ISOTOPIC GRAND UNIFICATION WITH THE INCLUSION OF GRAVITY

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Abstract

We introduce a duat lifting of unified gauge theories, the first characterized by the isotopies, which are axiom-preserving maps into broader structures with positive-definite generalized units used for the representation of matter under the isotopies of the Poincaré symmetry, and the second characterized by the isodualities, which are anti-isomorphic maps with negative-definite generalized units used for the representation of antimatter under the isodualities of the Poincaré symmetry. We then submit, apparently for the first time, a novel grand unification with the inclusion of gravity for matter embedded in the generalized positive-definite units of unified gauge theories while gravity for antimatter is embedded in the isodual isounit. We then show that the proposed grand unification provides reaalistic possibilities for a resolution of the axiomatic incompatibilities between gravitation and electroweak interactions due to curvature, antimatter and the fundamental space-time symmetries.

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1 Introduction

In this note we study the structural incompatibilities for an axiomatically consistent inclusion of gravitation[(1)] in the unifield gauge theories of electroweak interactions[(2)] due to:

(1)Curvature. In fact, electroweak theories are essentially structured on *Minkowskian* axioms, while gravitational theories are formulated via *Riemannian* axioms, a disparity which is magnified at the operator level because of known technical difficulties of quantum gravity[(3)], e.g., to provide a PCT theorem comparable to that of electroweak interactions.

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(2) Antimatter. In fact, electroweak theories are *bona fide* relativistic theories, thus characterizing antimatter via *negative-energy* solutions, while gravitation characterizes antimatter via *positive-definite* energy-momentum tensors.

(3) Fundamental space-time symmetries. In fact, the electroweak interactions are based on the axioms of the special relativity, thus verifying the fundamental *Poincaré symmetry* $P(3.1)$, while such a basic symmetry is absent in contemporary gravitation.

Without any claim of uniqueness (see, e.g., the recent studies on unified theories of monograph[2m] and references quoted therein), we present apparently for the first time a conceivable resolution of the above structural incompatibilities via the use of the following new methods:

(A) Isotopies. A baffling aspect in the inclusion of gravity in unified gauge theories is their apparent geometric incompatibility despite their individual beauty and experimental verifications.

The view we would like to convey is that the above structural incompatibility is *not* necessarily due to insufficiencies in Einstein's field equations,but rather to *insufficiencies in their mathematical treatment*. Stated in plain language, we believe that the achievement of axiomatic compatibility between gravitation and electroweak interactions requires a basically new mathematics, that is, basically new numbers, new space, etc.

In the hope of resolving in due time this first structural incompatibility, Santilli[4a] proposed back in 1978 when at the Department of Mathematics of Harvard University under DOE support, a new mathematics based on the so-called *isotopies*, and today known as *isomathematics*, which was then studied in numerous works[4−11] (see in particular the latest memoir[4e]).

The isotopies are nowadays referred to *maps* (also called *liftings*) *of any given linear, local and canonical or unitary theory into its most general possible nonlinear, nonlocal and noncanonical or nonunitary extensions, which are nevertheless capable of reconstructing linearity, locality and canonicity or unitarity on certain generalized spaces and fields, called isospaces and isofields.* From their Greek meaning, isotopies are therefore "axiom-preserving ".

The fundamental isotopy of this note is that of the 4-dimensional unit $I = Diag.(1,1,1,1)$ of the Minkowskian and Riemannian space-time into a $4\times4-$dimensional, everywhere invertible, Hermitian and positive-definite matrix $\hat{I}$ whose elements have an arbitrary functional dependence on the local space-time coordinates x, as any other needed variable,

$$I = Diag.(1,1,1,1) \rightarrow \hat{I}(x,\ldots) = (\hat{I}^{\mu}_{\nu}(x,\ldots)) = \hat{I}^{\dagger} = [\widehat{T}(x,\ldots)]^{-1} > 0, \qquad (1)$$

with corresponding lifting of the conventional associative product

$$A \times B \rightarrow A \widehat{\times} B = A \times \widehat{T} \times B, \qquad (2)$$

under which $\hat{I}(x,\ldots) = [\widehat{T}(x,\ldots)]^{-1}$ is the correct left and right unit of the new theory called *isounit*, in which case $\widehat{T}(x,\ldots)$ is called the *isotopic element*.

When applicable, liftings (1) and (2) require for consistency the reconstruction of *all* mathematical methods of contemporary physics, with no exception known to this author. In fact, they require new numbers and fields called *isonumbers* and *isofields* with an arbitrary (positive-definite) unit, new spaces over isofields called *isospaces*, the new *isodifferential calculus*, the new *isoeuclidean*, *isominkowskian* and *isosymplectic geometries*, etc. which we cannot possibly review in this note and must assume as known (see their latest formulation in memoir[(4e)]).

In a communication at the *VII Marcel Grossmann Meeting on General Relativity* (mg7) held in 1994 at Stanford University, Santilli[(5a)] showed that the isomathematics permits a novel classical and operator treatment of gravitation which, on one side, preserves Riemannian metrics, Einstein's field equations and related experimental verifications while, on the other side, verifies the abstract *Minkowskian* axioms.

The above reformulation is evidently fundamental for this note, inasmuch as it offers realistic possibilities of resolving the structural incompatibility between electroweak and gravitational interactions due to curvature, by reducing the latter to the axiomatic structure of the former *for the case of matter only* (see below for antimatter).

The main mechanism is that first presented at mg7 (Ref.[(5a)], p. 501) which is based on the factorization of any given Riemannian metric (e.g., Schwarzschild metric[(1c)]) $g(x)$ into the Minkowski metric $\eta(+1,+1,+1,-1)$

$$g(x) = T(x) \times \eta \tag{3}$$

where *the gravitational isotopic element* $T(x)$ is evidently a 4-dimensional matrix which is always positive-definite from the locally Minkowskian character of Riemann. The entire theory must then be reconstructed with respect to the *gravitational isounit*

$$\hat{I} = [\widehat{T}\,(x,\ldots)]^{-1} = \eta \times [g(x)]^{-1} > 0. \tag{4}$$

Note that the component truly representing curvature in the Riemannian geometry *is not* the Riemannian metric $g(x)$ but rather its isotopic component $T(x)$, trivially, because the remaining component η is flat. It is then easy to see that the *isotopic treatment of gravity formally eliminates curvature, thus rendering gravitation axiomatically compatible with the electroweak interactions.* In fact, curvature exists when the gravitational isotopic element $T(x)$ is referred to the conventional spacetime unit I, while curvature formally disappears when $T(x)$ is referred to a generalized unit which is its inverse $[T(x)]^{-1}$.

