TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 218, 1976

# ISOTROPIC TRANSPORT PROCESS ON A RIEMANNIAN MANIFOLD

BY

MARK A. PINSKY[1]

ABSTRACT. We construct a canonical Markov process on the tangent bundle of a complete Riemannian manifold, which generalizes the isotropic scattering transport process on Euclidean space. By inserting a small parameter it is proved that the transition semigroup converges to the Brownian motion semigroup provided that the latter preserves the class $C_0$. The special case of a manifold of negative curvature is considered as an illustration.

1. **Introduction.** In order to construct a diffusion process on a differentiable manifold it is necessary, in general, to solve stochastic differential equations in coordinate patches and then piece together the resulting local diffusions [4]. In the case of the sphere $S^{n-1}$, Stroock [10] has shown that the Brownian motion may be obtained by solving a single stochastic differential equation on $R^n$, whose solution stays on the sphere of its own accord.

The purpose of this paper is to show that, on a wide class of Riemannian manifolds, the Brownian motion can be approximated in law by a globally defined stochastic process—the isotropic transport process. The paths of this process are piecewise geodesic. The joint position-velocity motion is a Markov process on the tangent bundle of the manifold. In the case of $R^n$, it coincides with the usual transport process [11]. Using a theorem of Kurtz [8], we show that the Brownian motion on the manifold can be approximated, in the sense of weak convergence, by a sequence of isotropic transport processes. As a by-product we obtain a formula for the Laplace-Beltrami operator as the spherical average of the second covariant derivative along the geodesic flow field in the tangent bundle.

2. **Geodesic flow field.** Let $M$ be a complete $C^\infty$ Riemannian manifold of dimension $n$. $T_x(M)$ denotes the tangent space at $x \in M$; $T(M)$ denotes the tangent bundle over $M$. For $\widetilde{x} = (x, \xi) \in T(M)$, we consider the geodesic flow $X = \{X^{(x,\xi)}(t), t \geqslant 0\}$. This satisfies the properties that $X^{(x,\xi)}(0) = x$, $\dot{X}^{(x,\xi)}(0) = \xi$ and $\dot{X}^{(x,\xi)}(t)$ is a parallel vector field along $\{X^{(x,\xi)}(t), t \geqslant 0\}$.

Received by the editors May 11, 1975.

*AMS (MOS) subject classifications* (1970). Primary 60J25, 60J65, 53C20; Secondary 58G99, 47D05.

*Key words and phrases.* Isotropic transport process, geodesic flow, Brownian motion, contraction semigroup.

[1] Supported by the National Science Foundation MPS71-02838 A04.

We consider the lifting of $X$ to a curve $\widetilde{X}$ in $T(M)$, defined by $\widetilde{X}^{(x,\xi)}(t) = (X^{(x,\xi)}(t), \dot{X}^{(x,\xi)}(t))$. Let $Z$ be the tangent vector to this curve at $t = 0$. $Z$ is a vector field on $T(M)$ and is called the *geodesic flow field* [6, 132]. We let $D_Z$ denote the operation of (covariant) differentiation defined by $Z$.

Let $C_0(T(M))$ be the Banach space of continuous functions on $T(M)$ with $\lim_{d(x,x_0)\to\infty} f(x) = 0$. [Here $d$ is a metric consistent with the manifold topology and $x_0$ is a fixed point of $M$.] $C_0^k(T(M)) = \{f \in C_0(T(M))$: $f$ has $k$ continuous partial derivatives$\}$. $\|f\| = \sup_{\widetilde{x}\in T(M)} |f(\widetilde{x})|$. Finally, set

$$(T_t^0 f)(\widetilde{x}) = f(\widetilde{X}^{\widetilde{x}}(t)), \qquad (t > 0)$$

$$(R_\lambda^0 f)(\widetilde{x}) = \int_0^\infty e^{-\lambda t}(T_t^0 f)(\widetilde{x})\, dt \qquad (\lambda > 0).$$

