or simply a *modulus*) if, for all $p, q \in (0,1)$,

$$R(pq) \le R(p)\,R(q). \tag{2.1}$$

We use this definition also for a non-decreasing $R : (0,1) \to (0,1]$ which satisfies (2.1); then R can be extended by $R(1) = 1$.

LEMMA 2.2. *Let R be an arbitrary modulus. Then*

(a) $R(1) = 1$;
(b) *if $R(p) < 1$ for some $p \in (0,1)$, then $R(0^+) = 0$;*
(c) *if $R(p) < 1$ for all $p \in (0,1)$, then R is increasing in $(0,1]$;*
(d) *if the function $S(p) = 1 - R(1-p)$ satisfies $S(pq) \ge S(p)S(q)$ for all $p, q \in (0,1)$, then R is continuous in $(0,1)$. If, additionally, $R(p) < 1$ for some $p \in (0,1)$, then $R(p) < 1$ for all $p \in (0,1)$.*

Proof. (a), (b) and (c) are immediate. To prove (d), assume $R(p_0) < 1$ for some $p_0 \in (0,1)$, so that by (a), $R(0^+) = 0$, hence $S(1^-) = 1$. Given $p \in (0,1)$, let p_1 and q_1 tend respectively to p and 1 so that $p_1 > p$ and $p_1 q_1 < p$. Then from the inequality $S(p_1 q_1) \ge S(p_1)S(q_1)$ we deduce that $S(p^-) \ge S(p^+)$. Since S is non-decreasing, we get $S(p^-) = S(p^+)$, that is, S is continuous at the point p. To prove the last statement in (d), note that $S(q_0) > 0$ for $q_0 = 1 - p_0$. Therefore, for all positive integers n, $S(q_0^n) \ge S(q_0)^n > 0$. Since S is non-decreasing, $S(q) > 0$ for all $q \in (0,1)$, that is, $R(p) < 1$ for all $p \in (0,1)$.

LEMMA 2.3. *For any non-decreasing function f from $(0,1]$ into itself such that $f(p) \ge p$ for all $p \in (0,1]$, there exists a maximal modulus R majorized by f. Namely,*

$$R(p) = \inf f(p_1)\ldots f(p_n), \tag{2.2}$$

where the infimum is taken over all finite sets $\{p_1, \ldots, p_n\} \subset (0,1]$ with $p_1 \ldots p_n = p$. In particular, $R(p) \ge p$ for all $p \in (0,1]$. If f is concave, then so is R.

Proof. Obviously, the function R defined by (2.2) is the maximal modulus majorized by f, and $R(p) \ge p$ for all $p \in (0,1]$. Let f be concave. Since $R = \inf_n R_n$, where R_n is defined by (2.2) with fixed $n \ge 1$, it suffices to prove concavity of all R_n. Using induction, we only need to show that for any concave non-decreasing functions $f_1 \ge 0$ and $f_2 \ge 0$ on $(0,1]$, the function

$$R(p) = \inf f_1(p_1)\, f_2(p_2), \tag{2.3}$$

where the infimum is over all $p_1, p_2 \in (0,1]$ with $p_1 p_2 = p$, is also concave and non-decreasing on $(0,1]$. Moreover, since f_1 and f_2 can be written as pointwise infima over some families of non-decreasing affine functions, it suffices to consider the case $f_i(p) = a_i + b_i p$ with $a_i, b_i \ge 0$ $(i = 1,2)$. For such functions, (2.3) gives

$$R(p) = a_1 a_2 + b_1 b_2 p + 2\sqrt{a_1 a_2 b_1 b_2 p}$$

for $0 < p \le c = \min\{a_1 b_2/(a_2 b_1), a_2 b_1/(a_1 b_2)\}$, and R is affine for $p \ge c$ with $R'(c^+) = R'(c^-)$. Hence, R is concave and non-decreasing.

DEFINITION 2.4. In the following, we denote the modulus R from Lemma 2.3 by $\mathrm{mod}(f)$.

DEFINITION 2.5. A modulus R will be called a *perfect modulus* if $R(p) < 1$ for some $p \in (0,1)$, and for all $p, q \in (0,1)$ the following inequality holds:

$$S(pq) \ge S(p)S(q), \tag{2.4}$$

where $S(p) = 1 - R(1-p)$, $0 < p < 1$.