Reformulations (3) and (4) also imply the birth of a novel geometry, the *isominkowskian geometry*, first submitted by Santilli[(6a)] in 1983 which, in more recent studies[(5e)] has resulted to in a symbiotic unification of the Minkowskian and Riemannian geometries, because it verifies all the abstract axioms of the former, while preserving the machinery of the latter (covariant derivatives, connections, etc.). The formulation of gravity based on Eq.s (3) and (4) is then called *isominkowskian gravity.*

Allow us to stress for clarity that we are here referring to a mere *mathematical* reformulation of Einstein's historical field equations on the *mathematical* isominkowskian spaces (i.e., refer them to a new unit $\hat{I}$) because the projection of the treatment into the convectional space-time (i.e., when referred to the conventional space-time unit I) recovers the said historical equations in their totality. The above occurrences therefore offer realistic hopes of resolving the baffling occurrence indicated earlier, i.e., the apparent incompatibility between Einstein's majestic conception of gravitation with the geometric structure of electroweak interactions.

The reader should be aware that the proposed resolution works best where it is needed most, at the operator level. In fact, Santilli[5a] showed at mg7 that *the operator formulation of the isominkowskian representation of gravity verifies all abstract axioms and physical laws of conventional "relativistic" quantum mechanics (RQM).* The emerging new theory is called *operator isogravity (OIG)* and merely consists in embedding gravity in the *unit* of RQM.

The reader should be aware that the above classical and operator isotopies are supported by two, hitherto unknown symmetries, first presented in memoir[4f] under the tentative name of *isoselfscalar symmetries*, which are characterized by the transforms

$$\eta \rightarrow \widehat{\eta} = n^{-2} \times \eta, \; I \rightarrow \hat{I} = n^2 \times I, \tag{5}$$

where n is a parameter, and yields the symmetry of the conventional Minkowskian interval

$$x^2 = (x^\mu \times \eta_{\mu\nu} \times x^\nu) \times I = (x^\mu \times \widehat{\eta}_{\mu\nu} \times x^\nu) \times \hat{I} = x^{\hat{2}}, \tag{6}$$

with a corresponding invariance for the Hilbert space

$$\langle \Phi \,|\times|\, \Psi \rangle \times I = \big\langle \Phi \,\big|\times n^{-2} \times\big|\, \Psi \big\rangle \times (n^2 \times I) = \big\langle \Phi \,\big|\widehat{\times}\big|\, \Psi \big\rangle \times \hat{I}. \tag{7}$$

The isominkowskian representation of gravity then emerges from the above classical and quantum symmetries via the axiom-preserving addition of an x-dependence in the n-parameter, much along the transition from Abelian to non-Abelian gauge theories.

(B) Isodualities. Structural incompatibility (2) is only the symptom of deeper problems in the contemporary treatment of antimatter. To begin, matter is treated nowadays at *all* levels, from Newtonian to electroweak interactions, while antimater is treated only at the level of *second quantization.* Since there are serious indications that half of the universe could well be made up of antimatter,it is evident that a more effective theory of antimatter must also apply at all levels.

At any rate, recall that charge conjugation in quantum mechanics is an *anti-automorphic* map. As a result, no classical theory of antimatter can be axiomatically consistent via the mere change of the sign of the charge, because it must be an anti-automorphic (or, more generally, anti-isomorphic) image of that of matter (the alternative classical formulation of antimatter of Ref. [12c] has been recently brought to the author's attention).

The current dramatic disparity in the treatment of matter and antimatter also has its predictable problematic aspects. Since we currently use only one type of quantization (whether naive of symplectic), it is easy to see that *the operator image* *of the contemporary treatment of antimatter is not the correct charge conjugate state, but merely a conventional state of particles with a reversed sing of the charge.*

The view here submitted is that, as it is the case for curvature, the resolution of the above general shortcomings, including the achievement of compatibility in the treatment of antimatter between electroweak and gravitational interactions, requires a basically novel mathematics.

Santilli therefore entered into a second search for another novel mathematics under the uncompromisable condition of being an anti-isomorphic image of the preceding isomathematics. After inspecting a number of alternatives, this author[6c] submitted in 1985 the following map of an arbitrary quantity Q (i.e., a number, or a vector field or an operator) under the tentative name of *isoduality*

$$Q \to Q^d = -Q^\dagger \tag{8}$$

When applied to the *totality* of quantities and their operations of a given theory of matter, map (8) yields an anti-isomorphic image, as axiomatically needed for antimatter. Moreover, while charge conjugation is solely applicable within operator settings, isoduality (8) is applicable at *all* levels of study, beginning at the *Newtonian* level.

It is evident that map (8) implies a new mathematics, that with negative units called *isodual* mathematics[7], which includes new numbers, new spaces, new calculus, etc. In reality we have two different isodual mathematics, the fist anti-isomorphic image of the conventional mathematics used for matter, and the second is the anti-isomorphic image of the preceding isomathematics.

The above characteristics have permitted the construction of the novel *isodual theory of antimatter*[7] which is equivalent, although anti-isomorphic, to that of matter, and which therefore begins at the primitive *Newtonian* level and then continuous at the analytic and quantum levels, in which case it results in equivalence to charge conjugation for massive particles (see later on for photons).

Most importantly, the isodual theory of antimatter has resulted in agreement with all available classical and quantum experimental data on antimatter.

It is evident that isodualities offer a realistic possibility of resolving the second structural problem between electroweak and gravitational interactions because antimatter can be treated in both cases with *negative-energy*. This is due to the fact that isodualities imply the transition from the conventional space-time units of matter $I = Diag.(1,1,1,1) > 0$ to their negative image $I^d = -I < 0$. As a result, *all characteristics of matter change sing in the transmission to antimatter under isoduality*, thus yielding the correct conjugation of charge, as well as negative energy, negative energy-momentum tensor, and, inevitably, negative time. The historical objections against these negative values are inapplicable, because they are tacitly referred to the conventional positive units. In fact, negative energy and time referred to negative units are fully equivalent, although antiautomorphic, to the conventional positive energy and time referred to positive units.

The reader should also be aware that the isodual theory of antimatter was born from properties of the conventional Dirac equation:

$$[\gamma^{\mu} \times (p_{\mu} - e \times A_{\mu}/c) + i \times m] \times \psi(x) = 0, \tag{9a}$$

$$\gamma^{k} = \begin{pmatrix} 0 & \sigma^{k} \\ -\sigma^{k} & 0 \end{pmatrix}, \tag{9b}$$

$$\gamma^{4} = i \times \begin{pmatrix} I_s & 0 \\ 0 & -I_s \end{pmatrix}. \tag{9c}$$

In fact, as one can see, *the negative unit* $I_s^d = Diag.(-1,-1)$ *appears in the very structure of* γ_4. The isodual theory was then constructed precisely around Dirac's unit I_s^d.

