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ISOSINGULAR LOCI AND THE CARTESIAN PRODUCT STRUCTURE OF COMPLEX ANALYTIC SINGULARITIES

BY

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ABSTRACT. Let X be a (not necessarily reduced) complex analytic space, and let V be a germ of an analytic space. The locus of points q in X at which the germ X_q is complex analytically isomorphic to V is studied. If it is nonempty it is shown to be a locally closed submanifold of X, and X is locally a Cartesian product along this submanifold. This is used to define what amounts to a coarse partial ordering of singularities. This partial ordering is used to show that there is an essentially unique way to completely decompose an arbitrary reduced singularity as a cartesian product of lower dimensional singularities. This generalizes a result previously known only for irreducible singularities.

0. Introduction. Let X be a complex analytic space. For $q \in X$, X_q will denote the germ of X at q. In this paper I will study the isosingular loci defined by

DEFINITION 0.1. For $p \in X$ let

$$\operatorname{Iso}(X,p) = \{q \in X \mid X_q \cong X_p\}.$$

($\cong$ here and elsewhere will mean complex analytically isomorphic.) It will be shown that:

THEOREM 0.2. *For any $p \in X$, $\operatorname{Iso}(X,p)$ is a (possibly 0-dimensional) complex submanifold of some open subset of X. Moreover, for any $q \in \operatorname{Iso}(X,p)$ there is an open neighbornood U of q, and an analytic space Y such that $U \cong Y \times (U \cap \operatorname{Iso}(X,p))$. ($\times$ is the cartesian product in the category of analytic spaces.)*

This result is used to introduce what is, in effect, a partial ordering of complex analytic singularities in terms of their complexity. This, in turn, is used to study the ways in which a germ of an analytic space may be written as the cartesian product of other germs of analytic spaces. Let V be a germ of an analytic space (V not the reduced point). By a *decomposition of V of length*

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k I mean an unordered k-tuple $(V_1, \ldots, V_k)$ of germs of analytic spaces, no V_i being the reduced point, such that $V \cong V_1 \times \cdots \times V_k$. (Note that all the V_j will be reduced and positive dimensional if and only if V is reduced.) V will be called *indecomposable* if and only if V has no decomposition of length 2. Finally, V will be called *uniquely decomposable* if and only if (i) V has a decomposition $(V_1, \ldots, V_k)$ with all V_j indecomposable, and (ii) if $(V_1, \ldots, V_k)$ and $(W_1, \ldots, W_h)$ are two such decompositions of V, then $k = h$, and, after permuting the W_j's, one has $V_j \cong W_j$ for all j.

It will be shown that:

THEOREM 0.3. *If V is a positive dimensional germ of a reduced analytic space, then V is uniquely decomposable.*

This generalizes a result from [2]. It would be interesting to know if nonreduced singularities are uniquely decomposable. In particular, this would give a structure theorem for complex analytic Artin local rings.

Finally, let me remark that analogues of these definitions and results can also be formulated for reduced, irreducible germs of real analytic spaces, although the proofs are different [1]. For the purpose of the real analytic analogue of Theorem 0.3, a germ of real analytic space is said to be reduced if the natural map ${}_V\mathcal{Q} \to {}_V\mathcal{C}$ is injective. (${}_V\mathcal{Q}$ here is the real analytic local ring of V, and ${}_V\mathcal{C}$ is the ring of germs of continuous functions on V.) In particular, if V is a reduced irreducible germ of a complex space, then V is uniquely decomposable both complex analytically and real analytically, and these two decompositions are essentially the same [1]. It is not yet known if this is true for reducible V.

I wish to thank the referee for many helpful suggestions, especially for the simple proof he suggested for Lemma 1.5.

1. Preliminaries. Before proceeding with the proof of Theorem 0.2 I collect some useful preliminaries. The bulk of this section is well known, at least for reduced spaces.

I begin by giving the natural generalization of Whitney's first tangent cone [8] to arbitrary germs of analytic spaces. Let V be a germ of analytic space with local ring ${}_V\mathcal{O}$. Let ${}_V\mathfrak{m}$ denote the maximal ideal of ${}_V\mathcal{O}$ and e: ${}_V\mathcal{O} \to \mathbf{C} = {}_V\mathcal{O}/{}_V\mathfrak{m}$ be the natural evaluation map. Recall that one can define the Zariski tangent space of V, $TV = \{\mathbf{C}\text{-derivations } t\colon {}_V\mathcal{O} \to \mathbf{C}\}$.

DEFINITION 1.1. $C_1(V) = \{t \in TV \mid \text{there is a } \mathbf{C}\text{-derivation } \tau\colon {}_V\mathcal{O} \to {}_V\mathcal{O}$ satisfying $t = e \circ \tau\}$.

Clearly $C_1(V)$ is a complex linear subspace TV which is a bianalytic invariant of V. Let V be embedded as the germ at $0 \in \mathbf{C}^n$ of the complex analytic subspace of $\mathbf{C}^n$ defined by the ideal $\mathcal{I} \subset {}_n\mathcal{O}$ (${}_n\mathcal{O}$ is the ring of germs of holomorphic functions at $0 \in \mathbf{C}^n$). Then for $t \in TV$ we have $t \in C_1(V)$ if and only if there is a germ at $0 \in \mathbf{C}^n$ of a holomorphic vector field U such that $U(0) = t$ and $U\mathcal{I} \subset \mathcal{I}$.

Now suppose V and W are two germs of analytic spaces. Then, as is well known, we have natural inclusions

$$TV \subset T(V \times W) \quad \text{and} \quad TW \subset T(V \times W),$$

such that $TV \cap TW = \{0\}$ and $T(V \times W) = TV \oplus TW$. Moreover, we have

LEMMA 1.2. $C_1(V \times W) = C_1(V) \oplus C_1(W)$.

PROOF. The proof is easy and is left to the reader.

$C_1(V)$ is interesting because

LEMMA 1.3. *Let V be the germ at $0 \in \mathbf{C}^n$ of the analytic subspace of $\mathbf{C}^n$ defined by the ideal $\mathcal{I} \subset {}_n\mathcal{O}$. Then there is a germ of a holomorphic vector field U on $\mathbf{C}^n$ with $U(0) \neq 0$ and $U\mathcal{I} \subset \mathcal{I}$ if and only if there is a germ of an analytic space W such that $V \cong W \times \mathbf{C}_0$. ($\mathbf{C}_0$ denotes the germ of $\mathbf{C}$ at $0 \in \mathbf{C}$.)*

PROOF. See [3, §2.12].