By Lemma 2.2(c), necessarily $R(p) < 1$ for all $p < 1$. If one puts $R(0) = 0$, $S(1) = 1$, then again by Lemma 2.2, R and S are increasing bijections in $[0,1]$, and the inequalities (2.1) and (2.4) hold for all $p, q \in [0,1]$. Below we often mean that the perfect modulus is defined at 0. Denote by R^{-1} the inverse of R, so the "dual" function

$$R^*(p) = 1 - R^{-1}(1-p), \quad 0 \le p \le 1,$$

represents the inverse of S. Hence, (2.4) is equivalent to

$$R^*(pq) \le R^*(p)\,R^*(q), \quad p, q \in [0,1].$$

Thus, an increasing bijection R in [0,1] is a perfect modulus if and only if R and R^* are moduli simultaneously.

EXAMPLES. 1. The power functions $R(p) = p^\alpha$, with $0 < \alpha \le 1$, represent perfect moduli (Lemma 7.4). In this case, $R^*(p) = 1 - (1-p)^{1/\alpha}$.

2. Let F be the distribution function of the probability measure on $\mathbb{R}$ with even positive continuous density f. Then $R_h(p) = F(F^{-1}(p) + h)$ is a perfect modulus whenever $h > 0$ if and only if (1.9) is satisfied ([B2]). In this case $R_h^* = R_h$. Note also that R_h is concave for all $h > 0$ if and only if the function $I(p) = f(F^{-1}(p))$ is concave on $(0,1)$.

PROPOSITION 2.6. *For any non-decreasing concave function f from $(0,1]$ into itself such that $f(1) = 1$ and $f \ne 1$ identically, there exists a maximal perfect modulus majorized by f. This modulus is concave.*

Proof. We use the following simple properties of "dual" functions:

$$(R^*)^* = R; \tag{2.5}$$

$$\text{if } R_1 \le R_2, \text{ then } R_1^* \le R_2^*; \tag{2.6}$$

$$\text{if } R \text{ is concave, then } R^* \text{ is concave.} \tag{2.7}$$

Here R, R_1 and R_2 are increasing bijections in [0,1]. Now define functions R_λ by transfinite induction. Set $R_0 = \mathrm{mod}(f)$. By Lemmas 2.2 and 2.3, R_0 is an increasing concave bijection in $(0,1]$. If $\lambda = \xi + 1$ is not a limit ordinal number, then put

$$R_\lambda = \min\{\mathrm{mod}(R_\xi),\ (\mathrm{mod}(R_\xi^*))^*\}. \tag{2.8}$$

Otherwise, put $R_\lambda = \inf_{\xi<\lambda} R_\xi$. Assume in this definition that $\lambda < \lambda_0$, where λ_0 is large enough. Performing an induction step, assume that for each $\xi < \lambda$, R_ξ is an increasing concave bijection in $(0,1]$. Then, for $\lambda = \xi + 1$, the function R_λ in (2.8) is an increasing concave bijection in $(0,1]$ by Lemma 2.3 and the property (2.7). For limit ordinals λ, R_λ has the same properties as the infimum of increasing concave bijections in $(0,1]$.

Thus, we have defined a non-increasing transfinite sequence R_λ, $\lambda < \lambda_0$, of increasing concave bijections in $(0,1]$. If λ_0 is large enough, say, if $\mathrm{card}(\lambda_0) > 2^c$, then this sequence stabilizes at some $\lambda < \lambda_0$, that is, $R_{\lambda+1} = R_\lambda$. According to (2.8), this means that

$$R_\lambda = \mathrm{mod}(R_\lambda), \quad R_\lambda \le (\mathrm{mod}(R_\lambda^*))^*.$$

Therefore, R_λ is a modulus and, in view of (2.5) and (2.6),

$$R_\lambda^* \le ((\mathrm{mod}(R_\lambda^*))^*)^* = \mathrm{mod}(R_\lambda^*).$$

Hence, recalling that $\mathrm{mod}(g) \le g$, we deduce that R_λ^* is also a modulus. As a result, R_λ is a perfect modulus.

