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ISOTOPIES OF HOMEOMORPHISMS OF RIEMANN SURFACES AND A THEOREM ABOUT ARTIN'S BRAID GROUP

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Let $\tilde{X}$, X be orientable surfaces. Let $(p, \tilde{X}, X)$ be a regular covering space, possibly branched, with finitely many branch points and a finite group of covering transformations. We require also that every covering transformation leave the branch points fixed. A homeomorphism $\tilde{g}: \tilde{X} \to \tilde{X}$ is said to be "fiber-preserving" with respect to the triplet $(p, \tilde{X}, X)$ if for every pair of points $\tilde{x}, \tilde{x}' \in \tilde{X}$ the condition $p(\tilde{x}) = p(\tilde{x}')$ implies $p\tilde{g}(\tilde{x}) = p\tilde{g}(\tilde{x}')$. If $\tilde{g}$ is fiber-preserving and isotopic to the identity map via an isotopy $\tilde{g}_s$, then $\tilde{g}$ is said to be "fiber-isotopic to 1" if, for every $s \in [0, 1]$, the homeomorphism $\tilde{g}_s$ is fiber-preserving.

The condition that an isotopy be a fiber-isotopy imposes a symmetry which one feels, intuitively, is very restrictive. However, we find

THEOREM 1. *Let $g: \tilde{X} \to \tilde{X}$ be a fiber-preserving homeomorphism which is isotopic to the identity map. If the covering is branched, assume $\tilde{X}$ is not the closed sphere or torus. Then g is fiber-isotopic to the identity.*

Theorem 2 expresses a weaker result, which is true without exception.

THEOREM 2. *Let $\tilde{g}: \tilde{X} \to \tilde{X}$ be a fiber-preserving homeomorphism which is isotopic to the identity map. Then its projection g to X is also isotopic to the identity map; however, the isotopy may move branch points.*

A special case of Theorem 1 was established by the authors in an earlier paper [1] for the particular situation where X is a 2-sphere, and $\tilde{X}$ is a 2-sheeted covering of X with $2g + 2$ branch points. The proof given here is considerably simpler than the version in [1], and at the same time it holds in a much more general situation. The major tool that made this possible was the device of lifting maps to the universal covering space. The analogous problem in higher-dimensional manifolds has also been studied by the authors, and will be reported on separately.

Let $H(\tilde{X})$ be the group of all orientation-preserving homeomorphisms of $\tilde{X} \to \tilde{X}$, and let $D(\tilde{X})$ be the subgroup of those homeomorphisms which are isotopic to the identity map. Let $M(\tilde{X})$ be the quotient group $H(\tilde{X})/D(\tilde{X})$, that is the mapping class group of $\tilde{X}$. Assume that the

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covering $(p, \tilde{X}, X)$ is branched. Let $\tilde{P}_1, \ldots, \tilde{P}_n$ denote the set of branch points, and let $P_i = p(\tilde{P}_i)$, $i = 1, \ldots, n$. The branched covering has an associated unbranched covering which we denote by $(\tilde{p}, \tilde{X} - \tilde{P}_1, \ldots, \tilde{P}_n, X - P_1, \ldots, P_n)$, where $\tilde{p}$ is the restriction of p to $\tilde{X} - \tilde{P}_1, \ldots, \tilde{P}_n$. Let $M_p(\tilde{X})$ be the "symmetric" mapping class group of $\tilde{X}$, that is the subgroup of those elements in $M(\tilde{X})$ which can be represented by fiber-preserving maps. Theorem 1 can be rephrased: "*$M_p(\tilde{X})$ is canonically isomorphic to $M_p(\tilde{X} - \tilde{P}_1 \cup \ldots \cup \tilde{P}_n)$.*"

