Isotropic Solutions of the Einstein–Liouville Equations

J. EHLERS,* P. GEREN, AND R. K. SACHS*
The University of Texas, Austin, Texas

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The gravitational field generated by a gas whose one-particle distribution function obeys the Liouville equation is examined under the following assumptions: First, the distribution is locally isotropic in momentum space with respect to some world-velocity field; second, if the particles have rest-mass zero, the gas is irrotational. It is shown that the model is then either stationary or a Robertson-Walker model. The time dependence of the radius in the Robertson–Walker models is given in terms of integrals containing the distribution function.

1. INTRODUCTION

In galactic dynamics it is useful to relate the velocity dependence of the stellar distribution function to the spatial configuration of the galaxy and to the galaxy's gravitational field. In this paper we give some analogous general-relativistic results for the very simple case of a locally isotropic distribution function. We have in mind applications to cosmology.

Einstein's gravitational field equation

$$G_{ab} = -T_{ab} \tag{1.1}$$

relates the metric of space–time to the stress-energy–momentum distribution of matter. It is necessary to supplement (1.1) by assumptions about the structure of matter. We must specify the dependence of T_{ab} on the basic matter (or field) variables, and state the nongravitational equations of motion, constitutive equations, etc., which these additional variables are supposed to obey.

The model of matter used in this paper is that of kinetic theory. We imagine space–time contains a system of particles all having the same[1] proper mass m (≥ 0). We think of the metric g_{ab} in (1.1) as the macroscopic gravitational potential generated collectively by all the particles, and we assume that each particle moves as a test particle in this average field except during point collisions. Moreover, we restrict ourselves to two cases: either collisions are completely neglected—Case A; or there is collisional equilibrium (detailed balancing)—Case B.

Let $f(x, p)$ be the one-particle distribution function, defined on the seven-dimensional manifold of pairs (x, p), where x is a space–time point and p a tangent vector at x with $p^2 = -m^2$. [We use the signature $(+++-)$ for g_{ab}.] The function f determines the energy–momentum tensor via the equation

$$T_{ab}(x) = \int_{P_m(x)} p_a p_b f(x, p)\, dP_m; \tag{1.2}$$

here $P_m(x)$ denotes the mass hyperboloid $p^2 = -m^2$ in the tangent space of space–time at x, and dP_m is the Lorentz-invariant measure on $P_m(x)$.

Either Case A or B above implies that f satisfies the Liouville condition[2]

$$f[x(s), p(s)] = \text{const along each timelike (if } m > 0) \text{ or lightlike (if } m = 0) \text{ geodesic } \{x(s), p(s)\}. \tag{1.3}$$

The system of equations (1.1)–(1.3) is the general-relativistic analog of the basic equations of stellar dynamics; (1.1) corresponds to Poisson's equation and (1.3) corresponds to the collisionless Boltzmann equation with gravitational forces.

Equations (1.1)–(1.3) are not independent; either (1.1) or the pair (1.2) and (1.3) imply[2]

$$T^{ab}{}_{;b} = 0. \tag{1.4}$$

Real systems for which Case A above seems to be a reasonable model are the system of galaxies now[3] and the galaxies themselves, considered as systems of stars.[4] Case B, with $m = 0$, may be applicable to the early state of the universe in a big-bang model. In the latter case, pertaining to epochs earlier than 10^3 years, we may think of a mixture of photons, perhaps neutrinos and even gravitons, and some electrons and nucleons, with most of the energy due to rest-mass zero or to ultrarelativistic particles. For photons the

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[1] The assumption of equal masses could easily be relaxed; it is made here for simplicity and because of the special role played by a rest-mass-zero gas.

[2] G. E. Tauber and J. W. Weinberg, Phys. Rev. **122**, 1342 (1961).

[3] It is difficult to estimate reliably the relaxation time, but if one uses the usual Newtonian formulas (cf., e.g., Ref. 4) with a cutoff distance $\sim 10^{10}$ light years, one obtains relaxation times which are at least not short compared to the Hubble time.

[4] S. Chandrasekhar, *Principles of Stellar Dynamics* (Dover Publ. Inc., New York, 1960), especially Chap. II; see also the article by L. Woltjer in *Lectures in Applied Mathematics*, J. Ehlers, Ed. (American Mathematical Society, Providence, R.I., 1967), Vol. 9, especially Appendix I.

collisional equilibrium could be catalyzed by the electrons via scattering and free-free transitions; the average time a photon takes to Thomson-scatter at $t = 10^3$ years, assuming a temperature $T \approx 10^5$ °K and a mass density $\rho \approx 10^{-16}$ g/cm^3 (see Ref. 5), is of order 10^{-2} years, and this average collision time decreases rapidly if we consider still earlier epochs.

