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Isotropic Solutions of the Einstein-Boltzmann Equations

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Abstract. It is shown that in all solutions of the Einstein-Boltzmann equations in which the particle distribution function is isotropic about some 4-velocity field, the distortion of that velocity field vanishes; further, either its expansion or its rotation vanishes. We discuss briefly further kinetic solutions in which the energy-momentum tensor has a perfect fluid form.

1. Introduction

The General-Relativistic theory of a collision-dominated one-component gas is governed by the Einstein gravitational field equations[1]

$$R_{ab}-\tfrac{1}{2}Rg_{ab}+\Lambda g_{ab}=T_{ab} \tag{1.1}$$

where the energy-momentum tensor T_{ab} is obtained from a suitable model for the gas. A fluid model is appropriate if the collision dominance is assumed to imply that the gas is sufficiently close to equilibrium to allow the definition of a unique 4-velocity and the use of a conventional thermodynamic formalism. Included in this description is the perfect fluid approximation where one neglects the dissipative effects of heat conduction and shear viscosity.

Though inapplicable at very high densities, within its range of validity relativistic kinetic theory provides a more detailed description than the fluid model. For example it allows one to calculate the form of the transport equations which, in the macroscopic theory, are a phenomenological assumption. To a certain extent kinetic theory may therefore be used to examine the nature and validity of the fluid theory.

In this paper, we study kinetic theory when the one particle distribution function is everywhere isotropic, extending the results of EGS (Ehlers, Geren and Sachs [1]), and then consider the nature of the perfect fluid approximation. Preparatory to stating our precise result, we first briefly review the fluid and kinetic models.

[1] Latin indices run from 0 to 3, Greek indices from 1 to 3. We use square brackets to denote skew symmetrization, round brackets to denote symmetrization. The speed of light is normalized to unity.

In the *fluid model* (see e.g. [2]) the matter is regarded as a continuum with the average matter velocity at each point represented by the 4-velocity vector u^a, $u^a u_a = -1$. Its first covariant derivative $u_{a;b}$ may be expressed in the form ([2], [3])

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - \dot{u}_a u_b \qquad (1.2)$$

where $\dot{u}_a \equiv u_{a;b} u^b$ is the acceleration vector[2], $\omega_{ab} = \omega_{[ab]}$ is the vorticity tensor ($\omega_{ab} u^b = 0$), $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab} u^b = 0$, $\sigma^a{}_a = 0$) and $\theta \equiv u^a{}_{;a}$ is the expansion. The projection tensor h_{ab} is defined by $h_{ab} \equiv g_{ab} + u_a u_b$, so $h_{ab} u^b = 0$. The vorticity vector is defined by $\omega^a \equiv \frac{1}{2}\eta^{abcd} u_b \omega_{cd}$, and the magnitudes ω of the vorticity and σ of the shear by $\omega^2 \equiv \omega^a \omega_a = \frac{1}{2}\omega^{ab}\omega_{ab}$, $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$ respectively. As ω_{ab} and σ_{ab} are spacelike tensors, $\omega = 0 \Leftrightarrow \omega_{ab} = 0$ and $\sigma = 0 \Leftrightarrow \sigma_{ab} = 0$.

The fluid energy momentum tensor T_{ab} can be decomposed with respect to u_a in the form

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} \qquad (1.3)$$

where μ is the energy density of the fluid, p the (kinetic) pressure, q_a the heat flux vector ($q_a u^a = 0$), and π_{ab} is the shear viscosity term ($\pi_{ab} u^b = 0$, $\pi^a{}_a = 0$). Following Eckart ([4]) the temperature $T \geqq 0$ and entropy S are assumed to obey the Gibbs equation

$$T dS = du + p_t d\left(\frac{1}{\varrho}\right) \qquad (1.4)$$

where p_t is the thermodynamic pressure, ϱ is the effective rest mass density measured in the rest-frame of u^a, and the internal energy density u is defined by $\mu = \varrho(1+u)$.

Contracting the Bianchi identities $T^{ab}{}_{;b} = 0$ with u^a and using Eqs. (1.2) and (1.3) one finds[3]

$$\dot{\mu} + (\mu + p)\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + \dot{u}_a q^a = 0 \qquad (1.5)$$

which is the equation of conservation of thermal energy. Using (1.4) and the equation of conservation of matter, which is $\dot{\varrho} + \varrho\theta = 0$, it can be rewritten:

$$\left(\varrho S u^a + \frac{1}{T} q^a\right) = -(p - p_t)\theta - \frac{1}{T}\pi_{ab}\sigma^{ab} - \frac{1}{T^2}(T_{,a} + T\dot{u}_a) q^a . \quad (1.6)$$

Defining the entropy flux vector

$$S^a \equiv \varrho S u^a + \frac{1}{T} q^a , \qquad (1.7)$$

[2] For any function f, $\dot{f} \equiv f_{,a} u^a$; similarly for any tensor $S_{a\ldots b}$, $\dot{S}_{a\ldots b} \equiv S_{a\ldots b;c} u^c$.

[3] See previous footnote (page 2).

this yields the covariant second law of thermodynamics $S^a{}_{;a} \geqq 0$, provided a suitable choice is made of transport relations for the pressure difference, viscosity and heat flux. Such a choice, vindicated by kinetic theory, is

$$p - p_t = -\zeta\theta, \pi_{ab} = -\lambda\sigma_{ab}, q_a = -\kappa h_a{}^b(T_{,b} + T\dot{u}_b) \tag{1.8}$$

where ζ, λ, κ are the coefficients of bulk viscosity, shear viscosity and thermal conductivity respectively, and $\zeta \geqq 0, \lambda \geqq 0, \kappa \geqq 0$. In general, ζ, λ and κ are functions of the thermodynamic variables. To complete the description, some equation of state must be given determining p_t from the thermodynamic variables.

The *perfect fluid* approximation is usually introduced as the specialization $\lambda = \kappa = 0$; then

$$T_{ab} = \mu u_a u_b + p h_{ab} \tag{1.9}$$

and the contracted Bianchi identities $T^{ab}{}_{;b} = 0$ take the form

$$\dot{\mu} + (\mu + p)\theta = 0\,, \tag{1.10}$$

$$(\mu + p)\dot{u}_a + h_a{}^b p_{,b} = 0\,. \tag{1.11}$$

However Eqs. (1.6)–(1.8) show that if $\zeta \neq 0$ and the gas is expanding, it will still exhibit a dissipative effect and the flow will be irreversible. The recognition that this will be generally true is relatively new, and follows from kinetic theory studies by Ehlers ([2]) and Tauber and Weinberg ([5]), who showed that an expanding relativistic monatomic gas whose particles have non-zero rest mass could not be in equilibrium, and from work by Israel ([6]) who showed that such a gas generally possesses a bulk viscosity in contrast to the classical case where this phenomenon is essentially found only in polyatomic gases. If the particles have zero rest mass, the bulk viscosity vanishes and equilibrium is possible when expansion takes place.

In *kinetic theory* (see e.g. [7–8]) the gas is regarded as a distribution of particles, described by a distribution function $f(x^a, p^a)$, of proper mass m and 4-momentum p^a (so $p^a p_a = -m^2$). The number of particles in the volume element dx_a at x^a, with 4-momenta in the range dp^a about p^a, is then $f(x^a, p^a)\,|dx_b dp^b|$ so $f(x^a, p^a) \geqq 0$ for all x^a and all p^a. Along each particle world line $x^a(\tau)$, f obeys the relativistic Boltzmann equation

$$p^a f_{|a} = C \tag{1.12}$$

where the derivative $p^a f_{|a}$ is

$$p^a f_{|a} \equiv \frac{\partial f}{\partial x^a}\frac{dx^a}{d\tau} + \frac{\partial f}{dp^a}\frac{dp^a}{d\tau} = p^a\left(\frac{\partial f}{\partial x^a} - \Gamma^c_{ab} p^b \frac{\partial f}{\partial p^c}\right),$$

since the particles are assumed to move on geodesics between collisions. C is the collision term; restricting our attention to binary collisions[4], C has the form ([9])

$$C(x^a, p^a) = \tfrac{1}{2} \int\limits_{P'} \int\limits_{P''} \int\limits_{P'''} (f'''f'' - f'f)\, W(p, p'; p'', p''')\, \pi' \pi'' \pi''' \tag{1.13}$$

where for example $f' = f(x^a, p^{a'})$, P' is the mass shell $p'^a p'_a = -m^2$, and π' is the Lorentz invariant measure on that part of P' with future directed normal p'^a. W is a measure of the probability with which a collision $(p, p') \to (p'', p''')$ occurs; one assumes $W(p, p'; p'', p''') = W(p', p; p''', p'') = W(p'', p'''; p, p')$.

