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Isotropic Majority-Vote Model on a Square Lattice

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The stationary critical properties of the isotropic majority vote model on a square lattice are calculated by Monte Carlo simulations and finite size analysis. The critical exponents ν, γ, and β are found to be the same as those of the Ising model and the critical noise parameter is found to be $q_c = 0.075 \pm 0.001$.

KEY WORDS: Majority-vote models; stochastic spin systems; Monte Carlo simulation.

1. INTRODUCTION

It has been argued that nonequilibrium stochastic spin systems with up–down symmetry fall in the universality class of the equilibrium Ising model.(1) This conjecture has been found to be valid for several models that do not obey detailed balance.(2–6) Here we analyze, on a square lattice, a relatively simple nonequilibrium model with up–down symmetry, namely the isotropic majority vote model.(7,8) We have found, by Monte Carlo simulations and finite-size analysis, that its critical exponents, in the stationary state, are the same as those of the Ising model.

The majority vote model is defined as follows. Consider a regular lattice where at each site there is a spin variable $\sigma_i = \pm 1$. At each (discrete) time a spin is chosen at random. The chosen spin then adopts the majority sign of the spins in its neighborhood with probability p and the minority sign with probability $q = 1 - p$. In other words, the chosen spin flips with probability q if it agrees with the majority sign and flips with probability

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p if it does not. Many versions of the model can be set up by defining the neighborhood of a spin. Here we consider, on a square lattice, an isotropic version in which the neighborhood of a spin consists of its four nearest neighbors. Since this number is even, there are configurations in which the neighborhood may have an equal number of plus and minus signs. In those cases the chosen spin flips with probability one-half. At this point, the model differes from that defined by Liggett(7) and Gray,(8) who include the chosen spin in its neighborhood.

The spin flip probability $w_i(\sigma) = w_i(\{\sigma_i\})$ is then defined by

$$w_i(\sigma) = \frac{1}{2}\left[1 - (1-2q)\,\sigma_i S\left(\sum_\delta \sigma_{i+\delta}\right)\right] \tag{1}$$

where the summation is over the nearest neighbor sites and the function $S(x)$ is defined by $S(x) = \text{sign}(x)$ if $x \neq 0$ and $S(0) = 0$. The noise parameter q is restricted to the interval $0 \leqslant q \leqslant 1/2$, in which case the process is attractive. The case $1/2 \leqslant q \leqslant 1$ can be reduced to the attractive case by reversing all spins of one of the sublatices and replacing q by $1-q$. This transformation leaves $w_i(\sigma)$ invariant.

The majority vote model defined by (1) can be regarded as composed of a zero-temperature process (noiseless majority voting) and an infinite-temperature process (spin randomization). Consider an open Ising system with ferromagnetic nearest neighbor interactions connected to two heat reservoirs, one being a source and the other a sink of heat. Suppose that the heat baths are simulated by Glauber processes,(9) the one associated to the source occurring with probability b and the other with probability $1-b$. The open Ising system is then governed by a competing stochastic dynamics whose spin flip probability $w_i(\sigma)$ is given by(10,11)

$$w_i(\sigma) = \frac{1}{2}\left[1 - (1-b)\,\sigma_i \tanh\left(\beta_A \sum_\delta \sigma_{i+\delta}\right) - b\sigma_i \tanh\left(\beta_B \sum_\delta \sigma_{i+\delta}\right)\right] \tag{2}$$

where β_A and β_B are the inverse temperatures of the skin and the source, respectively. Now if we let the temperature of the sink be very small and that of the source be very high, so that $\beta_A \to \infty$ and $\beta_B = 0$, we obtain the spin flip probability given by (1) with $b = 2q$. With this interpretation there is, in the stationary state, a continuous flux of heat through the system when $0 < q < 1/2$.

As a manifestation of its dissipative nature, the model shows no microscopic reversibility in the stationary state. For instance, the probability of a closed path is, in general, different from that of the reversed path. As an example, consider a local configuration consisting of a nearest neighbor pair of up spins, one of them (spin 1) having no down spins as nearest neighbors and the other (spin 2) having one or two. From the spin flip probability, we find that the probability of the closed path in which these two spins are flipped in the sequence 1212 is greater than the reversed path by a factor $(1-q)/q$.