In this paper we consider those solutions of Eqs. (1.1), (1.2), and (1.3) in which the distribution is everywhere isotropic: There exists a timelike unit-vector field $u^a(x)$ such that $f(x, p)$ is, at any event x, invariant with respect to all those restricted homogeneous Lorentz transformations in the tangent space which leave u^a unchanged. In physical terms, this property means that there exists a preferred state of motion at each event x in the universe, with respect to which the peculiar motions of the particles near x are isotropically distributed. Analytically this means that f has the form $f(x, p) = h(x, -u(x) \cdot p)$. In Case B this isotropy follows from the assumed collisional equilibrium[6]; in Case A it is, of course, an independent assumption.

We show that this assumption (and, in the case $m = 0$, the additional assumption that either the acceleration or the rotation of the mean flow vanishes) leads, without any *a priori* assumptions about the symmetry of space-time, to a Robertson-Walker metric or to stationary space-times. In general-relativistic cosmology (we now have in mind Case A, $m > 0$) the cosmological principle and the Weyl postulate (see, e.g., Ref. 7) can, therefore, both be considered as consequences of the apparently weaker postulate of an isotropic distribution of peculiar velocities. The dependence of the scale factor $a(t)$ of the universe on the distribution function is given [Eq. (4.7)]; this corresponds to the dependence of $a(t)$ on the "equation of state" in hydrodynamical models.

Our result and the method of proof are extensions of the work of Tauber and Weinberg on general relativistic gases (Ref. 2). These authors have determined the restrictions imposed on the metric and the mean flow by the Liouville equation and the condition of isotropy; they did not consider the further restrictions imposed by the Einstein field equation. Because we want to point out the special role of rest-mass zero gases, and also because we need a more detailed description of the case of irrotational flows with expansion than that given in the paper mentioned, we shall rederive some of the relevant results.

[5] R. H. Dicke, P. J. E. Peebles, P. G. Roll, and D. T. Wilkinson, Astrophys. J. **142**, 414 (1965).
[6] K. Bichteler, Z. Physik **182**, 521 (1965).
[7] H. Bondi, *Cosmology* (Cambridge University Press, Cambridge, England, 1961).

2. GEOMETRICAL AND KINEMATICAL PRELIMINARIES

In this section we describe a few properties of congruences of timelike curves in normal hyperbolic Riemannian spaces. We use these properties in the proof of our main theorem.

Let u^a be the normalized tangent vector to a congruence of timelike curves $u_a u^a = -1$. The vector u^a may be interpreted physically as the local average particle world velocity.

The quantities ω_{ab}, σ_{ab}, $\dot{u}_a$, and θ, defined by

$$u_{a;b} = \omega_{ab} + \sigma_{ab} - \dot{u}_a u_b + \tfrac{1}{3}\theta(g_{ab} + u_a u_b), \qquad (2.1)$$

$$\omega_{(ab)} = \sigma_{[ab]} = \sigma^a{}_a = 0, \quad \omega_{ab} u^b = \sigma_{ab} u^b = 0, \quad (2.2)$$

are known, respectively, as the angular velocity (or vorticity tensor), the shear velocity, the acceleration, and the expansion velocity of the congruence (see, e.g., Refs. 8 and 9).

We use the brackets ( ) and [ ] for symmetrization and antisymmetrization, respectively, and use throughout the dot to indicate covariant differentiation in the u^a direction, e.g., $\dot{u}_a = u_{a;b} u^b$.