The particle proper number density n and average 4-velocity u^a are usually defined by

$$nu^a = \int\limits_P p^a f \pi, \quad u^a u_a = -1 . \tag{1.14}$$

(n is, up to a constant, the conserved density ϱ.) The energy-momentum tensor of the gas is given by

$$T_{ab} = \int\limits_P p_a p_b f \pi . \tag{1.15}$$

Then the Einstein-Boltzmann Eqs. (1.1), (1.12), (1.13), (1.15) furnish a complete description of the system[5].

The entropy flux vector is defined by

$$S^a = -\int\limits_P p^a f \log f \pi .$$

Ehlers ([2]) and Tauber and Weinberg ([5]) have shown from this that $S^a{}_{;a} \geqq 0$, and that the *equilibrium case* $S^a{}_{;a} = 0$ implies that $C = 0$ and that f has the form

$$f(x^a, p^a) = e^{\alpha(x^a) + \beta_b(x^a) p^b} \tag{1.16}$$

where β_a is a timelike Killing vector if $m > 0$, and a conformal Killing vector if $m = 0$. Substituting into (1.15) shows that T_{ab} has the perfect fluid form with $\beta_a = \frac{1}{T} u_a$; since β_a is a Killing vector when $m > 0$, this then implies that $\sigma_{ab} = \theta = 0$, so no expanding reversible flows are possible in this case. An examination of small deviations from equilibrium using the Chapman-Enskog ([6]) or Grad methods ([10–12]) enables one to justify Eqs. (1.7) and (1.8).

[4] This would not be adequate at very high densities.

[5] If $C = 0$, Eq. (1.12) becomes $p^a f_{|a} = 0$, and is called the Liouville equation.

The equilibrium solution (1.16) is a particular instance of an *isotropic* distribution function, i.e. there exists a vector field u^a such that f has the form

$$f = f(x^a, E), \quad E \equiv -u_a p^a . \tag{1.17}$$

For such distribution functions it follows that u^a obeys (1.14) and that T_{ab} has the perfect fluid form (1.9). Two problems naturally arise:

(1) What solutions of the Einstein-Boltzmann equations are possible when the distribution function is isotropic?

(2) What further solutions are there in which the energy-momentum tensor has the perfect fluid form ((1.9), (1.14) and (1.15) are satisfied) but f is *not* isotropic about u^a?

EGS [1] have partially answered (1) by showing that in a solution of the Einstein-Boltzmann equations in which (a) f is isotropic, (b) $C=0$, (c) $\omega=0$ when $m=0$, space-time is either stationary or a Robertson-Walker space-time. We shall give an extension of their results by showing that conditions (b) and (c) are largely inessential. More precisely, we shall prove

Theorem 1. *In a solution of the Einstein-Boltzmann equations with an isotropic distribution function, the velocity u^a obeys the conditions (i) $\sigma=0$, (ii) $\omega\theta=0$.*

While this result puts very strong restrictions on space-time, it does not give a complete extension of the results of EGS because it does not necessarily imply that u^a is a conformal Killing vector; however this extra condition *does* follow when $m=0$. Various authors have independently proved statement (i) of the theorem (see e.g. Stewart [11]).

We prove this theorem, except for two special cases, in Section 2; the two special cases are examined in the remaining sections. A special coordinate system is introduced in Theorem 2 of Section 3, and in Section 4 the field equations are reduced to a fairly simple form by use of these coordinates. The proof is completed in Theorem 3 of Section 5. We discuss our results in Section 6, briefly commenting on the implications for cosmological models, and on problem (2). Theorems 2 and 3 are phrased as results about general relativistic perfect fluids, independent of any kinetic assumptions, which are of some interest in their own right.

2. Isotropic Solutions of the Boltzmann Equation

Consider a collision-dominated one-component gas obeying the Einstein-Boltzmann equations, and with an isotropic distribution function. Then equations (1.1), (1.9)–(1.15) and (1.17) hold. Eqs. (1.13) and (1.17) imply the collision term has the form $C(x^a, E)$[6]. Using the method of EGS, we assume $\partial_E^x f \neq 0$[7] on some open interval of E (this holds in a physical situation except in the case of dust, which is collision free), and invert f on this interval to obtain $E = E(x^a, f)$. Then along any particle world line $x^a(\tau)$,

$$\frac{dE}{d\tau} \equiv p^a \partial_a^f E + \partial_f^x E p^a f_{|a} = p^a \partial_a^f E + C \partial_f^x E$$

by (1.12). But we also have $\frac{dE}{d\tau} = (-u^b p_b)_{;a} p^a = -p^a p^b u_{b;a}$ since $p^b{}_{;a} p^a = 0$. Therefore

$$-u_{a;b} p^a p^b = p^a \partial_a^f E + C \partial_f^x E \tag{2.1}$$

for all particle momenta p^a. Decomposing p^a as $p^a = E u^a + (E^2 - m^2)^{\frac{1}{2}} e^a$ where $e^a u_a = 0$, $e^a e_a = 1$, and substituting this and (1.2) into (2.1), one obtains

$$(E^2 - m^2)\sigma_{ab} e^a e^b + (E^2 - m^2)^{\frac{1}{2}} (E \dot{u}_a + \partial_a^f E) e^a + E\dot{E} + \tfrac{1}{3}\theta(E^2 - m^2) + C \partial_f^x E = 0$$

where $\dot{E} \equiv u^a \partial_a^f E$. This is valid for all directions e^a, so

$$\sigma_{ab} = 0, \tag{2.2}$$

$$E \dot{u}_a + \partial_a^f E = \alpha u_a, \tag{2.3}$$

$$E\dot{E} + \tfrac{1}{3}\theta(E^2 - m^2) + C \partial_f^x E = 0, \tag{2.4}$$

where α is some function $\alpha(x^a, p^a)$. Eq. (2.2) simply states that the solutions are shear-free; (2.3) and (2.4) may be combined to give the equivalent single equation

$$\dot{u}_a - \frac{1}{3}\theta u_a = -\partial_a^f \log E + u_a \left\{ \frac{3C\partial_f^x E - m^2\theta}{3E^2} \right\}. \tag{2.5}$$

Differentiating (2.5) with respect to f shows

$$0 = -\partial_f^x \partial_a^f \log E + u_a \partial_f^x \left\{ \frac{3C\partial_f^x E - m^2\theta}{3E^2} \right\}. \tag{2.6}$$

The *general case* is when $\partial_f^x \left\{ \frac{3C\partial_f^x E - m^2\theta}{3E^2} \right\}$ is not zero; we then write it as $\frac{1}{G(x^a, f)}$, and put $\partial_f^x \log E = H(x^a, f)$. Then (2.6) is

$$u_a = G \partial_a^f H. \tag{2.7}$$

[6] The results of this section hold for any other form of collision term as well as (1.13), provided only $C = C(x^a, E)$.

[7] $\partial_E^x f$ is partial differentiation with respect to E, keeping x^a constant, and so on.