Although there is no rigorous proof of phase transition for the isotropic majority vote model, it is possible to argue(8) that on a square lattice there must be two phases for sufficiently small q. Suppose that an island of up spins is formed on a sea of down spins. According to Gray,(8) the size of this island follows a birth-and-death process in which the death rate is larger than the birth rate. This would prevent the growth of the island, keeping the down spin phase stable. By symmetry there must be another phase with spins up. If, however, the up–down symmetry of the spin flip probability is broken, we expect no phase transition as in the Ising model, but unlike Toom's anisotropic voting model.(2)

When $q=1/2$ the spins flip independently, so that the stationary state is unique. Actually, one can prove(7,12) that the model has a unique stationary state on a square lattice when $1/4<q\leqslant 1/2$. This gives an upper bound for the critical noise, $q_c<1/4=0.25$, which is large compared to our numerical value $q_c=0.075\pm 0.001$. If one uses a pair approximation(11) one gets a critical noise $q_c=5/37\approx 0.135$, which presumably represents an upper bound on the correct value, as is typical for mean field approximations.

2. FINITE-SIZE SCALING

Suppose that, in an infinite system, a certain quantity $Q(\varepsilon)$ behaves, near the critical point, like $|\varepsilon|^{-\mu}$, where ε is the deviation of the external parameter from its critical value. Then, according to finite-size scaling theory,(13) one should have $Q_L(\varepsilon)=L^{\mu/\nu}\tilde{Q}(L^{1/\nu}\varepsilon)$ for a finite system of size L, where $\tilde{Q}(x)$ is a scaling function. $\tilde{Q}(x)$ is smooth function and analytical at the origin. Moreover, for a fixed boundary condition, it can be made universal by choosing appropriate metric factors. Here, we consider systems with $L\times L=N$ sites and periodic boundary conditions.

Define the variable m by $m=\sum_{i=1}^{N}\sigma_i/N$. We are interested in the following quantities(14): the "magnetization"

$$M_L=\langle |m| \rangle \tag{3}$$

the "susceptibility"

$$X_L=N\{\langle m^2\rangle-\langle |m|\rangle^2\} \tag{4}$$

and the reduced fourth-order cumulant

$$U_L = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2} \tag{5}$$

The averages are meant to be calculated in the stationary state.

These quantities are functions of the noise parameter q and obey the finite-size scaling relations

$$M_L(q) = L^{-\beta/\nu} \tilde{M}(L^{1/\nu}\varepsilon) \tag{6}$$

$$X_L(q) = L^{\gamma/\nu} \tilde{X}(L^{1/\nu}\varepsilon) \tag{7}$$

$$U_L(q) = \tilde{U}(L^{1/\nu}\varepsilon) \tag{8}$$

where $\varepsilon = q - q_c$. They can be derived by postulating a finite-size scaling relation for the stationary probability distribution $P_L(q, m)$ of m and using definitions (3)–(5). Following Binder,$^{(14)}$ we write

$$P_L(q, m) = L^{\beta/\nu} \tilde{P}(L^{1/\nu}\varepsilon, L^{\beta/\nu} m) \tag{9}$$

where $\tilde{P}(x, s)$ is a normalized scaling function. Notice that, in deriving relation (7), it follows that $\gamma/\nu = d - 2\beta/\nu$.

3. MONTE CARLO SIMULATION AND RESULTS

We have simulated the isotropic majority vote model on a square lattice with periodic boundary conditions. We used only square-shaped lattices with $L \times L = N$ sites, for several values of L ranging from $L = 5$ up

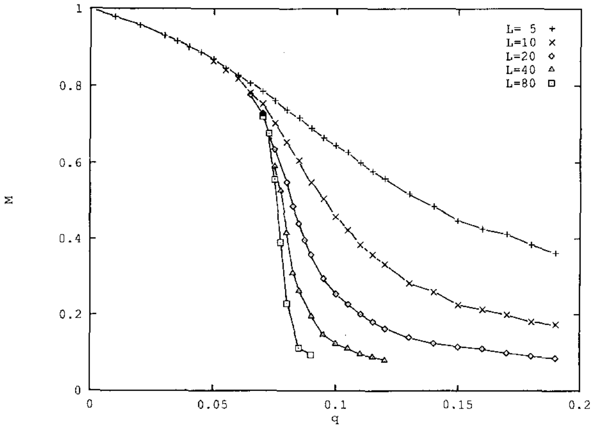

Fig. 1. Magnetization $M_L(q)$ as a function of the noise parameter q for several values of the system size L.