The definitions imply the following lemmas:

Lemma 1: A flow is irrotational, $\omega_{ab} = 0$, if and only if the streamlines are hypersurface-orthogonal, i.e., if and only if there exists a scalar t such that

$$\dot{t} u_a = -t_{,a} \neq 0. \qquad (2.3)$$

Lemma 2: The property

$$(\dot{u}_{[a} - \tfrac{1}{3}\theta u_{[a})_{,b]} = 0 \qquad (2.4)$$

is necessary and sufficient for the existence of a metric $\bar{g}_{ab}$ conformally related to g_{ab} such that the congruence is geodesic and expansion-free with respect to $\bar{g}_{ab}$; if (2.4) holds, we may put

$$\dot{u}_a - \tfrac{1}{3}\theta u_a = U_{,a}, \quad \bar{g}_{ab} = e^{-2U} g_{ab}. \qquad (2.5)$$

The properties discussed in these two lemmas are conformally invariant, that is, they are preserved under transformations

$$\bar{g}_{ab} = W^2 g_{ab}, \quad \bar{u}^a = W^{-1} u^a, \qquad (2.6)$$

where W is an arbitrary positive scalar field. The vanishing of shear, $\sigma_{ab} = 0$, is likewise conformally invariant.

[8] J. L. Synge, *Relativity: The General Theory* (North-Holland Publ. Co., Amsterdam, 1960).
[9] J. Ehlers, Akad. Wiss. Lit. (Mainz) Abhandl. Math.-Nat. Kl. **11**, 793 (1961).

By combining the preceding lemmas we obtain further:

Lemma 3: The curves of a congruence are the orbits of a one-dimensional (local) group of conformal mappings of space–time into itself if and only if the congruence is shearfree and satisfies (2.4); if these conditions are satisfied and U is defined by (2.5), $\xi^a = e^U u^a$ generates the group. If, in addition, $\theta = 0$, the mappings are isometries.

We shall now prove:

Lemma 4: If a congruence satisfies $\omega_{ab} = \sigma_{ab} = 0$ and (2.4), then the metric is conformally decomposable; that is, there exist coordinates $(x^a) = (x^\nu, t)$, $\nu = 1, 2, 3$, such that

$$G \overset{\text{DEF}}{=} g_{ab}\, dx^a\, dx^b = e^{2U}\{d\sigma^2 - dt^2\},$$
$$d\sigma^2 = \gamma_{\lambda\mu}(x^\nu)\, dx^\lambda\, dx^\mu, \quad u^a = e^{-U}\delta^a_4. \qquad (2.7)$$

In fact, if $\omega_{ab} = \sigma_{ab} = 0$ and (2.4) holds, we find from Lemma 2 that, with respect to $\bar{g}_{ab}$, $\bar{\omega}_{ab} = \bar{\sigma}_{ab} = \dot{\bar{u}}_a = \bar{\theta} = 0$, $\bar{u}^a$ is then covariant-constant with respect to $\bar{g}_{ab}$ by Eq. (2.1), and consequently $\bar{g}_{ab}$ is locally the direct product of a 3-space and a line (see Ref. 10, p. 286), so that g_{ab} can be written as in Eq. (2.7).

Finally, we shall establish two properties of Ricci proper congruences defined by

$$u^a R_{a[b} u_{c]} = 0. \qquad (2.8)$$

From the contracted Ricci identity $u^a{}_{;[ab]} = \frac{1}{2} R_{bc} u^c$ and Eq. (2.1), we compute

$$u^a R_{a[b} u_{c]} = \tfrac{2}{3}\theta_{,[b} u_{c]} + \text{terms containing } \omega_{ab} \text{ or } \sigma_{ab}.$$

Hence:

Lemma 5: If a Ricci proper congruence satisfies $\omega_{ab} = \sigma_{ab} = 0$, then its expansion velocity θ is constant on each hypersurface orthogonal to the streamlines, so that

$$\theta = \theta(t) \qquad (2.9)$$

with t as in Eq. (2.3).

If we specialize further by combining Lemmas 4 and 5 taking into account that, for the case (2.7), $\theta = 3e^{-U}(\partial U/\partial t)$, we get:

Lemma 6: If an irrotational, shearfree Ricci proper congruence satisfies Eq. (2.4), then coordinates exist such that (2.7) holds with

$$e^{-U} = X(t) + Y(x^\nu). \qquad (2.10)$$

[10] J. A. Schouten, *Ricci Calculus* (Springer-Verlag, Berlin, 1954).

3. ISOTROPIC SOLUTIONS OF LIOUVILLE'S EQUATION[11]

We now proceed to analyze Liouville's equation (1.3), ignoring the field equation (1.1) for the moment. We have to find $g_{ab}(x)$, $u_a(x)$, and $h(x, E)$ such that, for a given mass $m \geq 0$, the distribution function

$$f(x, p) = h[x, -u(x) \cdot p]$$

is constant on each geodesic $\{x^a(s), p^a(s)\}$ with $p^a = dx^a/ds$, $p^2 = g_{ab}p^a p^b = -m^2$. Here E is an auxiliary real variable ($E \geq m$) to be interpreted as the energy of a particle with respect to that local frame (with time axis u^a) with respect to which f is isotropic in momentum space.