Differentiating along a particle world line,

$$u_{a;b}p^b = (p^b \partial_b^f G + C\partial_f^x G)\,\partial_a^f H + G\{p^b \partial_b^f \partial_a^f H + C\partial_f^x(\partial_a^f H) - p^b \Gamma^n_{ab}\partial_n^f H\}\,.$$

However, differentiating (2.7) with respect to f shows

$$0 = \partial_f^x G \partial_a^f H + G\partial_f^x(\partial_a^f H)\,.$$

So

$$u_{a;b}p^b = \{\partial_b^f G \partial_a^f H + G(\partial_b^f \partial_a^f H - \Gamma^n_{ab}\partial_n^f H)\}\, p^b$$

holds for all particle 4-momenta p^b. Combined with (2.7), this shows $u_{[a;b}u_{c]} = 0$, i.e. these solutions are irrotational:

$$\omega_{ab} = 0\,. \tag{2.8}$$

Continuing for the moment in the manner of EGS, we note ([2]) that (2.8) implies the local existence of a function $t(x^a)$ such that $\dot{t}u_a = -t_{,a}$. Substituting into (2.7) gives $\dot{H}t_{,a} = \dot{t}\partial_a^f H$, which shows that $H \equiv \partial_f^x \log E = H(f, t)$. Integrating,

$$E = r(x^a)\,j(f, t) \tag{2.9}$$

where $\log j = \int H\,df$, and r is an arbitrary function of the x^a, so

$$f = f\left(t, \frac{E}{r}\right).$$

With this form of f, (1.9), (1.13) and (1.15) imply that μ, p and C have the form $\mu = \mu(t, r)$, $p = p(t, r)$ and $C = C(t, r, f)$. Substituting (2.9) into (2.3), (2.4), one obtains

$$\dot{u}_a = -h_a{}^b(\log r)_{,b}\,, \tag{2.10}$$

$$\frac{\dot{r}}{r} + \dot{t}\partial_t^f \log j + \frac{1}{3}\theta\left(1 - \frac{m^2}{j^2 r^2}\right) + \frac{C}{jr}\partial_f^x \log j = 0\,. \tag{2.11}$$

By virtue of (2.8) and (2.10) one can choose comoving coordinates $\{x^0, x^\alpha\}$ with $t = x^0$ and a metric of the form (3.5). Further by (2.2) and (2.8) the $(0, \nu)$ field equations (see [2] or [3]) reduce to $h^{ab}\theta_{,b} = 0$, i.e. $\theta = \theta(x^0)$ in these coordinates. Eq. (2.11) then shows that $r_{,0}$ is a function of x^0 and r, and therefore $r = r(x^0, \beta)$ where $\beta = r(0, x^\alpha)$, i.e. r depends only on time and the initial conditions β. Finally using (2.2), we see that the Boltzmann equation together with the results of § 3 implies that there are coordinates such that the metric has the form (3.4), (3.8) where $f_{\alpha\beta} = f_{\alpha\beta}(x^\sigma)$, $\theta(x^0) = -3r\dfrac{w_{,0}}{w}$, and $\dot{r}$ is determined by (2.11). For further information we must use the field equations. When $C = 0$, the right hand side of (2.5) is necessarily a gradient (EGS); then u_a is a conformal Killing vector and the space-time is a Robertson-Walker space-time (see EGS for details of the solutions). However when $C \neq 0$ this does not necessarily follow, since there exist solutions of Einstein's equations for a perfect fluid satisfying all these conditions (except (2.11) which is replaced by some equation of state), but with non-zero acceleration and u_a not a conformal Killing vector (see e.g. [13]).

The *special cases* are when $\partial^x_f \left\{ \frac{3C\partial^x_f E - m^2\theta}{3E^2} \right\} = 0$. Then (2.6) implies $\partial^x_f \partial^f_a \log E = 0$. Integrating,

$$E = j(f)\, r(x^a) \tag{2.12}$$

where j and r are arbitrary functions, and so

$$f = f\left(\frac{E}{r}\right).$$

Now μ and p have the form $\mu = \mu(r)$, $p = p(r)$ and hence there exists a relation $p = p(\mu)$ even though dissipative effects may be present. From (1.10), the functional form of μ, p implies

$$\theta = -\left(\frac{\dot{r}}{\mu + p}\right)\frac{d\mu}{dr}. \tag{2.13}$$

Substituting (2.12) into (2.3) and (2.4), one obtains (2.10) again and

$$j^2 r^2 (\log r)^{\cdot} + \tfrac{1}{3}\theta(j^2 r^2 - m^2) + Cjrd_f \log j = 0. \tag{2.14}$$

C has the form $C(r, f)$. Combining (2.13) and (2.14),

$$\dot{r}\left\{ j^2\left(r - \frac{1}{3}\,\frac{r^2}{\mu + p}\,\frac{d\mu}{dr}\right) + \frac{m^2}{3(\mu + p)}\,\frac{d\mu}{dr}\right\} = -\,Cjrd_f \log j.$$

The *first special case* is when the large bracket is not zero. Then $\dot{r}$ is a function of r, say $\dot{r} = \beta(r)$.

The *second special case* is when the large bracket is zero. Then the functional dependence implies we must have separately

$$m = 0, \qquad r\frac{d\mu}{dr} = 3(\mu + p), \tag{2.15}$$

whence also $C = 0$. Therefore the particles of the gas have zero rest mass. Now (1.15) implies $T^a{}_a = 0$ or, by (1.9), $p = \frac{1}{3}\mu$. Substituting this in (2.15) shows $\mu = \mu_0 r^4$ where μ_0 is a constant.

Both the special cases, considered macroscopically, are therefore shearfree perfect fluids with equations of state $p = p(\mu)$. In one case the acceleration potential r obeys a relation $\dot{r}=\beta(r)$, in the other case the equation of state is that of thermal radiation: $p=\frac{1}{3}\mu$. In Section 5 we show that both of these cases lead to the result $\theta\omega=0$.

3. Coordinates

In this section we show how to derive the local comoving coordinate system developed by Ehlers ([14]), Taub ([15]) and others for any perfect fluid with an acceleration potential r. To do so, consider the integral curves of the 4-velocity vector field u^a which intersect some surface S once in some open neighbourhood. Label the integral curves by arbitrary coordinates $\{x^\alpha\}$ assigned to them in this surface, and measure some arbitrary parameter x^0 from S along these curves. Then $\{x^0, x^\alpha\}$ are local *comoving coordinates;* in these coordinates the integral curves are the curves $\{x^\alpha=\text{constant}\}$, so $x^\alpha{}_{,a}u^a=u^\alpha=0$. Thus there is some function $v(x^a)$ such that

$$u^a=\frac{1}{v}\delta^a_0\,.$$

Under a coordinate transformation $x^{a'}=x^{a'}(x^a)$, $u^{a'}=\dfrac{\partial x^{a'}}{\partial x^b}u^b=\dfrac{1}{v}\dfrac{\partial x^{a'}}{\partial x^0}$. The transformations preserving the comoving form are therefore (a) *gauge transformations* $x^{0'}=x^{0'}(x^a)$, $x^{\alpha'}=x^\alpha$, corresponding to the freedom of choice of the surface $\{x^0=\text{constant}\}$; then $v'=v\Big/\dfrac{\partial x^{0'}}{\partial x^0}$; and (b) the freedom to *relabel the integral curves* by choosing new coordinates in an initial surface; then $x^{0'}=x^0$, $x^{\alpha'}=x^{\alpha'}(x^\beta)$.

Since $h_{ab}=g_{ab}+u_a u_b$,

$$ds^2\equiv g_{ab}dx^a dx^b=h_{ab}dx^a dx^b-(u_a dx^a)^2\,. \tag{3.1}$$

As $h_{ab}u^b=0$, in the comoving coordinates $h_{a0}=0$. Further $u_a=g_{ab}u^b=\dfrac{g_{a0}}{v}$, so $u^a u_a=-1\Rightarrow g_{00}=-v^2$, and $u_0=\dfrac{g_{00}}{v}=-v$.