It remains to show that R_λ is maximal among all perfect moduli R such that $R \le f$. This is equivalent to proving (by induction over ξ) that $R \le R_\xi$ for all $\xi < \lambda_0$. Since $R \le f$ and R is a modulus, we get $R \le \mathrm{mod}(f) = R_0$. Suppose that $R \le R_\eta$ for all $\eta < \xi$. If ξ is a limit ordinal, then $R \le \inf_{\eta<\xi} R_\eta = R_\xi$. If $\xi = \eta + 1$, we have $R \le \mathrm{mod}(R_\eta)$ since $R \le R_\eta$ and R is a modulus. By (2.6), $R^* \le R_\eta^*$, hence $R^* \le \mathrm{mod}(R_\eta^*)$, since R^* is a modulus. Again by (2.5) and (2.6), $R = (R^*)^* \le (\mathrm{mod}(R_\eta^*))^*$. Combining both estimates, we obtain

$$R \le \min\{\mathrm{mod}(R_\eta),\ (\mathrm{mod}(R_\eta^*))^*\} = R_\xi.$$

Thus, $R \le R_\xi$ for all $\xi < \lambda_0$. The proof is complete.

3. Moduli and integrals

PROPOSITION 3.1. *Let R be an increasing concave function from $[0,1]$ into itself such that $R(0) = 0$ and $R(1) = 1$. Then R represents a perfect modulus if and only if, for all $0 \le y \le x \le 1$ and $0 \le p \le 1$,*

$$R(px + (1-p)y) \le R(p)R(x) + (1 - R(p))R(y). \tag{3.1}$$

Equivalently and more generally, for any distribution function F of a probability measure on $[0,1]$,

$$R\Big(\int_0^1 R^{-1}\,dF\Big) \le \int_0^1 R(1 - F(t))\,dt. \tag{3.2}$$

Proof (see also [B3], Lemma). First note that (3.1) becomes (2.1) when $y = 0$ and becomes (2.4) when $x = 1$, while (3.2) becomes (3.1) at measures with two atoms. Conversely, given $c \in [0,1]$ and $p \in [0,1]$, the left hand side of (3.1) is constant on the segment

$$\Delta(c) = \{(x,y) : 0 \le y \le x \le 1,\ px + (1-p)y = c\},$$

while the right hand side represents a concave function, and on the two end points of $\Delta(c)$, (3.1) becomes (2.1) and (2.4). Hence, (3.1) holds for all points of $\Delta(c)$. To derive (3.2) from (3.1), again one can fix $c \in [0,1]$ and consider the convex compact set $M(c) = \{F : \int_0^1 R^{-1} dF = c\}$ on which the left hand side of (3.2) is constant, and the right hand side represents a concave functional. Hence, it suffices to check (3.2) for extremal "points" of $M(c)$, only. But these extremal measures have at most two atoms, and (3.2) leads us back to (3.1).

4. A functional form of isoperimetric inequalities. Let us now return to the notations of Section 1. Assume that, for all Borel sets $A \subset X$ with $0 < \mu(A) < 1$,

$$\mu(\mathrm{enl}(A)) \ge R(\mu(A)), \tag{4.1}$$

where R is some non-negative function defined on $(0,1)$. In particular, $0 \le R \le R_\mu$. We need a suitable functional form of (4.1). So, given a function $f : X \to [0,1]$, define its *enlargement* by

$$\mathrm{enl}\, f(x) = \sup_{y \in D(x)} f(y). \tag{4.2}$$

We observe the following elementary properties of this operation:

(a) $0 \le f \le \mathrm{enl}\, f \le 1$;
(b) $\{\mathrm{enl}\, f > t\} = \mathrm{enl}\{f > t\}$ for all $t \in \mathbb{R}$;
(c) therefore, the function $\mathrm{enl}\, f$ is lower semicontinuous (hence, Borel measurable);
(d) for indicator functions, $\mathrm{enl}\, \mathbf{1}_A = \mathbf{1}_{\mathrm{enl}(A)}$.

PROPOSITION 4.1. *Let R be a concave perfect modulus satisfying* (4.1). *Then, for any Borel measurable function $f : X \to [0,1]$,*

$$\mathbb{E}\, \mathrm{enl}\, f \ge R(\mathbb{E} R^{-1}(f)), \tag{4.3}$$

where R^{-1} is the inverse of R, and the expectations are with respect to μ.

