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ISOPERIMETRY AND HEAT KERNEL DECAY ON PERCOLATION CLUSTERS

BY PIERRE MATHIEU[1] AND ELISABETH REMY[2]

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We prove that the heat kernel on the infinite Bernoulli percolation cluster in $\mathbb{Z}^d$ almost surely decays faster than $t^{-d/2}$. We also derive estimates on the mixing time for the random walk confined to a finite box. Our approach is based on local isoperimetric inequalities. Some of the results of this paper were previously announced in the note of Mathieu and Remy [*C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001) 927–931].

1. Introduction. We deal separately with $2D$ site percolation and bond percolation in any dimension.

1.1. *Site percolation in* $2D$. Let ω be the random subgraph of $\mathbb{Z}^2$ obtained by keeping (resp. deleting) a point with probability p (resp. $1-p$), independently for different points of $\mathbb{Z}^2$. Call Q the law of ω. Points belonging to ω are called *open*. Two neighboring points of ω form an *open* edge. Let $\mathcal{C}$ denote the open cluster of the origin, that is, the connected component of ω containing 0, and define

$$p_c = \sup\{p;\ Q[\#\mathcal{C} = +\infty] = 0\}.$$

For a given choice of ω, we shall consider the usual random walk on $\mathcal{C}$, say $(X_t, t \ge 0)$: the random walker waits for an exponential time of parameter 1 and then chooses, uniformly at random, one of its neighbors in $\mathbb{Z}^2$, say y. If y belongs to $\mathcal{C}$ (the edge leading to y is open), then the walker moves to y; otherwise it stays still. Thus X_t defines a Markov chain on $\mathcal{C}$ which is reversible with respect to the counting measure on $\mathcal{C}$.

THEOREM 1.1. *For any $p > p_c$, there exists a constant $c_1 = c_1(p)$ such that Q a.s. on the set $\#\mathcal{C} = +\infty$, and for large enough t, we have*

$$\sup_{y\in\mathcal{C}} P_0^\omega[X_t = y] \le \frac{c_1}{t}. \tag{1}$$

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1.2. *Bond percolation in* $\mathbb{Z}^d$. Let ω be the random subgraph of $\mathbb{Z}^d$ obtained by keeping (resp. deleting) an edge with probability p (resp. $1-p$) independently for each bond. More precisely, for $x, y \in \mathbb{Z}^d$, we write: $x \sim y$ if x and y are neighbors in $\mathbb{Z}^d$, and $\mathbb{E}_d = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d, x \sim y\}$. We identify a subgraph of $\mathbb{Z}^d$ with an application $\omega : \mathbb{E}_d \to \{0, 1\}$, writing $\omega(x, y) = 1$ if the edge (x, y) is present in ω and writing $\omega(x, y) = 0$ otherwise. Edges in $\mathbb{E}_d$ will be called *open*. Let Q be the probability measure on $\{0, 1\}^{\mathbb{E}_d}$ under which the random variables $(\omega(e), e \in \mathbb{E}_d)$ are Bernouilli(p) independent variables. As before, let $p_c = \sup\{p; Q[\#\mathcal{C} = +\infty] = 0\}$ be the critical probability.

For a given subgraph ω, let $\mathcal{C}$ denote the open cluster at the origin, that is, $\mathcal{C}$ is the connected component of ω that contains 0. We still use the notation X_t to denote the random walk on $\mathcal{C}$: the random walker waits for an exponential time of parameter 1 and then chooses, uniformly at random, one of its neighbors in $\mathbb{Z}^d$, say y. If $\omega(x, y) = 1$, then the walker moves to y; otherwise it stays still. Thus X_t defines a Markov chain on $\mathcal{C}$ which is reversible with respect to the counting measure on $\mathcal{C}$. Let P_x^ω be the law of the chain $(X_t, t \geq 0)$ when started at point x.

THEOREM 1.2. *For any dimension* $d \geq 2$, *for any* $p > p_c$, *there exists a constant* $c_1 = c_1(p, d)$ *such that*, Q *a.s. on the set* $\#\mathcal{C} = +\infty$, *and for large enough* t, *we have*

$$\sup_{y \in \mathcal{C}} P_0^\omega[X_t = y] \leq \frac{c_1}{t^{d/2}}. \tag{2}$$

REMARKS. It is known that X_t satisfies a central limit theorem; see [5]. Our estimate (2) cannot be directly deduced from the C.L.T.

When $d \geq 3$, Theorem 1.2 implies that the walk is transient. This result was first proved in [7].

We comment a little on the lower bound for the kernel of X_t in Appendix D.

Heicklen and Hoffman [8] also obtained upper estimates for $P_0^\omega[X_t = y]$ using a different method than ours but they missed the correct limit behavior by a logarithmic factor.

1.3. *Isoperimetric inequalities.* It is known that the large time behavior of a Markov chain on a graph is related to the geometry at infinity of the graph. For instance, isoperimetric inequalities or Nash inequalities imply estimates on the heat kernel decay, that is, upper bounds on $\sup_{x,y} P_x[X_t = y]$ (cf. [4] or [12]).

We also use isoperimetric inequalities here, but, as we consider nonregular graphs such as percolation clusters, the classical approaches do not directly apply. In particular, since a percolation cluster contains, with probability one, an arbitrarily long linear piece, it is easy to see that Q a.s. as t tends to $+\infty$,

$$\sup_{x,y \in \mathcal{C}} P_x^\omega[X_t = y] = O(t^{-1/2}). \tag{3}$$

In the note [11], we sketched the proofs of Theorems 1.1 and 1.2 using local isoperimetric inequalities. Our proof here will be slightly different. Instead of estimating the decay of the kernel of the random walk X killed when leaving a big box, as in the note, we shall rather estimate the probability transitions of the random walk restricted to a finite box. The computation becomes a little heavier but we obtain estimates on the mixing time for the walk confined to a box that have their own interest.

Let us define $\mathcal{C}^n$ to be the connected component of $\mathcal{C} \cap [-n, n]^d$ that contains the origin. Note that, by this definition, a point is in $\mathcal{C}^n$ if and only if it belongs to $\mathcal{C}$ and can be reached from the origin by an open path contained in $[-n, n]^d$.

Let $\varepsilon > d$ and define the isoperimetric constant

$$I_\varepsilon(\mathcal{C}^n) = \inf_{A \subset \mathcal{C}^n, \#A \leq \#\mathcal{C}^n/2} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}},$$

where $\partial_{\mathcal{C}^n} A$ is the boundary of the set A in $\mathcal{C}^n$, that is, the set of nearest neighbor points $x \in \mathcal{C}^n$ and $y \in \mathcal{C}^n$ such that $\omega(x, y) = 1$ and with either $x \in A$ and $y \notin A$ or $x \notin A$ and $y \in A$. In Sections 2 and 3, we prove that under the assumptions of Theorem 1.1 or 1.2, for some constant β that depends only on p and d, Q a.s. on the set $\#\mathcal{C} = +\infty$, for large enough n, one has the inequality

$$I_{\varepsilon(n)}(\mathcal{C}^n) \geq \frac{\beta}{n^{1-d/\varepsilon(n)}}, \tag{4}$$

where $\varepsilon(n) = d + 2d \frac{\log\log n}{\log n}$.

1.4. *The random walk on a finite box.* Let $(X_t^n,\ t \geq 0)$ be the random walk X restricted to the set $\mathcal{C}^n$. The definition of X^n is the same as for X except that jumps outside $\mathcal{C}^n$ are now forbidden: the random walker waits for an exponential time of parameter 1 and then chooses, uniformly at random, one of its neighbors in $\mathbb{Z}^d$, say y. If $\omega(x, y) = 1$ and $y \in \mathcal{C}^n$, then the walker moves to y; otherwise it stays still. Thus X_t^n defines a Markov chain on $\mathcal{C}^n$ which is reversible with respect to the counting measure on $\mathcal{C}^n$.

It follows from general considerations on finite Markov chains (see [13]) that the isoperimetric inequality (4) yields different estimates on the kernel of X^n. Indeed (4) implies the Nash inequality: Under the assumptions of Theorem 1.1 or 1.2, there exists a constant β such that Q a.s. on the set $\#\mathcal{C} = +\infty$, $\exists\, n_0(\omega)$ and $\forall\, n \geq n_0(\omega)$,

$$\operatorname{Var}(g)^{1+2/\varepsilon(n)} \leq \frac{8}{\beta^2} n^{2(1-d/\varepsilon(n))} \mathcal{E}^n(g, g) \|g\|_1^{4/\varepsilon(n)}, \tag{5}$$

where $\mathcal{E}^n(\cdot, \cdot)$ is the Dirichlet form of the Markov chain X^n. The variance and the L_1 norms are computed with respect to the uniform probability measure on $\mathcal{C}^n$. Inequality (5) is a direct application of Theorem 3.3.11 of [13]. Besides (see

Theorem 2.3.1 of [13]), the Nash inequality (5) implies estimates on the transition probability:

$$\sup_{x,y\in\mathcal{C}^n}\left|\frac{1}{\#\mathcal{C}^n} - P_x^\omega[X_t^n = y]\right| \le \left(\frac{4\varepsilon(n)}{\beta^2}\right)^{\varepsilon(n)/2}\frac{n^{\varepsilon(n)-d}}{t^{\varepsilon(n)/2}}. \tag{6}$$

Another consequence of the isoperimetric inequality is a lower bound on the spectral gap: let λ^n be the lowest nonzero eigenvalue of the discrete Laplacian on $\mathcal{C}^n$. From Cheeger's inequality and the isoperimetric inequality (4), we deduce (see Lemma 3.3.7 of [13]):

THEOREM 1.3. *Under the assumptions of Theorem* 1.1 *or* 1.2, *there exists a constant β such that Q a.s. on the set $\#\mathcal{C} = +\infty$, for large enough n*

$$\lambda^n \ge \frac{\beta}{n^2}.$$

See also the paper of Benjamini and Mossel [2]. (Benjamini asked us to mention that there is a gap in the renormalization argument given in [2]. Compare with Sections 3.3 and 3.4 here.)

2. The isoperimetric inequality when $d = 2$. We consider the site percolation model in $\mathbb{Z}^2$, described in Section 1.1. For $x, y \in \mathbb{Z}^2$, we write: $x \sim y$ if x and y are neighbors, and $\mathbb{E} = \{(x, y) \in \mathcal{C} \times \mathcal{C}, x \sim y\}$.

2.1. *A preliminary result.* We consider the box $B_{m,n} \stackrel{\triangle}{=} ([0, m] \times [0, n]) \cap \mathbb{Z}^2$.

DEFINITION 2.1. A horizontal (resp. vertical) channel of $B_{m,n}$ (see Figure 1) is a path $(v_0, e_1, \dots, e_n, v_n)$, $v_i \in \mathbb{Z}^2$ and $e_i \in \mathbb{E}$ for all $0 \le i \le n$, such that:

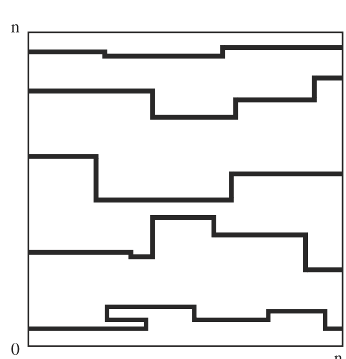

FIG. 1. *Example of horizontal channels.*

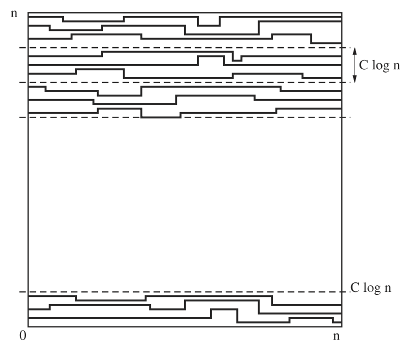

FIG. 2. *In each strip of width $C \log n$, we consider the Kesten's channels.*

- $(v_1, e_1, \dots, e_{n-1}, v_{n-1})$ is contained in the interior of $B_{m,n}$,
- $v_0 \in \{0\} \times [0, n]$ (resp. $v_0 \in [0, m] \times \{0\}$),
- $v_n \in \{m\} \times [0, n]$ (resp. $v_n \in [0, m] \times \{n\}$).