As there is an acceleration potential $r(x^a)$, (2.10) holds. (If there is an equation of state $p=p(\mu)$, the conservation Eq. (1.11) shows there necessarily *is* an acceleration potential

$$r=\exp\left(\int_{p_0}^{p}\frac{dp}{\mu+p}\right) \tag{3.2}$$

where μ_0 is a constant and $p_0=p(\mu_0)$. Since μ is a function of position, r is a function $r(x^a)$ in space-time.) Choose a gauge transformation so that $v' = \frac{1}{r}$; then (3.1) can be written

$$ds^2 = h_{\alpha\beta}(x^a)dx^\alpha dx^\beta - \frac{1}{r^2(x^a)}(dx^0 + a_\alpha(x^a)dx^\alpha)^2 \tag{3.3}$$

in these coordinates, where $a_\alpha = -ru_\alpha$. In the special case when the pressure vanishes, this definition sets $r=1$.

In these coordinates, $u^a = r\delta^a_0$ and $u_a = \left(-\frac{1}{r}, -\frac{a_\alpha}{r}\right)$. Computing $\dot{u}_a = u_{a,b}u^b - \Gamma^c_{ab}u_c u^b$ one obtains $\dot{u}_0 = 0$, $\dot{u}_\alpha = -a_{\alpha,0} + \frac{r_{,0}}{r}a_\alpha - \frac{r_{,\alpha}}{r}$. However the conservations Eqs. (2.10) are $\dot{u}_0 = 0$, $\dot{u}_\alpha = \frac{r_{,0}}{r}a_\alpha - \frac{r_{,\alpha}}{r}$. Therefore the conservation equations are equivalent to $a_{\alpha,0}=0$ and may be integrated to give

$$a_\alpha = a_\alpha(x^\beta)\,.$$

The vorticity tensor is $\omega_{ab} = u_{[a,b]} + \dot{u}_{[a}u_{b]}$, and so has components $\omega_{a0} = 0$, $\omega_{\alpha\beta} = -\frac{1}{r}a_{[\alpha,\beta]}$ in these coordinates. We make a gauge transformation $x^{0'} = x^0 + f(x^\alpha)$, $x^{\alpha'} = x^\alpha$, so that $dx^0 = dx^{0'} - \frac{\partial f}{\partial x^\alpha}dx^\alpha$ (which preserves (3.3)) and determine f as follows.

If $\underline{\omega = 0}$, $a_{[\alpha,\beta]} = 0$ which implies there is a function $g(x^a)$ such that $a_\alpha = g_{,\alpha}$. Choose $f=g$ to obtain the metric

$$ds^2 = h_{\alpha\beta}(x^a)dx^\alpha dx^\beta - \left(\frac{dx^0}{r(x^a)}\right)^2. \tag{3.4}$$

The surfaces $\{x^0 = \text{constant}\}$ are then orthogonal to u^a, and the remaining coordinate freedom is $x^{0'} = x^0 + C$, $x^{\alpha'} = x^{\alpha'}(x^\beta)$, where C is a constant.

If $\underline{\omega \neq 0}$, one cannot set $a_\alpha = 0$. Instead we choose f so that at each point in some hypersurface $\{x^0 = \text{constant}\}$, the vector ω^a lies in that surface, i.e. $\omega^0 = \frac{1}{2}\eta^{0\alpha\beta\gamma}a_\alpha a_{\beta,\gamma} = 0$. With this choice of f, $\varepsilon^{\alpha\beta\gamma}a_\alpha a_{\beta,\gamma} = 0$, which shows there exist functions $y(x^\alpha)$, $z(x^\alpha)$ such that $a_\alpha = yz_{,\alpha}$. These functions must be independent, as otherwise $\omega_{\alpha\beta} = 0$. Use the freedom of initial labelling to choose $x^3 = z$ and $x^2 = x^2(y,z)$ where $\frac{\partial x^2}{\partial y} \neq 0$; then (cf. [16])

$$a_\alpha(x^\beta) = y(x^2, x^3)\delta^3_\alpha$$

and the vorticity propagation equation (see [2, 3]), which takes the form

$$\omega^a{}_{,b}u^b - u^a{}_{,b}\omega^b = -\omega^a\left(\theta + \frac{\dot r}{r}\right) - u^a(\log r)_{,b}\omega^b$$

because of (2.10), is identically satisfied, as is the divergence relation $\omega^a{}_{;a} = 2\omega^a \dot u_a$.

With this form of a_α, the metric (3.3) in fact includes the case $\omega = 0$ as the special case $y = 0$. A more determinate system of coordinates can be obtained (cf. [14]) when $\omega \neq 0$ by choosing $x^2 = y$; then

$$ds^2 = h_{\alpha\beta}(x^a)dx^\alpha dx^\beta - \frac{1}{r^2(x^b)}(dx^0 + x^2 dx^3)^2 . \tag{3.5}$$

In these coordinates, $\omega_{ab} = \frac{1}{r}\delta^2_{[a}\delta^3_{b]} \neq 0$. The remaining *coordinate freedom* preserving this form is $x^{0'} = x^0 + k(x^2, x^3), x^{1'} = x^{1'}(x^1, x^2, x^3)$, $x^{2'} = x^{2'}(x^2, x^3), x^{3'} = x^{3'}(x^2, x^3)$, where the functions $k, x^{2'}, x^{3'}$ must satisfy the relations

$$\frac{\partial k}{\partial x^2} + x^{2'}\frac{\partial x^{3'}}{\partial x^2} = 0, \quad \frac{\partial k}{\partial x^3} + x^{2'}\frac{\partial x^{3'}}{\partial x^3} = x^2 , \tag{3.6}$$

which imply the integrability conditions $\frac{\partial x^{2'}}{\partial x^2}\frac{\partial x^{3'}}{\partial x^3} - \frac{\partial x^{2'}}{\partial x^3}\frac{\partial x^{3'}}{\partial x^2} = 1$. If k is chosen as an arbitrary C^2 function of x^2, x^3, then (see e.g. [17]), Chap. 2) functions $x^{2'}, x^{3'}$ can always be found to satisfy (3.6).

Finally, if one computes the *expansion tensor* $\theta_{ab} = u_{(a,b)} - \Gamma^c_{ab}u_c + \dot u_{(a}u_{b)}$ for these metrics, one obtains $\theta_{a0} = 0, \theta_{\alpha\beta} = \frac{r}{2}h_{\alpha\beta,0}$, so $\theta = \frac{r}{2}g^{\alpha\beta}h_{\alpha\beta,0}$. Defining $R^2 = \exp(\int \frac{1}{3}g^{\varrho\sigma}h_{\varrho\sigma,0}dx^0)$, where the integral is taken along the integral curves of u^a from $x^0 = 0$, one can re-express $h_{\alpha\beta}$ in the form $h_{\alpha\beta} = R^2(x^a) f_{\alpha\beta}(x^b)$; this relation defines $f_{\alpha\beta}$, and implies $g^{\alpha\beta}f_{\alpha\beta,0} = 0$. Then $\theta = 3r\frac{R_{,0}}{R}$ and the shear $\sigma_{ab} = \theta_{ab} - \frac{1}{3}\theta h_{ab}$ has components $\sigma_{a0} = 0$, $\sigma_{\alpha\beta} = \frac{r}{2}R^2 f_{\alpha\beta,0}$. To express R in terms of fluid variables, define w by

$$w = \exp\left(\int_{\mu_0}^{\mu}\frac{d\mu}{3(\mu + p)}\right). \tag{3.7}$$

(If the fluid does not obey an equation of state $p = p(\mu)$, this integral should be taken along the integral curves of u^a.) Then the conservation

Eq. (1.10) becomes $\frac{w_{,0}}{w} + \frac{R_{,0}}{R} = 0$, so $R = \frac{\lambda}{w}$ for some function $\lambda(x^\alpha)$. Relabelling $\lambda^2 f_{\alpha\beta}$ as $f_{\alpha\beta}$, one has

$$h_{\alpha\beta} = \frac{1}{w^2(x^a)} f_{\alpha\beta}(x^b), \quad C^{\alpha\beta} f_{\alpha\beta,0} = 0, \tag{3.8}$$

where $C^{\alpha\beta}(x^a)$ is the inverse of $f_{\alpha\beta}$, i.e. $C^{\alpha\beta} f_{\beta\gamma} = \delta^\alpha{}_\gamma$. Then $\theta = -3r \frac{w_{,0}}{w}$, $\sigma_{a0} = 0$, $\sigma_{\alpha\beta} = \frac{r}{2w^2} f_{\alpha\beta,0}$.