PROPOSITION 2.1. *$R_\lambda^0$ maps $C_0^1(T(M))$ into $C_0^1(T(M))$ and satisfies*

$$(\lambda - D_Z)R_\lambda^0 f = f \qquad (f \in C_0^1(T(M))). \tag{2.2}$$

PROOF. Let $f \in C_0^1(T(M))$. By the smooth dependence on initial conditions, $(x, \xi) \longrightarrow \widetilde{X}^{(x,\xi)}(t)$ is a $C^\infty$ mapping for each $t > 0$. Therefore $T_t^0 f \in C^1(T(M))$. The dominated convergence theorem implies that $R_\lambda^0 f \in C^1(T(M))$. To show that $R_\lambda^0 f$ vanishes at $\infty$, let $\epsilon > 0$. Choose $T > 0$ such that $e^{-\lambda T}\|f\|/\lambda \leqslant \epsilon/2$. Since $f \in C_0(T(M))$, we can find $R > 0$ such that $|f(\widetilde{x})| \leqslant \epsilon\lambda/2$ if $d(\widetilde{x}, \widetilde{x}_0) \geqslant R$. Now let $d(\widetilde{x}, \widetilde{x}_0) \geqslant R + T$; then $d(\widetilde{X}_t, \widetilde{x}_0) \geqslant R$ for $t \leqslant T$. Hence

$$|R_\lambda^0 f(x)| \leqslant \left|\int_0^T e^{-\lambda t} f(\widetilde{X}(t))\, dt\right| + \left|\int_T^\infty e^{-\lambda t} f(\widetilde{X}(t))\, dt\right|$$

$$\leqslant \frac{\epsilon\lambda}{2}\left(\frac{1 - e^{-\lambda T}}{\lambda}\right) + \frac{\epsilon}{2} \leqslant \epsilon,$$

for $d(\widetilde{x}, \widetilde{x}_0) \geqslant R + T$. Thus $R_\lambda^0 f \in C_0(T(M))$.

To verify (2.2) write, for $h > 0$,

$$T_h^0 R_\lambda^0 f = \int_0^\infty e^{-\lambda t} T_{t+h}^0 f\, dt = e^{\lambda h}\int_h^\infty e^{-\lambda u} T_u^0 f\, du$$

$$= e^{\lambda h}\left[R_\lambda^0 f - \int_0^h e^{-\lambda u}(T_u^0 f)\, du\right].$$

But $(T_u^0 g)(\widetilde{x}) = g(\widetilde{X}^{\widetilde{x}}(u))$. Therefore

$$D_Z(R_\lambda^0 f) \equiv \lim_{h\to 0} \frac{T_h^0(R_\lambda^0 f) - R_\lambda^0 f}{h}$$

$$= \lim_{h\to 0}\left[\frac{e^{\lambda h} - 1}{h} R_\lambda^0 f - \frac{1}{h}\int_0^h e^{-\lambda u}(T_u^0 f)\, du\right] = \lambda R_\lambda^0 f - f.$$

3. **Isotropic transport process.** Let $(\Omega, \mathcal{B}, P)$ be a probability space. On $\Omega$ we assume given a sequence $\{e_n\}_{n\geq 1}$ of independent random variables with the common distribution

$$P\{e_n > t\} = e^{-t} \qquad (t > 0, n = 1, 2, \ldots).$$

We set $\tau_0 = 0, \tau_n = e_1 + \cdots + e_n (n \geqslant 1)$. Let $\mu_x(d\eta)$ be the unique rotationally invariant probability measure on the unit sphere in $T_x(M)$. For $(x, \xi) \in T(M)$, we define a sequence of random variables $\{Y_1, \xi_1, Y_2, \xi_2, \ldots\}$ as follows:

$$Y_0 = x, \qquad \xi_0 = \xi.$$

Assuming that the joint distributions of $\{Y_1, \xi_1, \ldots, Y_{n-1}, \xi_{n-1}\}$ have been defined, we set