We say that two channels are *disjoint* if they do not have any vertex in common.

Let $N(m, n)$ be the maximal number of disjoint open horizontal channels in $B_{m,n}$.

THEOREM 2.2 ([9], Theorem 11.1). *Let $p > p_c$. For some constant $c(p) > 0$, and for some universal constants $0 < c_1, c_2, \delta < +\infty$, one has*

$$Q\big(N(m, n) \geq c(p)n\big) \geq 1 - c_1(m + 1) \exp\big(-c_2(p - p_c)^\delta n\big).$$

Construction of the Kesten grid. Theorem 2.2 gives us information about the "geometrical structure" of $\mathcal{C}^n$: The number of nonintersecting horizontal and vertical channels, which cross the box and belong to the infinite cluster, is proportional to the size of the box. They form a grid, that we call the *Kesten grid*. Let us construct and consider the following particular Kesten grid: we divide the box $[-n, n]^2$ in horizontal strips of width $C \log n$, with the constant C large enough so that the expression $c_1(n + 1) \exp(-c_2(p - p_c)^\delta C \log n)$ is summable (and therefore one may apply the Borel–Cantelli lemma). Then it follows from Theorem 2.2, applied in each strip, that there is a number proportional to $C \log n$ of horizontal channels. We do the same construction for vertical channels as well.

The Kesten grid so constructed is regularly spread all over the box $[-n, n]^2$ (cf. Figure 2).

2.2. *Proof of the isoperimetric inequality.* Choose $\varepsilon > 2$, which depends on n and satisfies the condition

$$\liminf_{n \to +\infty} (\varepsilon - 2) \frac{\log n}{\log \log n} \geq 4. \tag{7}$$

Condition (7) is in particular satisfied if $\varepsilon = 2 + 4\frac{\log\log n}{\log n}$. We want to prove the inequality (4) in the case $d = 2$, that is, there exists some constant β that depends only on p such that Q a.s. on the set $\#\mathcal{C} = +\infty$, for large enough n, one has the inequality

$$I_{\varepsilon(n)}(\mathcal{C}^n) \geq \frac{\beta}{n^{(\varepsilon-2)/\varepsilon}}.$$

We first remark that, without loss of generality, in the definition of the isoperimetric constant $I_\varepsilon(\mathcal{C}^n)$, we can take the infimum only on connected subsets of $\mathcal{C}^n$.

Let A be a finite connected subset of $\mathcal{C}^n$. We denote by N the cardinal of A ($N \stackrel{\triangle}{=} \#A$). We look for a lower bound on the cardinal of $\partial_{\mathcal{C}^n} A$. The following result will be very useful:

$\exists\, \alpha(p) > 0$ such that, Q a.s on the event $\#\mathcal{C} = +\infty$,

$$\alpha(p) \leq \liminf_{n\to\infty} \frac{\#\mathcal{C}^n}{n^2} \leq 1. \tag{8}$$

Indeed, the first inequality (left) is a consequence of Theorem 2.2, and the second one comes from $\#\mathcal{C}^n \leq n^2$.

We introduce several classes of sets A and derive a lower bound on $\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}}$ in each of them.

(i) Let $\mathcal{A}_0 = \{A \subset \mathcal{C}^n, A \text{ connected}, N \leq n^{(\varepsilon-2)/(\varepsilon-1)}\}$.

Obviously $\#\partial A \geq 1$, so we have

$$\min_{A\in\mathcal{A}_0} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{1}{N^{(\varepsilon-1)/\varepsilon}} \geq \frac{1}{n^{(\varepsilon-2)/\varepsilon}}. \tag{9}$$

(ii) Consider now sets A such that $n^{(\varepsilon-2)/(\varepsilon-1)} < N \leq \frac{\#\mathcal{C}^n}{2}$.

We consider the Kesten grid, constructed in Section 2.1. Two cases appear naturally: A contains either no horizontal channel or no vertical one, or A contains at least one horizontal and one vertical channel.

If R_A denotes the smallest rectangle which contains the set A, then R_A is strictly included in $[-n, n]^2$ in the first case, and $R_A = [-n, n]^2$ in the second one.

We call $\operatorname{diam}(A)$ the length of the longest side of R_A.

CASE 1 (*A contains either no horizontal channel or no vertical channel*). Let $\mathcal{A}_1 = \{A \subset \mathcal{C}^n, A \text{ connected}, n^{(\varepsilon-2)/(\varepsilon-1)} < N \leq \frac{\#\mathcal{C}^n}{2},\ \operatorname{diam}(A) < n\}$.

We know that every channel which intersects both the set A and its complementary $\mathcal{C}^n \setminus A$ contains at least one element of $\partial_{\mathcal{C}^n} A$.

Let us consider the N_A channels which intersect R_A [we consider the horizontal channels if $\operatorname{diam}(A)$ is the vertical side, the vertical channels otherwise]. Each

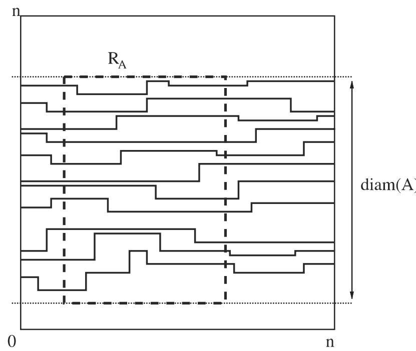

FIG. 3. *The number of horizontal channels which intersect R_A is proportional to* $\operatorname{diam}(A)$.

of these channels brings at least a contribution of 1 in the cardinal of A, so: $\#\partial_{\mathcal{C}^n} A \geq N_A$.

Thanks to condition (7),

$$\operatorname{diam}(A) \geq \sqrt{\#A} = \sqrt{N} \geq n^{(\varepsilon-2)/(2(\varepsilon-1))},$$

and this is larger than $C \log n$, for any constant C. Therefore Kesten's theorem can be applied, that is, $N_A \geq c_1(p) \operatorname{diam}(A)$ and $\#\partial_{\mathcal{C}^n} A \geq c_1(p) \operatorname{diam}(A) \geq c_1(p)\sqrt{N}$.

Thus, for any connected set $A \subset \mathcal{C}^n$, $n^{(\varepsilon-2)/(\varepsilon-1)} < N \leq \frac{\#\mathcal{C}^n}{2}$,

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} = \frac{\#(\partial_{\mathcal{C}^n} A)}{N^{(\varepsilon-1)/\varepsilon}} \geq \frac{c_1(p)\sqrt{N}}{N^{(\varepsilon-1)/\varepsilon}}.$$

This last quantity reaches its minimum for $N = \frac{\#\mathcal{C}^n}{2}$ and

$$\min_{A \in \mathcal{A}_1} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq 2^{(\varepsilon-2)/(2\varepsilon)} \frac{c_1(p)}{n^{(\varepsilon-2)/\varepsilon}}. \tag{10}$$

Let $\mathcal{A}_2 = \{A \subset \mathcal{C}^n,\ A \text{ connected}, n^{(\varepsilon-2)/(\varepsilon-1)} < N \leq \frac{\#\mathcal{C}^n}{2}, \operatorname{diam}(A) = n\}$.

We use the same reasoning as in the previous case, but here, all the horizontal (or vertical) channels of the Kesten grid intersect R_A, since $\operatorname{diam}(A) = n$ [and so, with Theorem 2.2, $N_A \geq c(p)n$]. Hence, $\#\partial_{\mathcal{C}^n} A \geq N_A \geq c(p)n$, and

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} = \frac{\#\partial_{\mathcal{C}^n} A}{N^{(\varepsilon-1)/\varepsilon}} \geq \frac{c(p)n}{N^{(\varepsilon-1)/\varepsilon}}.$$

The minimum is reached for $N = \frac{\#\mathcal{C}^n}{2}$:

$$\min_{A \in \mathcal{A}_2} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq 2^{(\varepsilon-1)/\varepsilon} \frac{c(p)}{n^{(\varepsilon-2)/\varepsilon}}. \tag{11}$$

CASE 2 (A contains at least one horizontal and one vertical channel). It implies that $R_A = [-n, n]^2$.

In this case, one has to take into account that Kesten's channels which are completely included in the set A do not contribute in $\#\partial_{\mathcal{C}^n} A$. Either there is, at least in one direction, a nonnegligeable proportion of Kesten's channels which are not completely included in A, or in both directions, almost all the channels are completely included in A. We consider separately these two possibilities.

The Kesten grid is constituted of $N(n) \geq c(p)n$ horizontal and vertical channels. We distinguish channels which are completely included in A (i.e., which have an empty intersection with $\mathcal{C}^n \setminus A$). Let

$$\delta_{\mathrm{H}} \stackrel{\triangle}{=} \frac{1}{n}\#\{\text{horizontal channels } \subset A\}, \qquad \delta_{\mathrm{V}} \stackrel{\triangle}{=} \frac{1}{n}\#\{\text{vertical channels } \subset A\}.$$

For some δ to be chosen later in the interval $]0, c(p)[$, we define

$$\mathcal{A}_3 = \Big\{A \subset \mathcal{C}^n, A \text{ connected}, n^{(\varepsilon-2)/(\varepsilon-1)} < N \leq \frac{\#\mathcal{C}^n}{2}, \\ R_A = [-n, n]^2, \delta_{\mathrm{H}} < \delta \text{ or } \delta_{\mathrm{V}} < \delta\Big\}.$$

At least in one direction, there are less than δn channels which are completely included in A. As the total number of channels in one direction is $c(p)n$, we have more than $(c(p) - \delta)n$ channels which are *not* completely included in A, and therefore intersect $\mathcal{C}^n \setminus A$. Thus, $\#\partial_{\mathcal{C}^n} A \geq (c(p) - \delta)n$, and

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{(c(p) - \delta)n}{N^{(\varepsilon-1)/\varepsilon}}.$$

The minimum is reached for $N = \frac{\#\mathcal{C}^n}{2}$:

$$\min_{A\in\mathcal{A}_3} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq 2^{(\varepsilon-1)/\varepsilon} \frac{(c(p) - \delta)}{n^{(\varepsilon-2)/\varepsilon}}. \tag{12}$$

Let

$$\mathcal{A}_4 = \Big\{A \subset \mathcal{C}^n, A \text{ connected}, n^{(\varepsilon-2)/(\varepsilon-1)} < N \leq \frac{\#\mathcal{C}^n}{2}, \\ R_A = [-n, n]^2, \delta_{\mathrm{H}} > \delta \text{ and } \delta_{\mathrm{V}} > \delta\Big\}.$$

Let us consider the subgrid of the Kesten grid, constituted only with the channels included in A. These channels divide $[-n, n]^2$ in different nonoverlapping regions, which will be called *domains*.

If we enumerate the horizontal channels of the subgrid by $H_1, H_2, \ldots$, and the vertical ones by $V_1, V_2, \ldots$, the domain $D_{i,j}$ is constituted of all the points of $\mathcal{C}^n$

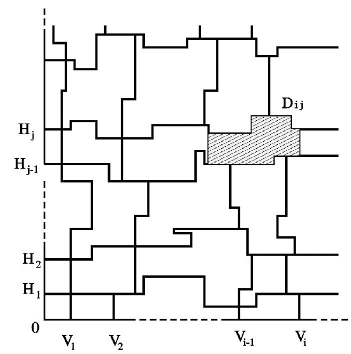

FIG. 4. *The domain $D_{i,j}$ is included in the gray part.*

included (strictly) between V_{i-1}, V_i, H_{j-1} and H_j (with: $H_0 = [0, n] \times 0$ and $V_0 = 0 \times [0, n]$). See Figure 4.