Proof. Let F be the distribution function of f. Applying (b) and (3.2), we obtain

$$\begin{aligned}\mathbb{E}\, \mathrm{enl}\, f &= \int_0^1 \mu(\mathrm{enl}\, f > t)\, dt = \int_0^1 \mu(\mathrm{enl}\{f > t\})\, dt \\ &\ge \int_0^1 R(\mu(f > t))\, dt = \int_0^1 R(1 - F(t))\, dt \\ &\ge R\Big(\int_0^1 R^{-1}\, dF\Big) = R(\mathbb{E} R^{-1}(f)).\end{aligned}$$

Clearly, (4.3) turns into (4.1) on indicator functions.

Remark 4.2. Let $X = \mathbb{R}^n$ be the Euclidean space with the enlargement $\mathrm{enl}(A) = A^h$ generated with the Euclidean metric, and for any $h > 0$, let (4.1) hold with the functions $R_h(p) = F_0(F_0^{-1}(p) + h)$, where F_0 is the distribution function of a symmetric log-concave probability measure whose

support is the whole real line and such that (1.9) is satisfied. Then all R_h are concave perfect moduli. Put $I(p) = f_0(F_0^{-1}(p))$, where f_0 is the density of F_0. Letting $h \to 0^+$ in (4.3), using Taylor expansion for the right hand side, and noting that $\mathrm{enl}\, f(x) = f(x) + |\nabla f(x)| h + O(h^2)$, we come to

$$I(\mathbb{E} f) - \mathbb{E} I(f) \le \mathbb{E} |\nabla f|, \tag{4.4}$$

for every smooth function f on $\mathbb{R}^n$ with values in $[0,1]$. In turn, (4.4) implies (4.1) with $R = R_h$. When μ is the canonical Gaussian measure, (4.1) holds for $R = R_\mu$ which is of the above form with f_0 being the density of the one-dimensional Gaussian measure, and then (4.4) represents a functional form of the Gaussian isoperimetric inequality (see [B3]). For another functional form of isoperimetric inequalities, see [BH2].

5. Extension to product spaces. When R is a concave perfect modulus, (4.1) and (4.3) are easily extended to product spaces (X^n, μ^n) for the uniform enlargement defined by (1.4). According to (4.2), given a function $f : X^n \to [0,1]$, one should put

$$\mathrm{enl}\, f(x_1, \ldots, x_n) = \sup_{1 \le i \le n} \sup_{y_i \in D(x_i)} f(y_1, \ldots, y_n). \tag{5.1}$$

PROPOSITION 5.1. *Let R be a concave perfect modulus satisfying* (4.1). *Then, for any Borel measurable function $f : X^n \to [0,1]$,*

$$\mathbb{E}\, \mathrm{enl}\, f \ge R(\mathbb{E} R^{-1}(f)), \tag{5.2}$$

where the expectations are with respect to μ^n. In particular, for any Borel set $A \subset X^n$,

$$\mu^n(\mathrm{enl}(A)) \ge R(\mu^n(A)). \tag{5.3}$$

Proof. To perform the induction step, assume we have two spaces (X, μ) and (X', μ') with families $D(x)$ and $D'(x')$, respectively, such that (4.1) holds in both spaces. Consider the product $(X \times X', \mu \times \mu')$ with the enlargement generated by the family of cubes $D(x) \times D'(x')$, $(x, x') \in X \times X'$, and consider a Borel measurable function $f : X \times X' \to [0,1]$. Introducing $V = R^{-1}$ and fixing $x' \in X'$, one can write (4.3) for the function $x \to f(x, x')$ as

$$\int_X V(f(x, x'))\, d\mu(x) \le V\Big(\int_X \sup_{y \in D(x)} f(y, x')\, d\mu(x)\Big). \tag{5.4}$$

Set

$$g(x') = \int_X \sup_{y \in D(x)} f(y, x')\, d\mu(x).$$

The function g is well-defined but it need not be measurable, so we introduce its measurable minorant g_*. By Fubini's theorem, the left hand side of (5.4) is μ'-measurable. In addition, $V(g_*)$ is a measurable minorant for $V(g)$. Therefore (5.4) may be written as