Since $h(x, E) > 0$ and $h(x, E) \to 0$ as $E \to \infty$ on physical grounds, we know that $h' = \partial h/\partial E \neq 0$ for some open E interval. For E in this interval let us put $h(x, E) = F$ and, for the solution with respect to E, write $E = g(x, F)$. Then Liouville's equation is equivalent to the statement that

$$\frac{dE}{ds} = -\frac{d}{ds}(u_a p^a) = -u_{a;b} p^a p^b = p^a g_{,a} \quad (3.1)$$

on each geodesic, where we define $g_{,a} = \partial g/\partial x^a$ with F fixed. If we split the 4-momentum in the form

$$p^a = Eu^a + (E^2 - m^2)^{\frac{1}{2}} e^a,$$
$$u_a e^a = 0, \quad e_a e^a = 1, \qquad (3.2)$$

and insert Eqs. (3.2) and (2.1) into Eq. (3.1), we obtain

$$g\dot{g} + \frac{\theta}{3}(g^2 - m^2) + (g^2 - m^2)^{\frac{1}{2}}(g\dot{u}_a + g_{,a})e^a + (g^2 - m^2)\sigma_{ab}e^a e^b = 0.$$

This equation has to hold identically in the seven independent variables x^a, F, e^a; e^a may be considered as a point on a Euclidean, two-dimensional unit sphere. Hence, since spherical harmonics of different degrees are linearly independent,

$$\sigma_{ab} = 0, \quad \dot{u}_a + (\log g)_{,a} = \alpha u_a, \quad \frac{\theta}{3} = \frac{-g\dot{g}}{g^2 - m^2}. \qquad (3.3)$$

The last two of these equations can be replaced by the single relation

$$\dot{u}_a - \tfrac{1}{3}\theta u_a = -(\log g)_{,a} - \frac{m^2\theta}{3g^2} u_a. \qquad (3.4)$$

Differentiating this equation with respect to F and inserting the resulting expression for u_a into Eq. (3.4), we obtain

$$\dot{u}_a - \tfrac{1}{3}\theta u_a = -\tfrac{1}{2}(\log gg')_{,a} = U_{,a}, \qquad (3.5)$$

[11] For this whole section, compare Ref. 2, Sec. III.

where $g' = \partial g/\partial F$ and $U(x)$ is defined by Eq. (3.5) up to an additive constant.

According to Lemma 3 of Sec. 2, *the congruence associated with an isotropic distribution is conformal* (Tauber and Weinberg, 1962).

The function $g(x, F)$ is related to $U(x)$ by $dU = -\frac{1}{2}d(\log gg')$, d referring to the variables x^a only with F treated as a parameter. Integrating gives

$$e^{2U}(g^2 - m^2) = l(F) - k(x) \tag{3.6}$$

with some functions l and k. But from (3.3) and (3.5)

$$\dot{U} = \frac{\theta}{3}\frac{-(g^2 - m^2)^{\cdot}}{2(g^2 - m^2)};$$

consequently, differentiation of Eq. (3.6) in the u^a direction gives

$$\dot{k} = 0; \tag{3.7}$$

thus k is constant on each streamline.

Combining Eqs. (3.4) and (3.5), we get a further condition:

$$\frac{m^2\theta}{3}u_a = m^2\dot{U}u_a = -g^2(U + \log g)_{,a}. \tag{3.8}$$

To summarize: Characterizing properties of an isotropic solution of Liouville's equation are Eqs. (3.6), (3.7), (3.8), and the conformal character of the congruence generated by u^a.

According to Eq. (3.8), two possibilities exist:

A. $m\theta = 0$: In this case (3.8) requires that g^2e^{2U} is a function of F only; then the distribution function has the form

$$f(x, p) = j(\xi_a(x)p^a), \tag{3.9}$$

where $\xi^a = e^U u^a$ generates a conformal group and j is some function. If $\theta = 0$, which is necessarily so if $m \neq 0$, the group is an isometry group.

It is well known that Eq. (3.9) gives first integrals for the equations of geodesics; the remarkable fact is that these are the only ones of the form $h(x^a, -u_b(x)\dot{x}^b)$.