Each of these domains which intersects $\mathcal{C}^n \setminus A$ increases by one unit the cardinal of $\#\partial_{\mathcal{C}^n} A$. Thus, one way to find a lower bound for $\#\partial_{\mathcal{C}^n} A$ is to count the number of such domains.

We classify domains into two categories: *big* and *small* ones. To do so, we need to define the *distance between two channels*. [Channels are irregular, but thanks to the construction of the channels in strips (cf. Figure 2), their fluctuations are not greater than $C \log n$. See Figure 5.]

DEFINITION 2.3. Let C_1 and C_2 be two channels in the box $[-n, n]^2$ (both in the same direction). We define the distance between C_1 and C_2 by:

$$d(C_1, C_2) \stackrel{\triangle}{=} \inf\{d(x, y); x \in C_1,\ y \in C_2\},$$

where $d : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}_+$ is the usual Euclidean distance.

We will consider that the domain $D_{i,j}$ is *big* if the distance $d(V_{i-1}, V_i)$ or

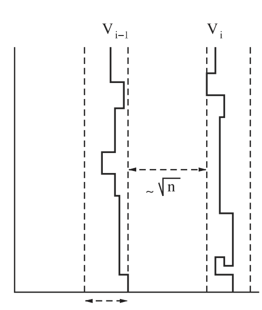

FIG. 5. *Fluctuations of channels are not greater than $C \log n$.*

$d(H_{j-1}, H_j)$ is greater than or equal to $\sqrt{n}$. The other domains are *small*. Let B be the set of *big domains*.

Let us evaluate the total volume of big domains. First, we consider the set B_V consisting of all the domains $D_{i,j}$ such that $d(V_{i-1}, V_i) \geq \sqrt{n}$. Clearly, the volume of B_V is smaller than the volume between all the channels V_{i-1} and V_i such that $d(V_{i-1}, V_i) \geq \sqrt{n}$. The domain delimited by the channels V_{i-1} and V_i contains a rectangle of size $n \times d(V_{i-1}, V_i)$, and it is strictly included in the rectangle $n \times [d(V_{i-1}, V_i) + 2C \log n]$, because the fluctuations of the channels are smaller than $C \log n$ (see Figure 5). Therefore we have the following bound:

$$\begin{aligned}\#B_V &\leq n\left[\sum_{\{i; d(V_{i-1},V_i)\geq\sqrt{n}\}} d(V_{i-1}, V_i) + 2C \log n\right] \\ &\leq n \sum_{\{i; d(V_{i-1},V_i)\geq\sqrt{n}\}} d(V_{i-1}, V_i) + 2n\sqrt{n}C \log n.\end{aligned}$$

On the scale $\sqrt{n}$, we may apply Theorem 2.2: If $d(V_{i-1}, V_i) \geq \sqrt{n}$, there is $c(p)d(V_{i-1}, V_i)$ Kesten channels between the channels V_{i-1} and V_i. These channels belong to the $(c(p) - \delta_{\mathrm{H}})n$ ones which have not been considered in the subgrid (i.e., which are not completely included in A). Therefore,

$$\sum_{\{i; d(V_{i-1},V_i)\geq\sqrt{n}\}} c(p)d(V_{i-1}, V_i) \leq \big(c(p) - \delta_{\mathrm{H}}\big)n \leq \big(c(p) - \delta\big)n$$

and

$$\sum_{\{i; d(V_{i-1},V_i)\geq\sqrt{n}\}} d(V_{i-1}, V_i) \leq \frac{(c(p) - \delta)}{c(p)} n.$$

Therefore,

$$\#B_V \leq \frac{(c(p) - \delta)}{c(p)} n^2 + 2n^{3/2} C \log n.$$

We find the same upper bound for the volume of the set B_H, constitued of all the domains $D_{i,j}$ such that $d(H_{j-1}, H_j) \geq \sqrt{n}$. So,

$$\#B \leq 2\frac{(c(p) - \delta)}{c(p)} n^2 + 4n^{3/2} C \log n.$$

It is now sufficient to take δ such that $\frac{(c(p)-\delta)}{c(p)} \leq \frac{\#\mathcal{C}^n}{9n^2}$ to see that, for large enough n,

$$\#B \leq \tfrac{3}{9}\#\mathcal{C}^n.$$

Therefore,

$$\#(B \cap \mathcal{C}^n) \leq \tfrac{3}{9}\#\mathcal{C}^n.$$

Call S the union of small domains. Thus $S \cap (\mathcal{C}^n \setminus A)$ covers an area of at least $\frac{6}{9}\#\mathcal{C}^n$. We have assumed that $\#(\mathcal{C}^n \setminus A) \geq \frac{\#\mathcal{C}^n}{2}$. Therefore,

$$\#\big(S \cap (\mathcal{C}^n \setminus A)\big) \geq \tfrac{1}{6}\#\mathcal{C}^n.$$

Besides, by definition, each small domain has volume smaller than n. Thus,

$$\#\{D_{i,j} : D_{i,j} \text{ small, and } D_{i,j} \cap (\mathcal{C}^n \setminus A) \neq \varnothing\} \geq \frac{\#\mathcal{C}^n}{6n}.$$

Since any such domain $D_{i,j}$ contributes by at least one in $\#\partial_{\mathcal{C}^n} A$, it follows that: $\#\partial_{\mathcal{C}^n} A \geq \frac{\#\mathcal{C}^n}{6n}$, and

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{\#\mathcal{C}^n}{6nN^{(\varepsilon-1)/\varepsilon}}.$$

We conclude that

$$\min_{A \in \mathcal{A}_4} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{2^{(\varepsilon-1)/\varepsilon}}{6n^{(\varepsilon-2)/\varepsilon}}. \tag{13}$$

To conclude, we just gather the inequalities (9)–(13).

3. The isoperimetric inequality when $d \geq 2$. We consider the bond percolation model in $\mathbb{Z}^d$, described in Section 1.2. For $x, y \in \mathbb{Z}^d$, we write: $x \sim y$ if x and y are neighbors in $\mathbb{Z}^d$, and $\mathbb{E}_d = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d, x \sim y\}$. The application $\omega : \mathbb{E}_d \to [0, 1]$ is called a configuration, and the random variables $(\omega(e), e \in \mathbb{E}_d)$ are Bernouilli(p) independent variables.

Let $\mathcal{B}^n = [-n, n]^d$. We have already defined the isoperimetric constant,

$$I_\varepsilon(\mathcal{C}^n) = \inf_{A \subset \mathcal{C}^n, \#A \leq \#\mathcal{C}^n/2} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}}, \tag{14}$$

with

$$\partial_{\mathcal{C}^n} A = \left\{(x, y) \in \mathbb{E}_d, \omega(x, y) = 1 : \begin{matrix} x \in A \cap \mathcal{C}^n \\ y \in A^C \cap \mathcal{C}^n \end{matrix} \text{ or } \begin{matrix} x \in A^C \cap \mathcal{C}^n \\ y \in A \cap \mathcal{C}^n \end{matrix}\right\}$$

and A^C the complement of A in $\mathbb{Z}^d$.

It will be convenient to also introduce the isoperimetric constants

$$I_\varepsilon^{(\alpha)}(\mathcal{C}^n) = \inf_{A \subset \mathcal{C}^n, \#A \leq (1-\alpha)\#\mathcal{C}^n} \frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}}, \tag{15}$$

where $\alpha \in]0, \frac{1}{2}]$. Note that $I_\varepsilon(\mathcal{C}^n) = I_\varepsilon^{(1/2)}(\mathcal{C}^n)$.

We claim that for $\alpha \in]0, \frac{1}{2}]$ and $p > p_c$, there exists a constant β such that, Q a.s. on the set $\#\mathcal{C} = \infty$, and for large enough n, one has

$$I_{\varepsilon(n)}^{(\alpha)}(\mathcal{C}^n) \geq \beta n^{d/\varepsilon(n)-1}, \tag{16}$$

where $\varepsilon(n) = d + 2d\frac{\log\log n}{\log n}$.

The proof proceeds in three steps.

3.1. *Geometric arguments.* We first prove that (16) is equivalent to (17). This proof is based on general arguments that would actually work on any graph. Next, we prove that (17) is implied by (18). This second step relies on specific properties of the underlying graph $\mathbb{Z}^d$. Although both proofs are rather classical, we give some details for the reader's convenience.

Choose $\alpha \in]0, \frac{1}{2}]$. Define

$$\bar{I}_\varepsilon(\mathcal{C}^n) = \inf_{A \subset \mathcal{C}^n} \frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A)\pi^n(\mathcal{C}^n \setminus A))^{(\varepsilon-1)/\varepsilon}},$$

where π^n is the uniform probability on $\mathcal{C}^n$ and, $\forall B \subset \mathbb{E}_d$, $Q^n(B) = \frac{1}{2d\#\mathcal{C}^n}\#\{e = (x, y) \in B, x, y \in \mathcal{C}^n, \omega(x, y) = 1\}$. Then

$$\bar{I}_\varepsilon(\mathcal{C}^n) \le \alpha^{1/\varepsilon-1} \inf_{A\subset\mathcal{C}^n, \#A \le (1-\alpha)\#\mathcal{C}^n} \frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A))^{(\varepsilon-1)/\varepsilon}} \le \alpha^{1/\varepsilon-1}\bar{I}_\varepsilon(\mathcal{C}^n).$$

The first inequality follows from the fact that, if $\#A \le (1-\alpha)\#\mathcal{C}^n$, then $\pi^n(\mathcal{C}^n \setminus A) \ge \alpha$. To get the second inequality note that, since $\alpha \in]0, \frac{1}{2}]$, for any set $A \subset \mathcal{C}^n$, either $\#A \le (1-\alpha)\#\mathcal{C}^n$ or $\#(\mathcal{C}^n \setminus A) \le (1-\alpha)\#\mathcal{C}^n$. Also, since $\pi^n(A) = \frac{\#A}{\#\mathcal{C}^n}$,

$$\inf_{A\subset\mathcal{C}^n, \#A \le (1-\alpha)\#\mathcal{C}^n} \frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A))^{(\varepsilon-1)/\varepsilon}} = \frac{(\#\mathcal{C}^n)^{-1/\varepsilon}}{2d} I_\varepsilon^{(\alpha)}(\mathcal{C}^n).$$

In particular, it is equivalent to prove (16) for all values of $\alpha \in]0, \frac{1}{2}]$ or for some given value of $\alpha \in]0, \frac{1}{2}]$.

Moreover, in the computation of $\bar{I}_\varepsilon(\mathcal{C}^n)$, the infimum is reached for sets A such that A and $\mathcal{C}^n \setminus A$ are connected in $\mathcal{C}^n$. Indeed, write $A = \bigcup_i A_i$, where each set A_i is connected in $\mathcal{C}^n$.