In essence, Dirac assumed that the negative-energy solutions of his historical equation behaved in an unphysical way because tacitly referred to the conventional mathematics of his time, that with *positive units* $I_s > 0$. Santilli(7) showed that, when the same negative-energy solutions are referred to the *negative units* $I_s^d < 0$, they behaved in a fully physical way. This eliminates the need of second quantization for the treatment of antiparticles (as expected in a theory of antimatter beginning at the Newtonian level), and permits the reformulation of the equation in the form

$$[\widetilde{\gamma}^{\mu} \times (p_{\mu} - e \times A/c) + i \times m] \times \widetilde{\psi}(x) = 0, \tag{10a}$$

$$\widetilde{\gamma}^{\ \ k} = \begin{pmatrix} 0 & \sigma_k^d \\ \sigma_k & 0 \end{pmatrix}, \ \widetilde{\gamma}^4 = i \begin{pmatrix} I_s & 0 \\ 0 & I_s^d \end{pmatrix} \tag{10b}$$

$$\{\widetilde{\gamma}_{\mu} \widetilde{\gamma}_{\nu}\} = 2\eta_{\mu\nu}, \quad \widetilde{\psi} = -\ \widetilde{\gamma}_4 \times \psi = i \times \begin{pmatrix} \Phi \\ \Phi^d \end{pmatrix}, \tag{10c}$$

where $\Phi(x)$ is now two-dimensional, which is fully symmetrized between particles and antiparticles.

As was the case for the preceding isotopies, the isodual theory of antimatter also sees its solid roots in two additional novel symmetries, also unknown until recently, and first presented in memoir(4f), the first holding for the conventional Minkowski interval

$$x^2 = \left(x^{\mu} \times \eta_{\mu\nu} \times x^{\nu}\right) \times I = \left[x^{\mu} \times \left(-n^{-2} \times \eta_{\mu\nu}\right) \times x^{\nu}\right] \times \left(-n^2 \times I\right) =$$

$$= \left(x^{\mu} \times \widehat{\eta}^d_{\mu\nu} \times x^{\nu}\right) \times \hat{I}^d = x^{d2d}, \tag{11}$$

and the second holding for the Hilbert space

$$\langle \phi | \times | \psi \rangle \times I = \langle \phi | \times \left(-n^{-2}\right) \times | \psi \rangle \times \left(-n^2 \times I\right) = \left\langle \phi \middle| \times \widehat{T}^d \times \middle| \psi \right\rangle \times \hat{I}^d, \tag{12}$$

which ensure that all physical laws for matter also hold for antiparticles under our isodual representation, with corresponding symmetries for the isodual expressions.

The axiom-preserving lifting of the n-parameter to an explicit x-dependence then yields the *isodual isominkowskian treatment of gravity for antimmatter* with basic structures,

$$g(x) = \widehat{T}(x) \times \eta \rightarrow g^d(x) = -g(x) = \widehat{T}^d(x) \times^d \eta^d, \eta \rightarrow \eta^d = -\eta, \tag{13a}$$

$$\hat{I}(x) = \left[\widehat{T}(x)\right]^{-1} \rightarrow \hat{I}^d(x) = \left[\widehat{T}^d(x)\right]^{-1}. \tag{13b}$$

As we shall see in the next section rules (3)-(4) and (13) can indeed be implemented within unified gauge theories.

(C) Isotopies of the Poincaré symmetry and their isoduals. Judging from the studies herein reported, the most severe problems of compatibility between gravitation and electoweak interactions for both matter and antimatter appeared precisely were expected, in the *fundamental space-time symmetries*, because of the disparity indicated earlier of the validity of Poincaré symmetry for electroweak interactions and its absence for gravitation.

The latter problems called for a third series of studies presented in Ref.s[(6)] on the isotopies and isotopies and isodualities of the Poincaré symmetry $\widehat{P}(3.1)$, today called the *Poincaré-Santilli isosymmetry and its isodual*[(8b−8e)], which include the isotopies and isodualities of: the rotational symmetry[(6c)]; the Lorentz symmetry in classical[(6a)] and operator version[(6b)]; the SU(2)-spin symmetry[(6d)]; the Poincaré symmetry[(6e)]; , and the spinorial covering of the Poincaré symmetry[(6f)] (see monographs[(6g)] for comprehensive studies).

We are here referring to the reconstruction of the conventional symmetries whit respect to an arbitrary positive-definite unit (I), for the isotopies, and with respect to an arbitrary negative-definite unit, for the isodualities. This reconstruction yields the most general known nonlinear, nonlocal and noncanonical liftings of conventional symmetries, while being locally isomorphic (for isotopies) or anti-isomorphic (for isodualities) to the original symmetries.

One should be aware that the above structures required the prior step-by step isotopies and isodualities of Lie's theory (enveloping associative algebras, Lie algebras, Lie groups, transformation and representation theory, etc.),originally proposed by Santilli[(4a)] in 1978, studied in numerous subsequent works (see monographs[(4c,6g)]) and today called the *Lie-Santilli isotheory and its isodual*[(8−10)].

It is evident that isopoincaré symmetry and its isodual have fundamental character for this note. In fact, one of their primary applications has been the achievement of the universal *symmetry* (rather than covariance) of all possible Riemannian line elements in their isominkowskian representation[(6)]. Once the unit of gauge theories is lifted to represent gravitation, electroweak interactions will also every the isopoincaré symmetry for matter and its isodual for antimatter, thus offering hopes for the resolution of the most difficult problem of compatibility, that for space-time symmetries.

Perhaps unexpectedly, the fundamental space-time symmetry of the grand unified theory inclusive of gravitation submitted in this note is the total symmetry of the *conventional* Dirac equation, here written with their underlying space and units

$$S_{Tot} = \{SL(2.C) \times T(3.1)\} \times \left\{SL^d(2.C^d) \times^d T^d(3.1)\right\}, \tag{14a}$$

$$M_{Tot} = \{M(x,\, \eta,\, R) \times S_{spin}\} \times \left\{M^d(x^d,\, \eta^d,\, R^d) \times^d S^d_{spin}\right\}, \tag{14b}$$

$$I_{Tot} = \{I_{orb} \times I_{spin}\} \times \left\{I^d_{orb} \times I^d_{spin}\right\}, \tag{14c}$$

which has recently emerged as being *twenty-two dimensional*.