(5.5) $$\int_X V(f(x,x'))\,d\mu(x) \le V(g_*(x')),$$

which is true for μ'-almost all $x' \in X'$. Integrating (5.5) over x' and applying (4.3) to g_*, we obtain

(5.6) $$\int_{X'}\int_X V(f(x,x'))\,d\mu(x)\,d\mu(x') \le V\Big(\int_{X'} \operatorname{enl} g_*(x')\,d\mu'(x')\Big).$$

It remains to note that $g_* \le g$, hence $\operatorname{enl} g_* \le \operatorname{enl} g$, and that

$$\operatorname{enl} g(x') = \sup_{y' \in D'(x')} g(y') = \sup_{y' \in D'(x')} \int_X \sup_{y \in D(x)} f(y,y')\,d\mu(x)$$
$$\le \int_X \sup_{y' \in D'(x')} \sup_{y \in D(x)} f(y,y')\,d\mu(x) = \int_X \operatorname{enl} f(x,x')\,d\mu(x).$$

The proof is complete.

6. Properties of R_{μ^∞}. Proof of Theorem 1.1

PROPOSITION 6.1. *If $R_\mu \ne 1$ identically then R_{μ^∞} is a perfect modulus. If, additionally, R_μ is concave then so is R_{μ^∞}.*

Proof. This is the only place where we use (1.1) (except the inessential property (b) from Section 4: the definition of $\operatorname{enl} f$ can be a little modified to satisfy (b)). Now, for any set $B \subset X$, define its "interior" by

(6.1) $$\operatorname{int}(B) = \{x \in X : D(x) \subset B\}.$$

Due to (1.1), whenever X is the disjoint union of sets A and B, then

(6.2) X is the disjoint union of $\operatorname{enl}(A)$ and $\operatorname{int}(B)$.

The family of cubes $D_n(x) = D(x_1) \times \ldots \times D(x_n)$, $x = (x_1, \ldots, x_n) \in X^n$, satisfies (1.1), therefore, (6.2) holds in X^n for the same notion of interior as in (6.1). In addition, for all sets $A_1 \subset X^n$ and $A_2 \subset X^k$, we have

(6.3) $$\operatorname{enl}(A_1 \times A_2) = \operatorname{enl}(A_1) \times \operatorname{enl}(A_2),$$

(6.4) $$\operatorname{int}(A_1 \times A_2) = \operatorname{int}(A_1) \times \operatorname{int}(A_2).$$

Recall the definition (1.5) of $R = R_\mu^\infty$,

(6.5) $$R(p) = \inf_n \inf_{\mu^n(A) \ge p} \mu^n(\operatorname{enl}(A)),$$

and set

(6.6) $$S(p) = \sup_n \sup_{\mu^n(B) \le p} \mu^n(\operatorname{int}(B)).$$

In view of (6.2), these functions are connected by the identity

$$R(p) + S(1-p) = 1, \quad 0 < p < 1.$$

Let us check (1.6)–(1.7) for R and S. Given $p, q \in (0,1)$ and $c > 1$, one can find according to (6.5) some integers $n, k \ge 1$ and some Borel sets $A_1 \subset X^n$ and $A_2 \subset X^k$ with $\mu^n(A_1) \ge p$ and $\mu^k(A_2) \ge q$ such that

(6.7) $$\mu^n(\operatorname{enl}(A_1)) \le cR(p), \quad \mu^k(\operatorname{enl}(A_2)) \le cR(q).$$

The set $A = A_1 \times A_2 \subset X^{n+k}$ is of measure $\mu^{n+k}(A) \ge pq$, hence by (6.3) and (6.5),

(6.8) $$R(pq) \le \mu^{n+k}(\operatorname{enl}(A)) = \mu^n(\operatorname{enl}(A_1))\,\mu^k(\operatorname{enl}(A_2)).$$

Combining (6.7) and (6.8), we get $R(pq) \le c^2 R(p)R(q)$, for any $c > 1$. Letting $c \to 1^+$, we obtain $R(pq) \le R(p)R(q)$ for all $p, q \in (0,1)$. The same argument by means of (6.4) and (6.6) yields $S(pq) \ge S(p)S(q)$. Recalling Lemma 2.2, one can conclude that R is an increasing bijection in (0,1), and moreover, it is a perfect modulus. Concavity of R will be seen from the argument below.