Let

$$\gamma = \inf_i \frac{Q^n(\partial_{\mathcal{C}^n} A_i)}{(\pi^n(A_i)\pi^n(\mathcal{C}^n \setminus A_i))^{(\varepsilon-1)/\varepsilon}}.$$

Then,

$$\begin{aligned}
Q^n(\partial_{\mathcal{C}^n} A) &= \sum_i Q^n(\partial_{\mathcal{C}^n} A_i) \ge \gamma \sum_i \bigl(\pi^n(A_i)\pi^n(\mathcal{C}^n \setminus A_i)\bigr)^{(\varepsilon-1)/\varepsilon} \\
&= \gamma \sum_i \bigl(\pi^n(A_i)\bigl(1 - \pi^n(A_i)\bigr)\bigr)^{(\varepsilon-1)/\varepsilon} \\
&\ge \gamma \left(\sum_i \pi^n(A_i)\bigl(1 - \pi^n(A_i)\bigr)\right)^{(\varepsilon-1)/\varepsilon} \\
&= \gamma \left(\pi^n(A) - \sum_i \pi^n(A_i)^2\right)^{(\varepsilon-1)/\varepsilon}.
\end{aligned}$$

From $\sum_i \pi^n(A_i)^2 \le (\sum_i \pi^n(A_i))^2 = (\pi^n(A))^2$, we get $Q^n(\partial_{\mathcal{C}^n} A) \ge \gamma(\pi^n(A) \times (1-\pi^n(A)))^{(\varepsilon-1)/\varepsilon}$, that is,

$$\frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A)\pi^n(\mathcal{C}^n \setminus A))^{(\varepsilon-1)/\varepsilon}} \ge \gamma = \inf_i \frac{Q^n(\partial_{\mathcal{C}^n} A_i)}{(\pi^n(A_i)\pi^n(\mathcal{C}^n \setminus A_i))^{(\varepsilon-1)/\varepsilon}}.$$

We can apply the same argument to $\mathcal{C}^n \setminus A_i$ instead of A and prove that

$$\frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A)\pi^n(\mathcal{C}^n \setminus A))^{(\varepsilon-1)/\varepsilon}} \ge \inf_{i,j} \frac{Q^n(\partial_{\mathcal{C}^n} A_{i,j})}{(\pi^n(A_{i,j})\pi^n(\mathcal{C}^n \setminus A_{i,j}))^{(\varepsilon-1)/\varepsilon}},$$

where, for all i, the sets $A_{i,j}$ are the connected components of $\mathcal{C}^n \setminus A_i$. Since the sets $A_{i,j}$ are connected and such that $\mathcal{C}^n \setminus A_{i,j}$ is connected, we have indeed proved that the infimum in the definition of $\bar{I}_\varepsilon(\mathcal{C}^n)$ is reached for sets A such that A and $\mathcal{C}^n \setminus A$ are connected in $\mathcal{C}^n$.

The same kind of remark applies to $I_\varepsilon(\mathcal{C}^n)$: choose $A \subset \mathcal{C}^n$ with $\#A \le (1-\alpha)\#\mathcal{C}^n$. There exists a connected subset of $\mathcal{C}^n$, say A_0, such that $\mathcal{C}^n \setminus A_0$ is connected and

$$\frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A)\pi^n(\mathcal{C}^n \setminus A))^{(\varepsilon-1)/\varepsilon}} \ge \frac{Q^n(\partial_{\mathcal{C}^n} A_0)}{(\pi^n(A_0)\pi^n(\mathcal{C}^n \setminus A_0))^{(\varepsilon-1)/\varepsilon}}.$$

Among the two sets A_0 and $\mathcal{C}^n \setminus A_0$, one has π^n measure smaller than $1-\alpha$, say $\pi^n(A_0) \le 1-\alpha$.

Since $\#A \le (1-\alpha)\#\mathcal{C}^n$, we have $\pi^n(\mathcal{C}^n \setminus A) \ge \alpha$ and

$$\frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A))^{(\varepsilon-1)/\varepsilon}} \ge \alpha^{1-1/\varepsilon} \frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A)\pi^n(\mathcal{C}^n \setminus A))^{(\varepsilon-1)/\varepsilon}}.$$

Besides,

$$\frac{Q^n(\partial_{\mathcal{C}^n} A_0)}{(\pi^n(A_0)\pi^n(\mathcal{C}^n \setminus A_0))^{(\varepsilon-1)/\varepsilon}} \ge \frac{Q^n(\partial_{\mathcal{C}^n} A_0)}{(\pi^n(A_0))^{(\varepsilon-1)/\varepsilon}}.$$

We conclude that

$$\frac{Q^n(\partial_{\mathcal{C}^n} A)}{(\pi^n(A))^{(\varepsilon-1)/\varepsilon}} \ge \alpha^{1-1/\varepsilon} \frac{Q^n(\partial_{\mathcal{C}^n} A_0)}{(\pi^n(A_0))^{(\varepsilon-1)/\varepsilon}},$$

or equivalently

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \ge \alpha^{1-1/\varepsilon} \frac{\#(\partial_{\mathcal{C}^n} A_0)}{(\#A_0)^{(\varepsilon-1)/\varepsilon}}.$$

We conclude that, in the definitions of the isoperimetric constants $I_\varepsilon(\mathcal{C}^n)$ and $I_\varepsilon^{(\alpha)}(\mathcal{C}^n)$, we may restrict our attention to connected sets A such that $\mathcal{C}^n \setminus A$ is connected. It will affect the value of these constants by at most the multiplicative constant $\alpha^{1-1/\varepsilon}$ and therefore only change the value of constant β in (16).

Since, for any set $A \subset \mathcal{C}^n$ with $\#A \le (1-\alpha)\#\mathcal{C}^n$, $\partial_{\mathcal{C}^n} A \ne \varnothing$, we have

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \ge \frac{1}{(\#A)^{(\varepsilon-1)/\varepsilon}} \ge \beta n^{d/\varepsilon - 1},$$

as soon as $\#A \le \frac{n^{(\varepsilon-d)/(\varepsilon-1)}}{\beta^{\varepsilon/(\varepsilon-1)}}$.

Thus, gathering these remarks, we see that an equivalent formulation of (16) is as follows:

For $c > 0$ and $p > p_c$, there exists a constant β such that, Q a.s. on the set $\#\mathcal{C} = \infty$, and for large enough n, one has

$$\text{(17)}\quad \inf\left\{\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}}, A \subset \mathcal{C}^n, \#A \le \frac{\#\mathcal{C}^n}{2}, A \text{ connected}, \right.$$
$$\left. \mathcal{C}^n \setminus A \text{ connected}, \#A \ge cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)}\right\} \ge \beta n^{d/\varepsilon(n)-1}.$$

From now on, let $A \subset \mathcal{C}^n$ be such that A and $\mathcal{C}^n \setminus A$ are connected sets and $\#A \le \frac{1}{2}\#\mathcal{C}^n$.

Let B be the unique $\mathcal{B}^n$-connected component of $\mathcal{B}^n \setminus A$ which contains $\mathcal{C}^n \setminus A$. (Remember that $\mathcal{B}^n = [-n, n]^d$.) Consider now D, the complementary set of B in $\mathcal{B}^n : D = \mathcal{B}^n \setminus B$ (cf. Figure 6). Then, we have:

(i) $D \cap \mathcal{C}^n = A$, $B \cap \mathcal{C}^n = (\mathcal{C}^n \setminus A)$,
(ii) D is connected in $\mathbb{Z}^d$ (because A is connected in $\mathbb{Z}^d$),
(iii) its complement $B = \mathcal{B}^n \setminus D$ is also connected in $\mathbb{Z}^d$,
(iv) the boundary of A in $\mathcal{C}^n$ satisfies $\partial_{\mathcal{C}^n} A = \{(x, y) \in \mathbb{E}_d | (x, y) \in \partial_{\mathcal{B}^n} D, \omega(x, y) = 1\}$.

PROOF. Consider $(x, y) \in \partial_{\mathcal{C}^n} A$, $x \in A$ and $y \in (\mathcal{C}^n \setminus A)$. Then, $x \in D$ and $y \notin D$, so $(x, y) \in \partial_{\mathcal{B}^n} D$ and $\omega(x, y) = 1$.

Consider $(x, y) \in \partial_{\mathcal{B}^n} D$ with $x \in D$ and $\omega(x, y) = 1$. Then, $x \in A$, $y \notin A$ but $y \in \mathcal{C}^n$ ($\mathcal{C}^n$ is connected). So, $(x, y) \in \partial_{\mathcal{C}^n} A$. $\square$

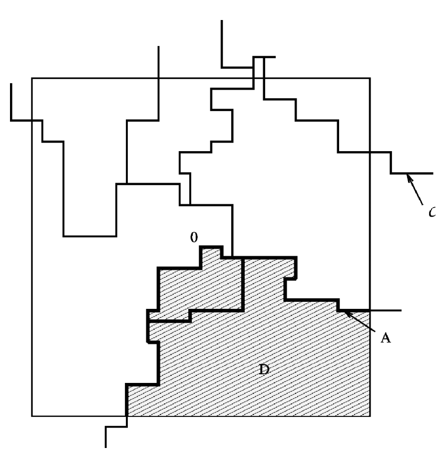

FIG. 6.

The two properties (ii) and (iii) imply that the boundary of D in $\mathcal{B}^n : \partial_{\mathcal{B}^n} D = \{(x, y) \in \mathbb{E}_d \cap \mathcal{B}^n | x \in D, y \notin D\}$ is $*$-connected in $\mathbb{Z}^d$ (cf. Appendix A). Property (iv) implies that $\#\partial_{\mathcal{C}^n} A = \sum_{\{e \in \partial_{\mathcal{B}^n} D\}} \mathbf{1}_{\omega(e)=1}$.

Therefore, we get

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{\sum_{\{e \in \partial_{\mathcal{B}^n} D\}} \mathbf{1}_{\omega(e)=1}}{(\#D)^{(\varepsilon-1)/\varepsilon}}.$$

Since $\#A \leq \frac{\#\mathcal{C}^n}{2}$, and by property (i),

$$\begin{aligned} \#D &\leq \#(\mathcal{B}^n \setminus \mathcal{C}^n) + \#A \\ &\leq \#\mathcal{B}^n - \#\mathcal{C}^n + \tfrac{1}{2}\#\mathcal{C}^n \leq (2n+1)^d - \tfrac{1}{2}\#\mathcal{C}^n. \end{aligned}$$

We know (cf. Appendix B) that there exists a constant, $\alpha > 0$ such that Q a.s. on the set $\#\mathcal{C} = +\infty$, for large n, $\#\mathcal{C}^n \geq 2\alpha(2n+1)^d$. Therefore,

$$\frac{\#D}{(2n+1)^d} \leq 1 - \frac{1}{2}\frac{\#\mathcal{C}^n}{(2n+1)^d} \leq 1 - \alpha.$$

Define

$$\mathcal{I}^{(\alpha)} = \inf_n \inf_{D \subset \mathcal{B}^n, \#D \leq (1-\alpha)(2n+1)^d} \frac{\#(\partial_{\mathcal{B}^n} D)}{(\#D)^{(d-1)/d}}.$$

The (classical) isoperimetric inequality states that $\mathcal{I}^{(\alpha)} > 0$. (As a matter of fact, it is well known that $\mathcal{I}^{(1/2)} > 0$. The general case $\alpha \in]0, \frac{1}{2}]$ follows just the same way we showed that (16) holds for any $\alpha \in]0, \frac{1}{2}]$ if and only if it holds for some $\alpha \in]0, \frac{1}{2}]$.)

Since, for large enough n, we have $\#D \leq (1-\alpha)(2n+1)^d$, we also have

$$(\#D)^{(\varepsilon-1)/\varepsilon} \leq \frac{\#(\partial_{\mathcal{B}^n} D)}{\mathcal{I}^{(\alpha)}}(\#D)^{(\varepsilon-d)/(\varepsilon d)} \leq \frac{\#(\partial_{\mathcal{B}^n} D)}{\mathcal{I}^{(\alpha)}}(2n+1)^{(\varepsilon-d)/\varepsilon}.$$

Thus, we have obtained the inequality

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{\sum_{\{e \in \partial_{\mathcal{B}^n} D\}} \mathbf{1}_{\omega(e)=1}}{\#(\partial_{\mathcal{B}^n} D)} \frac{\mathcal{I}^{(\alpha)}}{(2n+1)^{(\varepsilon-d)/\varepsilon}}.$$

Finally note that, if $\#A \geq cn^{(\varepsilon-d)/(\varepsilon-1)}$, then $\#D \geq cn^{(\varepsilon-d)/(\varepsilon-1)}$ and therefore $\#(\partial_{\mathcal{B}^n} D) \geq cn^{(\varepsilon-d)(d-1)/((\varepsilon-1)d)}$ for some other value of c.