To see the above occurrence, the reader should be aware that isodualities imply yet another new symmetry called *isoselfduality*[7)], which is the invariance under the isodual map (8). Dirac's gamma matrices verify indeed this new symmetry (from which the symmetry itself was derived in the first place), i.e., $\gamma_\mu \to \gamma^d_\mu = -\gamma^\dagger_\mu = \gamma_\mu$. As a result, contrary to a popular belief throughout this century, the Poincaré symmetry *cannot* be the total symmetry of Dirac's equations, evidently because it is not isoselfdual. For evident reasons of consistency, the total symmetry of Eq.s (9) must also be isoselfdual as the gamma-matrices are. This resulted in the identification of the total symmetry (14a) which is indeed isoselfdual.

To understand the dimensionality of symmetry one must first recall that isodual space are independent from conventional spaces. The doubling of the conventional dimensionality then yields *twenty* dimensions. The additional two dimensions are given by the novel isoselfscalarity, i.e., invariance (6)-(7) and their isoduals (11)-(12).

In short, the grand unification proposed in this note is based on the axiomatic structure of the conventional Dirac's equations, as emerged from the novel insights of memoir[(4f)], and merely subjected to axiom-preserving liftings, in which the inclusion of gravitation for matter is permitted by the novel isoselfscalar symmetries (6)-(7), and that for antimatter by the antiisomorphic images (11)-(12).

The reader should not be surprised that the four new invariances (6)-(7) and (11)-(12) remained undetected throughout this century. In fact, their identification required the prior discovery of new numbers, first the numbers with arbitrary positive units for invariance (6)-(7), and then the additional new numbers with arbitrary negative units for invariance (11)-(12).

2 Isotopic Gauge Theory

The isotopic of gauge theories were first studied in 1980's by Gasperini[(11a)], followed by Nishioka[(10b)], Karajannis and Jannussis[(11c)] and others, and ignored thereafter. These studies were defined on conventional spaces over conventional fields and via the conventional differential calculus. As such, they are not invariant, as we learned only recently in memoirs[(4f)].

In this section we shall introduce, apparently for the first time, *the isotopies of gauge theories, or isogauge theories* for short, formulated in an invariant way, that is, on isospaces over isospaces over isofields and characterized by the isodifferential calculus of memoir[4c]. The *isodual isogauge theories* are apparently introduced in this note for the first time.

The essential mathematical methods needed for an axiomatically consistent and invariant formulation of the isogauge theories are the following:

(1) Isofields[4d] of isoreal numbers $\widehat{R}(\widehat{n}, \widehat{+}, \widehat{\times})$ and isocomplex numbers $\widehat{C}(\widehat{c}, \widehat{+}, \widehat{\times})$ with: additive isounit $\widehat{0} = 0$; generalized multiplicative isounit $\hat{I}$ given by Eq. (1); elements, isosum, isoproduct and related generalized operations,

$$\hat{a} = a \times \hat{I},\ a = n,\ c,\ \hat{a}\widehat{+}\widehat{b} = (a + b) \times \hat{I},\ \ \hat{a}\widehat{\times}\widehat{b} = \hat{a} \times \widehat{T} \times \widehat{b} = (a \times b) \times \hat{I} \tag{15a}$$

$$\hat{a}^n = \hat{a}\widehat{\times}\hat{a}\widehat{\times} \ldots \widehat{\times}\hat{a},\ \hat{a}^{\hat{1/2}} = a^{1/2} \times \hat{I}^{1/2},\ \widehat{\hat{a}/\widehat{b}} = \left(\hat{a}/\widehat{b}\right) \times \hat{I},\ etc. \tag{15b}$$

(2) Isominkowski spaces[6a] $\widehat{M} = \widehat{M}\left(\widehat{x},\ \widehat{\eta},\ \widehat{R}\right)$ with isocoordinates $\widehat{x} = x \times \hat{I} = \{x^\mu\} \times \hat{I}$, isometric $\widehat{N} = \widehat{\eta} \times \hat{I} = [\widehat{T}\,(x, \ldots) \times \eta] \times \hat{I}$, and *isointerval* over the isoreals $\widehat{R}$

$$(\widehat{x} - \widehat{y})^{\hat{2}} = (\widehat{x} - \widehat{y})^\mu \,\widehat{\times}\widehat{N}_{\mu\nu}\widehat{\times}\, (\widehat{x} - \widehat{y})^\nu = \left[(x - y)^\mu \times \widehat{\eta}_{\mu\nu} \times (x - y)^\nu\right] \times \hat{I}, \tag{16}$$

equipped with Kadeisvili isocontinuity[10a] and Tsagas-Sourlas isotopology[10b] (see also Aslander and Keles[10d]). A more technical formulation of the isogauge theory can be done via the isobundle formalism on isogeometries recently reached by Vacaru[10c], which will be studied in a future work.

(3) Isodifferential calculus[4c] characterized by the following isodifferentials and isoderivatives

$$\widehat{d}\widehat{x}^\mu = \hat{I}^\mu_\nu \times d\widehat{x}^\nu,\ \widehat{d}\widehat{x}_\mu = \widehat{T}^\mu_\nu \times \widehat{x}_\nu,\ \widehat{\partial}_\mu \widehat{f} = \widehat{\partial \widehat{f}/\widehat{\partial}\widehat{x}^\mu} = \left(\widehat{T}^\mu_\nu \times \partial_\nu f\right) \times \hat{I}, \tag{17a}$$

$$\widehat{\partial}^\mu \widehat{f} = \left(\hat{I}^\mu_\nu \times \partial_\nu f\right) \times \hat{I},\ \widehat{\partial}\widehat{x}^\mu \widehat{/} \widehat{\partial}\widehat{x}^\nu = \widehat{\delta}^\mu_\nu = \delta^\mu_\nu \times \hat{I},\ \ etc. \tag{17b}$$

(4) Isofunctional isoanalysis[6q], including the reconstruction of all conventional and special functions and transforms into a form admitting of $\hat{I}$ as the left and the right unit. Since the isominkowskian geometry preserves the Minkowskian axioms, it allows the preservation of the notions of straight and intersecting lines, thus permitting the reconstruction of trigonometric and hyperbolic functions under the Riemannian metric $g\,(x) = \widehat{\eta}$[6q].