COROLLARY 1.4. *Let V be the germ at $0 \in \mathbf{C}^n$ of the analytic subspace of $\mathbf{C}^n$ defined by the ideal $\mathcal{I} \subset {}_n\mathcal{O}$. Then there are k germs of holomorphic vector fields $U_1, \ldots, U_k$ which preserve $\mathcal{I}$ and such that $U_1(0), \ldots, U_k(0)$ are linearly independent if and only if there is a germ of an analytic space W such that $V \cong W \times \mathbf{C}_0^k$. Also, $d = \dim_{\mathbf{C}} C_1(V)$ is the greatest such k.*

PROOF. The corollary follows from repeated applications of Lemma 1.3, the repeated applications being justified by Lemma 1.2. □

I finish the preliminaries with

LEMMA 1.5. *If V and W are germs of analytic spaces such that $V \times \mathbf{C}_0 \cong W \times \mathbf{C}_0$, then $V \cong W$.*

PROOF. The proof is based on an elementary remark. Let $Z \subset \mathbf{C}_0^n$ be any germ of an analytic space. Then $Z \times \mathbf{C}_0^k \subset \mathbf{C}_0^{n+k}$ in a natural way, and clearly $Z \cong (Z \times \mathbf{C}_0^k) \cap (\mathbf{C}_0^n \times \{0\})$. But more is true. If $M \subset \mathbf{C}_0^{n+k}$ is any germ of a complex n-manifold transverse to $\{0\} \times \mathbf{C}_0^k$, then $Z \cong M \cap (Z \times \mathbf{C}_0^k)$. To see this choose coordinates $(x_1, \ldots, x_n)$ on $\mathbf{C}_0^n$ and coordinates $(y_1, \ldots, y_k)$ on $\mathbf{C}_0^k$. Then M will be defined by equations $y_j - f_j(x_1, \ldots, x_n) = 0$, $j = 1, \ldots, k$. The mapping which sends $(x_1, \ldots, x_n, y_1, \ldots, y_k)$ to $(x_1, \ldots, x_n, y_1 - f_1(x_1, \ldots, x_n), \ldots, y_k - f_k(x_1, \ldots, x_n))$ is an isomorphism of $\mathbf{C}_0^{n+k}$ to itself which gives, by restriction, an explicit isomorphism

$$M \cap (Z \times \mathbf{C}_0^k) \cong (\mathbf{C}_0^n \times \{0\}) \cap (Z \times \mathbf{C}_0^k) \cong Z.$$

PROOF OF LEMMA 1.5. By Lemma 1.2, $\dim C_1(V) = \dim C_1(W) = k - 1$.

So, by Corollary 1.4, $V \cong V' \times \mathbf{C}_0^{k-1}$ and $W = W' \times \mathbf{C}_0^{k-1}$ for some V' and W' with $\dim C_1(V') = \dim C_1(W') = 0$. To prove $V \cong W$ it suffices to show $V' \cong W'$. By assumption $V' \times \mathbf{C}_0^k \cong W' \times \mathbf{C}_0^k$.

Suppose V' and W' are embedded as germs in $\mathbf{C}_0^n$. Then any isomorphism Ω: $V' \times \mathbf{C}_0^k \to W' \times \mathbf{C}_0^k$ extends to an isomorphism Ω: $\mathbf{C}_0^{n+k} \to \mathbf{C}_0^{n+k}$. Let $M \subset \mathbf{C}_0^{n+k}$ be a germ of a complex n-manifold transverse to $\{0\} \times \mathbf{C}_0^k$. Then by the above remark,

$$V' \cong M \cap \left(V' \times \mathbf{C}_0^k\right) \cong \Omega(M) \cap \Omega\left(V' \times \mathbf{C}_0^k\right) \cong \Omega(M) \cap \left(W' \times \mathbf{C}_0^k\right),$$

and it suffices to show that $\Omega(M)$ is transverse to $\{0\} \times \mathbf{C}_0^k$.

By construction

$$C_1\left(V' \times \mathbf{C}_0^k\right) = C_1\left(W' \times \mathbf{C}_0^k\right) = T\left(\{0\} \times \mathbf{C}_0^k\right).$$

The choice of M gives

$$TM \cap C_1\left(V' \times \mathbf{C}_0^k\right) = TM \cap T\left(\{0\} \times \mathbf{C}_0^k\right) = \{0\}.$$

Thus

$$\begin{aligned} T\Omega(M) \cap T\left(\{0\} \times \mathbf{C}_0^k\right) &= T\Omega(M) \cap C_1\left(W' \times \mathbf{C}_0^k\right) \\ &= T\Omega(M) \cap C_1\left(\Omega\left(V' \times \mathbf{C}_0^k\right)\right) = \{0\}, \end{aligned}$$

and we are done. $\square$

2. Proof of Theorem 0.2. I now turn my attention to Theorem 0.2. Let X be an analytic space and $p \in X$. Then clearly, for $q \in \operatorname{Iso}(X, p)$ one has $\operatorname{Iso}(X, q) = \operatorname{Iso}(X, p)$, so that Theorem 0.2 is purely local and may be restated as

THEOREM 2.1. *Let X be an analytic space and let $p \in X$. Then there is an open neighborhood U of p and an analytic space Y such that* $\operatorname{Iso}(X, p) \cap U$ *is a (possibly* 0*-dimensional) complex submanifold of U and*

$$U \cong Y \times (\operatorname{Iso}(X, p) \cap U).$$

PROOF. The proof of this theorem will take the rest of this section. It is convenient to begin with a definition.

DEFINITION 2.2. For $p \in X$ let $M(X, p)$ be the smallest germ at p of an analytic subspace of X such that $\operatorname{Iso}(X, p)_p \subset M(X, p)$. ($\operatorname{Iso}(X, p)_p$ denotes the germ at p of $\operatorname{Iso}(X, p)$.)

$M(X, p)$ certainly exists because the local ring ${}_X\mathcal{O}_p$ of X_p is noetherian. Moreover, $M(X, p)$ is a reduced germ because of its minimality. Also, if ψ: $X_p \to X_q$ is an isomorphism, then ψ induces an isomorphism ψ: $\operatorname{Iso}(X, p)_p \to \operatorname{Iso}(X, p)_q = \operatorname{Iso}(X, q)_q$, so that ψ must also induce an isomorphism ψ: $M(X, p) \to M(X, q)$.

LEMMA 2.3. *$M(X, p)$ is a (possibly* 0*-dimensional) germ of a submanifold of X_p.*

PROOF. Choose a neighborhood U of p small enough to find an analytic subspace M of U satisfying $M_p = M(X,p)$. By shrinking U we may assume that $\mathrm{Iso}(X,p) \cap U \subset M$, and also that $\dim M_q \leq \dim M(X,p)$ for all $q \in M$.

