

Research Article

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Isoperimetric Regions in \mathbb{R}^n with Density r^p

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Abstract: We show that the unique isoperimetric regions in \mathbb{R}^n with density r^p for $n \geq 3$ and $p > 0$ are balls with boundary through the origin.

Keywords: Isoperimetric; density

MSC: 53

1 Introduction

Recently, there has been a surge of interest in manifolds with density, partly because of their role in Perelman's proof of the Poincaré Conjecture. We consider the isoperimetric problem when volume and perimeter are weighted by the density function r^p and prove the following theorem:

Theorem 3.3. *In \mathbb{R}^n with density r^p , where $n \geq 3$ and $p > 0$, the unique isoperimetric regions, up to sets of measure zero, are balls with boundary through the origin.*

The density r^p is one of the simplest radial density functions, but it has some interesting properties. First, r^p is homogeneous in degree p , which means that given an isoperimetric region of one volume, we can scale it to get an isoperimetric region of a different volume. Second, r^p (or a constant multiple) is the only density for which spheres through the origin could be isoperimetric (see e.g. Rmk. 4.5). We can view our present problem as a venture either to prove a partial converse of this statement in the case that $p > 0$ or to extend the work of Dahlberg et al., who proved the result in \mathbb{R}^2 [2, Thm. 3.16]. Díaz et al. [3, Conj. 7.6] conjectured the generalization to \mathbb{R}^n and reduced the problem to analyzing planar curves. Recently, Chambers [1, Thm. 1.1] proved that balls centered at the origin are isoperimetric in \mathbb{R}^n with any radial *log-convex* density.

We adapt Chambers' proof to density r^p . Like Chambers, we first consider an isoperimetric region that is spherically symmetric (see Defn. 2.7), then prove the result in the general case. Given a spherically symmetric isoperimetric region, we prove that the generating curve for the boundary is a circle through the origin. The behavior of this curve is determined by a differential equation corresponding to the fact that isoperimetric hypersurfaces have constant generalized mean curvature [8, Defn. 2.3]. By spherical symmetry and regularity, the rightmost point of the curve is on the e_1 -axis, and the tangent vector at this point is vertical. Our Lemmas 4.6 and 4.8 show that if the osculating circle at the rightmost point of the curve, which we may assume to be $(1, 0)$, goes through the origin, then the curve is a circle through the origin.

We suppose for contradiction that the initial osculating circle does not pass through the origin, then take two cases according to whether its center is right or left of $(1/2, 0)$. We call these cases the right case and the left case, respectively. In the right case, the curve is like that in Chambers' proof in that the curvature


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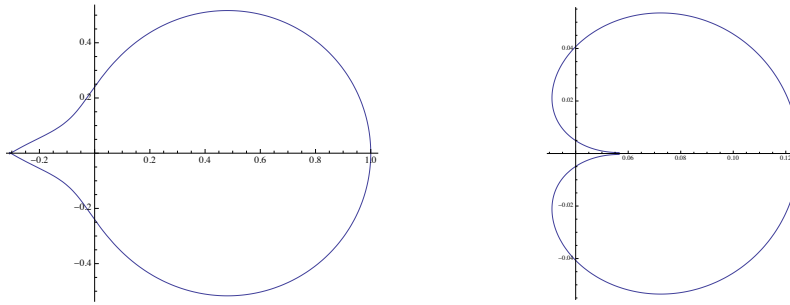


Figure 1: Sample curves in the left and right cases

is greater at a point above the e_1 -axis with tangent vector in the third quadrant than at the point of the same height with tangent vector in the second quadrant. As a result, the curve has a vertical tangent before it meets the e_1 -axis again and then curves in to meet the axis at an angle (Fig. 1, right). In the left case, the opposite inequality regarding curvatures holds, and, as a result, the curve never returns to vertical before reaching the axis (Fig. 1, left).

The left case presents the new challenge of showing that there is only one point on the upper half of the curve where the tangent vector is horizontal (Prop. 7.22). Additionally, although the curve in the right case is similar to that in Chambers, the proof is different in that we do not have the hypothesis that an isoperimetric hypersurface is mean convex, which is what Chambers used to prove that curvature was positive on the final segment of the curve ([1, Prop. 4.1]). We achieve the same result by computations that depend on the fact that our curve ends right of the e_2 -axis (Lemma 6.15), which is a property that may not hold for the generating curve in Chambers.

2 Existence, Regularity, and Symmetry

Definition 2.1. A region E is a measurable subset of \mathbb{R}^n . Its weighted volume is the integral of the density over E . Its boundary is the topological boundary. Its weighted perimeter is the integral of the density over the boundary with respect to $(n-1)$ -dimensional Hausdorff measure. We say a region is isoperimetric if it minimizes weighted perimeter for fixed weighted volume.

Theorem 2.2, a result of Morgan and Pratelli, guarantees the existence of isoperimetric regions of all volumes. After defining a regular point (Defn. 2.3), we state a standard result on the regularity of isoperimetric hypersurfaces.

Theorem 2.2. [8, Thm. 3.3] Assume that f is a (lower-semicontinuous) radial density that diverges to infinity. Then there exist isoperimetric sets of all volumes.

Definition 2.3. (Regular Point) Let E be an isoperimetric region. We say that a point $P \in \partial E$ is regular if there is an open set U containing P so that $\partial E \cap U$ is a smooth, embedded $(n-1)$ -dimensional manifold.

Proposition 2.4. [5, Cor. 3.8] Let S be an n -dimensional isoperimetric hypersurface in a manifold M with $C^{k-1, \alpha}$ ($k \geq 1$, $0 < \alpha < 1$) and Lipschitz Riemannian metric. Then except for a set of Hausdorff dimension at most $n-7$, S is locally a $C^{k, \alpha}$ submanifold; real analytic if the metric is real analytic.

By [5, Rmk. 3.10], the conclusion of Proposition 2.4 holds for a Riemannian manifold with density, provided that the density function is at least as smooth as the metric. In our case, the density r^p is smooth on

$\mathbb{R}^n - \{0\}$. Thus, if $E \subset \mathbb{R}^n$ is an isoperimetric region for density r^p , then ∂E is regular except on a set of Hausdorff dimension at most $n - 8$, after perhaps altering E by a negligible set of measure 0; henceforth we assume regions open. By the first variation formula, generalized mean curvature is constant on the set of regular points. The following proposition gives a sufficient condition for ∂E to be regular at a point.

Proposition 2.5. *If $P \in \partial E$ and E locally lies in a half-space to one side of a hyperplane through P , then ∂E is regular at P , provided that the density function is positive at P .*

Proof. Since E is an isoperimetric minimizer and the oriented tangent cone at P lies in a halfspace, the oriented tangent cone is a hyperplane. The result follows by [5, Prop. 3.5, Rmk. 3.10]. □

Corollary 2.6. All points in ∂E of maximal distance from the origin are regular.

Definition 2.7. (*Spherical Symmetrization*) *Given a region $E \subset \mathbb{R}^n$, let $A_E(r)$ denote the area of the intersection of E with S_r , the sphere of radius r centered at the origin. We define the spherical symmetrization of E to be the unique set E^* such that for all $r > 0$, $A_E(r) = A_{E^*}(r)$, and $E^* \cap S_r$ is a closed spherical cap that passes through $(r, 0, \dots, 0)$ and is rotationally symmetric about the e_1 -axis.*

Remark 2.8. *Since the set of singularities on the boundary of an isoperimetric region $E \subset \mathbb{R}^n$ has dimension at most $(n - 8)$, it follows that if E is spherically symmetric about the e_1 -axis, then all points in ∂E that are not on the e_1 -axis are regular.*

The following theorem demonstrates that for a radial density, spherical symmetrization preserves weighted volume but does not increase weighted perimeter. Moreover there are certain conditions under which the perimeter of a region remains the same after symmetrization only if the original region was spherically symmetric about some (oriented) line through the origin.

Theorem 2.9. [8, Thm. 6.2] Let f be a radial density on \mathbb{R}^n , and let E be a set of finite volume. Then the spherical symmetrization E^* satisfies

$$|E^*| = |E|$$

and

$$P(E^*) \leq P(E).$$

Suppose further that E is an open set of finite perimeter, and let $\nu(x)$ denote the normal vector at any $x \in \partial E$. If $\mathcal{H}^{n-1} \left(x \in \partial E : \nu(x) = \pm \frac{x}{|x|} \right) = 0$, and the set $I_E := \{r > 0 : 0 < \mathcal{H}^{n-1}(E \cap S_r) < \mathcal{H}^{n-1}(S_r)\}$ is an interval, then $P(E^*) = P(E)$ if and only if $E = E^*$ up to rotation about the origin.

It is immediate that if E is an isoperimetric region in Euclidean space with a radial density, then E^* is also isoperimetric.

3 Spheres Through The Origin Are Uniquely Minimizing

To prove our main result, Theorem 3.3, we begin by showing that any spherically symmetric isoperimetric region is a ball whose boundary is a sphere through the origin (Prop. 3.1). The proof of Proposition 3.1 comprises most of the paper, but we provide a sketch below. We apply this proposition to the symmetrized version of an arbitrary isoperimetric region to show that, in fact, any isoperimetric region is spherically symmetric about some oriented line through the origin (Prop. 3.2).

Proposition 3.1. Suppose that $E \subset \mathbb{R}^n$ is a spherically symmetric isoperimetric region in \mathbb{R}^n with density r^p . Then E is a ball whose boundary goes through the origin.

Proof. Assume without loss of generality that E is spherically symmetric about the positive e_1 -axis. Then E can be generated by rotating a planar set A about the e_1 -axis. Since E is spherically symmetric about the positive e_1 -axis, A is also spherically symmetric about the positive e_1 -axis. By regularity of ∂E (Defn. 2.3), we are assuming that A is open and that its boundary is a curve (possibly having multiple connected components). We define $\gamma \subset \partial A$ by beginning at the rightmost point on ∂A and following the curve through this point in both directions until it intersects the e_1 -axis again. This definition relies on regularity properties of ∂E ; see the beginning of Section 4 for more details.

We assume that $\gamma = (\gamma_1, \gamma_2) : [-\beta, \beta] \rightarrow \mathbb{R}^2$ is an arclength parameterization so that $\gamma(0)$ is the rightmost point on ∂A and $\gamma(\pm\beta)$ is the other intersection of γ with the e_1 -axis. Since r^p is homogeneous, all isoperimetric regions are similar, and we can assume without loss of generality that $\gamma(0) = (1, 0)$. We will show that γ is a circle through the origin. Given that γ is a circle through the origin, γ must comprise all of ∂A by spherical symmetrization. By Lemma 4.6, to prove that γ is a circle through the origin, it suffices to prove that there exists an s so that the associated canonical circle C_s (see Defn. 4.3) has the same curvature as γ at $\gamma(s)$ and C_s goes through the origin. By Lemma 4.8, the canonical circle C_0 at the rightmost point has the same curvature as γ at $\gamma(0)$. Therefore, it suffices to prove that C_0 passes through the origin, which occurs if and only if the center of C_0 is $(1/2, 0)$.

Suppose that the center of C_0 is right of $(1/2, 0)$. By Proposition 4.9, $\gamma_1(\beta) > 0$ and $\lim_{s \rightarrow \beta^-} \gamma_1'(s) > 0$. As a result, there exists $\varepsilon > 0$ so that $\gamma \cdot \gamma' > 0$ on $(\beta - \varepsilon, \beta)$, contradicting Lemma 4.2, which is a consequence of spherical symmetry.

Now suppose that the center of C_0 is left of $(1/2, 0)$. By Proposition 4.10, $\gamma_1(\beta) < 0$ and $\lim_{s \rightarrow \beta^-} \gamma_1'(s) < 0$, which results in the same contradiction of spherical symmetry.

The only remaining possibility is that γ is a circle through the origin. Thus, $\gamma = \partial A$ and, when rotated, γ generates a sphere through the origin. □

Given Proposition 3.1, we can prove our claim that any isoperimetric region in \mathbb{R}^n with density r^p is spherically symmetric.

Proposition 3.2. If E is an isoperimetric region in \mathbb{R}^n with density r^p , then $E = E^*$, up to a rotation about the origin.

Proof. By regularity (Defn. 2.3), we are assuming E is open. By Theorem 2.9, it suffices to show that I_E is an interval and that

$$\mathcal{H}^{n-1} \left(x \in \partial E : \nu(x) = \pm \frac{x}{|x|} \right) = 0. \tag{3.1}$$

We call a point x with $\nu(x) = \pm x/|x|$ *tangential*. Since symmetrization (Defn. 2.7) preserves weighted volume without increasing weighted perimeter, E^* is also isoperimetric. Applying Proposition 3.1, we conclude that E^* is a ball with boundary through the origin. It follows that I_E is an interval. Moreover, there exists no $r > 0$ such that the spherical cap $S_r \cap E$ is a full sphere. This will be important in our proof of (3.1). Suppose for contradiction that there exists a positive area subset of ∂E that is tangential. As in Morgan-Pratelli [8, Pf. of Cor. 6.4], at any smooth point of density of this tangential subset of ∂E , ∂E has the same generalized mean curvature as a sphere centered at the origin. It follows by uniqueness of solutions to elliptic partial differential equations that a component of ∂E is a sphere centered at the origin. E must contain an annular region centered at the origin with this spherical component as one of its bounding components. Thus, there exists an interval (r_0, r_1) such that for any r in (r_0, r_1) , $S_r \cap E$ is a full sphere, contradicting the fact that the boundary of E^* is a sphere through the origin. □

Combining Propositions 3.1 and 3.2 along with Theorem 2.2, we have proved:

Theorem 3.3. In \mathbb{R}^n with density r^p , where $n \geq 3$ and $p > 0$, the unique isoperimetric regions, up to sets of measure zero, are balls with boundary through the origin.

4 Structure of Proof

Sections 5, 6, and 7 are devoted to filling in the details of the proof of Proposition 3.1. Throughout these sections, we work within the following framework:

Let E be a spherically symmetric isoperimetric region. Then there is a set $A \subset \mathbb{R}^2$ such that E is the rotation of A about the e_1 -axis. We will analyze a certain curve on the boundary of A . We begin at the point P on the e_1 -axis that is the rightmost point on ∂A . By spherical symmetry, P is a point of E farthest from the origin, so ∂E is regular at P by Corollary 2.6. The tangent space to ∂A at P is spanned by e_2 . We follow ∂A , which has finite length, in both directions until it intersects the e_1 -axis at another point. The result is a Jordan curve $\gamma(s) : [-\beta, \beta] \rightarrow \mathbb{R}^2$ such that $\gamma(0) = P$ and $\gamma(\pm\beta)$ is the other intersection of the curve with the e_1 -axis (Fig. 2). Since r^p is homogeneous, all isoperimetric regions are similar to each other. Therefore, we may assume without loss of generality that $P = (1, 0)$. We assume that γ is a counterclockwise arclength parameterization. Let γ_1 and γ_2 denote the coordinates of γ . Then $\gamma_1(-s) = \gamma_1(s)$ and $\gamma_2(-s) = -\gamma_2(s)$ for all s . We let $\kappa(s)$ denote the curvature of γ at $\gamma(s)$.

By Corollary 2.6, γ is smooth at 0. By Remark 2.8, γ is smooth at all remaining points in $(-\beta, \beta)$. Since γ is smooth at 0 and 0 is a global maximum point of γ_1 , it follows that $\gamma'(0) = (0, 1)$ and that $\kappa(0) \geq 0$. In fact, 0 is a strict maximum point of γ_1 . To prove so, note that if there were an $s \neq 0$ so that $\gamma_1(s) > \gamma_1(0)$, then it would also be the case that $|\gamma(s)| > |\gamma(0)|$. However, there would be no point on ∂A that was on the positive e_1 -axis and was the same distance from the origin as $\gamma(s)$, contradicting spherical symmetry. Since 0 is a strict maximum point of γ_1 , $\kappa(0) > 0$. Moreover, since γ is symmetric over the e_1 -axis, $\kappa'(0) = 0$.