(5) Isominkowskian geometry[5e], i.e., the geometry of isomanifolds M over the isoreals $\widehat{R}$, which satisfies all abstract Minkovskian axioms because of the joint liftings $\eta \to \widehat{\eta} = T(x, \ldots) \times \eta$ and $I \to \widehat{I} = T^{-1}$, while preserving the machinery of Riemannian spaces (covariant derivatives, connections, etc.), although expressed in terms of the isodifferential calculus for consistency. In this new geometry *Riemannian* line elements are turned into identical *Minkovskian* forms via the embedding of gravity in the differentials, e.g., for the Schwarzschild exterior metric we have the isominkowskian reformulation (Ref.[5e], Eq.s (2.57)), where the space-time coordinates are assumed to be covariant,

$$\widehat{ds} = \widehat{d\hat{s}}^{\hat{2}} \hat{+} \hat{r}^{\hat{2}} \hat{\times} (\widehat{d\hat{\theta}}^{\hat{2}} \hat{+} iso \sin^{\hat{2}} \hat{\theta}) \hat{-} \widehat{d\hat{t}}^{\hat{2}}, \tag{18a}$$

$$\widehat{d\hat{r}} = \hat{T}_r \times d\hat{r},\ \widehat{d\hat{t}} = \hat{T}_t \times d\hat{t},\ \hat{T}_r = (1 - 2 \times M/r)^{-1},\ \hat{T}_t = 1 - 2 \times M/r. \tag{18b}$$

(6) Relativistic hadronic mechanics$^{(4f)}$ characterized by the *isohilbert space* $\hat{\mathcal{H}}$ first introduced by Myung and Santilli$^{(9c)}$ in 1982 with *isoinner product and isonormalization* over $\hat{C}$

$$\left\langle \hat{\phi} \hat{|} \hat{\psi} \right\rangle = \left\langle \hat{\phi} \left| \hat{\times} \right| \hat{\psi} \right\rangle \times \hat{I},\ \left\langle \hat{\psi} \hat{|} \hat{\psi} \right\rangle = \hat{I}. \tag{19}$$

Among the various properties we recall that: the *isohermiticity* on $\hat{\mathcal{H}}$ coincides with the conventional Hermiticity (thus, all conventional observables remain observables under isotopies); the isoeigenvalues of isohermitian operators are real and conventional (because of the identities $\hat{H} \hat{\times} \left| \hat{\psi} \right\rangle = \hat{E} \hat{\times} \left| \hat{\psi} \right\rangle = E \times \left| \hat{\psi} \right\rangle$); the condition of *isounitarity* on $\hat{\mathcal{H}}$ over $\hat{C}$ is given by $\hat{U} \hat{\times} \hat{U}^\dagger = \hat{U}^\dagger \hat{\times} \hat{U} = \hat{I}$ (see in memoir$^{(4f)}$ for details).

(7) The Lie-Santilli isotheory$^{(4,6,8d,10c)}$ with: conventional (ordered) basis of generators $X = (X_k)$, and parameters $\omega = (\omega_k)$ $k = 1, 2, \ldots, n$, only formulated in isospaces over isofields with a common isounit; universal enveloping isoassociative algebras $\hat{\xi}$ with infinite-dimensional basis characterized by the isotopic Poincaré-Birkhoff-Witt theorem$^{(4a,4c,6g)}$

$$\hat{I}, \hat{X}_i \hat{\times} \hat{X}_j,\ (i \le j),\ \hat{X}_i \hat{\times} \hat{X}_j \times \hat{X}_k,\ (i \le j \le k), \ldots \tag{20}$$

Lie-Santilli isoalgebras:

$$\left[\hat{X}_i \hat{,} \hat{X}_j \right] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i = \hat{C}_{ij}^k (x, \ldots) \hat{\times} \hat{X}_k, \tag{21}$$

where $\hat{C}_{ij}^n$ are the structure isofunctions; and isogroups characterized by isoexponentiation on $\hat{\xi}$ with structure

$$\hat{e}^{\hat{X}} = \hat{I} \hat{+} \widehat{\hat{X}/1!} \hat{+} \hat{X} \hat{\times} \widehat{\hat{X}/2!} \hat{+} \ldots = \left(e^{\hat{X} \times \hat{T}} \right) \times \hat{I} = \hat{I} \times \left(e^{\hat{T} \times \hat{X}} \right). \tag{22}$$

Despite the isomorhism between isotopic and conventional structures, the lifting of Lie's theory is nontrivial because of the appearance of the matrix $\hat{T}$ with nonlinear elements in the very *exponent* of the group structure, Eq.s (22). To avoid misrepresentations, one must therefore keep in mind that the isotopies of Lie's theory *were not* proposed to build "new algebras" (an impossible task since all simple Lie algebras are known from Cartan's classification), but to construct instead the most general possible nonlinear, nonlocal and noncanonical or nonunitary "realizations" of known Lie algebras.

Another important aspect the reader should keep in mind is that the isotopies are such to reconstruct linearity, locality and canonicity or unitarity on isospaces over isofields, called *isolinearity, isolocality and isocanonicity or isounitary.* As a result, the use of the conventional linear transformations on M over R, $X^{'} = A\,(a) \times x$ violates *isolinearity* on $\widehat{M}$ over $\widehat{R}$. In general, *any* use of conventional mathematics for isotopic theories leads to a number of inconsistencies which generally remain undetected by nonexperts in the field.

We are now minimally equipped to introduce the desired isogauge theory which can be characterized by an n-dimensional connected and non-abelian isosymmetry $\widehat{G}$ with: basic n-dimensional isounit (1); isohermitean operators $\widehat{X}$ on an isohilbert space $\widehat{\mathcal{H}}$ over the isofield $\widehat{C}(\widehat{c}, \widehat{+}, \widehat{\times})$; universal enveloping associative algebra $\widehat{\xi}$ with infinite isobasis (20); isocommutation rules (21); isogroup structure

$$\widehat{U} = \widehat{e}^{-i \times X_k \times \theta(x)_k} = (e^{-i \times X_k \times \widehat{T} \times \theta(x)_k}) \times \widehat{I},\ \widehat{U}^{\dagger} \widehat{\times} \widehat{U} = \widehat{I}; \tag{23}$$

where one should note the appearance of the gravitational isotopic elements in the exponent, and the parameters $\theta(x)_k$ now depend on the isominkowski space; isotransforms of the isostates on $\widehat{\mathcal{H}}$

$$\widehat{\psi}' = \widehat{U} \widehat{\times} \widehat{\psi} = (e^{-i \times X_k \times \widehat{T}(x,\ldots) \times \theta(x)_k}) \times \widehat{\psi}; \tag{24}$$

isocovariant derivatives[5c]

$$\widehat{D}_\mu \widehat{\psi} = (\widehat{\partial}_\mu - i \widehat{\times} \widehat{g} \widehat{\times} \widehat{A}(\widehat{x})^k_\mu \widehat{\times} \widehat{X}_k \widehat{\times} \widehat{\psi}; \tag{25}$$