We use the notation $F = \partial_{\mathcal{B}^n} D$. Let $\mathbb{E}^n$ be the set of edges in $\mathbb{E}_d$ with both end points in $\mathcal{B}^n$. Since D and $\mathcal{B}^n \setminus D$ are connected in $\mathcal{B}^n$, then F is $*$-connected, and we conclude that, in order to prove (16), it is sufficient to check that, for $c > 0$ and $p > p_c$, there exists a constant β such that, Q a.s. on the set $\#\mathcal{C} = \infty$, and for large enough n, one has

$$\inf\left\{\frac{\sum_{\{e \in F\}} \mathbf{1}_{\omega(e)=1}}{\#F}, F \subset \mathbb{E}^n, F \text{ } *\text{-connected}, \right.$$
$$\left. \#F \geq cn^{(\varepsilon(n)-d)(d-1)/((\varepsilon(n)-1)d)}\right\} \geq \beta.$$

Since, for large enough n, we have $cn^{(\varepsilon(n)-d)(d-1)/((\varepsilon(n)-1)d)} \geq (\log n)^{3/2}$, it might be easier to prove that, for $p > p_c$, there exists a constant β such that, Q a.s. on the set $\#\mathcal{C} = \infty$, and for large enough n, one has

$$(18)\quad \inf\left\{\frac{\sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1}}{\#F}, F \subset \mathbb{E}^n, F \ast\text{-connected}, \#F \geq (\log n)^{3/2}\right\} \geq \beta.$$

3.2. *Large values of p.* We use formulation (18), together with a contour argument, to check that (16) holds when p is close enough to 1.

For a given set of edges $F \subset \mathbb{E}^n$, since the random variables $(\omega(e), e \in F)$ are Bernoulli(p) and independent, we have

$$(19)\qquad Q\left[\frac{\sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1}}{\#F} \leq \beta\right] \leq e^{\lambda\beta\#F}\big(pe^{-\lambda} + (1-p)\big)^{\#F}.$$

On the other hand, we recall that the number of $\ast$-connected sets $F \subset \mathbb{E}^n$ of cardinality m such that $0 \in F$ is bounded by $\exp(am)$, for some constant a that depends on the dimension d only. See [14] for instance. Therefore

$$\begin{aligned} &Q\left[\exists\, F \subset \mathbb{E}^n, F \ast\text{-connected}, \#F \geq (\log n)^{3/2} \text{ and } \sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1} \leq \beta\#F\right] \\ &\quad\leq \sum_{m\geq(\log n)^{3/2}} (2n+1)^d e^{am+\lambda\beta m}\big(pe^{-\lambda} + (1-p)\big)^m. \end{aligned}$$

If p is close enough to 1, and if we choose β small enough, there will exist $\lambda > 0$ such that

$$-\xi = a + \lambda\beta + \log\big((1-p) + e^{-\lambda}p\big) < 0.$$

Then, for large enough n, one has

$$\begin{aligned} &Q\left[\exists\, F \subset \mathbb{E}^n, F \ast\text{-connected}, \#F \geq (\log n)^{3/2}, \text{ such that } \sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1} \leq \beta\#F\right] \\ &\quad\leq e^{-(\xi/2)(\log n)^{3/2}}. \end{aligned}$$

Note that this last expression is sumable in n. Therefore the Borel–Cantelli lemma implies that, Q a.s., for large enough n, for all $\ast$-connected set $F \subset \mathbb{E}^n$ with $\#F \geq (\log n)^{3/2}$, we have $\sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1} \geq \beta\#F$. In view of (18), we deduce that we have proved that there exists a number $p(d) < 1$ such that (16) holds for any $p > p(d)$ and any $\alpha \in]0, \frac{1}{2}]$.

REMARKS. We state here estimates of the tail of the distribution of the Cheeger constant:

$$\mathcal{I}_\infty(\mathcal{C}^n) = \inf_{A\subset\mathcal{C}^n, \#A\leq\#\mathcal{C}^n/2} \frac{\#(\partial_{\mathcal{C}^n} A)}{\#A}.$$

We wish to estimate $Q[\mathcal{I}_\infty(\mathcal{C}^n) \le \frac{\beta}{n}]$. As in the previous computation, we may restrict our attention to sets A such that $\#A \ge \frac{n}{\beta}$. Also as in previous computation, we see that, on the event $\#\mathcal{C}^n \ge 2\alpha(2n+1)^d$, then $\#D \le (1-\alpha)(2n+1)^d$.

Since $\#D \ge \#A \ge \frac{n}{\beta}$, we have

$$\#F \ge \mathcal{I}^{(\alpha)}(\#D)^{(d-1)/d} \ge \mathcal{I}^{(\alpha)}\left(\frac{n}{\beta}\right)^{(d-1)/d}.$$

On the other hand, because $\#A \le (2n+1)^d$, then

$$\#F \ge \mathcal{I}^{(\alpha)}(\#A)^{(d-1)/d} \ge \frac{\mathcal{I}^{(\alpha)}}{2n+1}\#A.$$

(We continue using the notation $F = \partial_{\mathcal{B}^n} D$.) Thus

$$\frac{\#(\partial_{\mathcal{C}^n} A)}{\#A} \ge \frac{\mathcal{I}^{(\alpha)}}{2n+1}\frac{\sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1}}{\#F}.$$

Thus $\#\mathcal{C}^n \ge 2\alpha(2n+1)^d$ and $\mathcal{I}_\infty(\mathcal{C}^n) \le \frac{\beta}{n}$ imply that there exists a $*$-connected set $F \subset \mathbb{E}^n$ such that $\#F \ge \mathcal{I}^{(\alpha)}(\frac{n}{\beta})^{(d-1)/d}$ and

$$\sum_{\{e\in F\}} \mathbf{1}_{\omega(e)=1} \le 3\frac{\beta}{\mathcal{I}^{(\alpha)}}\#F.$$

As in (19), we then get that there exists $p(d) < 1$ such that for any $\alpha > 0$, there exist constants $\beta > 0$ and $\xi > 0$ such that, for all $p \in]p(d), 1[$, one has

$$\begin{aligned} Q\left[\mathcal{I}_\infty(\mathcal{C}^n) \le \frac{\beta}{n} \text{ and } \#\mathcal{C}^n \ge 2\alpha(2n+1)^d\right] &\le \sum_{m \ge \mathcal{I}^{(\alpha)}(n/\beta)^{(d-1)/d}} (2n+1)^d e^{-2\xi m} \\ &\le \exp\{-\xi n^{(d-1)/d}\} \end{aligned}$$

for large enough n.

We also know (see [6]) that for some constants $\alpha > 0$ and $\xi > 0$, we have

$$Q[\#\mathcal{C}^n \le 2\alpha(2n+1)^d] \le \exp\{-\xi n^{(d-1)/d}\}.$$

Gathering the last two inequalities, we obtain the following.

THEOREM 3.1. *For any dimension $d \ge 2$, there exists $p(d) < 1$, and constants $\beta > 0$ and $\xi > 0$ such that, for all $p \in]p(d), 1[$ and $n \ge 1$,*

$$Q\left[\mathcal{I}_\infty(\mathcal{C}^n) \le \frac{\beta}{n}\right] \le \exp\{-\xi n^{(d-1)/d}\}.$$

As a consequence of this estimate and Cheeger's inequality, we obtain a lower bound for the spectral gap

$$Q\left[\lambda^n \le \frac{\beta}{n^2}\right] \le \exp\{-\xi n^{(d-1)/d}\}. \tag{20}$$

3.3. *Another isoperimetric inequality.* In order to prepare for the use of renormalization arguments, we shall need a more sophisticated version of the inequality (16), still for values of p close enough to 1. We shall also have to consider both site and bond percolation models.

As before, we let $\mathcal{B}^n$ be the box $[-n, n]^d$; $\mathcal{C}^n$ is the connected component of the random graph ω that contains the origin. We also use the notation $\mathcal{G}^n$ to be the set of vertices of $\mathcal{B}^n$ that belong to ω. Thus $\mathcal{C}^n$ is a subset of $\mathcal{G}^n$. Let $\mathcal{L}^n$ be the largest connected component of ω in $\mathcal{B}^n$.

Let A be a subset of $\mathcal{B}^n$. Define $n(A)$ to be the number of connected components of $\mathcal{B}^n \setminus \mathcal{L}^n$ that contain at least one connected component of A.

Let $p < 1$, $\alpha \in [\frac{1}{2}, 1[$ and $\beta > 0$ and define the event $\mathcal{A}$ by, for large enough n,

$$\inf\left\{\frac{n(A) + \#(\partial_{\mathcal{L}^n} A)}{(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}}, A \subset \mathcal{B}^n, \#A \le \alpha\#\mathcal{B}^n\right\} \ge \beta n^{d/\varepsilon(n)-1}. \tag{21}$$

We shall prove that there exists $p_0 < 1$ such that, for all $\alpha \in [\frac{1}{2}, 1[$, $p > p_0$, there exists $\beta > 0$ s.t. Q a.s., $\mathcal{A}$ holds, that is, we prove that

$$Q[\mathcal{A}] = 1. \tag{22}$$

Before beginning the proof, we state some simple remarks.

We first note that, on the set $\#\mathcal{C} = \infty$, for large enough n, $\mathcal{L}^n = \mathcal{C}^n$. [On the set $\#\mathcal{C} = \infty$, then $\#\mathcal{C}^n \ge 2\alpha(2n+1)^d$ for some $\alpha > 0$; see Appendix B. It is easy to show that, for p close enough to 1, there is at most one connected component of $\mathcal{G}^n$ of size larger than $(\log n)^{2d/(d-1)}$, see below.] Thus, once we have proved (22), it will follow that, for some constants $p < 1$ and $\beta > 0$, Q a.s. on the set $\#\mathcal{C} = \infty$, for large enough n, one has

$$\inf\left\{\frac{n(A) + \#(\partial_{\mathcal{C}^n} A)}{(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}}, A \subset \mathcal{B}^n, \#A \le \alpha\#\mathcal{B}^n\right\} \ge \beta n^{d/\varepsilon(n)-1}. \tag{23}$$

In this last statement, $n(A)$ may as well be defined as the number of connected components of $\mathcal{B}^n \setminus \mathcal{C}^n$ that contain at least one connected component of A. If, in (23), we restrict ourselves to sets A which are contained in $\mathcal{C}^n$, then $n(A) = 0$ and we retrieve the isoperimetric inequality (16).

$\mathcal{A}$ is an increasing event. Indeed, let $A \subset \mathcal{B}^n$ and suppose we add one edge to ω. Then $\#(\partial_{\mathcal{L}^n} A)$ will not decrease. Assume that $n(A)$ decreases by 1. It implies that at least one of the connected components of A did not intersect $\mathcal{L}^n$ before the addition of the extra edge and intersects $\mathcal{L}^n$ after the addition of the extra edge.

Then a new edge appeared in $\#(\partial_{\mathcal{L}^n} A)$. We conclude that the addition of one edge to ω does not decrease the sum $n(A) + \#(\partial_{\mathcal{L}^n} A)$.

As a consequence, we deduce that if (22) holds for some $p_0 < 1$, it then holds for all $p \in [p_0, 1]$. From the comparison theorems established in [10], it is equivalent to prove (22) for bond or site percolation.

We now turn to the proof of (22): by choosing p close enough to 1, we may, and will, always assume that $\#\mathcal{L}^n \geq \frac{1+\alpha}{2}\#\mathcal{B}^n$; see Appendix B. Since $\#A \leq \alpha\#\mathcal{B}^n$, $\mathcal{L}^n$ cannot be contained in A and therefore $n(A) + \#(\partial_{\mathcal{L}^n} A) \geq 1$. Therefore we may, and will, restrict ourselves to sets A such that $\#A \geq \beta^{-\varepsilon(n)/(\varepsilon(n)-1)} \times n^{(\varepsilon(n)-d)/(\varepsilon(n)-1)}$. For large enough n,

$$\beta^{-\varepsilon(n)/(\varepsilon(n)-1)} n^{(\varepsilon(n)-d)/(\varepsilon(n)-1)} \geq (\log n)^{3d/(2(d-1))}.$$

Therefore we may assume that $\#A \geq (\log n)^{3d/(2(d-1))}$.