In addition to analyzing the curvature of γ , we will also consider the generalized mean curvature of the surface generated by ∂A at a point $\gamma(s)$.

Definition 4.1. As in [8, Defn. 2.3], we define generalized mean curvature of a hypersurface in \mathbb{R}^n with density $f(x) = e^{\psi(x)}$ by

$$H_f = H_0 + \frac{\partial\psi}{\partial\nu}, \tag{4.1}$$

where H_0 is the unaveraged Riemannian mean curvature and ν is the outward unit normal vector. If $\psi(x) = g(|x|)$ for some smooth function g , then

$$H_f(x) = H_0(x) + g'(|x|) \frac{x}{|x|} \cdot \nu(x) \tag{4.2}$$

for any regular point x on the hypersurface with $x \neq 0$. In \mathbb{R}^n with density r^p , $g(r) = \log(r^p)$. Henceforth, we will denote

$$\frac{\partial\psi}{\partial\nu}(x)$$

by $H_1(x)$. For concision, given a point $\gamma(s)$, we refer to $H_1(\gamma(s))$ as $H_1(s)$ with analogous notation for the values of H_0 and H_f at $\gamma(s)$.

The following lemma of Chambers gives a useful result of spherical symmetrization.

Lemma 4.2. (Tangent Restriction) [1, Lemma 2.6] For every $s \in (0, \beta)$, $\gamma(s) \cdot \gamma'(s) \leq 0$.

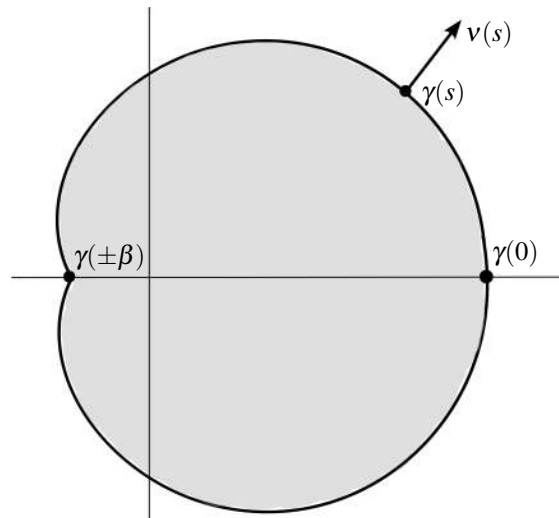


Figure 2: The generating curve γ and the outward unit normal vector at a point $\gamma(s)$

At each point on γ , we define a related circle that we call the canonical circle. We show in Proposition 5.1 that the curvature of the canonical circle accounts for one of two terms in a formula for the mean curvature of the surface of revolution.

Definition 4.3. [1, Defns. 3.1, 3.2] *Given $s \in (-\beta, \beta)$ with $s \neq 0$, let the canonical circle at s , denoted C_s , be the unique oriented circle centered on the e_1 -axis that passes through $\gamma(s)$ and has unit tangent vector at $\gamma(s)$ equal to $\gamma'(s)$. If $\gamma'(s)$ is a multiple of e_2 , then C_s is an oriented vertical line. We define C_0 to be $\lim_{s \rightarrow 0} C_s$. The regularity of the surface at $\gamma(0)$ guarantees the existence of this limit. We let $R(s)$ denote the radius of C_s and let $\lambda(s)$ denote its signed curvature. Then $\lambda(s) = 1/R(s)$ if C_s is counterclockwise oriented, and $\lambda(s) = -1/R(s)$ if C_s is clockwise oriented. Finally, we let $F(s)$ denote the abscissa of the center of C_s .*

The following lemma shows that spheres through the origin have constant generalized mean curvature. We apply this result to prove Lemmas 4.6 and 4.8, which imply that γ is a sphere through the origin, given that the curvature at the rightmost point is the same as the curvature of the circle through that point and the origin.

Proposition 4.4. In \mathbb{R}^n with density r^p , hyperspheres through the origin have constant generalized mean curvature.

Proof. Let S be a hypersphere through the origin, and assume without loss of generality that S can be obtained by rotating a circle C in the plane about the e_1 -axis. It suffices to prove that generalized mean curvature is the same at all points on C . H_0 is constant on C since it is constant on S . It remains to prove that H_1 is constant on C .

Let the center of C be $(a, 0)$ with $a > 0$. Then the polar coordinates equation for C is $r = 2a \cos \theta$. At a point $(r(\theta), \theta)$, the outward unit normal vector makes angle 2θ to the positive e_1 -axis, and the angle between the position vector and the outward unit normal vector is θ . Supposing that x has polar coordinates (r, θ) , we have

$$g'(|x|) \frac{x}{|x|} \cdot \nu(x) = \frac{p}{r} \cos \theta = \frac{p}{r} \frac{r}{2a} = \frac{p}{2a}.$$

Therefore, H_1 is constant on C , as required. □

Remark 4.5. *These computations show that the only density on $\mathbb{R}^2 - \{0\}$ ($\mathbb{R}^n - \{0\}$) for which circles (spheres) through the origin are isoperimetric is r^p , or a constant multiple thereof. On a circle C through the origin, parameterized by α , the quantity $\alpha(t)/|\alpha(t)| \cdot \nu(t)$ is a constant multiple of the magnitude of the position vector. Hence, for H_1 to be constant it must be the case that $g'(r)$ is inversely proportional to r . This occurs only if $g(r) = \log(r^p) + c$ for some p and some constant c .*

Lemma 4.6. (cf. [1, Lemma 3.2]) For any point $s \in [0, \beta)$, if C_s passes through the origin and $\kappa(s) = \lambda(s)$, then γ is a circle through the origin.

Proof. Supposing that C_s is arclength parameterized, to prove that C_s agrees with γ locally, it suffices by uniqueness theorems concerning solutions of ODEs to prove that both satisfy the differential equation $H_f = c$. This is clearly true since the tangent vectors of the two curves agree at $\gamma(s)$ and the generalized mean curvature of the surfaces generated by these curves is the same at $\gamma(s)$. To prove that $H_f = c$ at all points on C_s , it suffices to show that H_f is constant on C_s . This follows from the computations in Proposition 4.4. Having proved that γ and C_s coincide locally, we claim that, in fact, γ and C_s must coincide everywhere.

Let $S = \{t \in [-\beta, \beta] : \gamma([s, t]) \subset C_s\}$. Since γ and C_s agree near $\gamma(s)$, S is nonempty and therefore has a least upper bound m . Letting α be an arclength parameterization of C_s , it follows by smoothness of α and of γ that $m \in S$, that C_s is tangent to γ at $\gamma(m)$, and that $\kappa(m) = \lambda(s)$. (To conclude smoothness of γ at m , we are using our assumption that $m < \beta$.) By an identical argument to that in the first paragraph, there exists an open interval I containing m such that $\gamma(I) \subset C_s$, contradicting the fact that $m = \sup S$. We conclude that $m = \beta$. A similar argument shows that γ coincides with C_s on $[-\beta, s]$. \square

Remark 4.7. *By radial symmetry, spheres centered at the origin also have constant generalized mean curvature. Thus, if C_s is centered at the origin and $\kappa(s) = \lambda(s)$, then γ is a circle that is centered at the origin. We use this result to obtain contradictions in several places.*

Lemma 4.8. [1, p. 12] We have that $\kappa(0) = \lambda(0)$.

Proof. Showing that $\kappa(0) = \lambda(0)$ is equivalent to showing that $F(0) = 1 - 1/\kappa(0)$. If $\gamma_1'(s) \neq 0$, then

$$F(s) = \frac{\gamma(s) \cdot \gamma'(s)}{\gamma_1'(s)}.$$

Since $\kappa(0) > 0$ and κ is continuous at 0, there is a neighborhood of 0 on which $\gamma_1'(s) \neq 0$ except when $s = 0$. By definition,

$$F(0) = \lim_{s \rightarrow 0} F(s) = \lim_{s \rightarrow 0} \frac{\gamma(s) \cdot \gamma'(s)}{\gamma_1'(s)} = 1 - \frac{1}{\kappa(0)}.$$

\square

By Lemmas 4.6 and 4.8, if C_0 is a circle through the origin, then γ is a circle through the origin. This means that if $F(0) = 1/2$, then γ is a circle through the origin. We argue by contradiction, taking cases according to whether $F(0) > 1/2$ or $F(0) < 1/2$. In each case, we obtain a result that contradicts spherical symmetry. We state these results as the Right Tangent Lemma and the Left Tangent Lemma, and we devote a section to proving each.

Proposition 4.9. (Right Tangent Lemma) If $F(0) > 1/2$, then $\gamma_1(\beta) > 0$, $\lim_{s \rightarrow \beta^-} \gamma'(s)$ is in the fourth quadrant, and $\lim_{s \rightarrow \beta^-} \gamma'(s) \neq (0, -1)$.

Proposition 4.10. (Left Tangent Lemma) If $F(0) < 1/2$, then $\gamma_1(\beta) < 0$, $\lim_{s \rightarrow \beta^-} \gamma'(s)$ is in the third quadrant, and $\lim_{s \rightarrow \beta^-} \gamma'(s) \neq (0, -1)$.

5 Preliminary Lemmas

This section contains results relevant to both cases. Proposition 5.1 and Corollary 5.2 give expressions for the mean curvature and generalized mean curvature at a point on the hypersurface generated by γ in terms of the curvature of γ , the curvature of the canonical circle, and the normal derivative of the log of the density at that point. We then discuss computational techniques that we use to determine how these functions (and others) vary with arclength. Finally, Proposition 5.6 is used in both cases to compare curvatures at pairs of points on the curve that are at the same height.

Proposition 5.1. [1, Prop. 3.1] Given a point $s \in [0, \beta)$, we have that

$$H_0(s) = \kappa(s) + (n - 2)\lambda(s). \quad (5.1)$$

Proof. We consider the principal curvatures of the surface at a point $P = \gamma(s)$. We treat the case that $y = \gamma_2(s) > 0$ and that $\gamma'(s) \neq (0, \pm 1)$. A similar argument shows that (5.1) holds if $\gamma_2(s) < 0$ and $\gamma'(s) \neq (0, \pm 1)$. We claim that there exists no interval on which γ_2 is identically 0 or γ' is vertical; then it will follow by smoothness of γ that (5.1) holds at the remaining points.

To prove the claim, recall that γ is smooth at 0 and that, as a consequence of spherical symmetry, $\kappa(0) > 0$. Thus, γ_2 cannot be identically 0 on an interval including 0. On the other side of the curve, $\gamma(\beta)$ is defined to be the first point where the curve intersects the axis again, so even if a portion of the curve were a line segment along the e_1 -axis, that segment would not be parameterized by the function γ . The curve cannot have vertical tangent vector on an interval either. If a portion of the curve were a vertical line segment, then this vertical line segment, when rotated, would generate a portion of a hyperplane, which would have zero mean curvature. However, H_1 (the normal derivative of the log of the density) would vary as one moved up or down along the line segment, contradicting the fact that the surface has constant generalized mean curvature.

With this technical point out of the way, we proceed in the case that $y = \gamma_2(s) > 0$ and that $\gamma'(s) \neq (0, \pm 1)$. One of the principal curvatures at P is the the curvature of γ at this point. The cross section of the surface obtained by fixing the first coordinate is an $(n - 2)$ -dimensional sphere of revolution. The remaining principal curvatures of the surface are the principal curvatures of the sphere, which are equal. Thus, to compute one of the principal curvatures of the sphere, it is sufficient to compute the second principal curvature of a 2-dimensional surface in the $n = 3$ case. This second principal curvature is the normal curvature of a circle of revolution C .

By assumption that $y = \gamma_2(s) > 0$, the curvature of the circle C is $1/y$. We let n denote the inward unit normal vector to the surface and N denote the normal vector to the circle of revolution. Since $y > 0$, C_s is counterclockwise oriented if and only if n is downward (i.e. n has a negative e_2 -component). Thus,

$$\lambda(s) = \begin{cases} \frac{1}{R(s)}, & n \text{ downward} \\ \frac{-1}{R(s)}, & n \text{ upward.} \end{cases}$$

Meanwhile, by Meusnier's formula, the second principal curvature is given by

$$\kappa_2 = \frac{1}{y} \cos \phi,$$

where ϕ is the angle between n and N . Again, since $y > 0$,

$$\cos \phi = \begin{cases} \frac{y}{R(s)}, & n \text{ downward} \\ \frac{-y}{R(s)}, & n \text{ upward.} \end{cases}$$

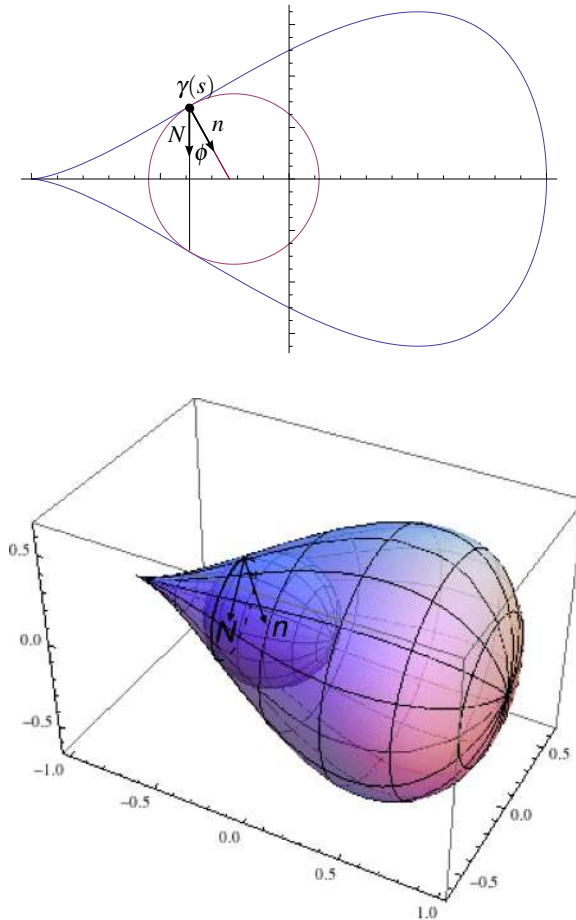


Figure 3: A cross section of the surface in the xy -plane and the inward unit normal vectors to the surface and to a circle of revolution

The first of these cases is depicted in Figure 3. In both cases, the second principal curvature is the curvature of the canonical circle.

Since one principal curvature of the surface at $\gamma(s)$ equals $\kappa(s)$ and all of the others equal $\lambda(s)$, mean curvature is given by

$$H_0(s) = \kappa(s) + (n - 2)\lambda(s).$$

□

Corollary 5.2. Since generalized mean curvature is constant on the set of regular points of ∂E , there is a constant c so that

$$c = H_f(s) = \kappa(s) + (n - 2)\lambda(s) + H_1(s). \tag{5.2}$$

for all $s \in (-\beta, \beta)$.

In the left and right cases delineated on p.1, for any $s \in [0, \beta)$ we can analyze how γ and related functions are instantaneously changing at $\gamma(s)$ by computing the requisite derivatives on the osculating circle to γ at $\gamma(s)$. A justification for this procedure will follow after we introduce some notation.