$$Y_n = X^{(Y_{n-1}, \xi_{n-1})}(e_n) \qquad (n = 1, 2, 3, \ldots),$$

$$P\{\xi_n \in d\eta\} = \mu_{Y_n}(d\eta) \qquad (n = 1, 2, 3, \ldots).$$

It is readily verified that the probability law of $(Y_1, \xi_1, \ldots)$ is well defined by these conditional distributions. We now define a stochastic process $\{Y(t), t \geqslant 0\}$ by

$$Y^{(x,\xi)}(t) = X^{(Y_n, \xi_n)}(t - \tau_n) \qquad (\tau_n \leqslant t \leqslant \tau_{n+1}).$$

The curve $t \longrightarrow Y(t) \in M$ is piecewise differentiable and therefore the tangent vector $\dot{Y}(t)$ is defined for $t \neq \tau_n$ $(n \geqslant 1)$. We normalize $\dot{Y}(t)$ so that $\dot{Y}(\tau_n) = \dot{Y}(\tau_n + 0)$. The pair $\widetilde{Y}(t) = (Y(t), \dot{Y}(t))$ will be called the *isotropic transport process* on $T(M)$.

Let

$$T_t f(\widetilde{x}) = E\{f(Y^{(x,\xi)}(t), \dot{Y}^{(x,\xi)}(t))\},$$

$$R_\lambda f(\widetilde{x}) = \int_0^\infty e^{-\lambda t} T_t f(\widetilde{x})\, dt.$$

THEOREM 3.1. *$\{T_t, t \geqslant 0\}$ is a strongly continuous contraction semigroup on $C_0(T(M))$. For $f \in C_0^1(T(M))$, the function $u = T_t f$ satisfies the differential equation*

$$\frac{\partial u}{\partial t}(t, x, \xi) = (D_Z u)(t, x, \xi) + \int_{T_x(M)} [u(t, x, \eta) - u(t, x, \xi)]\, \mu_x(d\eta).$$

This result will be established by working with the operators $\{R_\lambda, \lambda > 0\}$. Of course the semigroup property follows from the Markov property which can be established along the lines of [2, pp. 65–68]. But since we make no explicit use of the Markov property below, this proof is omitted.

PROPOSITION 3.2. *$R_\lambda$ maps $C_0(T(M))$ into $C_0(T(M))$ and satisfies*

(3.2a) $$R_\lambda f = R^0_{1+\lambda} f + R^0_{1+\lambda} P R_\lambda f \qquad (f \in C_0(T(M)))$$

*where* $Pf(x, \xi) = \int_{T_x(M)} f(x, \eta)\mu_x(d\eta)$.

PROOF. Following a similar argument above, we have

$$(R_\lambda f)(\tilde{x}) = \left\{\int_0^T + \int_T^\infty\right\} e^{-\lambda t} Ef(\tilde{Y}(t))\, dt.$$

Given $\epsilon > 0$, choose $T$ such that $e^{-\lambda T}\|f\|/T < \epsilon/2$. Choose $R$ such that $|f(\tilde{x})| \leqslant \epsilon$ for $d(x, x_0) \geqslant R$. Then for $d(x, x_0) \geqslant R + T$, the first integral is $\leqslant \epsilon/2$. Thus $|R_\lambda f(\tilde{x})| \leqslant \epsilon$ for $d(x, x_0) \geqslant R + T$. To prove (3.2a), we write

$$(R_\lambda f)(\tilde{x}) = E\left\{\int_0^{\tau_1} + \int_{\tau_1}^\infty\right\} e^{-\lambda t} f(\tilde{Y}(t))\, dt$$

$$= E\int_0^\infty I_{(t<\tau_1)} e^{-\lambda t} f(\tilde{X}(t))\, dt$$

$$+ E\left\{e^{-\lambda\tau_1}\int_0^\infty e^{-\lambda s} f(\tilde{X}(\tau_1 + s))\, ds\right\}.$$