Summing up, we have

Theorem 2. *Given a perfect fluid with equation of state* $p = p(\mu)$, *functions* $r(x^a)$ *and* $w(x^a)$ *are defined from* $\mu(x^a)$ *by* (3.2), (3.7). *Then local comoving coordinates can be found such that the space-time metric has the form* (3.4) *if* $\omega = 0$ *and* (3.5) *if* $\omega \neq 0$; *the conservation Eqs.* (1.11) *are then identically fulfilled. If* $h_{\alpha\beta}$ *is written in the form* (3.8), *the conservation Eq.* (1.10) *is identically fulfilled, and* $\theta = 0 \Leftrightarrow w = w(x^\alpha)$, $\sigma = 0 \Leftrightarrow f_{\alpha\beta} = f_{\alpha\beta}(x^\sigma)$.

In fact the proof has shown one can find such coordinates even when there is no relation $p = p(\mu)$, as long as there is an acceleration potential r. In these coordinates, the metric components are $g_{00} = -\frac{1}{r^2}$, $g_{0\nu} = -\frac{1}{r^2} a_\nu$, $g_{\mu\nu} = \frac{1}{w^2} f_{\mu\nu} - \frac{1}{r^2} a_\mu a_\nu$, and $g^{00} = -r^2 + w^2 C^{\alpha\beta} a_\alpha a_\beta$, $g^{0\nu} = -w^2 C^{\nu\mu} a_\mu$, $g^{\mu\nu} = C^{\mu\nu} w^2$ where $C^{\alpha\beta} f_{\beta\gamma} = \delta^\alpha{}_\gamma$ and $a_\nu = 0$ if $\omega = 0$, $a_\nu = x^2 \delta_\nu^3$ if $\omega \neq 0$. The determinants $g \equiv \det(g_{ab})$, $f \equiv \det(f_{\mu\nu})$ are related by $g = -\frac{f}{r^2 w^6}$. The fluid 4-velocity has components $u^a = r\delta_0^a$, $u_a = -\frac{1}{r}(\delta_a^0 + \delta_a^\alpha a_\alpha)$. In the case $\omega \neq 0$, $\omega^a = \frac{w^3}{2r\sqrt{f}} \delta_1^a$, and the magnitude ω of the vorticity is given by $\omega^2 = \frac{w^4 f_{11}}{4r^2 f}$ (so $f_{11} \neq 0$). Finally we remark that the definitions of r and w imply $(\mu + p) = (\mu_0 + p_0) r w^3$.

4. Field Equations

If follows from the previous section that for a shear-free perfect fluid with an equation of state $p = p(\mu)$ and with non-zero vorticity (i.e. $\sigma = 0$, $\omega \neq 0$), local comoving coordinates can be chosen so that the space-time metric is

$$ds^2 = \frac{1}{w^2(x^a)} \{f_{\alpha\beta}(x^\sigma)\, dx^\alpha dx^\beta - v^2(x^a)(dx^0 + x^2 dx^3)^2\} \tag{4.1}$$

where $v = \frac{w}{r}$, the functions w, r being defined by (3.2), (3.7). Their definitions imply $(\mu + p) = (\mu_0 + p_0)\frac{w^4}{v}$.

In this metric, the time dependence occurs only through the functions $w(x^a)$, $v(x^a)$. It is the purpose of this section to establish the field equations with the time (i.e. x^0) and 3-space (i.e. $f_{\alpha\beta}(x^\sigma)$) dependence exhibited explicitly. In doing so we shall take w as the basic time-dependent variable, regarding r (i.e. v) as a function of w.

The field Eqs. (1.1), (1.9) may be written in the form

$$R_{ab} = (\mu + p)\, u_a u_b + (\Lambda + \tfrac{1}{2}\mu - \tfrac{1}{2}p) g_{ab}\,.$$

It is convenient to perform the reduction of these equations in two stages. *First*, we write $g_{ab} = \frac{1}{w^2}\hat{g}_{ab}$, $g^{ab} = w^2 \hat{g}^{ab}$, and re-express the Ricci tensor R_{ab} in terms of the Christoffel symbols $\hat{\Gamma}^a_{bc}$, Ricci tensor $\hat{R}_{ab}$ and Ricci scalar $\hat{R}$ defined by the metric $\hat{g}_{ab}$, $\hat{g}^{ab}$. From the expression for $R_{ab} + \frac{w^2}{v^2} g_{ab} R_{00}$ and the field equations, one finds

$$\begin{aligned} &\hat{R}_{ab} + \frac{1}{v^2}\hat{R}_{00}\hat{g}_{ab} + \frac{2}{w} w_{,a,b} + \frac{2}{v^2 w} w_{,0,0}\hat{g}_{ab} \\ &\qquad - \frac{2}{w}\hat{\Gamma}^s_{ab} w_{,s} - \frac{2}{wv^2}\hat{\Gamma}^s_{00} w_{,s}\hat{g}_{ab} = \frac{\mu + p}{w^2} f_{\alpha\beta}\delta^\alpha_a \delta^\beta{}_b\,. \end{aligned} \tag{4.2}$$

Also from the expression for $v^2 g^{ab} R_{ab} + 6w^2 R_{00}$, one finds

$$\begin{aligned} &\hat{R}_{00} + \frac{v^2}{6}\hat{R} + \frac{2}{w} w_{,0,0} - \frac{2}{w}\hat{\Gamma}^s_{00} w_{,s} + \frac{v^2}{w^2}\hat{g}^{ns} w_{,n} w_{,s} \\ &\qquad = \frac{v^2}{3w^2}(3p + 2\mu - \Lambda)\,. \end{aligned} \tag{4.3}$$

Second, one reduces the equations further by expressing $\hat{g}_{ab}$ in terms of $f_{\alpha\beta}$ and w, so obtaining the field equations for (4.1) in terms of these quantities.

To express the results of this lengthy calculation, we define auxiliary variables from $f_{\mu\nu}(x^\sigma)$, $a_\nu(x^\sigma) \equiv x^2 \delta^3_\nu$, $w(x^a)$ and $v(w)$, as follows. The *time-varying* quantities are $w(x^a)$, $v(w)$, and

$$W(x^a) \equiv w_{,0}, \quad X_\alpha(x^a) \equiv w_{,\alpha} - a_\alpha W\,.$$

The expansion θ vanishes if and only if $W=0$. We write $\frac{dv}{dw}$ as v', and so on. The *purely spatial quantities* are defined from the metric $f_{\mu\nu}(x^\sigma)$ and its inverse $C^{\mu\nu}(x^\sigma)\,(f_{\mu\nu}C^{\nu\sigma}=\delta^\sigma_\mu)$. Firstly, there are the Christoffel symbols $\Gamma^{*\alpha}{}_{\beta\gamma}$, Ricci tensor $R^*{}_{\alpha\beta}$ and Ricci scalar R^* defined from $f_{\mu\nu}$:

$$\begin{aligned}
\Gamma^{*\alpha}{}_{\beta\gamma}(x^\sigma) &\equiv \tfrac{1}{2}C^{\alpha\sigma}(f_{\sigma\beta,\gamma}-f_{\beta\gamma,\sigma}+f_{\gamma\sigma,\beta}),\\
R^*{}_{\beta\delta}(x^\sigma) &\equiv \Gamma^{*\alpha}{}_{\beta\delta,\alpha}-\Gamma^{*\alpha}{}_{\beta\alpha,\delta}+\Gamma^{*\alpha}{}_{\nu\alpha}\Gamma^{*\nu}{}_{\beta\delta}-\Gamma^{*\alpha}{}_{\nu\delta}\Gamma^{*\nu}{}_{\beta\alpha},\\
R^*(x^\sigma) &\equiv C^{\alpha\beta}R^*{}_{\alpha\beta}\,.
\end{aligned}$$