Definition 5.3. Given s in $(-\beta, \beta)$, let A_s denote the unique oriented circle that is tangent to γ at $\gamma(s)$ and has curvature $\kappa(s)$. Note that if $\kappa(s) = 0$, then A_s is an oriented line with direction vector $\gamma'(s)$. For a fixed s , let α

be an arclength parameterization of A_s , and let \tilde{s} be the point in the domain of α so that $\alpha(\tilde{s}) = \gamma(s)$. For each t in the domain of α , let $\tilde{\kappa}(t)$ denote the signed curvature of α at $\alpha(t)$, and let

$$\tilde{H}_1(t) = \frac{\partial \psi}{\partial \nu}(\alpha(t)) = \frac{p}{|\alpha(t)|} \frac{\alpha(t)}{|\alpha(t)|} \cdot \nu(t),$$

where ν is the outward unit normal vector to α at $\alpha(t)$.

Since A_s is tangent to γ at $\gamma(s)$ and has curvature $\kappa(s)$, we have $\alpha'(\tilde{s}) = \gamma'(s)$ and $\alpha''(\tilde{s}) = \gamma''(s)$. Both $\tilde{\kappa}$ and \tilde{H}_1 are smooth functions on their domains.

We also consider circles tangent to α that are centered on the e_1 -axis, and we define analogues of the functions F , R , and λ introduced in Definition 4.3. We use these functions to approximate their counterparts on γ (cf. Lemma 6.6 and Lemma 7.6).

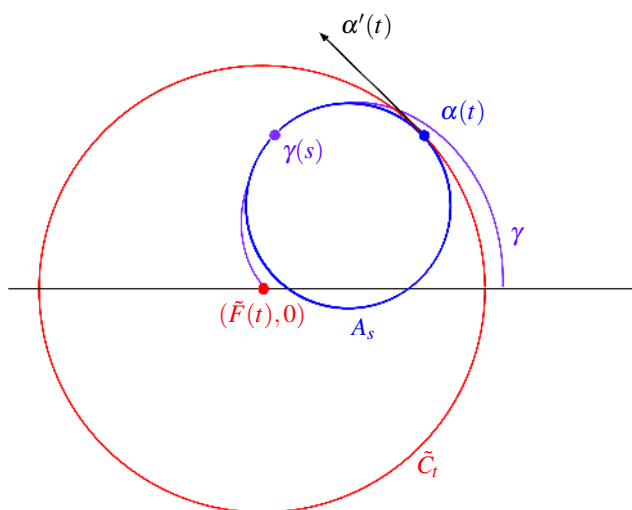


Figure 4: The osculating circle A_s at a point on γ and the canonical circle \tilde{C}_t at a point on A_s

Definition 5.4. Let A_s and α be as in Definition 5.3. Given t in the domain of α , let \tilde{C}_t denote the canonical circle to A_s at $\alpha(t)$, defined as follows: if $\alpha_2(t) \neq 0$, then we define \tilde{C}_t to be the unique oriented circle that has its center on the e_1 -axis and is tangent to A_s at the point $\alpha(t)$. If $\alpha_2(t) = 0$ and $\alpha'(t) = (0, \pm 1)$, then we define \tilde{C}_t to be A_s . If $\alpha_2(t) = 0$ and $\alpha'(t) \neq (0, \pm 1)$, then \tilde{C}_t is undefined. For each t so that \tilde{C}_t is defined, the canonical circle is defined on a neighborhood of t . We define the functions $\tilde{\lambda}$, \tilde{R} , and \tilde{F} by letting $\tilde{\lambda}(t)$, $\tilde{R}(t)$, and $\tilde{F}(t)$ be the signed curvature of \tilde{C}_t , the radius of \tilde{C}_t , and the abscissa of the center of \tilde{C}_t , respectively. Figure 4 shows the osculating circle A_s at a point on γ and the canonical circle \tilde{C}_t at a point on A_s .

For a given t , the canonical circle \tilde{C}_t depends only on $\alpha(t)$ and $\alpha'(t)$. It follows that \tilde{F} , \tilde{R} , and $\tilde{\lambda}$ can be computed in terms of α and α' and that their derivatives depend on α and its first two derivatives. In particular, since $\alpha'(\tilde{s}) = \gamma'(s)$ and $\alpha''(\tilde{s}) = \gamma''(s)$, we have

$$\tilde{F}'(\tilde{s}) = F'(s),$$

$$\tilde{R}'(\tilde{s}) = R'(s),$$

and

$$\tilde{\lambda}'(\tilde{s}) = \lambda'(s).$$

As well as analyzing these functions, we also consider the angle the tangent vector makes to the horizontal.

Definition 5.5. We define the function $\theta : S^1 \rightarrow (0, 2\pi]$ by letting $\theta(v)$ be the angle in the specified interval that v makes to the positive e_1 -axis.

The next proposition of Chambers concerns two C^2 functions on an interval (a, b) . Given $h : (a, b) \rightarrow \mathbb{R}_{\geq 0}$, we let $t_h(x)$ denote the unit tangent vector

$$\frac{(1, h'(x))}{|(1, h'(x))|},$$

and we let $\kappa_h(x)$ denote the upward curvature of the graph of h at x .

Proposition 5.6. [1, Prop 3.8] Consider two C^2 functions $f, g : (a, b) \rightarrow \mathbb{R}_{\geq 0}$ with $b > a$ that satisfy the following:

1. $\lim_{x \rightarrow b^-} t_f(x)$ and $\lim_{x \rightarrow b^-} t_g(x)$ exist,
2. $\lim_{x \rightarrow b^-} f(x)$ and $\lim_{x \rightarrow b^-} g(x)$ exist,
3. $f'(x) \geq 0$ and $g'(x) \geq 0$ for all $x \in (a, b)$,
4. $\lim_{x \rightarrow b^-} f(x) \leq \lim_{x \rightarrow b^-} g(x)$, and $\lim_{x \rightarrow b^-} \theta(t_f(x)) \geq \lim_{x \rightarrow b^-} \theta(t_g(x))$,
5. $\kappa_f(x) \leq \kappa_g(x)$ for all $x \in (a, b)$.

Then for every $x \in (a, b)$, $f(x) \leq g(x)$, and $\theta(t_f(x)) \geq \theta(t_g(x))$. Furthermore, if there exists a point $x_0 \in (a, b)$ such that $\kappa_f(x_0) < \kappa_g(x_0)$, then there is some $\phi > 0$ such that $\phi \leq \theta(t_f(x)) - \theta(t_g(x))$ for all $x \in (a, x_0)$.

6 Proof of Right Tangent Lemma

To prove Proposition 4.9, we assume that $F(0) > 1/2$. Then we consider two subintervals of $[0, \beta)$ that we call the upper curve and the lower curve after the objects of the same names in [1] (see Definitions 6.4 and 6.10). We will prove that the lower curve ends in a vertical tangent at a point right of the e_2 -axis (Lemma 6.15) and that, past this point, curvature is positive and the tangent vector is strictly in the fourth quadrant (Lemma 6.17). The end behavior of the curve is similar to that of the generating curve in [1] except that our curve must terminate right of the e_2 -axis, an additional feature which allows us to achieve a contradiction to spherical symmetry without an analogue of Chambers' Second Tangent Lemma [1, Lemma 2.5]. As such, many intermediate results are also similar to results in [1] and are cross-referenced.

Our analysis requires comparing curvatures at points of the same height on opposite sides of the curve. Specifically, we show that the curvature at the point on the left is strictly greater than the curvature at the corresponding point on the right (Prop. 6.14). By Corollary 5.2, it suffices to prove that λ , the canonical circle curvature, is less at the point on the left and H_1 , the normal derivative of the log of the density, is strictly less at the point on the left. For any $s \in [0, \beta)$, $\gamma(s) \neq (0, 0)$, so the normal derivative of $\log(r^p)$ at $\gamma(s)$ is given by

$$H_1(s) = \frac{p}{|\gamma(s)|} \frac{\gamma(s)}{|\gamma(s)|} \cdot \nu(s) = p \frac{\gamma(s)}{|\gamma(s)|^2} \cdot \nu(s). \tag{6.1}$$

More generally, given points $(x_1, y), (x_2, y) \in \mathbb{R}^2 - \{0\}$, and unit vectors ν_1 and ν_2 , one can compare the quantities

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot \nu_1^\perp \quad \text{and} \quad \frac{(x_2, y)}{|(x_2, y)|^2} \cdot \nu_2^\perp,$$

where ν_1^\perp and ν_2^\perp denote clockwise rotations of ν_1 and ν_2 by $\pi/2$ radians. (In our context, ν_1 and ν_2 will be tangent vectors to the curve at two points, so ν_1^\perp and ν_2^\perp will be the outward unit normal vectors.) We have discovered a set of sufficient conditions for the points (x_1, y) and (x_2, y) and the vectors ν_1 and ν_2 to satisfy

the inequality

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp > \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^\perp. \tag{6.2}$$

In Definition 6.1, we define two unit vectors v_1 and v_2 to be admissible with respect to (x_1, y) and (x_2, y) if they satisfy these conditions.

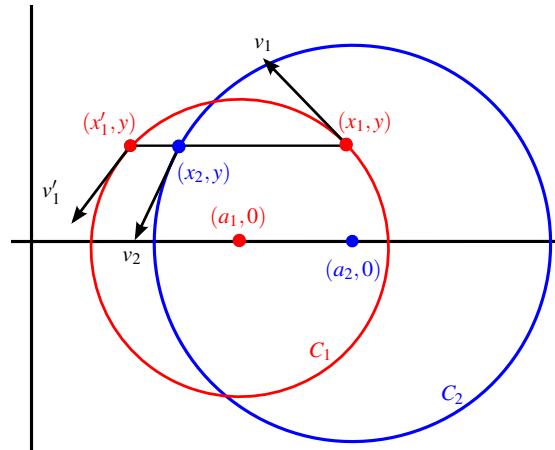


Figure 5: The vectors v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) .

Definition 6.1. Consider a pair of points (x_1, y) and (x_2, y) with $y > 0$, and a pair of unit vectors, v_1 and v_2 , which lie strictly in the second and third quadrants, respectively. Let v_1' denote the reflection of v_1 over the e_1 -axis. Let C_i denote the canonical circle with respect to v_i at (x_i, y) , with center $(a_i, 0)$ and radius R_i . As depicted in Figure 5, v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) if the following occur:

1. $a_1 > R_1$,
2. $\theta(v_2) \geq \theta(v_1')$,
3. $x_1 - a_1 \geq a_1 - x_2$.

Proposition 6.2. Consider a pair of points (x_1, y) and (x_2, y) in the upper half plane with $x_1 \geq x_2$. Let v_1 and v_2 be two unit vectors, and let v_1^\perp and v_2^\perp denote the clockwise rotations of these respective vectors through $\pi/2$ radians. If v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) , then

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp > \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^\perp.$$

Proof. Let (x_1', y) be the reflection of (x_1, y) over the vertical line $x = a_1$. It follows that $x_1' = a_1 - (x_1 - a_1)$. By symmetry, C_1 is also the canonical circle with respect to v_1' at (x_1', y) . We will show that

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp > \frac{(x_1', y)}{|(x_1', y)|^2} \cdot v_1'^\perp \tag{6.3}$$

and that

$$\frac{(x_1', y)}{|(x_1', y)|^2} \cdot v_1'^\perp \geq \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^\perp \tag{6.4}$$

To prove (6.3), we parameterize C_1 by $\alpha(t) = (a_1 + R_1 \cos t, R_1 \sin t)$ for t in $[0, 2\pi)$. Taking $t_1 \in (0, \pi/2)$ so that $\alpha(t_1) = (x_1, y)$, we have by symmetry that $(x_1', y) = \alpha(\pi - t_1)$. Using this parameterization to simplify

the quantities in (6.3), we have

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp = \frac{(a_1 + R_1 \cos t_1, R_1 \sin t_1)}{|\alpha(t_1)|^2} \cdot (\cos t_1, \sin t_1) = \frac{a_1 \cos t_1 + R_1}{|\alpha(t_1)|^2}$$

and

$$\frac{(x'_1, y)}{|(x'_1, y)|^2} \cdot v_1^{\perp} = \frac{(a_1 - R_1 \cos t_1, R_1 \sin t_1)}{|\alpha(\pi - t_1)|^2} \cdot (-\cos t_1, \sin t_1) = \frac{-a_1 \cos t_1 + R_1}{|\alpha(\pi - t_1)|^2},$$

whence

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp - \frac{(x'_1, y)}{|(x'_1, y)|^2} \cdot v_1^{\perp} = \frac{(a_1 \cos t_1 + R_1)|\alpha(\pi - t_1)|^2 - (-a_1 \cos t_1 + R_1)|\alpha(t_1)|^2}{|\alpha(t_1)|^2 |\alpha(\pi - t_1)|^2}.$$

The denominator is positive, so we need only show that the numerator is positive to conclude that (6.3) holds. A short computation reveals that

$$(a_1 \cos t_1 + R_1)|\alpha(\pi - t_1)|^2 - (-a_1 \cos t_1 + R_1)|\alpha(t_1)|^2 = 2a_1(a_1^2 - R_1^2) \cos t_1 > 0.$$

Since v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) , we have that $x_2 \geq a_1 - (x_1 - a_1) = x'_1$. Moreover, x'_1 must be positive, as $a_1 - (x_1 - a_1) > a_1 - R_1 > 0$. It follows that

$$\frac{1}{|(x'_1, y)|} \geq \frac{1}{|(x_2, y)|}.$$

Therefore, to prove (6.4), it suffices to show that

$$\frac{(x'_1, y)}{|(x'_1, y)|} \cdot v_1^\perp \geq \frac{(x_2, y)}{|(x_2, y)|} \cdot v_2^\perp. \tag{6.5}$$

We note that the left-hand side of (6.5) is $\cos(\theta(v_1^\perp) - \theta((x'_1, y)))$ and the right-hand side is equal to $\cos(\theta(v_2^\perp) - \theta((x_2, y)))$. Since v_1 is strictly in the second quadrant, v_2 is strictly in the third, and $x_2 \geq x'_1 > 0$, it follows that $0 < \theta(v_1^\perp) - \theta((x'_1, y))$, $\theta(v_2^\perp) - \theta((x_2, y)) < \pi$. As cosine is decreasing on $(0, \pi)$, it suffices to show that

$$\theta(v_2^\perp) - \theta((x_2, y)) \geq \theta(v_1^\perp) - \theta((x'_1, y)). \tag{6.6}$$

As noted above, $x_2 \geq x'_1$, so $\theta((x_2, y)) \leq \theta((x'_1, y))$. By the admissibility of v_1 and v_2 , we have that $\theta(v_2^\perp) \geq \theta(v_1^\perp)$. Combining these inequalities establishes (6.6), completing our proof of (6.4). \square

Having proved Proposition 6.2, we define the upper and lower curves and prove various properties that hold on these intervals. Our definition of the upper curve is motivated by the following observation.

Lemma 6.3. (cf. [1, Lemma 3.5]) Given that $F(0) > 1/2$, we have $\kappa''(0) > 0$.

Proof. Differentiating (5.2) and substituting 0 into the resulting equation, we have

$$0 = \kappa''(0) + (n - 2)\lambda''(0) + H_1''(0).$$

Since $\kappa'(0) = 0$, A_0 approximates γ up to fourth order at $\gamma(0)$. Thus, parameterizing A_0 by

$$\alpha(t) = \left(a + r \cos\left(\frac{t}{r}\right), r \sin\left(\frac{t}{r}\right) \right)$$

over $[-\pi r, \pi r)$, we have that $\lambda''(0) = \tilde{\lambda}''(0)$ and $H_1''(0) = \tilde{H}_1''(0)$. In particular, since $\tilde{\lambda}$ is constant, we have that $\lambda'(0) = \tilde{\lambda}'(0) = 0$ and $\lambda''(0) = \tilde{\lambda}''(0) = 0$. (One can deduce that $\tilde{\lambda}$ is constant as follows: recall that for each $t \in [-\pi r, \pi r)$, $\tilde{\lambda}(t)$ denotes the curvature of \tilde{C}_t , where \tilde{C}_t is defined to be the unique circle that has its center on the e_1 -axis and is tangent to A_0 at the point $\alpha(t)$. A_0 is a circle whose center is on the e_1 -axis. Thus, for each $t \in [-\pi r, \pi r)$, $\tilde{C}_t = A_0$.)