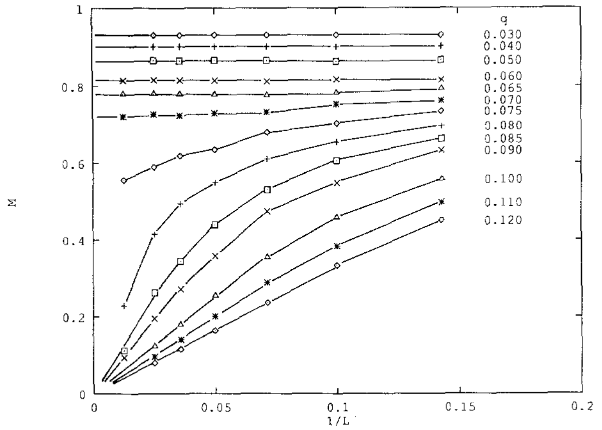

Fig. 2. Magnetization $M_L(q)$ as a function of $1/L$ for several values of q.

to $L = 80$. For each simulation we have started with a random configuration of spins. Given a certain configuration, the next one was obtained as follows. (a) Choose a spin at random, spin i, say. (b) Generate a random number r uniformly distributed between zero and unity. (c) If $r < w_i(\sigma)$, flip spin i, otherwise do not. After discarding the first configurations, so that the stationary regime was reached, we have calculated the quantities of interest. For $0.03 \leqslant q \leqslant 0.12$, we used 9×10^4 Monte Carlo steps to estimate the averages, for any size of the lattice. One Monte Carlo step equals L^2 spin-flip trials.

The magnetization is shown in Fig. 1 as a function of the noise parameter for several values of L. $M_L(q)$ never vanishes as long as L is

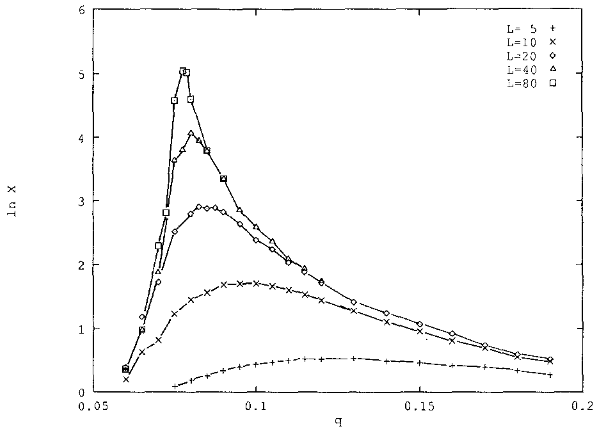

Fig. 3. Susceptibility $X_L(q)$ as a function of q for several values of L.

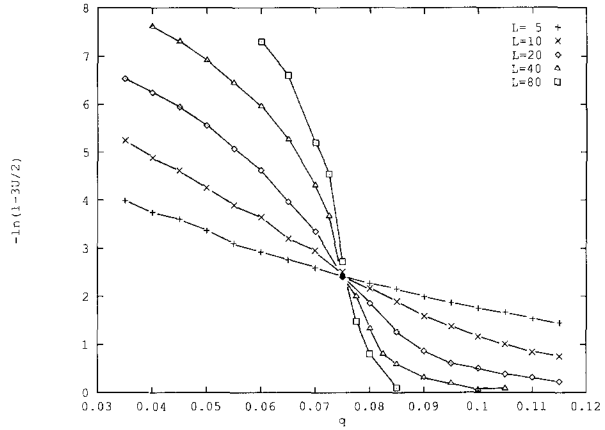

Fig. 4. Reduced fourth-order cumulant $U_L(q)$ as a function of q for several values of L. Within the accuracy of the data all curves intersect at $q_c = 0.075$. The value of $U_L(q)$ at the intersection is $U^* = 0.61$.

finite. However, if $q > q_c$, the magnetization behaves as $1/L$ and vanishes in the limit $L \to \infty$. When $q < q_c$, on the other hand, it approaches a nonzero value $M^*(q)$ in the limit $L \to \infty$. These two behaviors are shown in Fig. 2, where $M_L(q)$ is plotted as a function of $1/L$ for several values of q.