The case $\theta = 0$ is not of interest in cosmology, and we shall not consider it in detail.

B. $m\theta \neq 0$: In this case, Eq. (3.8) and Lemma 1 of Sec. 2 show that the congruence must be irrotational; consequently, Lemma 4 applies. Moreover, Eqs. (2.3) and (3.8) show that the preferred time variable t must be related to g and U by

$$\dot{t}g^2d(U + \log g) = m^2\dot{U}\,dt.$$

Hence, $e^{2U}g^2$ must depend functionally on t and F; this fact, together with Eqs. (3.6) and (3.7), restricts the functional relation to the form

$$e^{2U}g^2 = \bar{l}(F) - q(t) \tag{3.10}$$

with some functions $\bar{l}$, q. The distribution function is therefore

$$f(x, p) = j([\xi_a(x)p^a]^2 + q(t)). \tag{3.11}$$

Using the preferred coordinates of Eq. (2.7), we have, then, the result

$$G = \frac{k(x^\nu) - q(t)}{m^2}[d\sigma^2 - dt^2], \tag{3.12}$$

$$f(x, p) = j\left(\frac{k(x^\nu) - q(t)}{m^2}E^2 + q(t)\right), \tag{3.13}$$

E being the energy of p^a with respect to u^a.

When $m\theta \neq 0$, the irrotationality of the flow follows, as we have seen, from the Liouville equation and the isotropy condition. ω_{ab} might be different from zero if $m\theta = 0$, at least so long as no field equations are imposed. It is, however, of interest to note that, if the flow is geodesic and has expansion, $\dot{u}_a = 0 \neq \theta$, Eq. (3.5) and Lemma 1 show that $\omega_{ab} = 0$. For $m = 0$ and $\theta \neq 0$, we therefore have the subcases $A_1: \dot{u}_a = \omega_{ab} = 0$ and $A_2: \dot{u}_a \neq 0$. In the former, Lemma 4 applies again, and the metric can be written in the form (2.7).

4. SOLUTIONS OF THE FIELD EQUATION FOR ISOTROPIC DISTRIBUTIONS

We now ask which restrictions are imposed on the solutions $\{g_{ab}, f\}$ of Liouville's equation by the field equation (1.1) with the source (1.2). The isotropy of f with respect to u^a implies that

$$T^{ab} = (\mu + p)u^a u^b + pg^{ab}, \tag{4.1}$$

where the mean energy density μ and pressure p can be expressed in terms of f (see below). From (4.1) and (1.1) it is obvious that u^a is an eigenvector of the Ricci tensor, i.e., Eq. (2.8) holds. In Case B of the preceding section and also in Case A, if either $\dot{u}_a = 0$ or $\omega_{ab} = 0$ is assumed, we can apply Lemma 6 of Sec. 1; we then obtain the metric

$$[X(t) + Y(x^\nu)]^{-2}[d\sigma^2 - dt^2]. \tag{4.2}$$

In Case B, comparison of this expression with Eq. (3.12) shows that the conformal factor can depend only on t or on x^ν, but not on both variables. Since $\theta \neq 0$, we conclude that $k = \text{const}$, $Y = \text{const}$; hence, without loss of generality, $Y = 0$ in (4.2). The resulting metric satisfies the field equation with (4.1) only if it is a Robertson–Walker metric (see Ref. 12, p. 107)

$$a^2(t)\,d\sigma^2 - dt^2 \tag{4.3}$$

(t is a new time coordinate), where $d\sigma^2$ has constant curvature $\epsilon = \pm 1, 0$. From (3.13) the distribution

[12] P. Jordan, *Schwerkraft und Weltall* (Vieweg and Sohn, Braunschweig, Germany, 1955).

function is then of the form

$$f(x, p) = \frac{1}{4\pi} g(a^2(t)\mathbf{p}^2), \qquad (4.4)$$

where

$$\mathbf{p}^2 = (g_{ab} + u_a u_b)p^a p^b$$

is the squared 3-momentum of a particle relative to the preferred local frame defined by u^a, and g is some positive function of a real variable.