Secondly, there are the further quantities

$$\begin{aligned}
\alpha(x^\sigma) &\equiv \frac{f_{11}}{2f}=\frac{f_{11}}{2\det(f_{\alpha\beta})}=\frac{2\omega^2}{v^2w^2},\\
\psi_\beta{}^\alpha(x^\sigma) &\equiv \frac{1}{2}\,\varepsilon_{1\kappa\beta}C^{\kappa\alpha},\\
\phi_\nu(x^\sigma) &\equiv \frac{1}{4}\,C^{\alpha\sigma}(\Gamma^{*\kappa}{}_{\alpha 1}\varepsilon_{\kappa\nu\sigma}-\Gamma^{*\kappa}{}_{\alpha\kappa}\varepsilon_{1\nu\sigma})\,.
\end{aligned}$$

Eq. (4.2) and (4.3) can be used to obtain the field equations for the metric (4.1) in terms of these quantities. The result is:

$$\begin{aligned}
(0,0)\quad w\partial_0 W &= \left(\frac{3}{2}+\frac{v'w}{2}\right)W^2 - v^2\left(\frac{3}{2}-\frac{v'w}{2}\right)C^{\alpha\beta}X_\alpha X_\beta\\
&\quad -\frac{v^2w^2}{4}R^* - \frac{3v^4w^2}{4}\alpha + \frac{v^2}{2}(p-\Lambda) + v^2w^2\Theta\,,
\end{aligned} \tag{4.4}$$

$$(0,\nu)\quad \partial_0 X_\alpha = \frac{v'}{v}WX_\alpha + wv^2\phi_\alpha + \frac{wv^2}{2}\left(\frac{3v'}{v}-\frac{2}{w}\right)\psi_\alpha{}^\nu X_\nu\,, \tag{4.5}$$

$$\begin{aligned}
(\mu,\nu)\quad &\left(\frac{2}{w}-\frac{v'}{v}\right)\left\{\partial_\mu X_\nu - \Gamma^{*\alpha}{}_{\mu\nu}X_\alpha + \delta^2_{(\mu}\delta^3_{\nu)}\frac{W}{2} - a_{(\nu}\delta^3_{\mu)}\partial_0(X_3)\right\}\\
&-\frac{v''}{v}X_\mu X_\nu + R^*{}_{\mu\nu} - v^2\varepsilon_{1\mu\sigma}\psi_\nu{}^\sigma = f_{\mu\nu}\Theta\,,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
v^2w^2\Theta &= v^2(\mu+p) - v^4w^2\alpha - w^2v\,\{v''C^{\alpha\beta}X_\alpha X_\beta\\
&\quad + v'(C^{\alpha\beta}X_{\alpha,\beta} - a_\alpha C^{\alpha\beta}X_{\beta,0} - C^{\alpha\beta}\Gamma^{*\sigma}{}_{\alpha\beta}X_\sigma)\}\\
&\quad + 2wvv'C^{\alpha\beta}X_\alpha X_\beta + \frac{2wv'}{v}W^2 - 2w\partial_0 W\,.
\end{aligned} \tag{4.7}$$

In this form, the ten field equations, written with one complicated auxiliary function Θ, have the 'time' and 'space' dependence explicitly apparent in a convenient way.

5. The Special Cases

We now complete Theorem 1 by dealing with the two special cases of Section 2. More precisely, we prove

Theorem 3. *Consider a perfect fluid with* $\mu \neq 0$ *and for which either* (1) $p=0$, (2) $p=\frac{1}{3}\mu$, *or* (3) $p=p(\mu)$ *and* $\dot{r}=\beta(r)$ *for some function* β, *where* r *is the acceleration potential* (3.2). *Then*

$$\sigma=0 \Rightarrow \omega\theta=0$$

If $\theta \neq 0$, *the space-time is a Robertson-Walker space-time in cases* (1) *and* (2).

The *first case*, $p=0$, has been proved previously (see [16]). We consider now the *second case:* $p=\frac{1}{3}\mu$. We assume $\sigma=0$, $\mu\omega\theta \neq 0$, and obtain a contradiction; the discussion is analogous to that of ([16]).

Since we assume $\omega \neq 0$, one can use the coordinates and field equations of § 4. Substituting the equation of state $p=\frac{1}{3}\mu$ into (3.2), (3.7) shows $\mu=\mu_0 w^4$, $r=w$. Therefore $v=1$ and (4.1) shows that the solutions for which $\sigma=0$, $\omega \neq 0$ are conformal to a stationary space-time. Eliminating Θ from (4.4)–(4.7), the field equations for these solutions reduce to:

$$(0,0) \quad 2w\partial_0 W = W^2 - C^{\alpha\beta}X_\alpha X_\beta + \mu_0 w^4 - \frac{w^2}{6}(R^* + 7\alpha) - \frac{\Lambda}{3},$$

$$(0,\nu) \quad \partial_0 X_\nu = \phi_\nu w + \psi_\nu{}^\mu X_\mu,$$

$$(\mu,\nu) \quad \partial_\mu X_\nu = \Gamma^{*\beta}{}_{\mu\nu} X_\beta + a_\mu \psi_\nu{}^\beta X_\beta - \frac{w}{2}(R^*{}_{\mu\nu} + \alpha f_{\mu\nu} - \varepsilon_{1\mu\sigma}\psi_\nu{}^\sigma - 2a_\mu\phi_\nu) + f_{\mu\nu}\left(\frac{2\mu_0 w^3}{3} - \partial_0 W\right) - \frac{W}{2}\varepsilon_{1\mu\nu}.$$

The (0, 0) equation determines the propagation of W. The remaining field equations, after substitution for $\partial_0 W$ from (0, 0), are considered as constraints on W in a surface $\{x^0=\text{constant}\}$. These constraints must be satisfied during the entire time development of the system. The conditions for this are obtained by repeatedly differentiating the constraints with respect to x^0 and, at each stage, substituting for $\partial_0 W$ from (0, 0), thereby obtaining new constraints on W. Each constraint is simplified using the previous constraints, for example substituting for $\partial_0 X_\alpha$ from $(0, \alpha)$; it may reduce to an identity, but not all the constraints do so. It will be shown that four of the restrictions obtained in this way are inconsistent with the assumption $W \neq 0$, which is equivalent to assuming $\theta \neq 0$.