Figure 3 shows ln $X_L(q)$ as a function of q for several values of L. For each L, the susceptibility has a maximum whose position shifts toward the critical value as one increases L.[15]

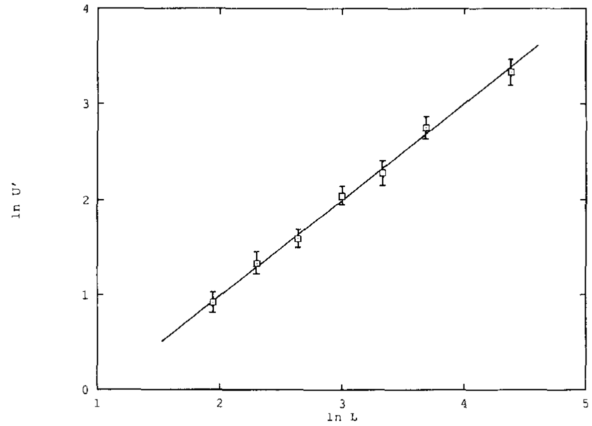

Fig. 5. Log-log plot of $dU_L(q)/dq$ at $q = q_c$ versus L. The solid line is the best fit with slope $1/\nu = 1.01$.

To locate the critical point, we plot the reduced fourth-order cumulant as a function of q for several values of L. All curves should intersect at $q = q_c$, since, according to the scaling relation (8), $U_L(q_c) = \tilde{U}(0) = U^*$, independent of L. From Fig. 4, we get $q_c = 0.075 \pm 0.001$ and $U^* = 0.61 \pm 0.01$. This value of U^* compares well with the value[16,17] $U^* = 0.611 \pm 0.001$ obtained for the square Ising model with periodic boundary condition.

To obtain an estimate of the critical exponent ν, we first obtained numerically $U'_L(q) = dU_L(q)/dq$ at the critical point for each value of L. From Eq. (8) we have the following scaling relation:

$$U'_L(q) = L^{1/\nu} \tilde{U}'(L^{1/\nu} \varepsilon) \tag{10}$$

so that $U'_L(q_c) = L^{1/\nu} \tilde{U}'(0)$. From a log-log plot of $U'_L(q_c)$ versus L, as shown in Fig. 5, we obtain $1/\nu$ as the slope of the straight line fitted to the data points. We obtain $\nu = 0.99 \pm 0.05$ and $\tilde{U}'(0) = 0.35 \pm 0.04$.

The ratio γ/ν is estimated from a log-log plot of $X_L(q_c)$ versus L. From Eq. (7) we have $X_L(q_c) = L^{\gamma/\nu} \tilde{X}(0)$, so that γ/ν is the slope of the straight line fitted to the data points as shown in Fig. 6. The best fit gives $\gamma/\nu = 1.73 \pm 0.05$ and $\tilde{X}(0) = 0.065 \pm 0.007$. Another estimate can be obtained from the log-log plot of the maximum value of the susceptibility X_L^* versus L. If we denote by q_L^* the value of q for which $X_L(q)$ is maximum, then from the scaling relation it follows that $q_L^* = q_c + x^*/L^{1/\nu}$, where x^* is independent of L and is the value of x for which $\tilde{X}(x)$ is maximum. Therefore $X_L^* = X_L(q_L^*) = L^{\gamma/\nu} \tilde{X}(x^*)$. From Fig. 6 the best fit gives $\gamma/\nu = 1.70 \pm 0.08$

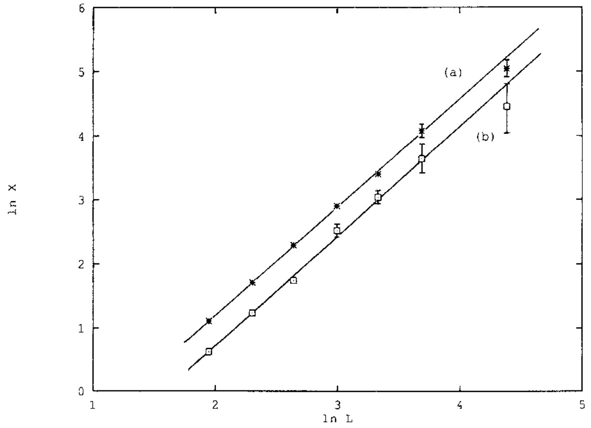

Fig. 6. Log-log plot of the susceptibility at (a) its maximum and (b) $q = q_c$, versus L. The solid lines are best fits with slopes $\gamma/\nu =$ (a) 1.70 and (b) 1.73.