Let A_1 be the union of the connected components of A that intersect $\mathcal{L}^n$. Let A_2 be the union of the connected components of A that do not intersect $\mathcal{L}^n$. Note that $\#(\partial_{\mathcal{L}^n} A) = \#(\partial_{\mathcal{L}^n} A_1)$.

Let us first assume that $\#A_1 \leq \frac{1}{2}\#A$. Then $\#A_2 \geq \frac{1}{2}\#A$ and therefore $\#A_2 \geq \frac{1}{2}(\log n)^{3d/(2(d-1))}$.

Let $c_1, \ldots, c_k$ be the different connected components of $\mathcal{B}^n \setminus \mathcal{L}^n$. The same contour argument as in Section 3.2 shows that, for any $a > 1$, if we choose p close enough to 1, then, for large enough n, $*$-connected sets of volume larger than $(\log n)^a$ intersect $\mathcal{L}^n$. Since, for all i, $\partial_{\mathcal{B}^n} c_i$ is $*$-connected, we have $\#\partial_{\mathcal{B}^n} c_i \leq (\log n)^a$. The classical isoperimetric inequality then implies that $\#c_i \leq (\log n)^{ad/(d-1)}$, for all $i = 1, \ldots, k$. Here we choose $a = 2$.

We then have

$$\begin{aligned} n(A) &= \sum_{i=1}^{k} \mathbf{1}_{A_2 \cap c_i \neq \varnothing} \\ &\geq \sum_{i=1}^{k} \frac{\#(A_2 \cap c_i)}{(\log n)^{2d/(d-1)}} \\ &= \frac{\#A_2}{(\log n)^{2d/(d-1)}} \\ &\geq n^{d/\varepsilon(n)-1} (\#A_2)^{(\varepsilon(n)-1)/\varepsilon(n)}, \end{aligned}$$

where the last inequality comes from the fact that $n^{1-d/\varepsilon(n)} \geq (\log n)^{2d/(d-1)}$ for large n. We conclude that $n(A) \geq (\frac{\#A}{2})^{(\varepsilon(n)-1)/\varepsilon(n)} n^{d/\varepsilon(n)-1}$.

Now assume that $\#A_1 \geq \frac{1}{2}\#A$. Then $\#A_1 \geq \frac{1}{2}(\log n)^{3d/(2(d-1))}$. We wish to show that

$$\frac{\#(\partial_{\mathcal{L}^n} A_1)}{(\#A_1)^{(\varepsilon(n)-1)/\varepsilon(n)}} \geq \beta n^{d/\varepsilon(n)-1}. \tag{24}$$

Without loss of generality, we may assume that A_1 is connected. Let us first check that we may also assume that all the connected components of $\mathcal{B}^n \setminus A_1$ intersect $\mathcal{L}^n$ and have size bounded by $\frac{1}{2}\#\mathcal{B}^n$. Indeed, let $c_1, \dots, c_k$ be the connected components of $\mathcal{B}^n \setminus A_1$. Suppose that some c_i does not intersect $\mathcal{L}^n$. Then $\#A_1 \leq \#(A_1 \cup c_i)$ and $\#\partial_{\mathcal{L}^n} A_1 = \partial_{\mathcal{L}^n}(A_1 \cup c_i)$. Also note that $\#(A_1 \cup c_i) \leq \frac{1+\alpha}{2}\#\mathcal{B}^n$. Therefore, changing the value of α to $\frac{1+\alpha}{2}$, it is sufficient to prove (24) for $A_1 \cup c_i$ instead of A_1. From now on, we shall assume that all c_i intersect $\mathcal{L}^n$.

Assume that $\#c_i > \frac{1}{2}\#\mathcal{B}^n$ for some i. Let $A_3 = \mathcal{B}^n \setminus c_i$. Note that A_3 is connected, $\mathcal{B}^n \setminus A_3$ is connected, A_3 intersects $\mathcal{L}^n$ (because A_1 intersects $\mathcal{L}^n$) and $\#A_3 \leq \frac{1}{2}\#\mathcal{B}^n$. Since $\partial_{\mathcal{L}^n} A_3 \subset \partial_{\mathcal{L}^n} A_1$ and $A_1 \subset A_3$, if we can prove (24) for A_3, we can prove it for A_1. Thus it is no loss of generality to assume that $\#c_i \leq \frac{1}{2}\#\mathcal{B}^n$.

We shall now use the fact that, for some choice of p and β, we have

$$\frac{\#\partial_{\mathcal{L}^n} c_i}{(\#c_i)^{(\varepsilon(n)-1)/\varepsilon(n)}} \geq \beta n^{d/\varepsilon(n)-1}. \tag{25}$$

Equation (25) follows from the fact that c_i is connected, $\mathcal{B}^n \setminus c_i$ is connected and $\partial_{\mathcal{L}^n} c_i \neq \varnothing$, since c_i intersects $\mathcal{L}^n$. Therefore the results of Section 3.2 can be applied.

We conclude by noticing that

$$\begin{aligned}
\#\partial_{\mathcal{L}^n} A_1 &= \sum_i \#\partial_{\mathcal{L}^n} c_i \\
&\geq \beta n^{d/\varepsilon(n)-1} \sum_i (\#c_i)^{(\varepsilon(n)-1)/\varepsilon(n)} \\
&\geq \beta n^{d/\varepsilon(n)-1} \left(\sum_i \#c_i\right)^{(\varepsilon(n)-1)/\varepsilon(n)} \\
&= \beta n^{d/\varepsilon(n)-1} (\#\mathcal{B}^n - \#A_1)^{(\varepsilon(n)-1)/\varepsilon(n)} \\
&\geq \beta n^{d/\varepsilon(n)-1} \left(\frac{1-\alpha}{\alpha}\right)^{(\varepsilon(n)-1)/\varepsilon(n)} (\#A_1)^{(\varepsilon(n)-1)/\varepsilon(n)},
\end{aligned}$$

since $\#A \leq \alpha\#\mathcal{B}^n$.

3.4. *Renormalization.* We now explain how to push the isoperimetric inequality from large values of p down to any $p > p_c$. To this end, we mainly rely on Proposition 2.1 in [1]. We shall also use some terminology from [1].

Choose $p > p_c$. N is an integer. We chop $\mathbb{Z}^d$ in a disjoint union of boxes of side length $2N+1$. Say $\mathbb{Z}^d = \bigcup_{\mathbf{i} \in \mathbb{Z}^d} B_{\mathbf{i}}$, where $B_{\mathbf{i}}$ is the box of center $(2N+1)\mathbf{i}$. Still following [1], let $B'_{\mathbf{i}}$ be the box of center $(2N+1)\mathbf{i}$ and side length $\frac{5}{2}N+1$. From now on, the word *box* will mean one of the boxes $B_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d$.

We say that a box B is *good* if B contains at least one edge from ω and the event $R_{\mathbf{i}}^{(N)}$ in equation (2.9) of [1] is satisfied. Otherwise, B is a *bad* box. We recall that the event $R_{\mathbf{i}}^{(N)}$ is defined by: there is a unique cluster, K, in $B'_{\mathbf{i}}$; all open paths contained in $B'_{\mathbf{i}}$ and of radius larger than $\frac{1}{10}N$ intersect K within $B'_{\mathbf{i}}$; K is crossing for each subbox $B \subset B'_{\mathbf{i}}$ of side larger than $\frac{1}{10}N$. See [1] for a more precise definition.

Let A be a subset of $\mathcal{C}^n$. We say that A *touches* the box B if $A \cap B \neq \varnothing$. We say that A *fills* B if A touches B and $A \cap B = B \cap \mathcal{C}^n$. We use the notation $\bar{n}_1$ (resp. n_1) to denote the number of boxes (resp. good boxes) touched by A but not filled by A. Similarly, let $\bar{n}_2$ (resp. n_2) denote the number of boxes (resp. good boxes) filled by A.

Following [1], we call *renormalized* process the percolation model obtained by taking the image of the initial percolation model by the application φ_N, see equation (2.11) in [1]. A site $\mathbf{i} \in \mathbb{Z}^d$ is thus declared *white* if the box $B_{\mathbf{i}}$ is good.

We choose N large enough so that the following two conditions are satisfied:

(i) the isoperimetric inequality (21) holds for the renormalized percolation model and $\alpha = \frac{4}{7}(1 + 10^{-3})$;

(ii) for large enough n and for all connected set $A \subset \mathcal{B}^n$ with $\#A \geq (2N+1)^d \times (\log n)^{3/2}$, we have

$$n_1 + n_2 \geq \tfrac{7}{8}(\bar{n}_1 + \bar{n}_2).$$

Such N exists. Indeed, we already know from Section 3.3 that p' can be chosen close enough to 1 so that the isoperimetric inequality (21) is satisfied for site percolation of parameter p'. From the comparison of Proposition 2.1 in [1], and the fact that event $\mathcal{A}$ in (21) is increasing, we deduce that (22) is also satisfied for the renormalized process if N is large enough.

Let us check that we can choose N so that (ii) is satisfied. For a connected set $A \subset \mathcal{B}^n$ with $\#A \geq (2N + 1)^d (\log n)^{3/2}$, we have

$$\bar{n}_1 + \bar{n}_2 \geq (2N + 1)^{-d} \#A \geq (\log n)^{3/2}.$$

Call $\tilde{A}$, the set of indices $\mathbf{i}$ such that $B_{\mathbf{i}}$ is touched by A. Note that $\bar{n}_1 + \bar{n}_2 = \#\tilde{A}$ and that $\tilde{A}$ is connected. Therefore, using Proposition 2.1 in [1], we only have to check the following property:

For any constant c, for p close enough to 1, and for site percolation, Q a.s., for large enough n, for any connected set $\tilde{A} \subset \mathcal{B}^n$ with $\#\tilde{A} \geq c(\log n)^{3/2}$, then

$$\#(\tilde{A} \cap \mathcal{G}^n) \geq \tfrac{7}{8}\#\tilde{A}. \tag{26}$$

(Remember than $\mathcal{G}^n$ is the set of vertices of ω in the box $\mathcal{B}^n$.)

Inequality (26) follows from a Borel–Cantelli argument based on the fact that the number of connected subsets of $\mathbb{Z}^d$ containing 0 and of cardinal m is bounded by $\exp(am)$, for some constant a that depends on the dimension only.

We want to check that claim (17) holds true.

First let A be a connected subset of $\mathcal{C}^n$ such that $\#A \ge cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)}$ and $n_1 \ge 2^d \beta(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)} n^{d/\varepsilon(n)-1}$.

Note that $cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)} \ge (2N+1)^d (\log n)^{3/2}$ if n is large enough. It follows that A is not entirely contained in one single box.

Each good box touched, but not filled, by A contributes by at least one edge the boundary of A. Indeed, assume that A touches but does not fill $B_{\mathbf{i}}$. Then we can find two points $x, y \in B_{\mathbf{i}} \cap \mathcal{C}^n$ such that $x \in A$ and $y \notin A$. Since A is not entirely contained in $B_{\mathbf{i}}$, there is a path in ω of length at least $N/5$ linking x to the boundary of $B_{\mathbf{i}}$. Similarly, there is a path in ω of length at least $N/5$ linking y to the boundary of $B_{\mathbf{i}}$. From the definition of a good box, it follows that these two open paths have to be connected to each other within $B'_{\mathbf{i}}$. Therefore, there is an open path linking x to y within $B'_{\mathbf{i}}$. On this open path, there is an edge, say (a,b), where a and b are neighbors, both are in $\mathcal{C}^n$ and $a \in A$, $b \notin A$. Thus we have found one edge in $\partial_{\mathcal{C}^n} A$. We can repeat that construction for each of the n_1 good boxes touched but not filled by A. Since a given edge cannot appear more that 2^d times, we get that $\#(\partial_{\mathcal{C}^n} A) \ge 2^{-d} n_1$. Therefore, since $n_1 \ge 2^d \beta(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)} n^{d/\varepsilon(n)-1}$, then we also have $\#(\partial_{\mathcal{C}^n} A) \ge \beta(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)} n^{d/\varepsilon(n)-1}$.