iso-Jacobi identity

$$\left[\widehat{D}_\alpha \widehat{,} \left[\widehat{D}_\beta \widehat{,} \widehat{D}_\gamma\right]\right] \widehat{+} \left[\widehat{D}_\beta \widehat{,} \left[\widehat{D}_\gamma \widehat{,} \widehat{D}_\alpha\right]\right] \widehat{+} \left[\widehat{D}_\gamma \widehat{,} \left[\widehat{D}_\alpha \widehat{,} \widehat{D}_\beta\right]\right] = 0; \tag{26}$$

where g and $\widehat{g} = g \times \widehat{I}$ are the conventional and isotopic coupling constants, $A(x)^k_\mu \times X_k$ and $\widehat{A}(\widehat{x})^k_\mu \widehat{\times} \widehat{X}_k = \left[A(x)^k_\mu \times X_k\right] \times \widehat{I}$ are the gauge and isogauge potentials; isocovariance

$$\left(\widehat{D}_\mu \widehat{\psi}\right)^{'} = \left(\widehat{\partial}_\mu \widehat{U}\right) \widehat{\times} \widehat{\psi} \widehat{+} \widehat{U} \widehat{\times} \left(\widehat{\partial}_\mu \widehat{\psi}\right) \widehat{-} \widehat{i} \widehat{\times} \widehat{g} \widehat{\times} \widehat{A}^{'} (\widehat{x})_\mu \widehat{\times} \widehat{\psi} = \widehat{U} \widehat{\times} \widehat{D}_\mu \widehat{\psi}, \tag{27a}$$

$$\hat{A}\, (\widehat{x})^{'}_\mu = -\widehat{g}^{\widehat{-1}} \widehat{\times} \left[\widehat{\partial}_\mu \widehat{U}\, (\widehat{x})\right] \widehat{\times} \widehat{U}\, (\widehat{x})^{\widehat{-1}}, \tag{27b}$$

$$\widehat{\delta} \hat{A}\, (\widehat{x})^k_\mu = -\widehat{g}^{\widehat{-1}} \widehat{\times} \widehat{\partial}_\mu \widehat{\theta}\, (\widehat{x})^k \,\widehat{+} \widehat{C}^k_{ij} \widehat{\times} \widehat{\theta}\, (\widehat{x})^i \,\widehat{\times} \hat{A}\, (\widehat{x})^j_\mu\,, \tag{27c}$$

$$\widehat{\delta} \widehat{\psi} = -\widehat{i} \widehat{\times} \widehat{g} \widehat{\times} \widehat{\theta}\, (\widehat{x})^k \,\widehat{\times} \widehat{X}_k \widehat{\times} \widehat{\psi}; \tag{27d}$$

non-abelian iso-Yang-Mills fields

$$\widehat{F}_{\mu\nu} = \widehat{i}\widehat{\times}\widehat{g}^{-\widehat{1}}\widehat{\times}\left[\widehat{D}_\mu, \widehat{D}_\nu\right]\widehat{\psi}, \tag{28a}$$

$$\widehat{F}^k_{\mu\nu} = \widehat{\partial}_\mu \hat{A}^k_\nu \widehat{-} \widehat{\partial}_\nu \hat{A}^k_\mu \widehat{+} \widehat{g}\widehat{\times}\widehat{C}^k_{ij}\widehat{\times}\hat{A}^i_\mu\widehat{\times}\hat{A}^j_\nu; \tag{28b}$$

related isocovariance properties

$$\widehat{F}_{\mu\nu} \to \widehat{F}'_{\mu\nu} = \widehat{U}\widehat{\times}\ \widehat{F}_{\mu\nu}\widehat{\times}\widehat{U}^{-1}, \tag{29a}$$

$$Isotr\left(\widehat{F}_{\mu\nu'}\widehat{\times}\widehat{F}^{\mu\nu'}\right) = Isotr\left(\widehat{F}_{\mu\nu}\widehat{\times}\widehat{F}^{\mu\nu}\right), \tag{29b}$$

$$\left[\widehat{D}_\alpha \widehat{,}\ \widehat{F}_{\beta\gamma}\right] \widehat{+} \left[\widehat{D}_\beta \widehat{,}\ \widehat{F}_{\gamma\alpha}\right] \widehat{+} \left[\widehat{D}_\gamma \widehat{,}\ \widehat{F}_{\alpha\beta}\right] \equiv 0; \tag{29c}$$

derivability from the isoaction

$$\widehat{S} = \widehat{\int} \widehat{d^4}\widehat{x}\left(-\widehat{F}_{\mu\nu}\widehat{\times}\widehat{F}^{\mu\nu}\widehat{/}\widehat{4}\right) = \widehat{\int}\widehat{d^4}\widehat{x}\left(-\widehat{F}^k_{\mu\nu}\widehat{\times}\widehat{F}^{\mu\nu}_k\widehat{/}\widehat{4}\right); \tag{30}$$

where $\widehat{\int} = \int \times \hat{I}$, plus all other familiar properties in isotopic formulation.

The *isodual isogauge theory* is the preceding theory following the application of the isodual map (8) to the totality of quantities and their operations. The latter theory is characterized by the isodual isogroup $\widehat{G}^d$ with isodual isounit $\hat{I}^d = -\hat{I}^\dagger = -\hat{I}$. The base fields are the field $\widehat{R}^d\left(\widehat{n}^d, \widehat{+}^d, \widehat{\times}^d\right)$ of isodual isoreal numbers $\widehat{n}^d = -\widehat{n} = -n\times\hat{I}$ and the field $\widehat{C}^d\left(\widehat{c}^d, \widehat{+}^d, \widehat{\times}^d\right)$of isodual isocomplex numbers $\widehat{c}^d = -\left(c \times \hat{I}\right)^\dagger = (n_1 - i \times n_2) \times \hat{I}^d = (n_1 + i \times n_2) \times \hat{I}$.

The carrier spaces are the isodual isominkowski space $\widehat{M}^d\left(\widehat{x}^d, \widehat{\eta}^d, \widehat{R}^d\right)$ on $\widehat{R}^d$ and the isodual isohilbert space $\mathcal{H}^d$ on $\widehat{C}^d$ with isodual isostates $\left|\widehat{\psi} >^d = -\right|\widehat{\psi} >^\dagger$ and isodual isoinner product $< \widehat{\phi}\Big|^d \times \widehat{T}^d \times \Big|\widehat{\psi} >^d \times \hat{I}^d$. It is instructive to verify that all eigenvalues of isodual isohermitean operators are *negative-definite* (when projected in our space-time), $\widehat{H}^d\widehat{\times}^d \left|\widehat{\psi} >^d = (-E) \times \right|\widehat{\psi} >$.