By the minimality of $M(X,p)$ we have $\mathrm{Iso}(X,p)_p \not\subset \mathrm{Sg}(M(X,p)) = (\mathrm{Sg}(M))_p$. Hence there is a $q \in \mathrm{Iso}(X,p) \cap U \subset M$ for which M_q is the germ of a manifold. Since $q \in \mathrm{Iso}(X,p)$ we have $\mathrm{Iso}(X,q)_q = \mathrm{Iso}(X,p)_q \subset M_q$ so that $M(X,q) \subset M_q$ and $\dim M(X,q) \leq \dim M_q$. But $q \in \mathrm{Iso}(X,p)$ also gives $M(X,q) \cong M(X,p)$ so that $\dim M(X,q) = \dim M(X,p) \geq \dim M_q$. Thus, in fact, $\dim M(X,q) = \dim M_q$. This, together with $M(X,q) \subset M_q$ and the fact that M_q is an irreducible germ, gives $M(X,q) = M_q$, which is a germ of a manifold. But $M(X,p) \cong M(X,q)$ and the lemma is proven. □

Note that $\dim M(X,p) = 0$ if and only if p is an isolated point of $\mathrm{Iso}(X,p)$, and in this case Theorem 2.1 is trivial. For the rest of this section I will assume $\dim M(X,p) = n \geq 1$.

REMARK 2.4. Since Theorem 2.1 is purely local in a neighborhood of p, we may shrink X by replacing X with a small open neighborhood of $p \in X$. This allows us to put X in a convenient form.

In this way we may suppose we have a connected submanifold $M \subset X$ such that $\mathrm{Iso}(X,p) \subset M$, and $M(X,p) = M_p$. Then, for all $q \in \mathrm{Iso}(X,p)$ we have $\mathrm{Iso}(X,q) = \mathrm{Iso}(X,p) \subset M$, and thus $M(X,q) \subset M_q$. But for $q \in \mathrm{Iso}(X,p)$ we have $M(X,q) \cong M(X,p) = M_p \cong M_q$, and we get $M(X,q) = M_q$ for all $q \in \mathrm{Iso}(X,p)$.

We may also assume that X is embedded as an analytic subspace of a polydisc Δ, $0 \in \Delta \subset \mathbf{C}^{n+m}$ (where $(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ give the coordinates on $\mathbf{C}^{n+m}$), and that the embedding is such that $p = 0 \in \mathbf{C}^{n+m}$ and $M = \Delta \cap (\mathbf{C}^n \times \{0\}) = \Delta \cap \{(x_1, \ldots, x_n, 0, \ldots, 0)\}$. Finally, we may also assume that we have holomorphic functions $f_1, \ldots, f_r$ on Δ which globally generate the coherent ideal sheaf defining X in Δ, and whose germs at $0 \in \mathbf{C}^{n+m}$, $f_{10}, \ldots, f_{r0}$, give a minimal set of generators for the defining ideal of the germ X_0. This setup will be fixed for the rest of this section. □

OBSERVATION 2.5. Theorem 2.1 will follow if it can be shown that $\dim C_1(X_0) \geq n = \dim M(X,0)$.

PROOF. It would then follow from Corollary 1.4 that there is an analytic space Y, a domain $D \subset \mathbf{C}^t$ ($t = \dim C_1(X_0)$), a neighborhood U of 0 in X, and an isomorphism $\psi \colon Y \times D \to U$. Let $(y_0, d_0) = \psi^{-1}(0) \in Y \times D$. Then $\{y_0\} \times D \subset \mathrm{Iso}(Y \times D, \psi^{-1}(0))$ so that $\psi(\{y_0\} \times D) \subset \mathrm{Iso}(X,0) \cap U \subset M \cap U$. Since $\psi(\{y_0\} \times D)$ and $M \cap U$ are submanifolds of U, and $\dim \psi(\{y_0\} \times D) = t \geq n = \dim(M \cap U)$, it follows that $t = n$ and $\psi(\{y_0\} \times D)$ is just the union of components of $M \cap U$. Shrinking Y, D, and U we can achieve $\psi(\{y_0\} \times D) = M \cap U$. But then $\mathrm{Iso}(X,0) \cap U = M \cap U$, a submanifold of U, and the result follows by using the isomorphism ψ: $\{y_0\} \times D \to M \cap U = \mathrm{Iso}(X, 0) \cap U$ to identify D and $\mathrm{Iso}(X, 0) \cap U$. $\square$

I now give a construction of Seidenberg [5], [6] which will be used to show $\dim C_1(X_0) \geqslant n$. Intuitively, the construction gives, for any natural number k, an algebraic variety whose points are certain k-jets of k-equivalences of X_0, and a constructible set whose points are certain "k-jets" of germs V k-equivalent to X_0. Recall that the germs V and W are k-equivalent if ${}_V\mathcal{O}/{}_V\mathfrak{m}^{k+1} \cong {}_W\mathcal{O}/{}_W\mathfrak{m}^{k+1}$ (${}_V\mathfrak{m}$ and ${}_W\mathfrak{m}$ are the maximal ideals in ${}_V\mathcal{O}$ and ${}_W\mathcal{O}$).

Let $g_1(P, x, y), \ldots, g_r(P, x, y)$ be polynomials of degree k in the variables (x, y) with indeterminant coefficients which I collectively denote by (P) (just as (x) collectively denotes $(x_1, \ldots, x_n)$). Let $a_{ij}(Q, x, y)$, $1 \leqslant i, j \leqslant r$, be polynomials of degree k in the (x, y) with indeterminant coefficients which I collectively denote by (Q). Let $\varphi_1(R, x, y), \ldots, \varphi_n(R, x, y)$, $\psi_1(R, x, y), \ldots, \psi_m(R, x, y)$ be polynomials of degree k in (x, y) such that $\varphi_i(R, 0, 0) = 0$, $1 \leqslant i \leqslant n$, and $\psi_j(R, x, 0) = 0$, $1 \leqslant j \leqslant m$, and having indeterminant coefficients which I collectively denote by (R). For convenience I let $\mathrm{Jac}(\varphi, \psi)(0)$ denote the jacobian of $(\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m)$ with respect to the (x, y) evaluated at $(x, y) = (0, 0)$. $\mathrm{Jac}(\varphi, \psi)(0)$ is a polynomial in the (R)'s. Finally let S be an indeterminant.

The (P) give coordinates on some affine space $\mathbf{C}^{M(k)}$. The (P, Q, R, S) give coordinates on some affine space $\mathbf{C}^{N(k)}$. Let π_P: $\mathbf{C}^{N(k)} \to \mathbf{C}^{M(k)}$ be defined by $\pi_P(P, Q, R, S) = (P)$.