We let now A be a connected subset of $\mathcal{C}^n$ with $n_2 \le \frac{1}{2}(\frac{2n+1}{2N+1})^d$, $n_1 \le 2^d \beta(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)} n^{d/\varepsilon(n)-1}$ and $\#A \ge cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)}$. Note that

$$cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)} \ge (2N+1)^d (\log n)^{3/2}$$

if n is large enough. It follows that A is not entirely contained in one single box.

Let $\tilde{A}$ be the set of indices $\mathbf{i} \in \mathbb{Z}^d$ such that $B_{\mathbf{i}}$ is touched by A. We wish to use the isoperimetric inequality (21) for the set $\tilde{A}$. By definition $(\bar{n}_1 + \bar{n}_2) = \#\tilde{A}$.

Since $(\bar{n}_1 + \bar{n}_2)(2N+1)^d \ge \#A$ and $\frac{7}{8}(\bar{n}_1 + \bar{n}_2) \le n_1 + n_2$, we then have

$$n_2 \ge (2N+1)^{-d} \tfrac{7}{8} \#A - 2^d \beta n^{d/\varepsilon(n)-1} (\#A)^{(\varepsilon(n)-1)/\varepsilon(n)} \ge (2N+1)^{-d} \tfrac{6}{8} \#A,$$

for large enough n. In particular it follows that, for large enough n, $n_1 \le 10^{-3} n_2$.

Remember that we have assumed that $n_2 \le \frac{1}{2}(\frac{2n+1}{2N+1})^d$. Thus

$$(\bar{n}_1 + \bar{n}_2) \le \frac{4}{7}(1 + 10^{-3}) \left(\frac{2n+1}{2N+1} \right)^d .$$

Therefore we may apply the isoperimetric inequality (21) to the set $\tilde{A}$ with $\alpha = \frac{4}{7}(1 + 10^{-3})$. Clearly $\tilde{A}$ is connected. We use the notation "$\tilde{\cdot}$" to indicate quantities defined at the level of the renormalized process.

Either $\tilde{A}$ is contained in one of the connected components of $\tilde{\mathcal{B}}^n \setminus \tilde{\mathcal{L}}^n$. Then $\tilde{n}(\tilde{A}) = 1$ and $\partial_{\tilde{\mathcal{L}}^n} \tilde{A} = \varnothing$ and therefore $1 \ge \beta n^{d/\varepsilon(n)-1} (\#\tilde{A})^{(\varepsilon(n)-1)/\varepsilon(n)}$. Since $\partial_{\mathcal{C}^n} A \ne \varnothing$, we then have

$$\#(\partial_{\mathcal{C}^n} A) \ge \beta n^{d/\varepsilon(n)-1} (\#\tilde{A})^{(\varepsilon(n)-1)/\varepsilon(n)}.$$

Let us now suppose that $\tilde{A}$ intersects $\tilde{\mathcal{L}}^n$. Then $\tilde{n}(\tilde{A}) = 0$ and

$$\#(\partial_{\tilde{\mathcal{L}}^n} \tilde{A}) \ge \beta n^{d/\varepsilon(n)-1} (\#\tilde{A})^{(\varepsilon(n)-1)/\varepsilon(n)}.$$

Let $(\mathbf{i},\mathbf{i}') \in \partial_{\tilde{\mathcal{L}}^n}\tilde{A}$. Then $B_{\mathbf{i}}$ and $B_{\mathbf{i}'}$ are good boxes; $\mathbf{i}$ and $\mathbf{i}'$ are neighbors; $\tilde{A}$ intersects $B_{\mathbf{i}}$ but not $B_{\mathbf{i}'}$. But note that each such couple $(\mathbf{i},\mathbf{i}')$ contributes by at least one edge to the boundary of A in $\mathcal{C}^n$. Indeed, we can find points $x \in B_{\mathbf{i}}$ and $y \in B_{\mathbf{i}'}$ such that $x \in A$ and $y \in \mathcal{C}^n \setminus A$. Because the two boxes $B_{\mathbf{i}}$ and $B_{\mathbf{i}'}$ are good, there is an open path linking x to y within $B'_{\mathbf{i}} \cup B'_{\mathbf{i}'}$, and, on this path, there must be an edge of $\partial_{\mathcal{C}^n} A$. As we perform this construction for different choices of $(\mathbf{i},\mathbf{i}')$, a given edge appears at most 2^d times. Therefore,

$$\#(\partial_{\mathcal{C}^n} A) \geq 2^{-d}\beta n^{d/\varepsilon(n)-1}(\#\tilde{A})^{(\varepsilon(n)-1)/\varepsilon(n)}.$$

Since $\#\tilde{A} \geq n_2 \geq (2N+1)^{-d}\frac{6}{8}\#A$, we have

$$\#(\partial_{\mathcal{C}^n} A) \geq \beta n^{d/\varepsilon(n)-1}(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}$$

with a different value of β.

Finally, let A be a connected subset of $\mathcal{C}^n$ such that $A' = \mathcal{C}^n \setminus A$ is also connected, $\#A \leq \#\mathcal{C}^n/2$, $n_2 \geq \frac{1}{2}(\frac{2n+1}{2N+1})^d$, $n_1 \leq 2^d\beta(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}n^{d/\varepsilon(n)-1}$ and $\#A \geq cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)}$. Let n'_1 and n'_2 be defined as n_1 and n_2 with A being replaced by A'. Note that $n'_1 = n_1 \leq 2^d\beta(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}n^{d/\varepsilon(n)-1}$.

Also $\#A' \geq \#\mathcal{C}^n/2 \geq cn^{(\varepsilon(n)-d)/(\varepsilon(n)-1)}$, see Appendix B. Since $n_2 + n'_2 \leq (\frac{2n+1}{2N+1})^d$, we must have $n'_2 \leq \frac{1}{2}(\frac{2n+1}{2N+1})^d$. Thus we may apply the previous isoperimetric inequality to A' and get that:

$$\#(\partial_{\mathcal{C}^n} A') \geq \beta n^{d/\varepsilon(n)-1}(\#A')^{(\varepsilon(n)-1)/\varepsilon(n)}.$$

But note that $\partial_{\mathcal{C}^n} A' = \partial_{\mathcal{C}^n} A$ and $\#A' \geq \frac{1}{2}\#\mathcal{C}^n \geq \#A$. Therefore

$$\#(\partial_{\mathcal{C}^n} A) \geq \beta n^{d/\varepsilon(n)-1}(\#A)^{(\varepsilon(n)-1)/\varepsilon(n)}.$$

4. Proofs of the theorems. We prove inequalities (1) and (2), that is, that

$$\sup_{x\in\mathcal{C}} P_0^\omega[X_t = x] \leq \frac{c_1}{t^{d/2}}.$$

We use the Carne–Varopoulos inequality [cf. inequality (28), Appendix C]:

$$P_0^\omega[X_t = x] \leq e^{-|x|^2/(4t)} + e^{-ct}.$$

CASE 1. If $|x|^2 \geq 2dt\log t$, then $e^{-|x|^2/(4t)} \leq Ct^{-d/2}$ and also $e^{-ct} \leq Ct^{-d/2}$, provided t is large enough.

CASE 2. If $|x|^2 < 2dt\log t$.

We choose $t\log t = bn^2$, with $b < \frac{1}{4d+2}$.

Let τ^n denote the exit time of $\mathcal{B}^{n-1}$ for X_t. Then

$$P_0^\omega[X_t = x] \leq P_0^\omega[X_t^n = x] + P_0^\omega[\tau^n \leq t].$$

Combining with the estimates (6) and (30), we get that

$$P_0^\omega[X_t = x] \le n^{-d} + \left(\frac{4\varepsilon}{\beta^2}\right)^{\varepsilon/2} \frac{n^{\varepsilon-d}}{t^{\varepsilon/2}} + 2tn^{d-1}e^{-n^2/(4t)} + e^{-ct}.$$

We consider the right-hand side of the inequality term by term:

(i) $n^{-d} = (\frac{t\log t}{b})^{-d/2} \le t^{-d/2}$, this will be satisfied as soon as t is large enough.

(ii)

$$\begin{aligned}
&\left(\frac{4\varepsilon}{\beta^2}\right)^{\varepsilon/2} \frac{n^{\varepsilon-d}}{t^{\varepsilon/2}} \le \frac{C}{t^{d/2}} \\
&\quad\iff\quad \left(\frac{4\varepsilon}{\beta^2}\right)^{\varepsilon/2} \left(\frac{n^2}{t}\right)^{(\varepsilon-d)/2} \le C \\
&\quad\iff\quad \left(\frac{4\varepsilon}{\beta^2}\right)^{\varepsilon/2} \left(\frac{\log t}{b}\right)^{(\varepsilon-d)/2} \le C \\
&\quad\iff\quad \log K + d\frac{\log\log n}{\log n}\log\left(\frac{\log t}{b}\right) \le \log C,
\end{aligned}$$

with $K = (\frac{4\varepsilon}{\beta^2})^{\varepsilon/2}$. Since $\frac{\log\log n}{\log n}\log\log t \to 0$, this is true for large t, and for some constant C.

(iii)

$$\begin{aligned}
&2tn^{d-1}e^{-n^2/(4t)} \le \frac{C}{t^{d/2}} \\
&\quad\iff\quad \log t + (d-1)\log n + \frac{d}{2}\log t \le \log\frac{C}{2} + \frac{n^2}{4t} \\
&\quad\iff\quad \log t + (d-1)\log n + \frac{d}{2}\log t \le \log\frac{C}{2} + \frac{\log t}{4b} \\
&\quad\iff\quad \log t\left[\frac{1}{4b} - 1 - \frac{d}{2}\right] + \log\frac{C}{2} \ge (d-1)\log n,
\end{aligned}$$

and this is OK if $2[\frac{1}{4b} - 1 - \frac{d}{2}] > d-1$, that is, if b satisfies $b < \frac{1}{4d+2}$.

(iv) The last term, e^{-ct}, clearly decays faster than $t^{-d/2}$.

APPENDIX A: CONNECTIVITY

We prove the following statement:

> Let $\mathcal{B}$ be a finite box of $\mathbb{Z}^d$. Let $A \subset \mathcal{B}$ such that A and $\mathcal{B} \setminus A$ are both connected. Then, $\partial_{\mathcal{B}} A$ is $*$-connected.

Let us recall what $*$-connectedness is (see [6]): we embed $\mathbb{Z}^d$ into $\mathbb{R}^d$ and endow $\mathbb{R}^d$ with the norm $\|x\| = \max_{i=1,\dots,d} |x_i|$. Consider two nearest neighbor edges in $\mathbb{Z}^d$, say $e = (x, y)$ and $e' = (x', y')$, where $x \sim y$ and $x' \sim y'$. We say that e and e' are $*$-neighbors if $\|\frac{x+y}{2} - \frac{x'+y'}{2}\| \leq 1$. A set of edges, say F, is said to be $*$-connected if for any $e \in F$ and $e' \in F$, there exists a sequence of edges starting at e, ending at e' and such that any two successive edges in this sequence are $*$-neighbors.