To prove that $\kappa''(0) > 0$, it now suffices to prove that $\tilde{H}_1''(0) < 0$. Since $a = F(0)$ and $r = R(0)$, our assumption that

$$F(0) > \frac{1}{2} = \frac{\gamma_1(0)}{2}$$

is equivalent to the inequality $a > r$. Thus, computing \tilde{H}_1'' , we have

$$\tilde{H}_1''(0) = \frac{p}{|\alpha(0)|^4} \frac{a}{r^2} (r^2 - a^2) < 0.$$

□

Definition 6.4. (cf. [1, Defn. 3.4]) Let the upper curve K be defined as the set of all $s \in [0, \beta]$ such that for all t in $[0, s]$ the following properties are satisfied:

1. $\gamma'(t)$ lies in the second quadrant,
2. $\kappa(t) \geq \lambda(t) > 0$.

Lemma 6.5. (cf. [1, Lemma 3.11]) We have that K is nonempty and that $\sup K > 0$.

Proof. Since $\gamma'(0) = (0, 1)$, $\kappa(0) > 0$, and κ is continuous, we can conclude that there exists $\rho_1 > 0$ so that $\gamma'(s)$ lies in the second quadrant for all $s \in [0, \rho_1]$. Meanwhile, recall that $\kappa(0) = \lambda(0) > 0$ (Prop. 4.8) and that $\kappa'(0) = 0$ by spherical symmetry. As deduced in the proof of Lemma 6.3, we have that $\lambda'(0) = \lambda''(0) = 0$. However, $\kappa''(0) > 0$. It follows by taking Taylor approximations that there exists $\rho_2 > 0$ so that $\kappa(s) \geq \lambda(s) > 0$ for all $s \in [0, \rho_2]$. Taking $\rho = \min(\rho_1, \rho_2)$, it follows that $[0, \rho] \subset K$. Thus, K is nonempty and $\sup K > 0$. □

Having proved that $\sup K > 0$, we let

$$\delta = \sup K.$$

The following lemma extends our assumption that $F(0) > R(0)$ and allows us to check the first condition of admissibility.

Lemma 6.6. If $s \in K$, then $F(s) > R(s)$.

Proof. By the assumptions defining the right case, $F(0) > R(0)$. We claim that $F' \geq 0$ on K and $R' \leq 0$ on K . To prove so, we will use a similar argument to that in [1, Lemma 5.3]: for a fixed $s \in K$, let

$$\alpha(t) = \left(a + r \cos\left(\frac{t}{r}\right), b + r \sin\left(\frac{t}{r}\right) \right) \quad (6.7)$$

be an arclength parameterization of A_s , and let \tilde{s} be the point in the domain of α so that $\alpha(\tilde{s}) = \gamma(s)$. Since $\kappa(s) \geq \lambda(s)$, it follows that $b \geq 0$. By the discussion following Definition 5.4, $F'(s) = \tilde{F}'(\tilde{s})$ and $R'(s) = \tilde{R}'(\tilde{s})$. Thus, we seek formulae for $\tilde{F}(t)$ and $\tilde{R}(t)$. We will only consider t for which $\alpha_2(t) > 0$.

Fix t with $\alpha_2(t) > 0$. As depicted in Figure 6, the vector from $\alpha(t)$ to the center of \tilde{C}_t is in the direction of the inward unit normal vector at $\alpha(t)$. An arclength parameterization of the line containing these points is given by

$$\beta(u) = \alpha(t) + u \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \alpha'(t) = \begin{bmatrix} a + r \cos\left(\frac{t}{r}\right) - u \cos\left(\frac{t}{r}\right) \\ b + r \sin\left(\frac{t}{r}\right) - u \sin\left(\frac{t}{r}\right) \end{bmatrix}.$$

We let u_0 be the value of u so that $\beta_2(u_0) = 0$. Then we have

$$u_0 = \frac{b + r \sin\left(\frac{t}{r}\right)}{\sin\left(\frac{t}{r}\right)}.$$

Since β is an arclength parameterization, u_0 is the distance from $\alpha(t)$ to the center of \tilde{C}_t , i.e.

$$\tilde{R}(t) = u_0 = \frac{b + r \sin\left(\frac{t}{r}\right)}{\sin\left(\frac{t}{r}\right)}. \quad (6.8)$$

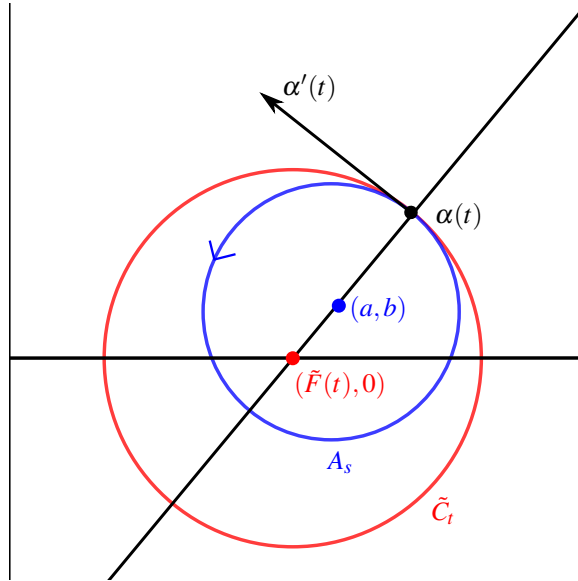


Figure 6: The canonical circle at $\alpha(t)$ and the line parameterized by β

Meanwhile,

$$\tilde{F}(t) = \beta_1(u_0) = a - b \cot\left(\frac{t}{r}\right). \tag{6.9}$$

Differentiating, we obtain

$$\tilde{F}'(t) = \frac{b}{r} \csc^2\left(\frac{t}{r}\right), \tag{6.10}$$

and

$$\tilde{R}'(t) = -\frac{b}{r} \csc^2\left(\frac{t}{r}\right) \cos\left(\frac{t}{r}\right). \tag{6.11}$$

Since $b \geq 0$, we have $\tilde{F}'(\tilde{s}) \geq 0$. Meanwhile, since $s \in K$, $\gamma'(s)$ is in the second quadrant. Thus, $\cos(\tilde{s}/r) \geq 0$, from which it follows that $\tilde{R}'(\tilde{s}) \leq 0$.

Since $F' \geq 0$ on K and $R' \leq 0$ on K , we have $F(s) \geq F(0) > R(0) \geq R(s)$ for any $s \in K$. □

We will soon prove several properties of δ , but first we require one more lemma.

Lemma 6.7. (cf. [1, Lemma 3.4]) Let $s \in (0, \delta)$. If $\kappa(s) = \lambda(s) > 0$, then $\lambda'(s) = 0$, but $\kappa'(s) > 0$.

Proof. Differentiating Equation (5.2) gives $\kappa'(s) + (n - 2)\lambda'(s) + H'_1(s) = 0$. By the hypothesis that $\kappa(s) = \lambda(s)$, we have that $A_s = C_s$. It follows that the canonical circle to A_s at each point is A_s , so $\tilde{\lambda}$ is constant. In particular, $\lambda'(s) = \tilde{\lambda}'(\tilde{s}) = 0$.

Given this result, to prove that $\kappa'(s) > 0$, it suffices to prove that $H'_1(s) < 0$. Parameterizing A_s as in (6.7), we compute that

$$\tilde{H}'_1(t) = \frac{-p(a^2 + b^2 - r^2)(-b \cos(\frac{t}{r}) + a \sin(\frac{t}{r}))}{r|\alpha(t)|^4}. \tag{6.12}$$

Since $A_s = C_s$, we have that $b = 0$, $a = F(s)$, and $r = R(s)$. By Lemma 6.6, $a > r > 0$. Finally, since $r \sin(\frac{\tilde{s}}{r}) = \gamma_2(s) > 0$, we have that $\sin(\frac{\tilde{s}}{r}) > 0$. Thus,

$$H'_1(s) = \tilde{H}'_1(\tilde{s}) = \frac{-p(a^2 - r^2)(a \sin(\frac{\tilde{s}}{r}))}{r|\alpha(\tilde{s})|^4} < 0.$$

□

Proposition 6.8. (cf. [1, Prop. 3.12]) The following properties of δ hold:

1. $\delta < \beta$,
2. $\delta \in K$,
3. $\gamma_1(\delta) \geq F(s)$ for any $s \in [0, \delta]$,
4. $\gamma_1(\delta) > 0$,
5. $\gamma'(\delta) = (-1, 0)$.

Proof. The proofs of (1)-(3) are identical to their counterparts in [1, Prop. 3.12]. Setting $s = 0$ in the inequality $\gamma_1(\delta) \geq F(s)$, we have

$$\gamma_1(\delta) \geq F(0) > \frac{\gamma_1(0)}{2} > 0.$$

To prove that $\gamma'(\delta) = (-1, 0)$, we argue by contradiction; specifically, we show that if $\gamma'(\delta) \neq (-1, 0)$, then there exists $\varepsilon > 0$ so that $[\delta, \delta + \varepsilon) \subset K$.

Suppose that $\gamma'(\delta) \neq (-1, 0)$. We have by Lemma 4.2 that $\gamma'(\delta)$ is strictly in the second quadrant. By continuity of γ' , there exists $\varepsilon_1 > 0$ so that $\gamma'(s)$ is in the second quadrant for all $s \in [\delta, \delta + \varepsilon_1)$. Since $\delta \in K$, $\lambda(\delta) > 0$. By continuity of λ , $\lambda > 0$ on an open interval containing δ . By reducing ε_1 if necessary, we can assume that $\lambda(s) > 0$ for all $s \in [\delta, \delta + \varepsilon_1)$.

From here, it suffices to show that there exists $\varepsilon_2 > 0$ so that $\kappa(s) \geq \lambda(s)$ for all $s \in [\delta, \delta + \varepsilon_2)$. To demonstrate the existence of such an ε_2 , we take two cases. Since $\delta \in K$, $\kappa(\delta) \geq \lambda(\delta)$. If $\kappa(\delta) > \lambda(\delta)$, then the existence of such an ε_2 follows by continuity of $\kappa - \lambda$. Meanwhile, if $\kappa(\delta) = \lambda(\delta)$, then we apply Lemma 6.7 to conclude that $\lambda'(\delta) = 0$, but $\kappa'(\delta) > 0$. It follows that there exists $\varepsilon_2 > 0$ so that $\kappa(s) \geq \lambda(s)$ for all $s \in [\delta, \delta + \varepsilon_2)$. In either case, taking $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ guarantees that $[\delta, \delta + \varepsilon) \subset K$, contradicting the fact that δ is an upper bound for K . □

Lemma 6.9. We have that $\kappa'(\delta) > 0$.

Proof. Differentiating the ODE $H_f = c$, we obtain $\kappa'(\delta) + (n - 2)\lambda'(\delta) + H'_1(\delta) = 0$. Let (a, b) be the center of A_δ and r be its radius. Since $\kappa(\delta) \geq \lambda(\delta)$, it follows that $b \geq 0$. Parameterizing A_δ as in (6.7), we see that $\gamma(\delta) = \alpha(\pi r/2)$. Thus, $\lambda'(\delta) = \tilde{\lambda}'(\pi r/2)$. By inverting (6.8) and differentiating, we conclude that $\tilde{\lambda}'(\pi r/2) = 0$. Since $H'_1(\delta) = \tilde{H}'_1(\pi r/2)$, it suffices to prove that $\tilde{H}'_1(\pi r/2) < 0$.

Looking to (6.12), we claim that $a^2 + b^2 > r^2$. To prove so, let $R = R(\delta)$ be the radius of C_δ . As depicted in Figure 7, since $\gamma'(\delta) = (-1, 0)$, we have that $R = r + b$ and $a = F(\delta)$. We apply Lemma 6.6 to give $a > R = r + b$. Since both sides of the inequality $a - b > r$ are positive, we may square to give $(a - b)^2 > r^2$. Since $b \geq 0$, this implies that $a^2 + b^2 > r^2$. Therefore, we have that

$$\tilde{H}'_1\left(\frac{\pi r}{2}\right) = \frac{-pa(a^2 + b^2 - r^2)}{r|\alpha(\frac{\pi r}{2})|^4} < 0.$$

□

Definition 6.10. (cf. [1, Defn. 3.5]) Let the lower curve L be defined as the set of all s in $[\delta, \beta)$ such that for all $t \in [\delta, s]$ the following hold:

1. $\gamma'(t)$ is in the third quadrant with $\gamma'(t) \neq (-1, 0)$ if $t > \delta$,
2. If \bar{t} is the unique point in K with $\gamma_2(\bar{t}) = \gamma_2(t)$, then $\kappa(\bar{t}) \leq \kappa(t)$.

Since $\gamma'(\delta) = (-1, 0)$, $\kappa(\delta) > 0$, and $\kappa'(\delta) > 0$, these conditions hold on an interval $[\delta, \delta + \varepsilon)$. Thus, L is nonempty and has a supremum, which we denote by η .

By condition (1) in Definition 6.10, $\gamma'(s) \neq (-1, 0)$ if $s \in (\delta, \eta)$. Similarly, there can be no $s_0 \in (0, \delta)$ with $\gamma'(s_0) = (-1, 0)$. (If there were such an s_0 , then we would have $\kappa(s_0) \geq \lambda(s_0) > 0$. Consequently, on an interval immediately following s_0 , γ' would be strictly in the third quadrant, contradicting the fact that δ is

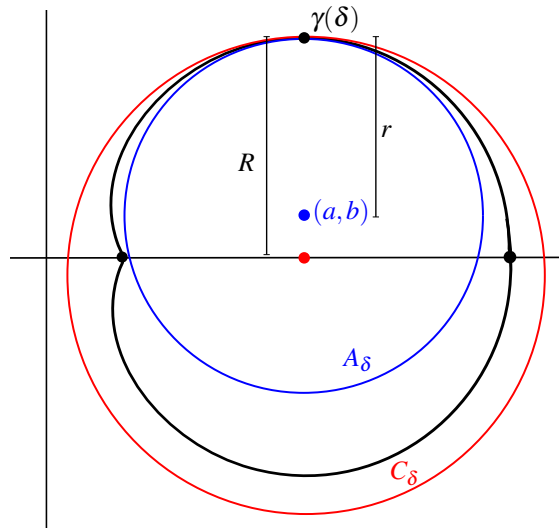


Figure 7: The canonical circle C_δ and the osculating circle A_δ

the least upper bound of K .) Since γ_2' does not vanish on $(0, \delta)$ or on (δ, η) , we can apply the Inverse Value Theorem to define a local inverse of γ_2 over each of these intervals.

Definition 6.11. We define $h : (\gamma_2(\eta), \gamma_2(\delta)) \rightarrow (0, \delta)$ by letting $h(y)$ be the unique $t \in (0, \delta)$ such that $\gamma_2(t) = y$. Similarly, we define $k : (\gamma_2(\eta), \gamma_2(\delta)) \rightarrow (\delta, \eta)$ by letting $k(y)$ be the unique $t \in (\delta, \eta)$ such that $\gamma_2(t) = y$.

Using these local inverse functions, we define functions $f, g : (\gamma_2(\eta), \gamma_2(\delta)) \rightarrow \mathbb{R}$ as follows.