Let $T_0 f_1, \ldots, T_0 f_r$ be the Taylor expansions about $0 \in \mathbf{C}^{n+m}$ of $f_1, \ldots, f_r$ (which are chosen as in Remark 2.4). Consider the conditions:

$$\begin{aligned} &g_i(P, \varphi(R, x, y), \psi(R, x, y)) - \textstyle\sum a_{ij}(Q, x, y) T_0 f_j(x, y) \\ &\text{are in the } (k+1)\text{st power of the ideal generated by the} \\ &(x_1, \ldots, x_n, y_1, \ldots, y_m) \text{ for } 1 \leqslant i \leqslant r. \end{aligned} \tag{2.6}$$

These conditions are equivalent to a finite number of polynomial equations in the (P, Q, R). These equations, together with the polynomial equation

$$S \cdot \mathrm{Jac}(\varphi, \psi)(0) \cdot \det\|a_{ij}(Q, 0, 0)\| = 1, \tag{2.7}$$

define an analytic subspace of $\mathbf{C}^{N(k)}$. Let $A(k)$ be the reduction of this analytic space. Then $A(k)$ is a finite union of affine subvarieties of $\mathbf{C}^{N(k)}$. Let $B(k) = \pi_P(A(k))$. Then $B(k)$ is a constructible subset of $\mathbf{C}^{M(k)}$ [4, p. 97].

Now, in $\mathbf{C}^n$ having coordinates $(z) = (z_1, \ldots, z_n)$ define a polydisc $\Delta' \subset \mathbf{C}^n$ by $\Delta' = \{(z) \in \mathbf{C}^n | (z, 0) \in \Delta \subset \mathbf{C}^{n+m}\}$. (Here Δ is as defined in Remark 2.4.) For a fixed $(z) \in \Delta'$ and for a function h holomorphic on a neighborhood of $(z, 0)$ in Δ, $h(z + x, y)$ is a function holomorphic near $0 \in \mathbf{C}^{n+m}$. I let $T_z h$ denote the Taylor expansion in the variables (x, y) centered at $(x, y) = (0, 0)$ of the function $h(z + x, y)$. I will let $T_z^k h$ denote the polynomial one gets from $T_z h$ by discarding all terms of order greater than k. Note that the coefficients of $T_z h$ are just the values at $(z, 0)$ of various derivatives of h. Thus, these coefficients vary holomorphically with (z).

Let $f_1, \ldots, f_r$ be as in Remark 2.4. Then the coefficients of $T_z^k f_1, \ldots, T_z^k f_r$ define a holomorphic map $T(k)\colon \Delta \to \mathbf{C}^{M(k)}$.

REMARK 2.8. If $(z) \in T(k)^{-1}(B(k))$, then $X_{(z,0)}$ is k-equivalent to X_0. In fact, any point of $A(k) \cap \pi_P^{-1}(T(k)(z))$ gives a germ of an isomorphism $(\varphi(k), \psi(k))\colon \mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$ (defined by polynomials) and an $r \times r$ matrix $\|a_{ij}(k)\| \in \mathrm{Gl}(r, {}_{n+m}\mathcal{O})$ (defined by polynomials) which satisfy

$$T_z^k f_i(\varphi(k), \psi(k)) - \sum a_{ij}(k) T_0 f_j \in {}_{n+m}\mathfrak{m}^{k+1}, \qquad 1 \leqslant i \leqslant r. \tag{2.9}$$

This is equivalent to

$$f_i(z + \varphi(k)(x, y), \psi(k)(x, y)) - \sum a_{ij}(k)(x, y) f_j(x, y) \in {}_{n+m}\mathfrak{m}^{k+1}, \quad 1 \leqslant i \leqslant r. \tag{2.10}$$

This shows that $(z + \varphi(k), \psi(k))\colon \mathbf{C}_0^{n+m} \to \mathbf{C}_{(z,0)}^{n+m}$ defines k-equivalence from X_0 to $X_{(z,0)}$.

REMARK 2.11. If $z \in T(k)^{-1}(B(k))$ for all k, then $X_0 \cong X_{(z,0)}$. Thus $(z, 0) \in \mathrm{Iso}(X, 0)$.

PROOF. By Remark 2.8, for each k we have a germ of an isomorphism $(\varphi(k), \psi(k))\colon \mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$ and an $\|a_{ij}(k)\| \in \mathrm{Gl}(r, {}_{n+m}\mathcal{O})$ satisfying (2.10). I apply Wavrik's [7] extension of Artin's theorem on solutions of analytic equations. By this result, for k sufficiently large we can find a germ of a map $(\varphi, \psi)\colon \mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$ and $a_{ij} \in {}_{n+m}\mathcal{O}$, $1 \leqslant i, j \leqslant r$, which satisfy

$$
\begin{aligned}
&\text{(a)} \quad \varphi_i - \varphi_i(k) \in {}_{n+m}\mathfrak{m}^2, \qquad 1 \leqslant i \leqslant n,\\
&\text{(b)} \quad \psi_j - \psi_j(k) \in {}_{n+m}\mathfrak{m}^2, \qquad 1 \leqslant j \leqslant m,\\
&\text{(c)} \quad a_{ij} - a_{ij}(k) \in {}_{n+m}\mathfrak{m}^2, \qquad 1 \leqslant i,j \leqslant r \quad \text{and}\\
&\text{(d)} \quad f_i(z + \varphi(x, y), \psi(x, y)) - \sum a_{ij}(x, y) f_j(x, y) = 0,\\
&\qquad\qquad 1 \leqslant i \leqslant r.
\end{aligned} \tag{2.12}
$$

(2.12)(a) and (b) show that $(\varphi, \psi)\colon \mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$ is a germ of an isomorphism and (2.12)(c) shows that $\|a_{ij}\| \in \mathrm{Gl}(r, {}_{n+m}\mathcal{O})$. With these facts in mind, (2.12)(d) shows that $(z + \varphi, \psi)\colon \mathbf{C}_0^{n+m} \to \mathbf{C}_{(z,0)}^{n+m}$ induces an isomorphism $X_0 \to X_{(z,0)}$. □

REMARK 2.13. If $(z, 0) \in \mathrm{Iso}(X, 0)$ then $z \in T(k)^{-1}(B(k))$ for all k.

I now finish the proof of Theorem 2.1 with

PROPOSITION 2.15. $\dim C_1(X_0) = \dim M(X, 0) = n$.

PROOF. The argument of Observation 2.5 shows that $\dim C_1(X_0) \leqslant n$. The other inequality must be proven.

Since $B(k)$ is constructible we can write $T(k)^{-1}(B(k)) \times \{0\} \subset M$ as a finite union $T(k)^{-1}(B(k)) \times \{0\} = \bigcup (F_i - G_i)$ where each F_i and G_i are analytic subsets of M, and G_i contains no irreducible component of F_i. By Remark 2.13 we have $\mathrm{Iso}(X, 0) \subset T(k)^{-1}(B(k)) \times \{0\} \subset \bigcup F_i \subset M$. But then $\mathrm{Iso}(X, 0)_0 \subset \bigcup F_{i0} \subset M_0 = M(X, 0)$. By the minimality of $M(X, 0)$ it follows that $\bigcup F_{i0} = M_0$. But, since M_0 is irreducible, there is an i, say $i = 1$, so that $F_{10} = M_0$. But then $F_1 = M$, G_1 is a proper analytic subset of M, and $M - G_1 \subset T(k)^{-1}(B(k)) \times \{0\}$.