PROOF. Remember that $\partial_{\mathcal{B}} A = \{(x, y); x \sim y,\ x \in A \text{ and } y \in \mathcal{B} \setminus A \text{ or } x \in \mathcal{B} \setminus A \text{ and } y \in A\}$. In particular, we do not take into account edges with an endpoint outside $\mathcal{B}$.

We identify a point $x \in \mathbb{Z}^d$ with the unit cube of $\mathbb{R}^d$ whose center is x. Similarly, we identify an edge $e = (x, y)$ with the common face of the cubes defined by x and y. By this way, $\mathcal{B}$ is identified with a cube in $\mathbb{R}^d$, say $\tilde{\mathcal{B}}$. A is identify with a connected subset of $\tilde{\mathcal{B}}$, say $\tilde{A}$. $\tilde{\mathcal{B}} \setminus \tilde{A}$ is connected, and $\partial_{\mathcal{B}} A$ is identified with the boundary of $\tilde{A}$ in $\tilde{\mathcal{B}}$, say $\partial_{\tilde{\mathcal{B}}} \tilde{A}$.

We shall prove that $\partial_{\tilde{\mathcal{B}}} \tilde{A}$ is connected.

Let $e = (a, b)$ and $e' = (a', b')$ be edges in $\partial_{\mathcal{B}} A$. Assume that $a \in A$, $a' \in A$, $b \in \mathcal{B} \setminus A$, $b' \in \mathcal{B} \setminus A$. Since A is connected, there is a path in A linking a with a'. In the same way, since $\mathcal{B} \setminus A$ is connected, there is a path in $\mathcal{B} \setminus A$ linking b with b'. Joining these two paths and the segments $[a, a']$ and $[b, b']$, we obtain a closed loop in $\tilde{\mathcal{B}}$, say γ, that contains points both in the interior of $\tilde{A}$ and $\tilde{\mathcal{B}} \setminus \tilde{A}$, and with exactly two points of intersection with $\partial_{\tilde{\mathcal{B}}} \tilde{A}$, say α and α'.

Let $(\gamma(t), t \in [0, 1])$ be a smooth deformation of γ such that $\gamma(0) = \gamma$, $\gamma(1)$ is reduced to one point and lies in the interior of $\tilde{A}$. Further, assume that $\gamma(t) \subset \tilde{\mathcal{B}}$ for all $t \in [0, 1]$.

Consider the evolution of the set $\gamma(t) \cap \partial_{\tilde{\mathcal{B}}} \tilde{A}$, and note that, as t goes from 0 to 1, $\gamma(t) \cap \partial_{\tilde{\mathcal{B}}} \tilde{A}$ is made of a finite number of points which are continuously moving on $\partial_{\tilde{\mathcal{B}}} \tilde{A}$. Two points may meet, and then disappear. A new point may also appear, and then split into two new points that start wandering around.

Call $(\alpha_t, t \in [0, 1])$ the trajectory in $\gamma(t) \cap \partial_{\tilde{\mathcal{B}}} \tilde{A}$ issued from α. We thus see that α_t cannot disappear from $\gamma(t) \cap \partial_{\tilde{\mathcal{B}}} \tilde{A}$ unless it meets another point in $\gamma(t) \cap \partial_{\tilde{\mathcal{B}}} \tilde{A}$. A moment of thought should convince the reader that this fact implies that there is an arc in $\bigcup_{t \in [0,1]} \gamma(t) \cap \partial_{\tilde{\mathcal{B}}} \tilde{A}$ joining α and α'.

Thus, $\partial_{\tilde{\mathcal{B}}} \tilde{A}$ is connected, which implies that $\partial_{\mathcal{B}} A$ is $*$-connected; see [6], Appendix A. $\square$

APPENDIX B: CARDINAL OF THE CLUSTER IN THE BOX

We establish that, for all $p > p_c : \exists \alpha$ such that Q a.s. on the set $\#\mathcal{C} = +\infty$, one has, for large enough n,

$$\#\mathcal{C}^n \geq 2\alpha(2n + 1)^d. \tag{27}$$

Choose $\rho \geq 1$. For $x \in \mathcal{C}$, let $D(0, x)$ denote the minimal length of an open path in $\mathcal{C}$ connecting 0 and x. Assume that there exists $x \in \mathcal{C} \setminus \mathcal{C}^n$ such that $|x| \leq n/\rho$. Without loss of generality, we may, and will, assume that $|x| \geq n/(2\rho)$. Thus the shortest path in $\mathcal{C}$ linking 0 to x must leave the box $\mathcal{B}^n$ and therefore $D(0, x) \geq n \geq \rho|x|$. From ([1], Theorem 1.1), we know that there exist a choice of ρ and a constant $\beta > 0$ such that

$$Q[x \in \mathcal{C},\ D(0, x) \geq \rho|x|] \leq e^{-\beta|x|},$$

as $|x| \to +\infty$. In particular,

$$\begin{aligned} Q&\left[\exists x \in \mathcal{C} \setminus \mathcal{C}^n,\ |x| \leq \frac{n}{\rho}\right] \\ &\leq Q\left[\exists x \in \mathcal{C},\ \frac{n}{2\rho} \leq |x| \leq \frac{n}{\rho},\ D(0, x) \geq \rho|x|\right] \\ &\leq \sum_{|x| \in [\frac{n}{2\rho}, \frac{n}{\rho}]} e^{-\beta|x|} \\ &\leq (2n + 1)^d e^{-\beta n/(2\rho)}. \end{aligned}$$

From the Borel–Cantelli lemma, we deduce that, Q a.s, for large enough n,

$$\mathcal{C} \cap \mathcal{B}^{n/\rho} \subset \mathcal{C}^n.$$

It directly follows from the ergodic theorem that $\#(\mathcal{C} \cap \mathcal{B}^{n/\rho})/(2n + 1)^d$ has an almost sure nonvanishing limit. Therefore $\#\mathcal{C}^n/(2n + 1)^d$ has an almost sure nonvanishing lim inf.

APPENDIX C: CARNE–VAROPOULOS BOUND

For a given subgraph of $\mathbb{Z}^d$, say ω, P_0^ω denotes the law of the continuous time random walk on ω started at 0, that is, under P_0^ω, the coordinate process $(X_t, t \geq 0)$ waits for an exponential time of parameter 1, then chooses uniformly at random one of its neighbors, say y, and moves to y if $y \in \omega$. Otherwise, X stays still.

We can also construct X_t as a time changed discrete parameter random walk on ω. Then $X_t = Y_{N_t}$, where N_t is a Poisson process of parameter 1 and $(Y_k, k \in \mathbb{N})$ is the discrete time random walk on ω defined by successively choosing, uniformly at random, one neighbor of the current position and moving to it if it belongs to ω.

From [3], we know that

$$P_0^\omega[Y_k = x] \leq e^{-|x|^2/(2k)}.$$

Therefore,

$$\begin{aligned} P_0^\omega[X_t = x] &\leq e^{-|x|^2/(4t)} + P_0^\omega[N_t \geq 2t] \\ &\leq e^{-|x|^2/(4t)} + e^{-ct}, \end{aligned} \tag{28}$$

where $c = \log 4 - 1$.

Let now τ^n be the exit time of X from $\mathcal{B}^{n-1}$. Thus $\sigma^n = N_{\tau^n}$, where σ^n is the exit time for the process Y. Then

$$\begin{aligned} P_0^\omega[\sigma^n \le k] &= \sum_{i=0}^{k} P_0^\omega[\sigma^n = i] \\ &\le \sum_{i=0}^{k} \sum_{y \in \mathcal{B}^n \setminus \mathcal{B}^{n-1}} P_0^\omega[Y_i = y] \\ &\le \sum_{i=0}^{k} \sum_{y \in \mathcal{B}^n \setminus \mathcal{B}^{n-1}} e^{-|y|^2/(2i)} \end{aligned}$$

(with the Carne–Varopoulos inequality). Now, as $y \in \mathcal{B}^n \setminus \mathcal{B}^{n-1} \Rightarrow |y| = n$, we obtain the following upper bound:

$$P_0^\omega[\sigma^n \le k] \le kn^{d-1} e^{-n^2/(2k)}.$$

Thus

$$\begin{aligned} P_0^\omega[\tau^n \le t] &\le P_0^\omega[\sigma^n \le 2t] + P_0^\omega[N_t \ge 2t] \\ &\le 2tn^{d-1} e^{-n^2/(4t)} + e^{-ct}, \end{aligned} \tag{29}$$

for some constant $c > 0$.

APPENDIX D: LOWER BOUND FOR $P_0^\omega[X_t = 0]$

To conclude, we briefly discuss the lower bound issue. We shall only consider the mean: $Q[P_0^\omega[X_t = 0] | \#\mathcal{C} = \infty]$. Q a.s. lower bounds, that is, almost-sure lower bounds for $P_0^\omega[X_t = 0]$ are being investigated at the present time. Our result is as follows:

For all $p > p_c$, there exists a constant c such that, for all $t > 0$,

$$Q\big[P_0^\omega[X_t = 0] | \#\mathcal{C} = \infty\big] \ge \frac{c}{t^{d/2}}. \tag{30}$$

For $p > p_c$, there is, with Q probability 1, a unique infinite cluster in ω, say $\mathcal{G}$ and $Q[P_0^\omega[X_t = 0] | \#\mathcal{C} = \infty] = Q[P_0^\omega[X_t = 0] | 0 \in \mathcal{G}]$.

In the next sequence of inequalities, we use the reversibility of the process X_t and the translation invariance, that is, the fact that $Q[P_x^\omega[X_t = x]; x \in \mathcal{G}]$ does not depend on x:

$$\begin{aligned}
Q\big[P_0^\omega[X_t = y]; 0 \in \mathcal{G}\big] &= Q\big[P_0^\omega[X_t = y]; 0, y \in \mathcal{G}\big] \\
&= Q\left[\sum_z P_0^\omega[X_{t/2} = z] P_z^\omega[X_{t/2} = y]; 0, y \in \mathcal{G}\right] \\
&= Q\left[\sum_z P_0^\omega[X_{t/2} = z] P_y^\omega[X_{t/2} = z]; 0, y \in \mathcal{G}\right] \\
&\le Q\left[\sqrt{\sum_z P_0^\omega[X_{t/2} = z]^2}\sqrt{\sum_z P_y^\omega[X_{t/2} = z]^2}; 0, y \in \mathcal{G}\right] \\
&= Q\Big[\sqrt{P_0^\omega[X_t = 0]}\sqrt{P_y^\omega[X_t = y]}; 0, y \in \mathcal{G}\Big] \\
&\le \sqrt{Q[P_0^\omega[X_t = 0]; 0 \in \mathcal{G}]}\sqrt{Q[P_y^\omega[X_t = y]; y \in \mathcal{G}]} \\
&= Q\big[P_0^\omega[X_t = 0]; 0 \in \mathcal{G}\big].
\end{aligned}$$

From the invariance principle (see [5]) it follows that there exists $a > 0$ such that, for all $t > 0$, we have

$$Q\big[P_0^\omega\big[X_t \in \mathcal{B}^{a\sqrt{t}}\big]\big|0 \in \mathcal{G}\big] \ge \tfrac{1}{2}.$$

In particular,

$$\sup_{y \in \mathcal{B}^{a\sqrt{t}}} Q\big[P_0^\omega[X_t = y]\big|0 \in \mathcal{G}\big] \ge c t^{-d/2},$$

for some constant c that depends only on the dimension.

Since $\sup_{y \in \mathcal{B}^{a\sqrt{t}}} Q[P_0^\omega[X_t = y]|0 \in \mathcal{G}] = Q[P_0^\omega[X_t = 0]|0 \in \mathcal{G}]$, (30) is proved.

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CMI
39 RUE JOLIOT-CURIE
13013 MARSEILLE
A FRANCE
E-MAIL: pierre.mathieu@cmi.univ-mrs.fr

IML
CAMPUS DE LUMINY
13009 MARSEILLE
FRANCE
E-MAIL: remy@iml.univ-mrs.fr