To effect this programme we need the first time derivative of (0, 0). After substitution from $(0, \alpha)$ and using $X_\alpha X_\beta C^{\alpha\nu}\psi_\nu{}^\beta = 0$, this is found to be:

$$(0,0)_{,0} \quad \partial_0\left(\frac{2\mu_0 w^3}{3} - \partial_0 W\right) = C^{\alpha\beta}\phi_\alpha X_\beta + W\left(\frac{R^*}{6} + \frac{7\alpha}{6}\right).$$

Next, differentiating (2, 3) with respect to x^0 and substituting from $(0, \alpha)$, $(2, \alpha)$ and $(0, 0)_{,0}$, one obtains

$$(2,3)_{,0} \quad \left(f_{23}C^{\alpha\beta}\phi_\alpha + \Gamma^{*\alpha}{}_{23}\psi_\alpha{}^\beta - \phi_3\delta_2^\beta - \frac{1}{2}\partial_2 C^{2\beta} - \frac{1}{2}C^{2\alpha}\Gamma^{*\beta}{}_{2\alpha}\right)X_\beta$$
$$= \frac{W}{2}\left(R^*{}_{23} - \frac{R^*}{3}f_{23} - \frac{4\alpha}{3}f_{23} - C^{23}\right)$$
$$+ w\left(\partial_2\phi_3 - \Gamma^{*\alpha}{}_{23}\phi_\alpha - \frac{1}{4}C^{2\alpha}R^*{}_{2\alpha} - \frac{\alpha}{2}\right) + \frac{\mu_0 w^3}{3}.$$

Finally, differentiating $(0, \nu)$ with respect to x^0, substituting from (0, 0), $(0, \nu)$ and (ν, α), and using $X_\alpha X_\beta(C^{\beta\alpha}{}_{,\nu} + 2C^{\alpha\sigma}\Gamma^{*\beta}{}_{\nu\sigma}) = 0$, $C^{\alpha\beta}\psi_\alpha{}^\nu = -C^{\alpha\nu}\psi_\alpha{}^\beta$, one obtains

$$(0,\nu)_{,0} \quad \left\{C^{\alpha\beta}R^*{}_{\nu\alpha} - 4\psi_\nu{}^\alpha\psi_\alpha{}^\beta - \left(\frac{4\alpha}{3} + \frac{R^*}{3}\right)\delta_\nu^\beta\right\}X_\beta$$
$$+ \frac{8}{3}\mu_0 w^2 X_\nu = w\left\{\partial_\nu\left(\frac{R^*}{6} + \frac{7\alpha}{6}\right) + 2\psi_\nu{}^\alpha\phi_\alpha\right\}.$$

These equations have the form:

$$(2,3)_{,0} \quad b^\nu X_\nu = b^4 w + b^5 W + \frac{\mu_0 w^3}{3},$$

$$(0,\nu)_0 \quad d_\nu{}^\mu X_\mu + \frac{8}{3}\mu_0 w^2 X_\nu = w e_\nu$$

where the b^ν, b^4, b^5, $d_\nu{}^\mu$ and e_ν are functions of x^α only. We regard these as algebraic equations for X_α. Now assuming θ, or equivalently W, is non-zero, we have the following *lemma:* if a polynomial P in w of the form

$$P = \sum_{n=1}^{m} a_n(x^\alpha) w^n$$

vanishes, then each coefficient a_n must be zero. (The proof is by differentiating m times with respect to x^0, and cancelling W after each differentiation. This shows $a_m = 0$. The same procedure is applied to each coefficient in turn.) Considering $(0, 1)_{,0}$, $(0, 2)_{,0}$ and $(0, 3)_{,0}$, the determinant Δ of the left hand sides cannot be zero, as the lemma would then imply in particular that the leading coefficient of w, namely $(\frac{8}{3}\mu_0)^3$, was zero. Therefore we can solve these equations for X_α in the form

$$X_\alpha = \frac{A_\alpha}{\Delta} \tag{5.1}$$

where the A_α are polynomials of the form P and degree 5 in w, and Δ is of degree 6 in w.

Substitute (5.1) in $(2, 3)_{,0}$:

$$b^5 W = \frac{b^\alpha A_\alpha - b^4 w \Delta - \frac{1}{3}\mu_0 w^3 \Delta}{\Delta} \equiv \frac{B}{\Delta} \tag{5.2}$$

where B is a polynomial of form P and of degree 9 in w. Now multiplying $(0, 0)$ by $(b^5)^2$ and substituting from (5.1) and (5.2),

$$\begin{aligned} 2wB(\Delta B' - B\Delta') - B^2\Delta + \Delta(b^5)^2 C^{\alpha\beta} A_\alpha A_\beta - \Delta^3 (b^5)^2 \mu_0 w^4 \\ - \Delta^3 (b^5)^2 \frac{w^2}{6}(R^* + 7\alpha) + \frac{A}{3}\Delta^3 (b^5)^2 = 0 \end{aligned}$$

where $'$ denotes differentiation with respect to w. The whole expression is a polynomial of form P, so in particular its leading coefficient, namely that of $2wB(\Delta B' - B\Delta') - B^2\Delta$, must vanish; however this is $5\left(\frac{\mu_0}{3}\right)^2 \left(\frac{8\mu_0}{3}\right)^9$, and so we have the desired contradiction. When $\theta \neq 0$ the space-time is conformal to a static space-time and must be a Robertson-Walker space time (see [1], Section 4).

Finally we outline the proof of the *third case:* the equation of state $p(\mu)$ is not known, but there is a relation $\dot{r} = \beta(r)$, or equivalently $\dot{w} = \lambda(w)$, for some functions β, λ. In our coordinates, this equation is

$$\frac{w}{v} w_{,0} = \lambda(w).$$

Assuming $w_{,0} \neq 0$, this implies there exists a function $v(w) = \int \frac{w}{v\lambda} dw$ such that $\frac{dv}{dw} \neq 0$ and $v_{,0} = 1$. Hence $v = x^0 + k(x^\alpha)$ where k is an arbitrary function. Inverting one obtains

$$w = w(x^0 + k)$$

It is now convenient to consider two possibilities:

(i) k is independent of x^1.

Then one can use the coordinate freedom $x^{0'}=x^0+k(x^2,x^3)$ (see Section 3) to set $w=w(x^0)$, and therefore $v=v(x^0)$ and $X_\nu=-a_\nu W$. The (0, 1) field equation shows that $\phi_1=0 \Leftrightarrow \varepsilon_{\alpha\beta\gamma} f_{1\alpha} f_{1\beta,\gamma}=0$, so there exist functions $\sigma(x^\alpha)$, $\varrho(x^\alpha)$ such that $f_{1\alpha}=\sigma\varrho_{,\alpha}$. Now ϱ cannot be independent of x^1, as otherwise $f_{11}=0$ which implies $\omega=0$ (see Section 3); so one can set $x^1=\varrho$. Then $f_{12}=f_{13}=0$, the field equations simplify considerably, and it can be shown they imply $\theta=0$.

(ii) k depends on x^1.

We can set $x^1=k$, and again it can be shown that the field equations imply $\theta=0$.

This concludes our treatment of the special cases.

6. Discussion

We have seen that where there is a distribution function isotropic about some 4-velocity field u^a, and the Boltzmann equation is satisfied, then the motion is distortion-free ($\sigma=0$). If further the Einstein equations are satisfied with the energy-momentum tensor determining the space-time curvature being that given by the distribution function, then either the motion is non-expanding ($\theta=0$) or non-rotating ($\omega=0$).

These results are of some interest at two different stages in the evolution of a cosmological model. If the *early stages* of evolution of the universe were highly collision dominated, one might suppose ([1]) that to a first approximation, (i) the matter and radiation mixture was an ultrarelativistic gas, i.e. could be regarded as a perfect fluid with equation of state $p=\frac{1}{3}\mu$, and (ii) the collision dominance implied the distribution function was isotropic.

The space-time would be a Robertson-Walker space-time (by theorems 1, 3) if these conditions were exactly fulfilled. However difficulties arise with galaxy formation in such a universe, cf. [3], so these conditions cannot be exactly fulfilled. If $\sigma\neq 0$ the collisions would tend to isotropize f and could be expected to lead to a distribution function which was nearly, but not exactly, isotropic. The question of the stability of our result then arises: if an isotropic f implies isotropic expansion, does an almost isotropic f imply that the expansion anisotropy is small? We believe that this is not so; i.e. that consistent Einstein-Boltzmann solutions can be obtained with arbitrarily large anisotropy of expansion, even if the anisotropy of f is small. Nevertheless one might expect that the particle collisions would tend to isotropize not only f but also the expansion (cf. Misner [18]) and so the solution would tend towards a Robertson-Walker solution even if it were very anisotropic initially.