G_1, of course, depends on k. Putting this dependency into the notation, write $G(k) = G_1$. Let H denote the union of all the $G(k)$. Then $M - H$ is dense in M and $M - H \subset T(k)^{-1}(B(k)) \times \{0\}$ for every k. Thus, by Remark 2.11, we have $M - H \subset \mathrm{Iso}(X, 0)$. Thus, $\mathrm{Iso}(X, 0)$ is dense in M.

Now for a positive integer k define $E(k) \subset \Delta' \times \mathbf{C}^{N(k)}$ by

$$E(k) = \{(z, P, Q, R, S) | (P, Q, R, S) \in A(k) \text{ and } T(k)(z) = P\}.$$

$E(k)$ carries in a natural way the structure of a reduced analytic space, and I will suppose it is so endowed. Let π_1: $E(k) \to \Delta'$ be defined by $\pi_1(z, P, Q, R, S) = (z)$. $E(k)$ is constructed so that $\pi_1(E(k)) = T(k)^{-1}(B(k))$. But then $M - G(k) \subset \pi_1(E(k)) \times \{0\}$ so that $\pi_1(E(k))$ contains a dense open subset of Δ'. Since $E(k)$ is second countable, it follows that $D(k) = \{$regular points of $E(k)$ at which $\mathrm{rank}(\pi_1|_{E(k)}) = n\}$ is a nonempty open subset of $E(k)$ [9, Chapter 4, Theorem 8D]. $\pi_1(D(k))$ is an open (in fact a dense open) subset of Δ'. Note that for any $(w) \in \pi_1(D(k))$ we can find a section of $\pi_1|_{E(k)}$ on a neighborhood of (w) (by the implicit function theorem).

Since $\mathrm{Iso}(X, 0)$ is dense in M, we can find a $(w) \in \pi_1(D(k))$ such that $(w, 0) \in \mathrm{Iso}(X, 0)$. Choose a section of $\pi_1|_{E(k)}$ over a neighborhod U of (w). This section gives holomorphic functions on $U \times \mathbf{C}^{n+m}$, $a_{ij}(k)(z, x, y)$, $1 \leq i, j \leq r$; $\varphi_i(k)(z, x, y)$, $1 \leq i \leq n$; and $\psi_j(k)(z, x, y)$, $1 \leq j \leq m$. All these are polynomials in (x, y) of degree k with coefficients being holomorphic functions on U. Moreover,

(2.16)(a) $\det\|a_{ij}(k)(z, 0, 0)\|$ is a nonvanishing holomorphic function on U.

(b) $\varphi_i(k)(z, 0, 0) = 0$, $1 \leq i \leq n$; $\psi_j(k)(z, x, 0) = 0$, $1 \leq j \leq m$. And for each fixed $z \in U$, $(\varphi(k), \psi(k)) = (\varphi_1(k), \dots, \varphi_n(k), \psi_1(k), \dots, \psi_m(k))$ defines a germ of an isomorphism $\mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$.

(c) For each i, $1 \leq i \leq r$,

$$f_i(z + \varphi(k)(z, x, y), \psi(k)(z, x, y)) - \sum a_{ij}(k)(z, x, y) f_j(x, y)$$

is in the $(k + 1)$st power of the ideal generated by the (x, y).

I want to transfer this information from $(w) \in \Delta'$ to $(0) \in \Delta'$. This can be done because $(w, 0) \in \mathrm{Iso}(X, 0)$. We have $\|a_{ij}\| \in \mathrm{Gl}(r, {}_{n+m}\mathcal{O})$ and a germ of an isomorphism (φ, ψ): $\mathbf{C}_0^{n+m} \to C_0^{n+m}$ such that

$$f_i(w+\varphi(x,y),\psi(x,y)) = \sum a_{ij}(x,y)f_j(x,y), \qquad 1 \leqslant i \leqslant r, \quad (2.17)$$

and by Remark 2.4 we automatically have $\psi_1(x,0) = \cdots = \psi_m(x,0) = 0$.

Let $\|b_{ij}\| \in \mathrm{Gl}(r, {}_{n+m}\mathcal{O})$ be defined by the condition $\|b_{ij}(\varphi(x,y),\psi(x,y))\| = \|a_{ij}(x,y)\|^{-1}$. Let

$$(\lambda,\tau) = (\lambda_1,\dots,\lambda_n,\tau_1,\dots,\tau_m): \mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m} \qquad (2.18)$$

be the inverse of (φ,ψ): $\mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$. Since $\psi(x,0) = 0$, it follows that $\tau(x,0) = 0$ and also that $\lambda(\varphi(x,0),0) = (x)$. For convenience I write $\mu(x) = \varphi(x,0)$, so that $\lambda(\mu(x),0) = (x)$. Equations (2.17) are equivalent to

$$f_i(\lambda(x,y),\tau(x,y)) = \sum b_{ij}(x,y)f_j(w+x,y), \qquad 1 \leqslant i \leqslant r.$$

Let $(t) = (t_1,\dots,t_n)$, so that (t,x,y) give coordinates on $\mathbf{C}^{2n+m}$. Then $\varphi_i(k)(w+\mu(t),x,y)$, $1 \leqslant i \leqslant n$, and $\psi_j(k)(w+\mu(t),x,y)$, $1 \leqslant j \leqslant m$, define germs in ${}_{2n+m}\mathcal{O}$, and $\|a_{ij}(k)(w+\mu(t),x,y)\| \in \mathrm{Gl}(r, {}_{2n+m}\mathcal{O})$.

Define $\|\alpha_{ij}(k)(t,x,y)\| \in \mathrm{Gl}(r, {}_{2n+m}\mathcal{O})$ to be the product of the matrices

$$\left\|b_{il}\big(\mu(t)+\varphi(k)(w+\mu(t),x,y),\psi(k)(w+\mu(t),x,y)\big)\right\|$$

on the left and $\|a_{lj}(k)(w+\mu(t),x,y)\|$ on the right. Define

$$\omega(k)(t,x,y) = \lambda\big(\mu(t)+\varphi(k)(w+\mu(t),x,y),\psi(k)(w+\mu(t),x,y)\big) - (t),$$

and define

$$\gamma(k)(t,x,y) = \tau\big(\mu(t)+\varphi(k)(w+\mu(t),x,y),\psi(k)(w+\mu(t),x,y)\big).$$

Now, using the equations $\lambda(0,0) = \tau(x,0) = 0$, $\lambda(\mu(x),0) = (x)$, (2.16)(b), and the fact that a composite of isomorphisms is an isomorphism we get

$\omega(k)(t,0,0) = 0$, $\gamma(k)(t,x,0) = 0$ and, the map $(\omega(k)(0,x,y),\gamma(k)(0,x,y))$: $\mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$ is a germ of an isomorphism. (2.19)

Now, replacing (x) in (2.18) by $\mu(t)+\varphi(k)(w+\mu(t),x,y)$ and (y) by $\psi(k)(w+\mu(t),x,y)$, and using (2.16)(c) with (z) replaced by $(w+\mu(t))$, we get

$$f_i(t+\omega(k)(t,x,y),\gamma(k)(t,x,y)) - \sum \alpha_{ij}(k)(t,x,y)f_j(x,y) \in {}_{2n+m}\mathfrak{m}^{k+1} \quad \text{for } 1 \leqslant i \leqslant r. \qquad (2.20)$$

This completes the transfer of the information of (2.16) from $(w) \in \Delta'$ to $0 \in \Delta'$.