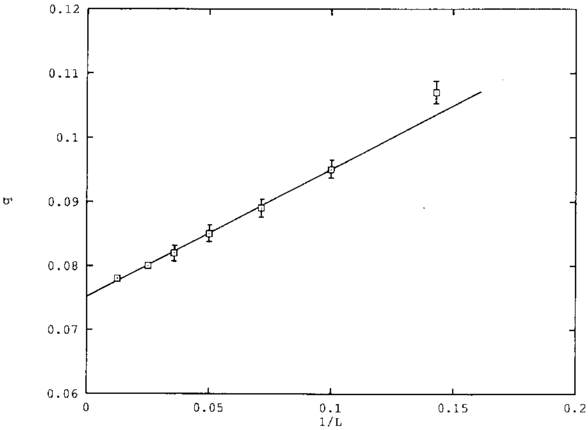

Fig. 7. Extrapolation of q_L^*, the value of q when the susceptibility is maximum, versus $1/L$. The extrapolation gives $q_c = 0.075$.

and $\tilde{X}(x^*) = 0.112 \pm 0.010$. Figure 7 shows q_L^* as a function of $1/L$. We can verify that indeed $q_c = 0.075 \pm 0.01$ and we obtain $x^* = 0.20 \pm 0.02$.

To obtain an estimate of β/ν, we have proceeded in a similar way. Since from Eq. (6) we have $M_L(q_c) = L^{-\beta/\nu}\tilde{M}(0)$, a log-log plot of the magnetization at $q = q_c$ versus L gives $-\beta/\nu$ as the slope of the straight line fitted to the data points. From Fig. 8 we obtain $\beta/\nu = 0.125 \pm 0.005$ and also $\tilde{M}(0) = 0.94 \pm 0.01$.

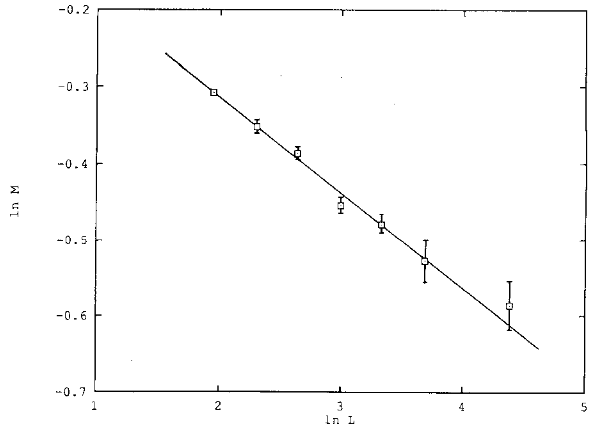

Fig. 8. Log-log plot of the magnetization at $q = q_c$ versus L. The solid line is the best fit with slope $-\beta/\nu = -0.125$.

All the amplitudes estimated above are not universal. However, one can make a universal quantity by taking the ratio between the susceptibility at its maximum and at $q = q_c$. From the results above we get $\tilde{X}(x^*)/\tilde{X}(0) = 1.7 \pm 0.2$.

4. CONCLUSION

We have simulated the isotropic majority vote model on a square lattice with periodic boundary conditions. The estimates of the critical exponents $\nu = 0.99 \pm 0.05$, $\gamma/\nu = 1.73 \pm 0.05$, and $\beta/\nu = 0.125 \pm 0.005$ compare well with the exact values $\nu = 1$, $\gamma = 7/4$, and $\beta = 1/8$ for the equilibrium Ising model. The value for the reduced fourth-order cumulant $U^* = 0.61 \pm 0.01$ is in accordance with the best value $U^* = 0.611 \pm 0.001$ obtained for this quantity for the square Ising model with periodic boundary conditions. These numerical results indicate that the isotropic majority vote model has the same universal critical behavior as the equilibrium Ising model.

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