From (1.2) and (4.4), introducing $x = a\,|\mathbf{p}|$, we get

$$\mu = a^{-4}\int_0^\infty x^2 g(x^2)(a^2m^2 + x^2)^{\frac{1}{2}}\,dx,$$
$$p = \tfrac{1}{3}a^{-4}\int_0^\infty x^4 g(x^2)(a^2m^2 + x^2)^{-\frac{1}{2}}\,dx. \qquad (4.5)$$

These relations imply, as is well known,[2] energy conservation, $(\mu a^3)^{\cdot} + p(a^3)^{\cdot} = 0$, and therefore the only remaining field equation is

$$3a^2(\dot{a}^2 + \epsilon) = \int_0^\infty x^2 g(x^2)(m^2a^2 + x^2)^{\frac{1}{2}}\,dx. \qquad (4.6)$$

Since all these universes have, according to Raychaudhuri's theorem, a singular state $a = 0$ which we may take as the t origin, the time development of a generalized Friedmann model is determined by the function g, the distribution, through

$$t = \frac{\sqrt{3}}{2}\int_0^{a^2}\left[-3\epsilon u + \int_0^\infty x^2 g(x^2)(m^2u + x^2)^{\frac{1}{2}}dx\right]^{-\frac{1}{2}}du. \qquad (4.7)$$

Equations (4.3), (4.4), (4.5), and (4.7) determine completely the model universe in Case B.

We now return to Case A and restrict attention to the subcase $\theta \neq 0$ so that $m = 0$. Since, in this case, $T^a{}_a = 0$ from (1.2), in Eq. (4.1) we have

$$p = \tfrac{1}{3}\mu. \qquad (4.8)$$

Independently of kinetic theory, it follows that, for an energy-momentum tensor (4.1) together with (4.8), the conservation law $T^{ab}{}_{;b} = 0$ is equivalent to the relation

$$\dot{u}_a - \tfrac{1}{3}\theta u_a = -\tfrac{1}{4}(\log\mu)_{,a}, \qquad (4.9)$$

which implies the conservation law $\dot{\mu} + \frac{4}{3}\mu\theta = 0$. Its geometrical meaning is described in Lemma 2 of Sec. 2. (The quantity whose density is $\mu^{\frac{3}{4}}$ is conserved during the motion. For thermal radiation, this conserved quantity is the entropy.)

Combining (4.9) with the arguments which led to the metric (4.2) [cf. Eqs. (2.5) and (2.10)], we see that in the case $m = 0$ the source quantity μ is related to the conformal factor by

$$\mu = [X(t) + Y(x^\nu)]^4. \qquad (4.10)$$

Now we use the "4,4 component" of the field equation (1.1):

$$G_{ab}u^a u^b = -\mu,$$

where the left-hand side can easily be computed from (4.2) by means of the equations for conformal transformations,[13] and the right-hand side is given by Eq. (4.10). We obtain

$$6\left(\frac{dx}{dt}\right)^2 - 2(X + Y)^4 - \bar{R}(X + Y)^2 - 4\Delta Y(X + Y) + 6DY = 0. \qquad (4.11)$$

Here $\bar{R}$ is the Ricci scalar of $d\sigma^2$, Δ is the Laplace operator of $d\sigma^2$, and $DY = \gamma^{\lambda\mu}Y_{,\lambda}Y_{,\mu}$. Since $\theta \neq 0$ implies $dX/dt \neq 0$, we can introduce $t' = X(t)$ as a new time variable and write $(dX/dt)^2 = F(t')$. Then (4.11) becomes

$$6F(t') = 2(t' + Y)^4 + \bar{R}(t' + Y)^2 + 4\Delta Y(t' + Y) + 6DY.$$

This equation holds identically in t' and x^ν; the left-hand side is independent of x^ν; therefore, the right-hand side (in particular, the coefficient $8Y$ of t'^3) is independent of x^ν; then $Y = \text{const}$. We absorb Y into $X(t)$ so that $Y = 0$. The further analysis is identical with the one in Case A, following Eq. (4.3), with the specialization $m = 0$ in Eqs. (4.5) to (4.7). Then the models are precisely the Tolman models.

We have proven the following:

Theorem 1: The most general solution of the Einstein–Liouville equations (1.1), (1.2), and (1.3) with an isotropic distribution function for particles with nonvanishing mass is either stationary or a generalized Friedmann model {(4.3), (4.4), (4.7)}; for particles with vanishing mass, the solution is either stationary, or a Tolman model, or nonstationary with $\dot{u}_a \neq 0 \neq \omega_{ab}$.[14]

If one looks at the proof, one recognizes that a result can also be formulated which is independent of kinetic-theory assumptions.