Now, since we have the $\omega(k)$, $\gamma(k)$, and $\|\alpha_{ij}(k)\|$ satisfying (2.20) for every k, we can again apply Wavrik [7]. Then, choosing a sufficiently large k, we get $\alpha_{ij} \in {}_{2n+m}\mathcal{O}$, $1 \leqslant i, j \leqslant r$, $\omega_i \in {}_{2n+m}\mathcal{O}$, $1 \leqslant i \leqslant n$, and $\gamma_j \in {}_{2n+m}\mathcal{O}$, $1 \leqslant j \leqslant m$, such that

(a) $\alpha_{ij} - \alpha_{ij}(k) \in {}_{2n+m}\mathfrak{m}^2, \qquad 1 \leqslant i,j \leqslant r,$

(b) $\omega_i - \omega_i(k) \in {}_{2n+m}\mathfrak{m}^2, \qquad 1 \leqslant i \leqslant n,$ and

$\gamma_j - \gamma_j(k) \in {}_{2n+m}\mathfrak{m}^2, \qquad 1 \leqslant j \leqslant m,$

(c) $f_i(t + \omega(t, x, y), \gamma(t, x, y)) = \sum \alpha_{ij}(t, x, y) f_j(x, y),$

$$1 \leqslant i \leqslant r. \qquad (2.21)$$

Using (2.19) and (2.21)(b) we see that the map $(\omega(0, x, y), \gamma(0, x, y))$: $\mathbf{C}_0^{n+m} \to \mathbf{C}_0^{n+m}$ is a germ of an isomorphism. But a trivial calculation shows that the value of the jacobian of this map at $0 \in \mathbf{C}^{n+m}$ is the same as the value of the jacobian of the map $(t, \omega(t, x, y), \gamma(t, x, y))$: $\mathbf{C}_0^{2n+m} \to \mathbf{C}_0^{2n+m}$ at $0 \in \mathbf{C}^{2n+m}$. Denoting this later map by Ω, we see that $\Omega = (t, \omega, \gamma)$: $\mathbf{C}_0^{2n+m} \to \mathbf{C}_0^{2n+m}$ is a germ of an isomorphism.

Let $X_0' \subset \mathbf{C}_0^{2n+m}$ be the germ of an analytic space defined by the ideal in ${}_{2n+m}\mathcal{O}$ generated by the germs at $0 \in \mathbf{C}^{2n+m}$ of $f_1(t + x, y), \ldots, f_r(t + x, y)$. We have $\mathbf{C}_0^n \times X_0 \subset \mathbf{C}_0^{2n+m}$ is defined by the ideal in ${}_{2n+m}\mathcal{O}$ generated by the germs at $0 \in \mathbf{C}^{2n+m}$ of $f_1(x, y), \ldots, f_r(x, y)$. By (2.21)(c) we see that $\Omega = (t, \omega, \gamma)$ induces a map Ω: $\mathbf{C}_0^n \times X_0 \to X_0'$.

Since $\|\alpha_{ij}(k)\| \in \mathrm{Gl}(r, {}_{2n+m}\mathcal{O})$, (2.21)(a) shows that $\|\alpha_{ij}\| \in \mathrm{Gl}(r, {}_{2n+m}\mathcal{O})$. Using this, and the fact that Ω: $\mathbf{C}_0^{2n+m} \to \mathbf{C}_0^{2n+m}$ is an isomorphism, (2.21)(c) actually give Ω induces an isomorphism Ω: $\mathbf{C}_0^n \times X_0 \to X_0'$.

Now the holomorphic vector fields $\partial/\partial t_1, \ldots, \partial/\partial t_n$ clearly preserve the ideal generated by $f_1(x, y), \ldots, f_r(x, y)$. Since Ω: $\mathbf{C}_0^{2n+m} \to \mathbf{C}_0^{2n+m}$ is an isomorphism we can push these vector fields forward to get germs of holomorphic vector fields $\Omega_*(\partial/\partial t_1), \ldots, \Omega_*(\partial/\partial t_n)$. Since Ω induces an isomorphism Ω: $\mathbf{C}_0^n \times X_0 \to X_0'$ we see that the $\Omega_*(\partial/\partial t_i)$, $1 \leqslant i \leqslant n$, all preserve the ideal generated by $f_1(t + x, y), \ldots, f_r(t + x, y)$. Clearly, the holomorphic vector fields $(\partial/\partial t_i - \partial/\partial x_i)$, $1 \leqslant i \leqslant n$, also preserve this ideal. Using (2.21)(b) and (2.19) we easily calculate

$$\Omega_*\left(\frac{\partial}{\partial t_i}\right)(0) = \left.\frac{\partial}{\partial t_i}\right|_0, \qquad 1 \leqslant i \leqslant n.$$

Since $\partial/\partial t_1|_0, \ldots, \partial/\partial t_n|_0$, $(\partial/\partial t_1 - \partial/\partial x_1)|_0, \ldots, (\partial/\partial t_n - \partial/\partial x_n)|_0$ are linearly independent we get $\dim C_1(X_0') \geqslant 2n$. Since $\mathbf{C}_0^n \times X_0 \cong X_0'$, Lemma 1.2 gives $\dim C_1(X_0) \geqslant n$. □

The proof of Theorem 2.1 is now complete. Using it we see $\mathrm{Iso}(X, p)_p = M(X, p)$. Thus, by Proposition 2.15, $\dim \mathrm{Iso}(X, p)_p = \dim C_1(X_p)$. In particular, we get

REMARK 2.22. p is an isolated point of $\mathrm{Iso}(X, p)$ if and only if $\dim C_1(X_p) = 0$.

3. Clustering. Let V be a germ of an analytic space. Recall that by a representative of V one means a pair (X, p) consisting of an analytic space X, and a point $p \in X$ such that $V \cong X_p$.

DEFINITION 3.1. Let V and W be germs of analytic spaces. I will say that W clusters in V if and only if there is a representative (X, p) for V and a sequence $q_i \in X - \{p\}$ such that the q_i converge to p, and every pair (X, q_i) is a representative for W.

Note that if W clusters in V and if (X', p') is any representative of V, then one can find such a sequence $q_i' \in X' - \{p'\}$. Also, clustering is transitive; if V_1 clusters in V_2, and V_2 clusters in V_3 then V_1 clusters in V_3. Finally, if V clusters in V then $\dim C_1(V) \geqslant 1$. This last observation follows from Remark 2.22.