$\widehat{G}^d$ is characterized by the isodual Lie-Santilli isotheory with isodual generators $\widehat{X}^d = -\widehat{X}$, isodual isoassociative product $\widehat{A}^d\widehat{\times}^d\widehat{B}^d = \widehat{A}^d \times \widehat{T}^d \times \widehat{B}^d$, $\widehat{T}^d = -\widehat{T}$ and related isodual isoenveloping and Lie-Santilli isoalgebra. The elements of $\widehat{G}^d$ are the isodual isounitary isooperators $\widehat{U}^d\left(\widehat{\theta}^d\left(\widehat{x}^d\right)\right) = -\widehat{U}\left(-\widehat{\theta}\left(-\widehat{x}\right)\right)$. In this way, the isodual isogauge theory is seen to be an anti-isomorphic image of the preceding theory, as desired.

It is an instructive exercise for the reader interested in learning the new techniques to study first the isodualities of the *conventional* gauge theory (rather than of their isotopies), and show that they essentially provide a mere reinterpretation of the usually discarded, advanced solutions as characterizing antiparticles. Therefore, in the isoselfdual theory with total gauge symmetry $\widehat{G} \times \widehat{G}^d$, isotopic retarded solutions are associated with particles and advanced isodual solutions are associated with antiparticles.

No numerical difference is expected in the above reformulation because in conventional theories particles and antiparticles are represented with retarded solutions while advanced solutions are generally discarded. By comparison, in the isodual theory retarded solutions are solely used for particles and advanced solutions are solely used for antiparticles, the two solutions being formulated in their respective different spaces over different fields.

It is also recommendable for the interested reader to verify that the isotopies are indeed equivalent to charge conjugation for all massive particles, with the exception of the photon(7c). In fact, isodual theories predict that the antihydrogen atom(12b) emits a new photon, tentatively called by this author the *isodual photon*, which coincides with the conventional photon for all possible interactions, thus including electroweak interactions, *except gravitation*(7c). This indicates that the isodual map is inclusive of charge conjugation for massive particles, but it is broader that the latter.

Isodual theory in general, thus including the proposed grand unification, predict that all *stable* isodual particles, such as the isodual photon, the isodual electron (positron), the isodual photon (antiproton) and their bound states (such as the antihydrogen atom), experience *antigravity* in the field of Earth (defined as the reversal of the sign of the curvature tensor). If confirmed, the prediction may offer the possibility in the future to ascertain whether far away galaxies and quasars are made-up of matter or of antimatter.

Known objections against antigravity are inapplicable because they are tacitly referred to positive units and also because the isodual theory predicts that particle-antiparticle bound states such as the positronium, experience attraction in both fields of matter and antimatter(7). The latter predictions are currently under experimental study by Mills(11a) and others(11b).

We also note that the isotopies leave unrestricted the functional dependence of the isounit (1), provided that it is positive-definite. In this note we use only the x-dependence to represent *exterior* gravitational problems in vacuum. The isotheory also admits an arbitrary nonlinearity in the velocities and other variables which is used for the study of *interior* gravitational problems. The isotheory naturally admits a dependence of the isounit on the *wavefunctions* and their derivatives while preserving isolinearity in isospace (thus preserving the superposition principle, as needed for a consistent representation of composite systems). For these and other aspects we refer the reader to memoir(4f).

We finally note that the isomathematics is a particular cases of the broader *genomathematics*(4a, 4c, 4f), which occurs for non-Hermitian generalized units and is used for an axiomatization of irreversibility. In turn, the genomathematics is a particular case of the *hypemathematics*(4c, 4f), which occurs when the generalized units are given by ordered *sets* of non-Hermitian quantities and is used for the representation of multivalued complex systems (e.g., biological) in irreversible conditions. Evidently both the genomathematics and hypermathematics admit an anti-isomorphic image under isoduality (an outline of these novel mathematics can be found in Page 18 of Web Site[9y]).

In conclusion the methods in this note permit the study of *seven* liftings of conventional gauge theories; (1) the *isodual gauge theories* for the treatment of antimatter without gravitation in vacuum; (2, 3), the *isogauge theories and their isoduals*, for the inclusion of gravity for matter and antimatter in reversible conditions in vacuum (exterior gravitational problem); (4, 5) the *genogauge theories and their isoduals*, for the inclusion of gravity for matter and antimatter in irreversible interior conditions (interior gravitational problems); and (6, 7) the *hypergauge theories and their isoduals*, for multivalued and irreversible generalizations. This note is restricted to theories (1, 2, 3).

3 Iso-Grand-Unification

In this note we have submitted, apparently for the first time, an Iso-Grand-Unification (IGU) with the inclusion of gravity characterized by the total isoselfdual symmetry

$$\widehat{S}_{Tot} = \left(\widehat{P}\,(3.1)\,\widehat{\times}\widehat{G}\right) \times \left(\widehat{P}\,(3.1)^{d}\,\widehat{\times}^{d}\widehat{G}^{d}\right) =$$

$$= \left[\widehat{SL}\left(2,\,\widehat{C}\right)\widehat{\times}\widehat{T}\,(3.1)\right] \times \left[\widehat{SL}^{d}\left(2,\,\widehat{C}^{d}\right)\widehat{\times}\widehat{T}^{d}\,(3.1)\right], \qquad (31)$$

where $\widehat{\mathcal{P}}$ is the Poincaré-Santilli isosymmetry[10c] in its isospinorial realization[6f], $\widehat{G}$ is the isogauge symmetry of the preceding section and the remaining structures are the corresponding isoduals.

Without any claim of a final solution, it appears that the proposed IGU does indeed offer realistic possibilities of at least resolving the axiomatic incompatibilities (1), (2) and (3) between gravitational and electroweak interactions indicated in Sect. 1. In fact, IGU represents gravitation in a form geometrically compatible with that of the electroweak interactions, represents antimatter at all levels via negative-energy solutions, and characterizes both gravitation as well as electroweak interactions via the universal isopoincaré symmetry.

It should be indicated that we are referring here to the *axiomatic* consistency. The *physical* consistency is a separate problem which cannot possibly be investigate in this introductory note and will be investigated in future works. At this point we merely mention the general rule according to which isotopic liftings preserve not only the original axioms, but also the original numerical values[6g] (as an example, the image in isominkowskian space over the isoreals of the light cone, not only is a perfect cone, but a cone why the original characteristic angle, preserving the speed of light in vacuum as the maximal causal speed in isominkowskian space). This occurrence provides realistic hopes for the joint achievement of axiomatic and physical consistency.