Let $T_{g,n}$ denote a Riemann surface of genus g with n points removed. The groups $M(T_{g,0})$ can be expected to play an important role in understanding the topology of 3 manifolds, in Teichmüller theory, and again in the theory of automorphisms of infinite groups. However, for the cases $g \geqq 3$, very little is known about these groups. As a step in this direction, we ask how the subgroups $M_p(T_{g,0}) \subset M(T_{g,0})$ can be characterized algebraically. Restricting our attention to the case where the group of covering transformations is generated by a single element t of order k, we observe that a homeomorphism $h: T_{g,0} \to T_{g,0}$ is fiber-preserving iff $\tilde{h}\tilde{t}\tilde{h}^{-1} = \tilde{t}^s$ for some $1 \leqq s \leqq k - 1$. Hence the subgroup $M_p(T_{g,0})$ is necessarily included in the normalizer of the cyclic subgroup T generated by the isotopy class $[\tilde{t}]$ of the element t. To determine whether $M_p(T_{g,0})$ coincides with the normalizer of T, we first show

THEOREM 3. *Let $[\tilde{t}]$, $[\tilde{h}] = M(T_{g,0})$, where $[\tilde{t}]$ has finite order k. Suppose that $[\tilde{h}]$ belongs to the normalizer of $[\tilde{t}]$, i.e.,*

$$[\tilde{h}]\,[\tilde{t}]\,[\tilde{h}]^{-1} = [\tilde{t}]^s, \qquad 1 \leqq s \leqq k - 1, (s, k) = 1.$$

Then $[\tilde{h}]$ and $[\tilde{t}]$ can be represented by topological mappings $\tilde{h}, \tilde{t}$ which have the properties

$$\tilde{h}\tilde{t}\tilde{h}^{-1} = \tilde{t}^s \quad \text{and} \quad \tilde{t}^k = 1.$$

Using Theorem 3, we were then able to establish that for the case where the group T is cyclic:

THEOREM 4. *The symmetric subgroups $M_p(T_{g,0})$ are precisely the normalizers of the cyclic subgroups T generated by any element $[\tilde{t}]$ of finite order in $M(T_{g,0})$.*

To apply Theorem 4, we restrict our attention to the case of k-sheeted cyclic coverings $(p, T_{g,0}, T_{0,0})$ of the sphere $T_{0,0}$ by the closed surface $T_{g,0}$. In this situation, the covering will have n branch points, where k, n and g are related by the formula $2g = (k - 1)(n - 2)$.

For this special case we find

THEOREM 5. *Projecting fiber-preserving homeomorphisms induces an isomorphism i between the groups $M_p(T_{g,0})/T$ and $M(T_{0,n})$.*

Since generators and defining relations are known for the group $M(T_{0,n})$ for every integer n (see [3], [4]) we can use this result (which is constructive) to determine explicit presentations for all of the groups $M_p(T_{g,0})$.

For the special case $g = 2$, $k = 2$, $n = 6$, it was shown in [1] that the group $M_p(T_{2,0})$ coincides with the full mapping class group $M(T_{2,0})$. We find that this situation was special indeed, and, in fact,

THEOREM 6. *If $g \geqq 3$, there does not exist any finite cyclic covering with the property that $M_p(T_{g,0})$ coincides with $M(T_{g,0})$.*

As an application of the above results, we discuss and settle a conjecture about Artin's braid group B_n. The braid group can be defined as that group of automorphisms of a free group $F_n = \langle x_1, \ldots, x_n \rangle$ of rank n which maps every generator x_i into a conjugate of itself, and preserves the product $x_1 x_2 \cdots x_n$ [4]. Let k be any integer $\geqq 2$, and let N_k be the normal closure in F_n of the n elements $x_1^k, \ldots, x_n^k$. Then the elements in B_n induce a group of automorphisms of F_n/N_k, which we denote by $B_{n,k}$. We show that our geometric results imply

THEOREM 7. *$B_{n,k}$ is canonically isomorphic to B_n.*

A detailed report giving proofs of the theorems stated above will be published in another journal. The methods used were a combination of geometric and algebraic arguments and also, in the case of Theorem 3, techniques in Teichmüller theory, similar to those used in [2].

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