Theorem 2: The only solution of the Einstein field equation (1.1) with a "perfect-radiation" source

$$T^{ab} = \frac{\mu}{3}(4u^a u^b + g^{ab})$$

[13] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, N.J., 1956); P. Jordan, J. Ehlers, and W. Kundt, Akad. Wiss. Lit. (Mainz) Abhandl. Math.-Nat. Kl. No. **2**, 23 (1960).

[14] Whether the last case actually admits solutions is not known at present. Some perturbation calculations suggest this case is empty. Of course stationary solutions are known: see O. Klein, Arkiv Mat. Astr. Fys. **34A**, Paper 19 (1947).

in shearfree, irrotational motion is the Tolman universe.

We also note the following:

Corollary: The gravitational field generated by a spherically symmetric "perfect-radiation" source in shearfree motion is either static or the Tolman universe.

In fact, a timelike vector field u^a, invariant under the group O_3 (acting on spacelike spheres), is automatically hypersurface-orthogonal; the gas is then irrotational, and the corollary follows from Theorem 2.

We end this section with a few additional remarks:

(1) Equation (4.5) can be considered as a parameter representation of an "equation of state" $\mu = \varphi(p)$ determined by the distribution g. If $m = 0$, $\mu = 3p$ for all g's.

(2) The original Friedmann universes, i.e., the dust models ($p = 0$), are contained in {(4.3), (4.4), (4.7)} as the limiting case in which

$$g(x^2) = \frac{4M}{m}\frac{\delta(x^2)}{x}, \quad \mu a^3 = M = \text{const};$$

they are the only models without any random particle motions.

(3) For $t \to 0$, and hence $a \to 0$, all the models (except the dust model) behave, according to Eqs. (4.5) and (4.7), asymptotically like a Tolman radiation universe; if a model expands indefinitely, it behaves for $t \to \infty$ and $a \to \infty$ asymptotically like a dust model; more precisely, one has $\mu \sim a^{-3}$ and $p/\mu \sim a^{-2}$.

(4) A Planck distribution

$$f(x, p) = \frac{2}{h^3}\left[\exp\left(\frac{-u_a p^a}{kT}\right) - 1\right]^{-1}$$

is rigorously compatible with (4.4) if $m = 0$ and $T \sim a^{-1}$; an equilibrium distribution for $m > 0$, however, is incompatible with an isotropically expanding universe.[9]

According to Eq. (3.13), the general solution of Liouville's equation in a Robertson–Walker universe has the form $f(x, p) = j(a^2(t)\mathbf{p}^2)$; hence if at $t = t_0$ we have, say, a (relativistic) Boltzmann distribution

$$c \exp\left(\frac{-E}{kT_0}\right) = c \exp\left(\frac{-(m^2 + \mathbf{p}^2)^{\frac{1}{2}}}{kT_0}\right),$$

then we obtain later

$$f(x, p) = c \exp\left(\frac{-1}{kT_0}\left\{m^2 + \left[\frac{a(t)}{a(t_0)}\right]^2 \mathbf{p}^2\right\}^{\frac{1}{2}}\right),$$

which is *not* an exact equilibrium distribution. For $(a(t)/a(t_0))^2\mathbf{p}^2 \ll m^2$ we have, however, approximately

$$f(x, p) \approx c' \exp(\mathbf{p}^2/2mkT),$$

with

$$T = T_0(a(t_0)/a(t))^2,$$

which is a (nonrelativistic) Boltzmann distribution with a temperature $T \sim a^{-2}$ (compare Ref. 5).

5. DISCUSSION

Unfortunately, the result presented cannot be taken to mean that the universe in its earliest stages was necessarily a Friedmann model with detailed balance established by rapid collisions of a gas whose particles have zero or negligible rest mass. There are various difficulties. First, nothing is known as yet about the case where a rest-mass zero gas rotates, not even if time-dependent detailed-balance rotational solutions exist. Second, it is known that in a Friedmann model there are particle horizons.[15] For example, with the parameters mentioned in the Introduction a given particle has had time at $t = 10^3$ years to communicate with only about 10^{14} solar masses of matter. There must be particle horizons in more general models as well; we can hardly suppose that portions of the gas which have not had time to communicate have been able to establish detailed balance. More generally, our equilibrium considerations do not indicate how quickly detailed balance is established, if at all.

[15] W. Rindler, Monthly Notices Roy. Astron. Soc. **116**, 662, 1956.