The reader should be aware that the methods of the recent memoir[4f] permit a truly elementary, explicit construction of the proposed IGU. As well known, the transition from the Minkowskian metric η to Riemannian metrics $g\,(x)$ is a *noncanonical* *transform* at the classical level and, therefore, a *nonunitary transform* at the operator level. The method herein considered for turning a gauge theory into an IGU consists in the following representation of the selected gravitational model, e.g., Schwarzschild's model,

$$g(x) = T(x) \times \eta,\ T(x) = \left(U \times U^\dagger\right)^{-1}, \tag{32a}$$

$$\left(U \times U^\dagger\right) = Diag.\left[(1 - 2 \times M/r) \times Diag.(1,1,1), (1 - 2 \times M/r)^{-1}\right], \tag{32b}$$

and then subjecting the *totality* of the gauge theory to the nonunitary transform $U \times U^\dagger$. The method then yields: the isounit $I \rightarrow \hat{I} = U \times I \times U^\dagger$; the isonumbers $a \rightarrow \hat{a} = U \times a \times U^\dagger = a \times \left(U \times U^\dagger\right) = a \times \hat{I}$, $a = n, c$; the isoproduct with the correct expression and Hermiticity of the isotopic element, $A \times B \rightarrow U \times (A \times B) \times U^\dagger = \left(U \times A \times U^\dagger\right) \times \left(U \times U^\dagger\right)^{-1} \times \left(U \times B \times U^\dagger\right) = \widehat{A} \times \widehat{T} \times \widehat{B} = \widehat{A}\widehat{\times}\widehat{B}$; the correct form of the isohilbert product on $\widehat{C}$, $< \phi| \times |\psi > \rightarrow U \times < \phi| \times |\psi > \times U^\dagger = (< \phi| \times U^\dagger) \times \left(U \times U^\dagger\right)^{-1} \times (U \times |\psi\rangle) \times \left(U \times U^\dagger\right) = < \widehat{\phi}\Big| \times \widehat{T} \times \Big|\widehat{\psi} > \times \hat{I}$; the correct Lie-Santilli isoalgebra $A \times B - B \times A \rightarrow \widehat{A}\widehat{\times}\widehat{B} - \widehat{B}\widehat{\times}\widehat{A}$; the correct isogroup $U \times \left(e^X\right) \times U^\dagger = \left(e^{X \times \widehat{T}}\right) \times \hat{I}$, the isopoincaré symmetry $\mathcal{P} \rightarrow \widehat{\mathcal{P}}$, and the isogauge group $G \rightarrow \widehat{G}$.

It is then easy to verify that the emerging IGU is indeed invariant under all possible additional nonunitary transforms $W \times W^\dagger = \hat{I}$ provided that, for evident reasons of consistency, they are written in their identical isounitary form, $W = \widehat{W} \times \widehat{T}^{1/2}$, $W \times W^\dagger = \widehat{W}\widehat{\times}\widehat{W}^\dagger = \widehat{W}^\dagger\widehat{\times}\widehat{W} = \hat{I}$. In fact, we have the invariance of the isounit $\hat{I} = \hat{I}' = \widehat{W}\widehat{\times}\hat{I}\widehat{\times}\widehat{W}^\dagger = \hat{I}$, the invariance of the isoproduct $\widehat{A}\widehat{\times}\widehat{B} \rightarrow \widehat{W}\widehat{\times}\left(\widehat{A}\widehat{\times}\widehat{B}\right)\widehat{\times}\widehat{W}^\dagger = \widehat{A}'\widehat{\times}\widehat{B}'$, etc. Note that the isounit is *numerically* preserved, as it is the case for the conventional unit I under unitary transform, and that the selection of a nonunitary transform $W \times W^\dagger = \hat{I}'$ with value different from $\hat{I}$ evidently implies the transition to a different gravitational model.

Note that the lack of implementation of the above nonunitary-isounitary lifting to only *one* aspect of the original gauge theory (e.g., the preservation of the old numbers or of the old differential calculus) implies the loss of the invariance of the theory[(4f)]. The assumption of the negative-definite isounit $\hat{I}^d = -(U \times U^\dagger)$ then yields the isodual component of the IGU.

In closing, *the most significant possibility we would like to convey is that gravitation has always been present in unified gauge theories. It did creep in un-noticed because embedded where nobody looked for, in the "unit" of gauge theories.* In fact, the isogauge theory of Sect. 2 coincides with the conventional theory at the abstract level to such an extent that we could have presented the proposed IGU with exactly the same symbols of the conventional gauge theories without the "hats", and merely subjecting the same symbols to a more general realization.

Also, the isounit representing gravitation as per rule (32) verifies all the properties of the conventional unit I of gauge theories, $\hat{I}^{\hat{n}} = \hat{I}$, $\hat{I}^{1\hat{/}2} = \hat{I}$, $d\hat{I}/dt = \hat{I}\widehat{\times}\widehat{H} - \widehat{H}\widehat{\times}\hat{I} =$

$\widehat{H} - \widehat{H} = 0$, etc. The "hidden" character of gravitation in conventional gauge theories is then confirmed by the isoexpectation value[(4f)] of the isounit which recovers the conventional unit I of gauge theories, $\widehat{<}\hat{I}\widehat{>} =< \widehat{\psi}\Big| \times \widehat{T} \times \hat{I} \times \widehat{T} \times \Big|\widehat{\psi} > / < \widehat{\psi}\Big| \times \widehat{T} \times \Big|\widehat{\psi} >= I$.

It then follows that the proposed IGU constitutes an explicit and concrete realization of the theory of "hidden variables"[(13a)] $\lambda = T(x) = g(x)/\eta$, $\widehat{H}\widehat{\times}\Big|\widehat{\psi} >= \widehat{H} \times \lambda \times \Big|\widehat{\psi} >= E_\lambda \times \Big|\widehat{\psi} >$, and the theory is correctly reconstructed with respect to the new unit $\hat{I} = \lambda^{-1}$, in which von Neumann's Theorem[(13b)] and Bell's inequalities[(13c)] do not apply, evidently because of the nonunitary character of the theory (see Vol. II of Refs. [6g] for details).

In conclusion, as indicated beginning with the title of the recent memoir[(4f)], the proposed inclusion of gravitation in unified gauge theories is essentially along the teaching of Einstein, Podolsky and Rosen[(14)] on the "lack of completion" of quantum mechanics, only applied to gauge theories.

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