Definition 6.12. Given $y \in (\gamma_2(\eta), \gamma_2(\delta))$, let

$$f(y) = 2\gamma_1(\delta) - \gamma_1(h(y)),$$

and let

$$g(y) = \gamma_1(k(y)).$$

The function g gives the e_1 -coordinate of a point in $\gamma(L)$ with a given e_2 -coordinate. If we begin with the point in $\gamma(K)$ with a given e_2 -coordinate, then f gives the e_1 -coordinate of the reflection of this point over the line $x = \delta$. We can use these functions to prove two properties of the lower curve.

Lemma 6.13. (cf. [1, Lemma 3.13]) For each $s \in [\delta, \eta]$, let \bar{s} be the unique point in K so that $\gamma_2(\bar{s}) = \gamma_2(s)$. Then the following hold:

$$\gamma_1(\bar{s}) - \gamma_1(\delta) \geq \gamma_1(\delta) - \gamma_1(s), \tag{6.13}$$

$$\theta(\gamma'(s)) \geq 2\pi - \theta(\gamma'(\bar{s})). \tag{6.14}$$

Proof. Both inequalities are trivially true if $s = \delta$. Now let $s \in (\delta, \eta)$ be fixed, and let $y = \gamma_2(s)$. By the definition of L (Defn. 6.10), f and g satisfy the hypotheses of Proposition 5.6. From the inequality $f \leq g$ in Proposition 5.6, (6.13) above is immediate. To arrive at (6.14), let $t_f(y)$ and $t_g(y)$ denote the unit tangent vectors to the graphs of f and g at y . Note that $\gamma((\delta, \eta))$ is the set $\{(g(y), y) : y \in (\gamma_2(\eta), \gamma_2(\delta))\}$, and the reflection of $\gamma((0, \delta))$ over the line $x = \delta$ is the set $\{(f(y), y) : y \in (\gamma_2(\eta), \gamma_2(\delta))\}$. Let $y = \gamma_2(s)$. Then we obtain the tangent

vector $t_g(y)$ from $\gamma'(s)$ by rotating $\gamma'(s)$ clockwise through π radians and reflecting the resulting vector in the first quadrant over the line $y = x$. Therefore, we have

$$\theta(t_g(y)) = \frac{\pi}{2} - (\theta(\gamma'(s)) - \pi) = \frac{3\pi}{2} - \theta(\gamma'(s)).$$

Similarly, we obtain $t_f(y)$ from $\gamma'(\bar{s})$ by reflecting over the line $x = \delta$ and reflecting over the line $y = x$. Thus,

$$\theta(t_f(y)) = \frac{\pi}{2} - (\pi - \theta(\gamma'(\bar{s}))) = \theta(\gamma'(\bar{s})) - \frac{\pi}{2}.$$

Substituting these results into the second inequality in Proposition 5.6 completes the proof. □

Proposition 6.14. Let $s \in (\delta, \eta)$, and suppose that $\gamma'(s) \neq (0, -1)$. If \bar{s} is the unique point in K so that $\gamma_2(\bar{s}) = \gamma_2(s)$, then $\kappa(s) > \kappa(\bar{s})$.

Proof. Since H_f is constant,

$$\kappa(s) + (n - 2)\lambda(s) + H_1(s) = \kappa(\bar{s}) + (n - 2)\lambda(\bar{s}) + H_1(\bar{s}).$$

It can be shown using right triangle trigonometry and (6.14) from Lemma 6.13 that $\lambda(s) \leq \lambda(\bar{s})$. Thus, to prove that $\kappa(s) > \kappa(\bar{s})$, it suffices to prove that $H_1(s) < H_1(\bar{s})$. We show that $\gamma'(\bar{s})$ and $\gamma'(s)$ are admissible with respect to $\gamma(\bar{s})$ and $\gamma(s)$ and then appeal to Proposition 6.2. Since $\gamma'(s)$ is not equal to $(0, -1)$, $\gamma'(s)$ lies strictly in the third quadrant. By Lemma 6.6, $F(\bar{s}) > R(\bar{s})$. Thus the first condition in the definition of admissibility is met.

By Lemma 6.13, $\theta(\gamma'(s)) \geq 2\pi - \theta(\gamma'(\bar{s}))$, satisfying the second condition of admissibility. Furthermore, by the same lemma, we have $\gamma_1(\bar{s}) - \gamma_1(\delta) \geq \gamma_1(\delta) - \gamma_1(s)$. By Proposition 6.8, $\gamma_1(\delta) \geq F(\bar{s})$, so $\gamma_1(\bar{s}) - F(\bar{s}) \geq F(\bar{s}) - \gamma_1(s)$, and the final condition for admissibility is satisfied.

Because $\gamma'(\bar{s})$ and $\gamma'(s)$ are admissible with respect to $\gamma(\bar{s})$ and $\gamma(s)$, we conclude by Proposition 6.2 that

$$\frac{\gamma(\bar{s})}{|\gamma(\bar{s})|^2} \cdot \gamma'(\bar{s})^\perp > \frac{\gamma(s)}{|\gamma(s)|^2} \cdot \gamma'(s)^\perp.$$

By (6.1), it follows that $H_1(s) < H_1(\bar{s})$, as required. □

By a similar argument to that in [1, Lemma 3.14] along with Proposition 6.14, $\eta < \beta$, $\eta \in L$, and $\gamma'(\eta) = (0, -1)$. In addition to these properties of η , we can also show using the curvature comparison that $\gamma_1(\eta) > 0$. Then proving that $\gamma_1(\beta) > 0$ is a matter of showing that γ_1 is increasing on (η, β) . To establish the second claim of the Right Tangent Lemma, we consider the functions κ and γ' on (η, β) . Lemma 6.16 gives a computational result regarding κ , whereas Lemma 6.17 extends this result as well as showing that γ' is strictly in the fourth quadrant on (η, β) .

Lemma 6.15. We have that $\gamma_1(\eta) > 0$.

Proof. By Lemma 6.13, $\gamma_1(\bar{\eta}) - \gamma_1(\delta) \geq \gamma_1(\delta) - \gamma_1(\eta)$. Furthermore, $\gamma_1(\delta) = F(\delta) \geq F(\bar{\eta})$. Therefore, $\gamma_1(\bar{\eta}) - F(\bar{\eta}) \geq \gamma_1(\bar{\eta}) - \gamma_1(\delta) \geq \gamma_1(\delta) - \gamma_1(\eta) \geq F(\bar{\eta}) - \gamma_1(\eta)$.

Finally, $\gamma_1(\bar{\eta}) - F(\bar{\eta}) < R(\bar{\eta})$, because $R(\bar{\eta})$ is the distance from $(F(\bar{\eta}), 0)$ to $\gamma(\bar{\eta})$. It follows that $\gamma_1(\eta) \geq F(\bar{\eta}) - (\gamma_1(\bar{\eta}) - F(\bar{\eta})) > F(\bar{\eta}) - R(\bar{\eta})$. By Lemma 6.6, this final expression is positive. □

Lemma 6.16. Let $s \in (0, \beta)$. If $\gamma_1(s) \geq 0$ and $\gamma'(s)$ is in the fourth quadrant, then $\kappa(s) > 0$.

Proof. Since $\gamma'(s)$ is in the fourth quadrant and $\gamma_2(s) > 0$, $\lambda(s) \leq 0$. Since $\gamma(s)$ is in the first quadrant and $v(s)$ is in the third, $\gamma(s) \cdot v(s) \leq 0$, which implies that $H_1(s) \leq 0$. Meanwhile, we have that $H_f(s) = H_f(0) > 0$, because $H_1(0) > 0$, $\kappa(0) > 0$ by spherical symmetry, and $\kappa(0) = \lambda(0)$ (Proposition 4.8). Hence, it must be the case that $\kappa(s) > 0$. □

Lemma 6.17. (cf. [1, Prop. 4.1]) For $s \in (\eta, \beta)$, $\gamma'(s)$ lies strictly in the fourth quadrant, and $\kappa(s) > 0$.

Proof. Define $A \subset (\eta, \beta)$ so that $s \in A$ if and only if for all $t \in (\eta, s)$, $\gamma'(t)$ lies strictly in the fourth quadrant and $\kappa(t) > 0$. Note that A is nonempty because $\kappa(\eta) > 0$, $\gamma'(\eta) = (0, -1)$, and κ is continuous at η . Thus, A has a supremum ω . To prove the lemma we show that $\omega = \beta$.

Suppose for contradiction that $\omega < \beta$. Then γ is smooth at ω ; in particular, $\gamma'(\omega)$ and $\kappa(\omega)$ are defined. Since $\gamma'(t)$ lies in the fourth quadrant for all $t \in (\eta, \omega)$, $\gamma'(\omega)$ is in the fourth quadrant. Since $\kappa > 0$ on (η, ω) , $\gamma'(\omega)$ is not equal to $(0, -1)$. Furthermore, $\gamma_1(\omega) > 0$, as $\gamma_1(\eta) > 0$ (Lemma 6.15) and γ' lies in the fourth quadrant on (η, ω) . If $\gamma'(\omega)$ were equal to $(1, 0)$, then we would have $\gamma'(\omega) \cdot \gamma(\omega) = \gamma_1(\omega) > 0$, contradicting the Tangent Restriction Lemma (Lemma 4.2). Thus $\gamma'(\omega)$ lies strictly in the fourth quadrant. By Lemma 6.16, $\kappa(\omega) > 0$. Thus, by continuity of γ' and κ on $[0, \beta]$, A could be extended past ω , contradicting the definition of ω . \square

Proof of the Right Tangent Lemma (Lemma 4.9). It follows from Lemma 6.17 that $\gamma_1(\beta) > 0$, as $\gamma'(s)$ lies strictly in the fourth quadrant for all $s \in (\eta, \beta)$, and $\gamma_1(\eta) > 0$. As $\kappa > 0$ and γ' is in the fourth quadrant on (η, β) , the angle $\theta(s)$ that $\gamma'(s)$ makes with the e_1 -axis, measured counterclockwise in radians, must be a strictly increasing function on (η, β) that is bounded above by 2π . Therefore, $\lim_{s \rightarrow \beta^-} \theta(s)$ exists and is in $(\theta(\eta), 2\pi]$. It follows that $\lim_{s \rightarrow \beta^-} \gamma'(s)$ exists, lies in the fourth quadrant, and is not $(0, -1)$. \square

7 Proof of Left Tangent Lemma

In the previous section, the key to proving the Right Tangent Lemma was to show that the curvature was greater at a point on the lower curve than at its corresponding point on the upper curve, allowing us to find $\eta < \beta$ where $\gamma'(\eta) = (0, -1)$. Now, in the left case (Prop. 4.10), we will prove the opposite inequality concerning curvatures at corresponding points, with the aim of showing that $\lim_{s \rightarrow \beta^-} \gamma'(s)$ is in the third quadrant and not equal to $(0, -1)$. This case, however, presents new obstacles. One difficulty we eliminate is the possibility that there are multiple points on the portion of γ parameterized by $[0, \beta]$ where the tangent vector is $(-1, 0)$. In the right case, the lower curve naturally terminated at a point where the tangent vector was $(0, -1)$. However, the goal in the left case will be to show that the lower curve does not terminate before β (Lemma 7.17), allowing us to apply the curvature comparison all the way up to β . We begin with a new definition of admissibility for the left case and an analogue of Proposition 6.2.

Definition 7.1. Consider two points (x_1, y) and (x_2, y) and two unit vectors v_1 and v_2 , strictly in the second and third quadrants, respectively. Let C_i , a_i , R_i , x'_1 , and v'_1 be as in Definition 6.1. Finally, let $(x^*, 0)$ be the unique point on the e_1 -axis so that v_2 is tangent at (x^*, y) to the circle centered at the origin that passes through (x^*, y) . We say that v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) if the following hold:

1. $0 < a_1 < R_1$,
2. $\theta(v_2) \leq \theta(v'_1)$,
3. $R_2 \leq R_1$,
4. $x_2 \in [x^*, x'_1]$.

Figures 8 and 9 depict vectors v_1 and v_2 that are admissible with respect to (x_1, y) and (x_2, y) when $x_2 \geq 0$ and when $x_2 < 0$.

Proposition 7.2. If v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) , then H_1 is larger at (x_2, y) with respect to v_2 than at (x_1, y) with respect to v_1 , i.e.

$$\frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^\perp > \frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp.$$

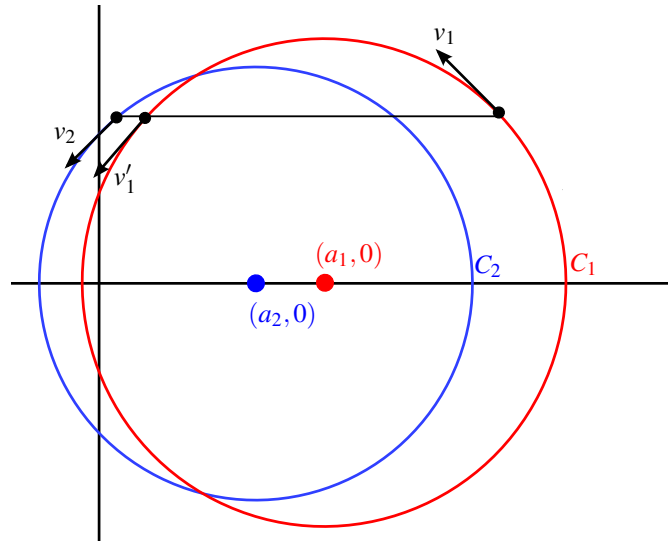


Figure 8: The vectors v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) with $x_2 \geq 0$.

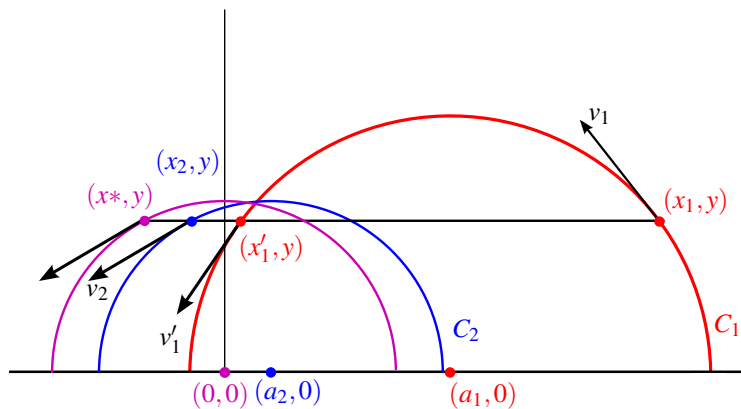


Figure 9: The vectors v_1 and v_2 are admissible with respect to (x_1, y) and (x_2, y) with $x_2 < 0$.