LEMMA 3.2. *If $\{V_1, \ldots, V_k\}$ is a finite set of germs of analytic spaces with* $\dim C_1(V_i) = 0$, $1 \leqslant i \leqslant k$, *then one can find an $i \in \{1, \ldots, k\}$ such that V_i does not cluster in any of the V_j, $1 \leqslant j \leqslant k$.*

PROOF. If not, we can find a map $\varphi\colon \{1, 2, \ldots, k+1\} \to \{V_1, \ldots, V_k\}$ such that $\varphi(i)$ clusters in $\varphi(i+1)$ for $1 \leqslant i \leqslant k$. But φ cannot be injective. Let i and j, $i < j$, be such that $\varphi(i) = \varphi(j)$. Using the transitivity of clustering we get $\varphi(i)$ clusters in itself, so that, by the previous lemma, $\dim C_1(\varphi(i)) \geqslant 1$. This is a contradiction. □

We shall not need, but it is interesting to note,

PROPOSITION 3.3. *If V and W are germs of analytic spaces, and if V clusters in W and W clusters in V, then $V \cong W$ and* $\dim C_1(V) \geqslant 1$.

PROOF. The proof is left to the reader.

4. Decompositions. Throughout this section, all analytic spaces and all germs of analytic spaces will be taken to be reduced. I will use V, W, V_1, etc. to denote reduced germs of analytic spaces. I will use X, Y, X_1, etc. to denote reduced analytic spaces. Before proving Theorem 0.3, I will collect some elementary but useful facts.

If $V = \bigcup V_i$ is the decomposition of V into irreducible components and $W = \bigcup W_j$ is the decomposition of W into irreducible components, then $V \times W = \bigcup (V_i \times W_j)$ is the decomposition of $V \times W$ into irreducible components. For a germ V and an integer d we define $N(V, d)$ to be the number of irreducible components of V of dimension d, and we define a polynomial $P(V, t) = \sum N(V, d) t^d$. Then for any d we get $N(V \times W, d) = \sum N(V, i) N(W, d-i)$ so that $P(V \times W, t) = P(V, t) P(W, t)$. It follows that if $V_1 \times W \cong V_2 \times W$ then $P(V_1, t) = P(V_2, t)$. Finally we have the important observation that if $V \subset W$ and $V \neq W$ then there is a d such that

$N(V, d) < N(W, d)$. Thus, if V is isomorphic to $W_1 \subset W$ and $P(V, t) = P(W, t)$ then $W_1 = W$ and $V \cong W$.

I now prove

THEOREM 0.3. *If V is a positive dimensional germ of a reduced analytic space, then V is uniquely decomposable.*

PROOF. The existence of a decomposition of V into indecomposables is trivial. Any decomposition of maximal length will do. (Note that the length of any decomposition of $V \leqslant \dim V$.) Only the uniqueness must be proven. The proof will proceed by induction on $\dim V$.

If $\dim V = 1$, then V is indecomposable and there is nothing to prove.

Now suppose $\dim V > 1$ and Theorem 0.3 has been proven for all germs of dimension $< \dim V$. We must prove the uniqueness for V. I begin with two reductions.

REDUCTION 1. We may assume V is not indecomposable because if V is indecomposable there is nothing to prove. □

Now suppose $(V_1, \ldots, V_k)$ and $(W_1, \ldots, W_l)$ are two decompositions of V with all V_i and all W_j indecomposable. By Reduction 1 we may assume $k \geqslant 2$ and $l \geqslant 2$.

REDUCTION 2. We may assume $\dim C_1(V) = 0$.

PROOF OF REDUCTION 2. Suppose $\dim C_1(V) > 0$. Then, by Lemma 1.2, we may reorder the V_i's and the W_j's to achieve $\dim C_1(V_1) > 0$ and $\dim C_1(W_1) > 0$. Since V_1 and W_1 are indecomposable we get, by applying Lemma 1.3, that $V_1 \cong W_1 \cong \mathbf{C}_0$. But then, we can use Lemma 1.5 to conclude that $(V_2, \ldots, V_k)$ and $(W_2, \ldots, W_l)$ give two decompositions into indecomposables of some germ V'. Since $\dim V' = \dim V - 1$ the unique decomposability of V follows from the induction hypothesis. □

Making use of both reductions (and of Lemma 1.2), let $(V_1, \ldots, V_k)$ and $(W_1, \ldots, W_l)$ be two decompositions with all V_i and W_j indecomposable. Then $k \geqslant 2$, $l \geqslant 2$, $\dim C_1(V_i) = 0$, $1 \leqslant i \leqslant k$, and $\dim C_1(W_j) = 0$, $1 \leqslant j \leqslant l$.

Let $n = \max\{\dim V_1, \ldots, \dim V_k, \dim W_1, \ldots, \dim W_l\}$ and let $A = \{V_i | \dim V_i = n\} \cup \{W_j | \dim W_j = n\}$. By Lemma 3.2, I can find a $V' \in A$ which does not cluster in any element of A. Since V' clearly cannot cluster in any W with $\dim W < \dim V' = n$, we get, in fact, that V' does not cluster in any V_i, $1 \leqslant i \leqslant k$, and V' does not cluster in any W_j, $1 \leqslant j \leqslant l$.

We may assume that V' is isomorphic to the first r of the V_i's and to the first s of the W_j's, and that no other V_i or W_j is isomorphic to V'.

We may also assume that $r \geqslant s$. Then $r \geqslant 1$. We set $\gamma = r$ if $r < k$, and $\gamma = k - 1$ if $r = k$. Note that $1 \leqslant \gamma \leqslant k - 1$.

Let $(X_1, p(1)), \ldots, (X_k, p(k))$ be representatives for $V_1, \ldots, V_k$ and let $(Y_1, q(1)), \ldots, (Y_l, q(l))$ be representatives for $W_1, \ldots, W_l$.

Shrinking the Y_j's (by replacing each Y_j by a small open neighborhood of $q(j) \in Y_j$) we may assume that for each j, $1 \leq j \leq l$, we have $\dim(Y_j)_q \leq \dim W_j$ for every $q \in Y_j$. Since V' does not cluster in any of the W_j's, we may also assume (by further shrinking the Y_j's) that for each j, $1 \leq j \leq l$, we have V' is not isomorphic to $(Y_j)_q$ for any $q \in Y_j - \{q(j)\}$.

Since $V_1 \times \cdots \times V_k \cong W_1 \times \cdots \times W_l$, we can find (after shrinking the X_i's) an isomorphism $\psi: X_1 \times \cdots \times X_k \to U$ (an open neighborhood of $(q(1), \ldots, q(l))$ in $Y_1 \times \cdots \times Y_l$) such that $\psi(p(1), \ldots, p(k)) = (q(1), \ldots, q(l))$. This isomorphism will first be used to show $\gamma \leq s$.