Proof of Theorem 1.1. By Proposition 2.6 with $f = R_\mu$, there exists a maximal modulus R majorized by R_μ. Therefore, since R_{μ^∞} is a perfect modulus, we have $R_{\mu^\infty} \le R$. Again by Proposition 2.6, R is concave, hence Proposition 5.1 may be applied to R: by (5.3), for any $n \ge 1$ and for any Borel set $A \subset X^n$,

$$\mu^n(\operatorname{enl}(A)) \ge R(\mu^n(A)).$$

But this means that $R_{\mu^\infty} \ge R$. As a result, $R_{\mu^\infty} = R$.

Proof of Corollary 1.2. It remains to show that $R_{\mu^2} = R_\mu$ implies (1.6) and (1.7). As in the proof of Proposition 6.1, we get

$$R_{\mu^2}(pq) \le R_\mu(p)R_\mu(q), \quad S_{\mu^2}(pq) \ge S_\mu(p)S_\mu(q),$$

where $S_\mu(p) = 1 - R_\mu(1-p)$, $S_{\mu^2}(p) = 1 - R_{\mu^2}(1-p)$. Therefore, in case $R_{\mu^2} = R_\mu$, the function R_μ is a perfect modulus.

7. Proof of Theorem 1.3.

First we make several simple statements on moduli that will enable us to see how to apply Theorem 1.1. Denote by $\mathcal{F}_+$ (respectively, $\mathcal{F}_-$) the family of all non-decreasing functions $f : (0,1] \to (0,1]$ with $f(1) = 1$ such that

$$T_f(x) = -\log f(\exp(-x)), \quad x \ge 0,$$

is convex (resp., concave) on $[0,\infty)$. For f smooth in $(0,1]$, this means that the function

$$L_f(p) = pf'(p)/f(p)$$

is non-increasing (resp., non-decreasing) in $(0,1]$. Clearly, if $f_1, f_2 \in \mathcal{F}_+$, then $f = \min(f_1, f_2) \in \mathcal{F}_+$, since $T_f = \max(T_{f_1}, T_{f_2})$.

LEMMA 7.1. *If $R \in \mathcal{F}_+$, then R is a modulus.*

LEMMA 7.2. *If $f \in \mathcal{F}_-$, then the function $R = \mathrm{mod}(f)$ is representable as*

$$R(p) = \inf_{n\geq 1} f(p^{1/n})^n, \qquad 0 < p \leq 1. \tag{7.1}$$

Proof. First let $R \in \mathcal{F}_+$. The function $T_R(x) + T_R(a-x)$ is convex in $0 \leq x \leq a$, and therefore attains its maximum either at $x = 0$, or at $x = a$. Hence, for all $x, y \geq 0$, $T_R(x+y) \geq T_R(x) + T_R(y)$, that is, $R(pq) \leq R(p)R(q)$ for all $p, q \in (0,1]$. Thus R is a modulus. Now let $R \in \mathcal{F}_-$. In accordance with (2.2), we should minimize the products $f(p_1)\ldots f(p_n)$ under the condition $p_1 \cdots p_n = p$, or in terms of T_f, we should maximize the function

$$u(x_1, \ldots, x_n) = T_f(x_1) + \ldots + T_f(x_n)$$

on the symplex $x_1 + \ldots + x_n = x$, $x_i \geq 0$ (with $x = -\log p$). By Jensen's inequality, $T_f(x_1) + \ldots + T_f(x_n) \leq nT_f(x/n)$, that is, u attains its maximum at the point $(x/n, \ldots, x/n)$. Thus,

$$\sup u = nT_f(x/n) = -n \log f(p^{1/n}).$$

It remains to maximize over all n.

COROLLARY 7.3. *Let $\alpha \in [0,1]$. For the function $f(p) = (1-\alpha) + \alpha p$, we have $\mathrm{mod}(f)(p) = p^\alpha$.*

Indeed, $f \in \mathcal{F}_-$ and $\inf_n f(p^{1/n})^n = \lim_{n\to\infty} f(p^{1/n})^n = p^\alpha$.