At *late stages* corresponding to the present epoch in the universe, one can regard the cosmic background radiation as effectively non-interacting, and so as obeying the Liouville equation. The very high degree of isotropy of the background radiation which is observed suggests one can represent this radiation by a distribution function f which is isotropic about the average velocity u^a of the matter, which can be reasonably represented as 'dust' (i.e. a pressure-free perfect fluid). Then the results of Section 2 show that the shear of u^a vanishes and since $\dot{u}=0$ for a non-interacting pressure-free perfect fluid by (1.11), Eq. (2.3) shows $\omega=0$. Therefore the universe would be a Robertson-Walker universe (cf. [1, 19]). In fact f cannot be *exactly* isotropic, so the question of stability of the result again arises: if radiation propagates with a distribution function which is almost isotropic about the 4-velocity of matter, which is represented as dust, is the universe approximately a Robertson-Walker universe? Studies of perturbed Robertson-Walker models (e.g. [20]) and of anisotropic universe models (e.g. [18, 21, 22]) seem to indicate that this *is* so; that in fact, therefore, the high isotropy of the background radiation *is* a very good indication that the present day universe is very like a Robertson-Walker universe.

If one accepts a Robertson-Walker model of the universe, the discussion of Section 1 shows that although the energy momentum tensor is that of a 'perfect fluid', kinetic theory indicates that the motion will in general (i.e. when $m \neq 0$ and $C \neq 0$) be irreversible, corresponding to a non-zero bulk viscosity. To see what effect this has, we consider a Robertson-Walker universe with flat spatial sections, in which the equation of state is $p=(\gamma-1)\mu-\zeta\theta$, where γ and the coefficient ζ of bulk viscosity are constants ($\zeta \geqq 0, 1<\gamma<\frac{4}{3}$). When $\zeta \neq 0$ the field equations can be solved to give

$$(R(t))^{\frac{3\gamma}{2}} = \exp\left(\frac{3\zeta t}{4}\right) \sinh\left(\sqrt{\frac{9\zeta^2}{16}+\frac{3\Lambda\gamma^2}{4}}\, t\right)$$

for $\Lambda \neq 0$, and

$$(R(t))^{\frac{3\gamma}{2}} = \frac{2}{3\zeta}\left(\exp\left(\frac{3\zeta t}{2}\right)-1\right)$$

for $\Lambda=0$, where $R(t)$ is the Robertson-Walker radius function. When $\zeta=0$, the solutions are

$$(R(t))^{\frac{3\gamma}{2}} = \sinh\left(\sqrt{\frac{3\Lambda\gamma^2}{4}}\, t\right)$$

for $\Lambda \neq 0$, and

$$(R(t))^{\frac{3\gamma}{2}} = t$$

for $\Lambda = 0$. The effect of bulk viscosity is to *increase* the expansion rate $\theta \equiv \frac{3\dot{R}}{R}$ at any given value of R. Although in reality ζ would not be a constant, these solutions still indicate the qualitative effect of a non-zero bulk viscosity. In particular if we consider oscillating models where $\Lambda^* \equiv -\Lambda > \frac{3\zeta^2}{4\gamma^2}$, then

$$(R(t))^{\frac{3\gamma}{2}} = \exp\left(\frac{3\zeta t}{4}\right) \sin\left(\sqrt{\frac{3\Lambda^*\gamma^2}{4} - \frac{9\zeta^2}{16}}\, t\right).$$

This shows that the effect of the bulk viscosity is to *increase* the radius attained in one cycle and to *increase* the cycle period, (cf. [23], Section 175). In a more realistic calculation one would need at least a 2-component model for the gas; the bulk viscosity would be appreciable only at about 10^{10} °K. The quantitative effect would not be large ([10]) but the qualitative effect would be the same.

Our results show that the requirement that the distribution function be everywhere isotropic, is a very strong restriction. The second question we posed in Section 1 was: how strong a restriction is it to assume that the energy momentum tensor (1.15) has exactly the perfect fluid form (1.9) where u^a is given by (1.14)? An indication is given by (1.8) which shows that if κ and λ are $\neq 0$, T_{ab} has the perfect fluid form at all times only if $\sigma = 0$ and $\dot{u}_a = -h_a{}^b(\log T)_{,b}$. The values of λ and κ depend on the collision term. However calculations when f is near equilibrium show ([6, 11]) that $\lambda > 0$ and $\kappa > 0$ for reasonable collision terms. Further in Misner's calculations ([18]) for collision free radiation in a Bianchi I universe the radiation energy-momentum tensor T_{ab} only has a perfect fluid form at all times if the shear vanishes. (q^a vanishes identically because of the space-time geometry.) These results suggest that an exact Einstein-Boltzmann solution can have a perfect fluid energy-momentum tensor only under very restricted circumstances.

One might speculate that T_{ab} could have a perfect fluid form at all points only if f was isotropic. However this is not so, for consider a Robertson-Walker universe filled with zero-rest mass particles having a distribution function

$$f = a \exp\left(+\frac{u_a p^a}{kT}\right)\left\{1 + b(\xi_a p^a)^2 \left(20k^2 - \frac{10k}{T} u_b p^b + \frac{1}{T^2}(u_b p^b)^2\right)\right\}$$

where ξ_a is any killing vector, T is the temperature (proportional to the inverse Robertson-Walker radius), and a, b are constants, b being chosen small enough to make f positive. Then T_{ab} has the form (1.9) with $p=\frac{1}{3}\mu$ and f obeys the Liouville equation but is clearly not isotropic. One has an exact Einstein-Liouville solution when $R(t)$ obeys the Friedmann equation for $p=\frac{1}{3}\mu$ (one constant in the solution is determined by a; but the space-time is independent of the value of b).

We *conjecture* that a solution of the Einstein-Liouville equations can have an energy-momentum tensor T_{ab} with the perfect fluid form (1.9) at all points, and with u_a given by (1.14), only if $\sigma=0$ and there is an acceleration potential r for $\dot{u}_a$ (i.e. $\dot{u}_a = -h_a{}^b(\log r)_{,b}$)[8]; and further, that this will also be true for Einstein-Boltzmann solutions with reasonable collision cross-sections. If true, this would mean that the perfect fluid description was strictly applicable only under very restricted conditions. The statement would not merely be that a realistic fluid would have a small but finite viscosity and heat conduction, but[9] rather that one could not, for almost all fluid flows, have *any* self-consistent kinetic description leading to a perfect fluid form of T_{ab}; in particular, the fluid shear would have to vanish. While this would not imply the resulting physical effects were large, it nevertheless seems an interesting question of principle.

Whether or not this is true, a further question of interest is the determination of all perfect fluid solutions of Einstein's equations with $\sigma=0$. Theorem 3 shows that $\omega\theta=0$ for three distinct sets of conditions; one would like to know precisely what the general conditions are for which this result is true. (It is conceivably true for all perfect fluid solutions, or for all perfect solutions with an equation of state of the form $p=p(\mu)$.) If it does not hold for all perfect fluids it would be interesting to obtain exact solutions with $\sigma=0$, $\theta\omega\neq 0$.

Finally, we note that the Newtonian situation corresponding to the EGS theorem, i.e. the case of an isotropic solution of the Poisson and Liouville equations, has been solved by Ehlers and Rienstra ([24]). The Newtonian solutions are less restricted than the corresponding relativistic ones, for while they are also shear-free, the result $\omega\theta=0$ does not obtain in the Newtonian theory. Further the restriction to vanishing shear does not imply restrictions on space-time, as it does in the relativistic theory. No serious difficulty should arise in extending the Newtonian results to include a collision term.

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[8] Then one would introduce the coordinates and use the field equations given in Section 4.

[9] Just as one cannot have a reversible fluid flow in most space-times.

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