Choose $x(i) \in \operatorname{Reg}(X_i)$, $\gamma + 1 \leq i \leq k$. Let

$$\psi\big((p(1), \ldots, p(\gamma), x(\gamma + 1), \ldots, x(k))\big) = \big(y(1), \ldots, y(l)\big).$$

Then, using $(X_i)_{p(i)} \cong V_i$, we have

$$V_1 \times \cdots \times V_\gamma \times \left(X_{\gamma+1}\right)_{x(\gamma+1)} \times \cdots \times \left(X_k\right)_{x(k)} \cong \left(Y_1\right)_{y(1)} \times \cdots \times \left(Y_l\right)_{y(l)}. \tag{4.1}$$

Let $h = \dim(X_{\gamma+1})_{x(\gamma+1)} + \cdots + \dim(X_k)_{x(k)}$. For each j, $1 \leq j \leq l$, let $m(j) = \dim C_1((Y_j)_{y(j)})$. Then we can write $(Y_j)_{y(j)} = W_j' \times \mathbf{C}_0^{m(j)}$ where $\dim C_1(W_j') = 0$. Setting $m = \sum m(j)$, (4.1) becomes

$$V_1 \times \cdots \times V_\gamma \times \mathbf{C}_0^h \cong W_1' \times \cdots \times W_l' \times \mathbf{C}_0^m. \tag{4.2}$$

Using Lemma 1.2 we get $h = m$. Using Lemma 1.5 repeatedly we get

$$V_1 \times \cdots \times V_\gamma \cong W_1' \times \cdots \times W_l'. \tag{4.3}$$

By the construction of the Y_j's and our choice of V' we get

$$\dim V' \geq \dim W_j \geq \dim\big(\left(Y_j\right)_{y(j)}\big) \geq \dim W_j', \qquad 1 \leq j \leq l. \tag{4.4}$$

It is worth noting that $\dim V' = \dim W_j'$ only if $(Y_j)_{y(j)} \cong W_j'$.

Since $V_1 \cong \cdots \cong V_\gamma \cong V'$, we get from (4.4),

$$\gamma(\dim V') = \sum \dim W_j'. \tag{4.5}$$

Let $L = \{j \mid 1 \leq j \leq l \text{ and } \dim W_j' > 0\}$. From (4.4) and (4.5) we see that L contains at least γ integers.

On the other hand, $\dim V > \dim(V_1 \times \cdots \times V_\gamma)$, so by our induction hypothesis $V_1 \times \cdots \times V_\gamma$ is uniquely decomposable. Since each V_i is indecomposable, it follows that no decompositions of $V_1 \times \cdots \times V_\gamma$ can have length greater than γ. This shows that L contains at most γ integers. Thus, L contains precisely γ integers, and the W_j', $j \in L$, give the terms of a decomposition of $V_1 \times \cdots \times V_\gamma$ and all the W_j', $j \in L$, are indecomposable. Using the fact that $V_1 \times \cdots \times V_\gamma$ is uniquely decomposable, and using $V_i \cong V'$ for $1 \leq i \leq \gamma$, we see

$$V' \cong W_j' \quad \text{for all } j \in L. \tag{4.6}$$

Thus $\dim V' = \dim W_j'$ for $j \in L$. But as noted above, this gives $(Y_j)_{y(j)} \cong W_j'$ and thus $(Y_j)_{y(j)} \cong V'$ for $j \in L$. By our construction of the Y_j's, we see that, for $j \in L$, $y(j) \notin Y_j - \{q(j)\}$ so that $y(j) = q(j)$. Thus for $j \in L$ we get $V' \cong (Y_j)_{q(j)} \cong W_j$. This gives $s \geqslant \gamma = \operatorname{Card}(L)$, and $L \subset \{j | 1 \leqslant j \leqslant s\}$.

The proof of the theorem is now reduced to two cases.

Case 1. $s = \gamma$. In this case $L = \{j | 1 \leqslant j \leqslant s\}$. Then, what we have just seen is that for $x(i) \in \operatorname{Reg}(X_i)$, $s + 1 \leqslant i \leqslant k$,

$$\psi\big((p(1), \ldots, p(s), x(s+1), \ldots, x(k))\big) = (q(1), \ldots, q(s), y(s+1), \ldots, y(l)).$$

Since $\operatorname{Reg}(X_{s+1}) \times \cdots \times \operatorname{Reg}(X_k)$ is dense in $X_{s+1} \times \cdots \times X_k$, it follows that $\psi(\{(p(1), \ldots, p(s))\} \times X_{s+1} \times \cdots \times X_k)$ is contained in $\{(q(1), \ldots, q(s))\} \times Y_{s+1} \times \cdots \times Y_l$.

In other words, we have just established

FACT 4.7. $V_{s+1} \times \cdots \times V_k$ is isomorphic to some $W' \subset W_{s+1} \times \cdots \times W_l$.

Since $V_i \cong W_i \cong V'$ for $1 \leqslant i \leqslant s$ we get $V_1 \times \cdots \times V_s \cong W_1 \times \cdots \times W_s$. We also have

$$\begin{aligned} V &\cong (V_1 \times \cdots \times V_s) \times (V_{s+1} \times \cdots \times V_k) \\ &\cong (W_1 \times \cdots \times W_s) \times (W_{s+1} \times \cdots \times W_l). \end{aligned}$$

But then, the introductory remarks to this section show that we have established

FACT 4.8. $P(V_{s+1} \times \cdots \times V_k, t) = P(W_{s+1} \times \cdots \times W_l, t)$.

Again applying those introductory remarks we may conclude $V_{s+1} \times \cdots \times V_k \cong W_{s+1} \times \cdots \times W_l$. Since $\dim V > \dim(V_{s+1} \times \cdots \times V_k)$ we may apply the induction hypothesis to conclude $k - s = l - s$ (so that $k = l$), and (after permuting the $W_{s+1}, \ldots, W_k$) we have $V_i \cong W_i$ for $s + 1 \leqslant i \leqslant k$. We already had $V_i \cong W_i \cong V'$ for $1 \leqslant i \leqslant s$. This completes the proof of the theorem in Case 1.

Case 2. $s > \gamma$. In this case $r > \gamma$. But, by the definition of γ we see that this implies $r = k$ and $\gamma = k - 1$. But then we also have $s = k$.

In this case we have $V = V_1 \times \cdots \times V_s$ with all $V_i \cong V'$. Then $\dim V = s(\dim V')$. Since $\Sigma \dim W_j = s(\dim V')$, and since s of the W_j are isomorphic to V', we see that there are no W_j's except those isomorphic to V'. Thus $l = s$ so that $k = l$, and for each i, $1 \leqslant i \leqslant k$, $V_i \cong W_i \cong V'$.

This completes the induction step and the proof. □

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