Proof. We take cases according to whether $x_2 \geq 0$ or $x_2 < 0$. In the case that $x_2 \geq 0$, $|(x_2, y)| \leq |(x'_1, y)|$, and the result follows by a similar argument to that in Proposition 6.2. In the case case that $x_2 < 0$, we will prove two inequalities:

$$\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^\perp < \frac{(x^*, y)}{|(x^*, y)|^2} \cdot v_2^\perp, \tag{7.1}$$

$$\frac{(x^*, y)}{|(x^*, y)|^2} \cdot v_2^\perp \leq \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_1^\perp. \tag{7.2}$$

Beginning with (7.1), note that since $a_1 > 0$, we must have that $|(x_1, y)| > R_1$. Additionally, $|(x^*, y)| = R_2$. Combining these observations with the inequality $R_1 \geq R_2$, we have $|(x_1, y)| > R_1 \geq R_2 = |(x^*, y)|$. It follows that

$$\frac{1}{|(x_1, y)|} < \frac{1}{|(x^*, y)|},$$

so proving (7.1) has been reduced to showing that

$$\frac{(x_1, y)}{|(x_1, y)|} \cdot v_1^\perp \leq \frac{(x^*, y)}{|(x^*, y)|} \cdot v_2^\perp \neq 0.$$

This inequality is immediate when we recognize that

$$\frac{(x^*, y)}{|(x^*, y)|} \cdot v_2^\perp = \cos(\theta(v_2^\perp) - \theta((x^*, y))) = \cos(0) = 1.$$

To prove (7.2), we will rewrite the right side of the inequality using the subtraction identity for cosine. As noted above, $\theta(v_2^\perp) = \theta((x^*, y))$, so

$$\cos(\theta(v_2^\perp)) = \frac{x^*}{|(x^*, y)|} \quad \text{and} \quad \sin(\theta(v_2^\perp)) = \frac{y}{|(x^*, y)|}.$$

Hence, we have

$$\begin{aligned} \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^\perp &= \frac{1}{|(x_2, y)|} \cos(\theta(v_2^\perp) - \theta((x_2, y))) \\ &= \frac{1}{|(x_2, y)|} \left(\cos(\theta(v_2^\perp)) \cos(\theta((x_2, y))) + \sin(v_2^\perp) \sin(\theta((x_2, y))) \right) \\ &= \frac{1}{|(x_2, y)|} \left(\frac{x^*}{|(x^*, y)|} \frac{x_2}{|(x_2, y)|} + \frac{y}{|(x^*, y)|} \frac{y}{|(x_2, y)|} \right) \\ &= \frac{1}{|(x^*, y)|} \left(\frac{x^* x_2}{|(x_2, y)|^2} + \frac{y^2}{|(x_2, y)|^2} \right). \end{aligned}$$

By (4) in Definition 7.1 and the assumption that $x_2 < 0$, we have $x^* \leq x_2 < 0$. We multiply through by x_2 to obtain $x^* x_2 \geq x_2^2 > 0$. Substituting this into the above equation, we have

$$\frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^\perp \geq \frac{1}{|(x^*, y)|} \left(\frac{x_2^2 + y^2}{|(x_2, y)|^2} \right) = \frac{1}{|(x^*, y)|} = \frac{(x^*, y)}{|(x^*, y)|^2} \cdot v_2^\perp,$$

completing the second case. \square

Before we define the upper and lower curves, we require several lemmas. Propositions 7.9 and 7.10, which concern points where the unit tangent vector is in the second quadrant, are later used to check the conditions for admissibility. Meanwhile, we determine some properties that hold at points on the curve with positive first coordinates.

Lemma 7.3. Suppose that $s \in (0, \beta)$ and that $\gamma_1(s) \geq 0$. Then $\gamma_1'(s) < 0$.

Proof. Suppose for contradiction that $\gamma_1'(s) \geq 0$. If $\gamma'(s)$ were in the first quadrant and not equal to $(1, 0)$, this would violate the Tangent Restriction Lemma (Lemma 4.2). If $\gamma'(s) = (1, 0)$, then, by Lemma 6.17, $\kappa(s) > 0$, which implies by continuity of γ' that there exists $t > s$ so that $\gamma_1(t) > 0$, $\gamma_2(t) > 0$, and $\gamma'(t)$ is strictly in the first quadrant, producing the same contradiction to the Tangent Restriction Lemma. Thus, if $\gamma_1'(s) \geq 0$, then $\gamma'(s)$ must be in the fourth quadrant and not equal to $(1, 0)$. However, this also yields a contradiction, because, replacing η with s , we could then apply Lemmas 6.17 and 6.16 to achieve the same contradiction as in the right case. These lemmas would apply because $\gamma_1(s) \geq 0$. \square

Lemma 7.4. Suppose that $s \in (0, \beta)$ and that $\gamma_1(s) \geq 0$. Then $\gamma_1(t) > 0$ for all $t \in [0, s)$.

Proof. It is clear that $\gamma_1(0) > 0$. To prove the result on $(0, s)$, consider the set $C = \{t \in [0, s) : \gamma_1(u) > 0 \text{ for all } u \in [t, s)\}$. By Lemma 7.3, $\gamma_1'(s) < 0$, so there exists $\varepsilon > 0$ such that $(s - \varepsilon, s) \subset C$. Since C is nonempty and bounded below, it has a greatest lower bound. Let $c = \inf C$. It suffices to prove that $c = 0$. Suppose for contradiction that $c > 0$. By continuity of γ_1 , $\gamma_1(c) = 0$; if $\gamma_1(c)$ were positive, then we could extend C farther back, whereas if it were negative, then c would not be the greatest lower bound. By Lemma 7.3, $\gamma_1'(c) < 0$. It follows that there exists $\varepsilon' > 0$ such that $\gamma_1 < 0$ on $(c, c + \varepsilon')$, contradicting the fact that c is the greatest lower bound of C . \square

Now we consider the initial canonical circle C_0 . By spherical symmetry, $F(0) \geq 0$. It must actually be the case that $F(0) > 0$; otherwise, by Remark 4.7, γ would be a circle centered at the origin, contradicting the fact that balls centered at the origin are not stable ([9, Thm. 3.10]). Given this strict inequality, it follows by the computations in the proof of Lemma 6.3 that $\kappa''(0) < 0$.

A natural next step would be to extend the inequality $F(0) < R(0)$. We will eventually prove that $F(s) < R(s)$ for all s with $\gamma_1(s) \geq 0$ (Proposition 7.10). Since R' may alternate signs, this is slightly more complicated than merely reversing the inequalities in the proof of Lemma 6.6. To show that the sign of R' does not matter, we define an auxiliary function that keeps track of the discrepancy between F and R .

Definition 7.5. Define $G : (-\beta, \beta) \rightarrow \mathbb{R}$ by letting $G(s)$ be the e_1 -coordinate of the leftmost point on C_s ; i.e. $G(s) = F(s) - R(s)$. Likewise, for a fixed s , if α, \tilde{F} , and \tilde{R} are as in Definition 5.4, then we let $\tilde{G} = \tilde{F} - \tilde{R}$.

For a given s , we can compute the derivatives of \tilde{F} and \tilde{G} on the approximating circle A_s to prove the following lemma.

Lemma 7.6. (cf. [1, Lemma 5.3]) Let $s \in [0, \beta)$. If $\gamma_1(s) \geq 0$ and $\kappa(s) \leq \lambda(s)$, then $F'(s) \leq 0$ and $G'(s) \leq 0$.

Proof. We take three cases according to whether $\kappa(s) > 0$, $\kappa(s) = 0$, or $\kappa(s) < 0$. If $\kappa(s) = 0$, then A_s is the oriented line through $\gamma(s)$ that has direction vector $\gamma'(s)$. By Lemma 7.3, $\gamma_1'(s) < 0$. We parameterize A_s by $\alpha(t) = \gamma(s) + t\gamma'(s)$. Let $\tilde{F}(t)$ denote the e_1 -coordinate of the center of the canonical circle to A_s at $\alpha(t)$, and let $\tilde{R}(t)$ denote its radius. Then we can compute that

$$\tilde{F}'(t) = \frac{1}{\gamma_1'(s)} < 0 \quad \text{and} \quad \tilde{G}'(t) = \frac{1 + \gamma_2'(s)}{\gamma_1'(s)}.$$

Since γ is an arclength parameterization, the numerator of \tilde{G}' is necessarily nonnegative. Thus, $\tilde{G}' \leq 0$.

If $\kappa(s) \neq 0$, let (a, b) be the center of A_s , and let r be the radius. If $\kappa(s) > 0$, then $b \leq 0$, and we parameterize A_s as in (6.7). By (6.10) and (6.11), we have

$$\tilde{F}'(t) = \frac{b}{r} \csc^2\left(\frac{t}{r}\right)$$

and

$$\tilde{G}'(t) = \frac{b}{r} \csc^2\left(\frac{t}{r}\right) \left(1 + \cos\left(\frac{t}{r}\right)\right)$$

for all t with $\alpha_2(t) > 0$. Since $b \leq 0$, $\tilde{F}'(\tilde{s}) \leq 0$ and $\tilde{G}'(\tilde{s}) \leq 0$.

Finally, if $\kappa(s) < 0$, then $b > 0$. We now parameterize A_s by

$$\alpha(t) = \left(a + r \cos\left(\frac{t}{r}\right), b - r \sin\left(\frac{t}{r}\right)\right).$$

For a given t with $\alpha_2(t) > 0$, the line segment from $\alpha(t)$ to the center of \tilde{C}_t is in the direction of the outward unit normal vector to A_s at $\alpha(t)$, as shown in Figure 10. By similar computations to those in the proof of Lemma 6.6, we have that

$$\tilde{F}'(t) = -\frac{b}{r} \csc^2\left(\frac{t}{r}\right),$$

and

$$\tilde{G}'(t) = -\frac{b}{r} \csc^2\left(\frac{t}{r}\right) \left(1 - \cos\left(\frac{t}{r}\right)\right).$$

Since $b > 0$, we conclude that $\tilde{F}'(\tilde{s}) \leq 0$ and $\tilde{G}'(\tilde{s}) \leq 0$. □

Although we used both hypotheses of Lemma 7.6 in the proof, it is actually the case that the first hypothesis implies the second, as we prove below.

Lemma 7.7. Let $s \in [0, \beta)$. If $\gamma_1(s) \geq 0$, then $\kappa(s) \leq \lambda(s)$.

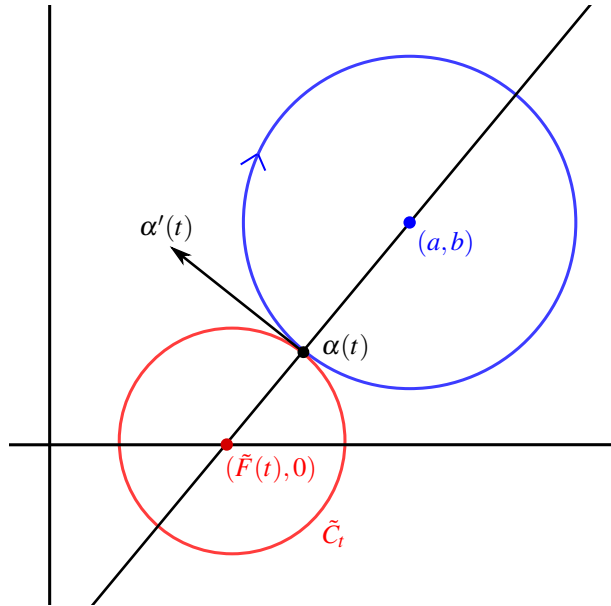


Figure 10: The canonical circle at $\alpha(t)$ in the case that A_s is oriented clockwise

Proof. Due to Lemma 4.8 and the fact that $\kappa'(0) = 0$, λ and κ are equal up to order two at 0. However, $\lambda''(0) = 0$, whereas $\kappa''(0) < 0$. Hence, there exists $t > 0$ so that $\kappa \leq \lambda$ on $[0, t]$. Let $S = \{t \in [0, \beta) : \kappa \leq \lambda \text{ and } \gamma_1 \geq 0 \text{ on } [0, t]\}$, and let $u = \sup S$. Since the inequalities that define S are not strict, it follows by smoothness of γ that $u \in S$. If $\gamma_1(u) = 0$, then, by Lemma 7.4, $\gamma_1(s) \geq 0$ only if $s \in [0, u]$. Thus, to prove that $\kappa(s) \leq \lambda(s)$ for all s with $\gamma_1(s) \geq 0$, it suffices to prove that $\gamma_1(u) = 0$.

Suppose for contradiction that $\gamma_1(u) > 0$. We will show that u is not an upper bound for S , but, instead, that there exists $\varepsilon > 0$ so that $[u, u + \varepsilon) \subset S$. We can obviously find $\varepsilon_1 > 0$ so that $\gamma_1 \geq 0$ on $[u, u + \varepsilon_1)$. It remains to show that there exists $\varepsilon_2 > 0$ so that $\kappa \leq \lambda$ on $[u, u + \varepsilon_2)$. The proof will be similar to that of Lemma 3.4 in [C].

First, we can prove by contradiction that $\kappa(u) = \lambda(u)$. Given this equation, we have that $C_u = A_u$, so $\lambda'(u) = \tilde{\lambda}'(\tilde{u}) = 0$. Thus, to guarantee the existence of a $\varepsilon_2 > 0$ so that $\kappa \leq \lambda$ on $[u, u + \varepsilon_2)$, it suffices to show that $\kappa'(u) < 0$. Since $\kappa \leq \lambda$ and $\gamma_1 \geq 0$ on $[0, u]$, it follows from Proposition 7.6 that $G' \leq 0$ on $[0, u]$. Therefore, $G(u) \leq G(0) < 0$ by assumption that $F(0) < R(0)$, and it follows by a similar argument to that in the proof of Lemma 6.7 that $\kappa'(u) < 0$. While the inequality $a > 0$ was immediate in the case that $a > r$, here it is more subtle. The fact that $a \geq 0$ follows from a similar argument to that in [1, Lemma 3.3]. To prove strict inequality, note that if $a = 0$, then γ is a circle centered at the origin, which contradicts the fact that balls centered at the origin are not stable ([9, Thm. 3.10]). □

We use Lemmas 7.6 and 7.7 to prove two propositions used in checking the conditions for admissibility (Props. 7.9 and 7.10), but first we require one additional lemma.

Lemma 7.8. Suppose that $s \in (0, \beta)$ and that $\gamma'(s)$ is in the second quadrant. Then $\gamma_1(s) > 0$.

Proof. By Lemma 4.2, $\gamma'(s) \neq (0, 1)$. Thus, $\gamma_2'(s) \geq 0$ and $\gamma_1'(s) < 0$. If $\gamma_1(s) < 0$ or s satisfies both $\gamma_1(s) = 0$ and $\gamma_2'(s) > 0$, then we can obtain a contradiction to Lemma 4.2. It remains to cover the case in which $\gamma_1(s) = 0$ and $\gamma'(s) = (-1, 0)$. By Proposition 7.7, $\kappa(s) \leq \lambda(s)$. If $\kappa(s) = \lambda(s)$, then γ is a circle centered at the origin, contradicting the fact that centered balls are not stable ([9, Thm. 3.10]). Now, suppose that $\kappa(s) < \lambda(s)$. Since $|\gamma(t)|$ is a non-increasing function of t and C_s is centered at the origin, $\gamma(t)$ must be contained in C_s for $t \geq s$. However, since $\kappa(s) < \lambda(s)$, the curve locally leaves the disk bounded by C_s . □

Proposition 7.9. Let $s \in [0, \beta)$. If $\gamma'(s)$ is in the second quadrant, then $F(s) > 0$.

Proof. We know that γ must eventually curve down and arrive at the e_1 -axis. Thus, there are points where γ' is in the third or fourth quadrant, and, by the Intermediate Value Theorem, combined with the fact that $\gamma' \neq (0, 1)$ on $(0, \beta)$ (Lemma 4.2), there is a point $t \geq s$ such that $\gamma'(t) = (-1, 0)$. By Lemma 7.8, $\gamma_1(t) > 0$; moreover, by Lemma 7.4, $\gamma_1 > 0$ on the interval $[s, t]$. Therefore, $F' \leq 0$ on $[s, t]$, from which it follows that $F(s) \geq F(t) = \gamma_1(t) > 0$. □

Proposition 7.10. If $\gamma'(s)$ is in the second quadrant, then $F(s) < R(s)$.

Proof. Since $\gamma'(s)$ is in the second quadrant, $\gamma_1(s) > 0$. In fact, for all $t \in [0, s]$, $\gamma_1(t) > 0$, so $\kappa \leq \lambda$ on $[0, s]$. Consequently, by Lemma 7.6, $G' \leq 0$ on $[0, s]$. By hypothesis that $F(0) < \gamma_1(0)/2$, we have that $G(0) < 0$. Therefore, $G(s) \leq G(0) < 0$. □

Having proved the propositions necessary for checking the conditions of admissibility, we define the upper and lower curves and prove that the curvature at a point on the lower curve is less than the curvature at its counterpart on the upper curve (Prop. 7.16).