LEMMA 7.4. *The function $R(p) = 1-(1-p)^{1/\alpha}$ belongs to $\mathcal{F}_+$ $(0 < \alpha \leq 1)$.*

Indeed, for $\alpha \in (0,1)$, set $\beta = 1/\alpha > 1$, and $q = 1-p$. Then

$$\alpha L_R(p) = \frac{p(1-p)^{\beta-1}}{1-(1-p)^\beta} = 1 - \frac{1-q^{\beta-1}}{1-q^\beta}$$

decreases in $p \in (0,1)$ if and only if the function $u(t) = (1-t^\gamma)/(1-t)$ decreases in $t \in (0,1)$ (here $\gamma = (\beta-1)/\beta \in (0,1)$ and $t = q^\beta$). The last statement is obvious.

Since the function $f(p) = p^\alpha$ for $\alpha \geq 0$ belongs to $\mathcal{F}_+$, from Lemmas 7.4 and 7.1 we obtain

COROLLARY 7.5. *Let $\alpha \in (0,1]$. The function*

$$R(p) = \min\{p^\alpha, 1-(1-p)^{1/\alpha}\}, \qquad p \in (0,1],$$

belongs to $\mathcal{F}_+$. Since $R^* = R$, R is thus a perfect modulus.

For the exponential measure ν, R_ν is given by (1.10). Hence, to prove Theorem 1.3, it remains to show the following:

PROPOSITION 7.6. *For $\alpha \in (0,1]$, the function $R(p)$ of Corollary 7.5 is the maximal perfect modulus majorized by the function*

$$f(p) = \min\{(1-\alpha) + \alpha p,\ p/\alpha\}.$$

Proof. Let R_1 be the maximal perfect modulus majorized by f. By Corollary 7.5, $R \leq R_1$, and we need to show the converse inequality. Indeed, $R_1(p) \leq f_1(p) = (1-\alpha) + \alpha p$, hence, according to Corollary 7.3, $R_1(p) \leq \mathrm{mod}(f_1)(p) = p^\alpha$. Recalling the property (2.6), we obtain

$$R_1^*(p) \leq \mathrm{mod}(f_1)^*(p) = 1 - (1-p)^{1/\alpha}.$$

Both inequalities give $R_1 \leq R = \min(\mathrm{mod}(f_1), \mathrm{mod}(f_1)^*)$. The statement follows.

8. Remarks. For the supremum-distance, the isoperimetric problem was earlier considered in [AM], where some general estimates for $R_{\mu^n}(1/2, h)$ were obtained and applied to the case of the d-sphere $X = S^d$, and in [B4], where the problem was studied on the class of monotone sets; see also [B2] which was a continuation of [B1]. As concerns other types of enlargement, the extremal sets are known in some special cases only. For the Euclidean distance in $X = \mathbb{R}^n$, these cases are:

1) Lebesgue measure: the extremal sets are balls (the classical isoperimetric theorem);

2) uniform distribution on the sphere: the extremal sets are again the balls (theorem of P. Lévy [Lev] and E. Schmidt [Sch]);

3) Gaussian measure: the extremal sets are half-spaces (theorem of V. N. Sudakov and B. S. Tsirel'son [ST] and C. Borell [Bor]); note that the half-spaces can be viewed as balls of infinite radius;

4) uniform distribution on the ball: the extremal sets are balls with specific centers and radii (Yu. D. Burago and V. A. Zalgaller [BZ]);

5) uniform distribution on the discrete cube $X_0 = \{0,1\}^n$: the extremal sets have been found by L. Harper [H]; in particular, they are the balls $A = D(a,k) \cap X_0$ in X_0 with center at $a \in X_0$ when such balls exist for the given measure p, that is, when

$$p = 2^{-n} \sum_{i=0}^{k} C_n^k$$

for some $0 \leq k \leq n-1$ (usually, Harper's theorem is formulated for the Hamming distance).

These classical results inspire the idea that, under some weak general assumptions, the extremal sets in the isoperimetric problem should be among balls and their complements (including set-limits of balls like half-spaces in the Euclidean space). In this note (as well as in [B2]), we tried to confirm such a conjecture on the example of the supremum distance (in which case the balls are cubes). It should, however, be pointed out that the uniform enlargement of sets is relatively large, and therefore the appropriate isoperimetric inequalities, even when they are sharp, represent weak estimates in the context of the concentration of measure phenomenon (see, e.g., [Tal1], [Tal2], [Tal3], [Led], [BH3]).

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