Definition 7.11. A point s is in the upper curve $K \subset (0, \beta)$ if and only if for all $t \in (0, s)$, $\gamma'(t)$ is strictly in the second quadrant.

Note that K is nonempty because $\gamma'(0) = (0, 1)$ and $\kappa(0) > 0$ (both consequences of spherical symmetry) and because κ is continuous at 0. Thus, K has a least upper bound δ . Since γ' is strictly in the second quadrant on $(0, \delta)$, $\gamma_2(\delta) > 0$, so $\delta < \beta$, from which it follows that γ is smooth at δ . In particular, γ' is continuous at δ . We apply the Intermediate Value Theorem, along with Lemma 7.3, to conclude that $\gamma'(\delta) = (-1, 0)$.

Definition 7.12. We define the lower curve $L \subset [\delta, \beta)$ as follows: $s \in L$ if and only if for all $t \in [\delta, s]$, the following hold:

1. $\gamma'(t)$ is in the third quadrant, with $\gamma'(t) \neq (-1, 0)$ if $t > \delta$,
2. If \bar{t} is the unique point in K so that $\gamma_2(\bar{t}) = \gamma_2(t)$, then $\kappa(t) \leq \kappa(\bar{t})$.

Since $\delta \in L$, L is nonempty and therefore has a supremum, which we denote by η .

By Proposition 7.22, δ is the only point in $[0, \beta)$ at which the tangent vector is $(-1, 0)$. We can use this fact to prove that $\eta > \delta$. In addition to Proposition 7.22, our proof that $\eta > \delta$ utilizes the following lemma, which shows that at any point on γ where the tangent vector is $(-1, 0)$ and the curvature is 0, the curvature has a negative derivative.

Proposition 7.13. Let $s \in (0, \beta)$, and suppose that $\gamma'(s) = (-1, 0)$. If $\kappa(s) \geq 0$, then $\kappa'(s) < 0$.

Proof. In the case that $\kappa(s) > 0$, the result follows by a similar argument to the proof of Lemma 6.9. Now, suppose that $\kappa(s) = 0$. The osculating circle to γ at $\gamma(s)$ is an oriented horizontal line which we parameterize by $\alpha(t) = \gamma(s) + t(-1, 0)$. For each t , $\tilde{R}(t) = \gamma_2(s)$, so $\tilde{\lambda}(t)$ is constant; in particular, $\tilde{\lambda}'(s) = \tilde{\lambda}'(0) = 0$. Meanwhile, for all t ,

$$\tilde{H}_1(t) = \frac{p}{|\alpha(t)|^2} \gamma_2(s) \tag{7.3}$$

By Lemma 7.8, $\gamma_1(s) > 0$. Differentiating (7.3), we have

$$H_1'(s) = \tilde{H}_1'(0) = -2 \frac{p \gamma_2(s)}{|\alpha(0)|^4} (-\gamma_1(s)) > 0.$$

Thus, $\kappa'(s) < 0$. □

Lemma 7.14. Given that $\gamma([0, \beta))$ has tangent vector $(-1, 0)$ only at δ , we have $\eta > \delta$.

Proof. It suffices to prove that there exist $\varepsilon_1, \varepsilon_2 > 0$ so that for all $s \in (\delta, \delta + \varepsilon_1)$, $\gamma'(s)$ is in the third quadrant with $\gamma'(s) \neq (-1, 0)$, and for all $s \in [\delta, \delta + \varepsilon_2)$, $\kappa(s) \leq \kappa(\bar{s})$. For the existence of such an ε_2 , we observe that since $\gamma'(s)$ is in the second quadrant for all $s \in [0, \delta]$, $\kappa(\delta) \geq 0$. Therefore, by Proposition 7.13, $\kappa'(\delta) < 0$. To prove that there is an $\varepsilon_1 > 0$ as described, it suffices to prove the strict inequality $\kappa(\delta) > 0$. By Proposition 7.13, if $\kappa(\delta) = 0$, then $\kappa'(\delta) < 0$. Hence, there exists $q \in (\delta, \beta)$ with $\gamma_2(q) > \gamma_2(\delta)$. By the Intermediate Value Theorem, applied to γ_2 on $[0, \beta]$, there is a later point at the same height as $\gamma(\delta)$. Since $\gamma' \neq (0, 1)$ on $(0, \beta)$ this implies the existence of $q' \geq q$ with $\gamma'(q') = (-1, 0)$, a contradiction. \square

Proposition 7.15. Given that $\gamma([0, \beta])$ has tangent vector $(-1, 0)$ only at δ , let $s \in L$ with $s > \delta$, and let \bar{s} be the unique point in K so that $\gamma_2(\bar{s}) = \gamma_2(s)$. Then the following two inequalities hold:

$$\gamma_1(\bar{s}) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(s), \tag{7.4}$$

$$\theta(\gamma'(s)) \leq 2\pi - \theta(\gamma'(\bar{s})). \tag{7.5}$$

Proof. By Definitions 7.11 and 7.12, γ_2' does not vanish on $(0, \delta)$ or on (δ, η) . Thus, we define k, h as in the proof of Lemma 6.13. However, to apply Proposition 5.6, we must define f and g in a different way than we defined them in Definition 6.12: now we define $f, g : (\gamma_2(\eta), \gamma_2(\delta)) \rightarrow \mathbb{R}$ by $f(y) = \gamma_1(k(y))$ and $g(y) = 2\gamma_1(\delta) - \gamma_1(h(y))$. \square

Proposition 7.16. Given that $\gamma([0, \beta])$ has tangent vector $(-1, 0)$ only at δ , let $s \in L$ with $s > \delta$ and $\gamma'(s) \neq (0, -1)$. Letting \bar{s} be the unique point in K so that $\gamma_2(s) = \gamma_2(\bar{s})$, we have $\kappa(s) < \kappa(\bar{s})$.

Proof. Since generalized mean curvature is constant on γ , we have $\kappa(s) + (n - 2)\lambda(s) + H_1(s) = \kappa(\bar{s}) + (n - 2)\lambda(\bar{s}) + H_1(\bar{s})$. By (7.5) and right triangle trigonometry, $\lambda(s) \geq \lambda(\bar{s})$. Therefore, it suffices by Proposition 7.2 to prove that $\gamma'(\bar{s})$ and $\gamma'(s)$ are admissible with respect to $\gamma(\bar{s})$ and $\gamma(s)$. Letting $(x_1, y) = \gamma(\bar{s})$, $(x_2, y) = \gamma(s)$, $v_1 = \gamma'(\bar{s})$, and $v_2 = \gamma'(s)$, we proceed to check each condition in the definition of admissibility.

Condition (1) follows from Proposition 7.9 and from Proposition 7.10. Recognizing that $\theta(v_1) = 2\pi - \theta(v_2)$, we can derive condition (2) from the second inequality in Proposition 7.15. Condition (3) follows by inverting the inequality $\lambda(s) \geq \lambda(\bar{s})$.

To verify that condition (4) holds, we must show that $x_2 \leq x_1'$ and that $x_2 \geq x^*$. The first inequality can be proved using the inequality $\gamma_1(\bar{s}) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(s)$ along with the fact that $F' \leq 0$ on $(0, \delta)$ (which is a consequence of Lemmas 7.6 and 7.7). To prove that $x_2 \geq x^*$, note that since $\gamma'(s)$ is tangent to the circle centered at the origin that passes through (x^*, y) , we have $0 = (x^*, y) \cdot \gamma'(s) = x^* \gamma_1'(s) + y \gamma_2'(s)$. Meanwhile, by Lemma 4.2, $0 \geq \gamma(s) \cdot \gamma'(s) = \gamma_1(s) \gamma_1'(s) + \gamma_2(s) \gamma_2'(s)$. Since $y = \gamma_2(s)$, it follows that $x^* \gamma_1'(s) \geq \gamma_1(s) \gamma_1'(s)$. Dividing through by $\gamma_1'(s)$ gives $x^* \leq \gamma_1(s)$. \square

Proposition 7.17. If there exists no $s \neq \delta$ so that $\gamma'(s) = (-1, 0)$, then $\eta = \beta$.

Proof. Suppose for contradiction that $\eta < \beta$. We will show that η is not an upper bound for L , but instead, that L can be extended. Recall that by defining a local inverse function $h : (\gamma_2(\eta), \gamma_2(\delta)) \rightarrow (0, \delta)$ as on p. 13, we can explicitly write $\bar{s} = h(\gamma_2(s))$. By continuity of γ' , of κ , and of $\kappa \circ h \circ \gamma_2$, $\gamma'(\eta)$ is in the third quadrant, and $\kappa(\eta) \leq \kappa(\bar{\eta})$. To show that $\eta \in L$, we need only show that $\gamma'(\eta) \neq (-1, 0)$. If $\gamma'(\eta) = (-1, 0)$, this would contradict the fact that γ does not have multiple horizontal tangents. Meanwhile, if $\gamma'(\eta) = (0, -1)$, this would contradict (7.5), which holds at η by continuity of γ on $(0, \beta)$ and by our assumption that $\eta < \beta$. Thus, $\gamma'(\eta)$ is strictly in the third quadrant. Finally, by an identical argument to that in Proposition 7.16, $\kappa(\eta) < \kappa(\bar{\eta})$. \square

Having shown that $\eta = \beta$, we are near to proving Lemma 4.10 with the assumption that δ is the only point at which γ' equals $(-1, 0)$. First, we show that $\gamma_1(\beta) < 0$. In order to discuss $\lim_{s \rightarrow \beta} \gamma'(s)$, we must first show

that the limit exists. For this, we prove in Proposition 7.20 that κ is eventually negative. The proof requires Proposition 7.18 as well as a lemma giving a bound on γ_1' (Lemma 7.19).

Proposition 7.18. Given that $F(0) < 1/2$ and that $\gamma([0, \beta])$ has tangent vector $(-1, 0)$ only at δ , we have that $\gamma_1(\beta) < 0$.

Proof. To prove that $\gamma_1(\beta) < 0$, we begin with Proposition 7.15, which states that if $s \in L$, and \bar{s} is the corresponding point in K such that $\gamma_2(s) = \gamma_2(\bar{s})$, then $\gamma_1(\bar{s}) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(s)$. Since $\beta = \sup L$ and γ is continuous at β , this inequality also holds for $s = \beta$. Noting that $\bar{\beta} = 0$, we have $\gamma_1(0) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(\beta)$. Since $\gamma'(\delta) = (-1, 0)$, $\gamma_1(\delta) = F(\delta)$. In turn, since F is non-increasing on the upper curve, $F(\delta) \leq F(0)$. Consequently, $\gamma_1(0) - F(0) \leq \gamma_1(0) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(\beta) \leq F(0) - \gamma_1(\beta)$. Rearranging gives $\gamma_1(\beta) \leq 2F(0) - \gamma_1(0) < 0$. □

Lemma 7.19. Given that $\gamma([0, \beta])$ has tangent vector $(-1, 0)$ only at δ , there exists $\xi > 0$ so that $\gamma_1'(s) \leq -\xi$ for all $s \in L$.

Proof. It suffices to prove that there exists $\tau > 0$ such that $\theta(\gamma'(s)) < 3\pi/2 - \tau$ for all s in L . Since $\gamma'(\delta) = (-1, 0)$ and γ' is continuous on $[0, \beta]$, there exists $s_0 > \delta$ such that $\theta \circ \gamma' < 5\pi/4$ on $[\delta, s_0]$. By Proposition 7.16, $\kappa(s_0) < \kappa(\bar{s}_0)$. Letting $y_0 = \gamma_2(s_0)$, we have that the upward curvatures of the graphs of the functions f and g defined in Proposition 7.15 satisfy $\kappa_f(y_0) < \kappa_g(y_0)$. Therefore, by Proposition 5.6, there exists $\phi > 0$ such that

$$\theta(t_f(y)) \geq \theta(t_g(y)) + \phi \tag{7.6}$$

for all $y \in (0, y_0)$. Take $\tau = \min\{\phi, \pi/4\}$. If $s \in [\delta, s_0]$, then $\theta(\gamma'(s)) < 5\pi/4 = 3\pi/2 - \pi/4 \leq 3\pi/2 - \tau$. If $s \in (s_0, \beta)$, let $y = \gamma_2(s)$. Then $\theta(t_f(y)) = 3\pi/2 - \theta(\gamma'(s))$, and $\theta(t_g(y)) = \theta(\gamma'(\bar{s})) - \pi/2$. Substituting into (7.6), we obtain $\theta(\gamma'(s)) \leq 2\pi - \theta(\gamma'(\bar{s})) - \phi$. Finally, since $y > 0$, it follows by Lemma 7.3 that $\theta(\gamma'(\bar{s})) > \pi/2$. Therefore, $\theta(\gamma'(s)) < 3\pi/2 - \phi \leq 3\pi/2 - \tau$. □

Proposition 7.20. Given that $\gamma([0, \beta])$ has tangent vector $(-1, 0)$ only at δ , there exists $\varepsilon > 0$ such that $\kappa < 0$ on $(\beta - \varepsilon, \beta)$.

Proof. We show that for s close to β , we can make $(n - 2)\lambda(s) + H_1(s)$ larger than c , the constant of the differential equation $H_f = c$. First, we show that by taking s sufficiently close to β , we can make $\lambda(s)$ large. The radius of the canonical circle at $\gamma(s)$ satisfies

$$R(s)^2 = (\gamma_1(s) - F(s))^2 + \gamma_2(s)^2 = \left(\gamma_1(s) - \frac{\gamma(s) \cdot \gamma'(s)}{\gamma_1'(s)} \right)^2 + \gamma_2(s)^2 = \frac{\gamma_2(s)^2}{\gamma_1'(s)^2}.$$

Since $\gamma_1'(s) < 0$, we have $R(s) = \frac{\gamma_2(s)}{-\gamma_1'(s)}$ and $\lambda(s) = \frac{-\gamma_1'(s)}{\gamma_2(s)}$.

By Lemma 7.19, $\lambda(s) \geq \xi/\gamma_2(s)$. Since $\gamma_2(\beta) = 0$ and γ is continuous, there exists $\varepsilon_1 > 0$ so that if $s \in (\beta - \varepsilon_1, \beta)$, then $\gamma_2(s) < \xi/c$. Consequently, for all $s \in (\beta - \varepsilon_1, \beta)$, $\lambda(s) > c$.

Now we will show that for s sufficiently large, $H_1(s)$ is positive. By Proposition 7.18 and continuity of γ , there exists $\varepsilon_2 > 0$ such that $\gamma_1 < 0$ on $(\beta - \varepsilon_2, \beta)$. For any s in this interval, $\gamma(s)$ and $v(s)$ are both strictly in the second quadrant, so $H_1(s) > 0$.

Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, and suppose that $s \in (\beta - \varepsilon, \beta)$. By our observations above and our assumption that $n \geq 3$, we have $(n - 2)\lambda(s) + H_1(s) > (n - 2)\lambda(s) \geq \lambda(s) > c$. Therefore, $\kappa(s)$ must be less than 0 to compensate. □

Proof of the Left Tangent Lemma (Lemma 4.10). By Proposition 7.18, $\gamma_1(\beta) < 0$. By Proposition 7.20, there exists $\varepsilon > 0$ such that $\kappa < 0$ on $(\beta - \varepsilon, \beta)$. On this interval, $\theta \circ \gamma'$ is a decreasing function of s . Since $\theta \circ \gamma'$ is decreasing and bounded below by π , $\lim_{s \rightarrow \beta^-} \gamma'(s)$ exists. Moreover, since γ' is strictly in the third quadrant on (δ, η) and $\theta \circ \gamma'$ is decreasing on $(\beta - \varepsilon, \beta)$, $\lim_{s \rightarrow \beta^-} \gamma'(s)$ is in the third quadrant but not equal to $(0, -1)$. □

7.1 Proof That There is Only One Horizontal Tangent

Finally, we supply a proof of the result used from Proposition 7.14 onward that δ is the only point in $[0, \beta]$ with tangent vector $(-1, 0)$. It is expedient to consider the sets $T = \{s \in [0, \beta] : \gamma'(s) = (-1, 0) \text{ and } \kappa(s) \geq 0\}$ and $U = \{s \in [0, \beta] : \gamma'(s) = (-1, 0)\}$. Consider the supremum δ of the upper curve K (Defn. 7.11). In the proof of Lemma 7.14, by assuming that δ was the only point in $[0, \beta]$ where the tangent vector was $(-1, 0)$ (the fact that we are about to prove), we could show that $\kappa(\delta) > 0$. However, even without this assumption, it must be the case that $\kappa(\delta) \geq 0$, because γ' is strictly in the second quadrant on $(0, \delta)$ (cf. proof of Lemma 7.14). Thus, $\delta \in T$. Since T is nonempty, it has a least upper bound v .

Lemma 7.21. The supremum of T satisfies the following:

1. $v < \beta$,
2. v is the maximum element of U ,
3. $\kappa(v) > 0$.

Proof. To prove that $v < \beta$, it suffices to show that there exists $\varepsilon_0 > 0$ so that if $s \in (\beta - \varepsilon_0, \beta)$ and $\gamma'(s) = (-1, 0)$, then $\kappa(s) < 0$. To achieve this result, we consider the ODE $H_f = c$. We know that the constant c is positive, because $H_1(0) = p$, $\kappa(0) > 0$, and $\lambda(0) = \kappa(0)$ by Proposition 4.8.

Since $\gamma_2(\beta) = 0$ and the curve is continuous at β , there exists $\varepsilon_0 > 0$ so that for any s in $(\beta - \varepsilon_0, \beta)$, we have $\gamma_2(s) < 1/c$. Let $s \in (\beta - \varepsilon_0, \beta)$ and suppose that $\gamma'(s) = (-1, 0)$. Then $\lambda(s) = 1/\gamma_2(s) > c$. Meanwhile, the outward unit normal at s is $\nu(s) = (0, 1)$, so

$$H_1(s) = \frac{p}{|\gamma(s)|^2} (\gamma_1(s), \gamma_2(s)) \cdot (0, 1) = \frac{p}{|\gamma(s)|^2} \gamma_2(s) > 0.$$

Given that $n \geq 3$, we have that $(n - 2)\lambda(s) + H_1(s) \geq \lambda(s) + H_1(s) > c$, which means that $\kappa(s)$ must be negative to compensate.

Given that $v < \beta$, it can be shown by continuity of γ' and κ on $(0, \beta)$ that $\gamma'(v) = (-1, 0)$ and that $\kappa(v) \geq 0$. Since $\gamma'(v) = (-1, 0)$, $v \in U = \{s \in [0, \beta] : \gamma'(s) = (-1, 0)\}$. We claim that v is the largest point in U . By definition of T , there exists no $s > v$ so that $\gamma'(s) = (-1, 0)$ and $\kappa(s) \geq 0$. Meanwhile, if there were an $s > v$ so that $\gamma'(s) = (-1, 0)$ and $\kappa(s) < 0$, then s would be a local minimum point of γ_2 . Since $\gamma_2(\beta) = 0$, there would exist $t > s$ so that t was a local maximum point of γ_2 . Since $\gamma' \neq (0, 1)$ on $(0, \beta)$ (Lemma 4.2), $\gamma'(t) \neq (1, 0)$. Thus, it must be the case that $\gamma'(t) = (-1, 0)$, contradicting the fact that $v = \sup T$. We conclude that v is the maximum element of U . Again, since $\gamma' \neq (0, 1)$ on $(0, \beta)$, this means that $\gamma_2' < 0$ on (v, β) .

Finally, to prove that $\kappa(v) > 0$, suppose for contradiction that $\kappa(v) = 0$. By Lemma 7.13, there exists $\varepsilon > 0$ so that $\kappa < 0$ on $(v, v + \varepsilon)$. Since $\gamma'(v) = (-1, 0)$, this would imply the existence of an interval following v on which the tangent vector was strictly in the second quadrant, contradicting the fact that $\gamma_2' < 0$ on (v, β) (cf. proof of Lemma 7.14). Thus, $\kappa(v) > 0$. \square

Proposition 7.22. There is only one point $\delta \in [0, \beta]$ so that $\gamma'(\delta) = (-1, 0)$.

Proof. Suppose for contradiction that $U - \{v\}$ is nonempty. Since $\gamma'(v) = (-1, 0)$ and $\kappa(v) > 0$, there exists $\varepsilon > 0$ so that γ' is strictly in the second quadrant on $(v - \varepsilon, v)$ and γ' is strictly in the third quadrant on $(v, v + \varepsilon)$. Since γ' is strictly in the second quadrant on $(v - \varepsilon, v)$, $U - \{v\} = \{s \in [0, v - \varepsilon] : \gamma'(s) = (-1, 0)\}$; that is, $U - \{v\}$ is a level set of the restriction of γ' to $[0, v - \varepsilon]$. As such, $U - \{v\}$ is closed in $[0, v - \varepsilon]$, which means that $U - \{v\}$ is a compact subset of \mathbb{R} and has a maximum element u .

We claim that $\gamma'(s)$ is strictly in the second quadrant for all $s \in (u, v)$. To prove so, suppose for contradiction that there exists $s \in (u, v)$ so that $\gamma'(s)$ is not strictly in the second quadrant. By Lemma 7.4, $\gamma_1(s) > 0$. Hence, we apply Lemma 7.3 to conclude that $\gamma'(s)$ is in the third quadrant. Since $\gamma_1' < 0$ on $(0, \delta]$ (Lemma 7.3) and γ' is strictly in the second quadrant on $(v - \varepsilon, v)$, there exists $t \in [s, v)$ so that $\gamma'(t) = (-1, 0)$, contradicting maximality of u in $U - \{v\}$.

We define w to be the unique point in (v, β) so that $\gamma_2(w) = \gamma_2(u)$. We will ultimately achieve a con-

tradiction by showing that $\gamma'(w) = (-1, 0)$. In turn, we will accomplish this by curvature comparison. Let $s \in (v, w)$, and let \bar{s} be the unique point in (u, v) so that $\gamma_2(\bar{s}) = \gamma_2(s)$. We claim that $\kappa(s) \leq \kappa(\bar{s})$. Since $\kappa'(v) < 0$ (Lemma 7.13), we already know that this inequality holds for all s sufficiently close to v . Additionally, recall that there exists $\varepsilon > 0$ so that $\gamma'(s)$ is strictly in the third quadrant for all s in $(v, v + \varepsilon)$. We will prove that $\gamma'(s)$ is strictly in the third quadrant for all $s \in (v, w)$.

Let $W = \{s \in (v, w) : \gamma'(t) \text{ is strictly in the third quadrant and } \kappa(t) \leq \kappa(\bar{t}) \text{ for all } t \text{ in } (v, s]\}$. Since W is nonempty and bounded above, W has a supremum, which we shall denote by z . As in Proposition 7.15, the following inequalities hold for all s in (v, z) :

$$\gamma_1(\bar{s}) - \gamma_1(v) \leq \gamma_1(v) - \gamma_1(s), \quad (7.7)$$

$$\theta(\gamma'(s)) \leq 2\pi - \theta(\gamma'(\bar{s})). \quad (7.8)$$

It can also be proved that $\lambda(s) \geq \lambda(\bar{s})$ for all s in (v, z) . By continuity of all relevant quantities on $(0, \beta)$, it follows that these inequalities hold at z as well.

Finally, since γ' is strictly in the second quadrant on (u, v) and strictly in the third quadrant on W , it can be proved by a similar argument to that in Proposition 7.16 that $w = z$. It follows that the inequalities (7.7) and (7.8) hold for all s in $(v, w]$. By (7.8), $\theta(\gamma'(w)) \leq 2\pi - \theta(\gamma'(\bar{w})) = 2\pi - \theta(\gamma'(u)) = \pi$. Since $\theta \circ \gamma' \in (\pi, 3\pi/2)$ on (v, w) , it must be the case that $\theta(\gamma'(w)) = \pi$. That is, $\gamma'(w) = (-1, 0)$, contradicting the fact that there exists no $s > v$ with $\gamma'(s) = (-1, 0)$. \square

8 Glossary of Notation

Throughout this section, we assume, as at the beginning of Section 3, that E is a spherically symmetric isoperimetric region, and that $A \subset \mathbb{R}^2$ is a spherically symmetric set that generates E when rotated about the e_1 -axis. We first summarize the meanings that we have assigned to characters of the Latin alphabet, then proceed through the characters of the Greek alphabet that are used in the article. Characters used only in Section 7.1 are excluded.

A_s	Given an $s \in (-\beta, \beta)$, A_s denotes the osculating circle to γ at $\gamma(s)$ (see Defn. 5.3).
C_s	Given an $s \in (-\beta, \beta)$, C_s denotes the canonical circle to γ at $\gamma(s)$, i.e. the unique oriented circle that is tangent to γ at $\gamma(s)$ and has its center on the e_1 -axis (see Defn. 4.3).
\tilde{C}_t	For a fixed s , let α be an arclength parameterization of A_s . Given t in the domain of α such that $\alpha_2(t) \neq 0$ or $\alpha'(t) = (0, \pm 1)$, \tilde{C}_t denotes the canonical circle to α at $\alpha(t)$, i.e. the unique oriented circle that is tangent to α at $\alpha(t)$ and has its center on the e_1 -axis (see Defn. 5.4).
F	Given $s \in (-\beta, \beta)$, $F(s)$ denotes the abscissa of the center of C_s (see Defn. 4.3).
\tilde{F}	For a fixed s , let α be an arclength parameterization of A_s . For any t such that \tilde{C}_t exists, $\tilde{F}(t)$ denotes the abscissa of the center of \tilde{C}_t (see Defn. 5.4).
G	We define the function G on $(-\beta, \beta)$ by $G(s) = F(s) - R(s)$ (see Defn. 7.5).
\tilde{G}	For a fixed s , let α be an arclength parameterization of A_s . For any t such that \tilde{C}_t exists, let $\tilde{G}(t) = \tilde{F}(t) - \tilde{R}(t)$ (see Defn. 7.5).

- H_0 Given a regular point $x \in \partial E$, $H_0(x)$ denotes the unaveraged mean curvature of ∂E at x (i.e. the sum of the principal curvatures of ∂E at x). After parameterizing (the rightmost component of) ∂A , we also consider H_0 as a function of arclength: given $s \in (-\beta, \beta)$, we let $H_0(s)$ denote the unaveraged mean curvature of ∂E at $\gamma(s)$ (see Defn. 4.1).
- H_1 Given a regular point $x \in \partial E$, $H_1(x)$ denotes the directional derivative of the log of the density function in the direction of the outward unit normal vector to ∂E at x . Meanwhile, given $s \in (-\beta, \beta)$, we let $H_1(s)$ denote the directional derivative of the log of the density function in the direction of the outward unit normal vector to ∂E at $\gamma(s)$ (see Defn. 4.1).
- \tilde{H}_1 For a fixed s , let α be an arclength parameterization of A_s . For each t in the domain of α , let
- $$\tilde{H}_1(t) = \frac{p}{|\alpha(t)|} \frac{\alpha(t)}{|\alpha(t)|} \cdot \nu(t),$$
- where $\nu(t)$ is the outward unit normal vector to α at $\alpha(t)$ (see Defn. 5.3).
- H_f Given a regular point $x \in \partial E$, H_f denotes the generalized mean curvature of ∂E at x . Given $s \in (-\beta, \beta)$, we let $H_f(s)$ denote the generalized mean curvature of ∂E at $\gamma(s)$ (see Defn. 4.1).
- h In both cases, h denotes a local inverse function for γ_2 with codomain $(0, \delta)$: if $y \in (\gamma_2(\eta), \gamma_2(\delta))$, then $h(y)$ is the unique $t \in (0, \delta)$ so that $\gamma_2(t) = y$ (see Defn. 6.11).
- K In both cases, K denotes the subset of $[0, \beta)$ that we call the upper curve. In the right case, the upper curve is defined as the set of $s \in [0, \beta)$ so that $\gamma'(t)$ lies in the second quadrant and $\kappa(t) \geq \lambda(t) > 0$ for all $t \in [0, s]$ (see Defn. 6.4). In the left case, the upper curve is defined as the set of $s \in [0, \beta)$ so that $\gamma'(t)$ is strictly in the second quadrant for all $t \in (0, s)$ (see Defn. 7.11).
- k In both cases, k denotes a local inverse function for γ_2 with codomain (δ, η) : if $y \in (\gamma_2(\eta), \gamma_2(\delta))$, then $k(y)$ is the unique $t \in (\delta, \eta)$ so that $\gamma_2(t) = y$ (see Defn. 6.11).
- L In both cases, L denotes the subinterval of $[0, \beta)$ that we call the lower curve. In each case, the definition of L is rather technical, so we refer the reader to Definition 6.10 in the right case (Section 6) and to Definition 7.12 in the left case (Section 7).
- R Given $s \in (-\beta, \beta)$, we let $R(s)$ denote the radius of C_s (see Defn. 4.3).
- \tilde{R} For a fixed s , let α be an arclength parameterization of A_s . For any t such that \tilde{C}_t exists, we let $\tilde{R}(t)$ denote the radius of \tilde{C}_t (see Defn. 5.4).
- \bar{s} In each case, if $s \in L$, we let \bar{s} denote the unique point in K so that $\gamma_2(\bar{s}) = \gamma_2(s)$ (see Prop. 6.13, Prop. 7.16).
- \tilde{s} For a fixed s , let α be an arclength parameterization of A_s . We let \tilde{s} denote the point in the domain of α such that $\alpha(\tilde{s}) = \gamma(s)$ (see Defn. 5.3).

α	For a fixed s , we let α denote an arclength parameterization of A_s (see Defn. 5.3).
$\pm\beta$	Endpoints of the domain of γ
γ	Denotes an arclength parameterization of a component of ∂A (which, in fact, turns out to be the only component of ∂A ; see the beginning of Section 3).
δ	In each case, δ denotes the supremum of the upper curve. (In the right case (Section 6), see Defn. 6.4 and following. In the left case (Section 7), see Defn. 7.11 and following.)
η	In each case, η denotes the supremum of the lower curve. (In the right case (Section 6), see Defn. 6.10. In the left case (Section 7), see Defn. 7.12.)
θ	We define $\theta : S^1 \rightarrow (0, 2\pi]$ by letting $\theta(v)$ be the angle in the specified interval that v makes with the positive e_1 -axis (see Defn. 5.5).
κ	Given $s \in (-\beta, \beta)$, $\kappa(s)$ denotes the signed curvature of γ at $\gamma(s)$.
$\tilde{\kappa}$	For a fixed s , let α be an arclength parameterization of A_s . For any t in the domain of α , we let $\tilde{\kappa}(t)$ denote the signed curvature of A_s at $\alpha(t)$ (see Defn. 5.3).
λ	Given $s \in (-\beta, \beta)$, $\lambda(s)$ denotes the signed curvature of C_s (see Defn. 4.3).
$\tilde{\lambda}$	For a fixed s , let α be an arclength parameterization of A_s . For any t such that \tilde{C}_t exists, $\tilde{\lambda}(t)$ denotes the signed curvature of \tilde{C}_t (see Defn. 5.4).

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