The first integral $= \int_0^\infty e^{-t}e^{-\lambda t} f(\tilde{X}(t))\, dt = (R^0_{1+\lambda} f)(\tilde{x})$. For the second, note that $\tilde{X}(\tau_1 + s) = (X^{(Y_1,\xi_1)}(s), X^{(Y_1,\xi_1)}(s))$. Taking the conditional expectation with respect to $(Y_1, \xi_1)$ and noting that $Y_1 = X^{(x,\xi)}(\tau_1)$, we have

$$E\left\{e^{-\lambda\tau_1}\int_0^\infty e^{-\lambda s} f(\tilde{X}(\tau_1 + s))\, ds\right\} = E\{e^{-\lambda\tau_1}(R_\lambda f)(Y_1, \xi_1)\}$$

$$= E\{e^{-\lambda\tau_1}(PR_\lambda f)(Y_1)\} = \int_0^\infty e^{-\lambda s}(PR_\lambda f)(X^{(x,\xi)}(s))e^{-s}\, ds$$

$$= (R^0_{1+\lambda} PR_\lambda f)(x, \xi).$$

PROPOSITION 3.3. *Let* $f \in C_0^1(T(M))$. *Then* $R_\lambda f \in C_0^1(T(M))$ *and*

$$(3.3a) \qquad (\lambda - D_Z - P + I)(R_\lambda f) = f.$$

PROOF. Using Proposition 3.2, we iterate (3.2a), obtaining

$$R_\lambda f = R^0_{1+\lambda} f + \sum_{n=1}^\infty (R^0_{1+\lambda} P)^n (R^0_{1+\lambda} f).$$

The series converges uniformly due to the estimation $\|R^0_{1+\lambda} P\| \leqslant 1/(1 + \lambda)$. Thus $R_\lambda f \in C_0(T(M))$. For $f \in C_0^1(T(M))$, this series may be differentiated term-by-term and the differentiated series also converges uniformly. Hence $R_\lambda f \in C_0^1(T(M))$ which was to be shown.

To prove (3.3a), we apply the operator $(1 + \lambda - D_Z)$ to both sides of (3.2a) and use (2.2). Thus

$$(1 + \lambda - D_Z)R_\lambda f = f + PR_\lambda f,$$

which was to be proved.

PROPOSITION 3.4. *Let* $u \in C_0^1(T(M))$, $(\lambda - D_Z - P + I)u = 0$ *for some* $\lambda > 0$. *Then* $u \equiv 0$.

PROOF. Assume that $\sup_{\tilde{x} \in T(M)} u(x) > 0$. Then this sup is assumed at some $\tilde{x}_0 \in M$, for otherwise $\exists\, \tilde{x}_n \longrightarrow \infty$ such that $u(\tilde{x}_n) \longrightarrow \sup u(x) > 0$ which contradicts $f \in C_0(T(M))$. Now at $\tilde{x}_0$, $D_Z u(\tilde{x}_0) = 0$ and $Pu(\tilde{x}_0) - u(\tilde{x}_0) \leqslant 0$. But $Pu(\tilde{x}_0) - u(\tilde{x}_0) = \lambda u(\tilde{x}_0) > 0$, a contradiction. Therefore $u(\tilde{x}) \leqslant 0$ on $T(M)$. Applying the argument to $-u$, we see that $u(\tilde{x}) \geqslant 0$ on $T(M)$. Thus $u \equiv 0$.

To complete the proof of Theorem 3.1, we appeal to the Hille-Yosida theorem. $u = R_\lambda f$ is the unique solution of the equation $(\lambda - D_Z - P + I)u = f$ and satisfies $\|u\| \leqslant \|f\|/\lambda$. Hence there exists a strongly continuous contraction semigroup $\{\overline{T}_t, t \geqslant 0\}$ whose resolvent operators are given by $\{R_\lambda, \lambda > 0\}$. By the uniqueness of Laplace transform, we conclude that $\overline{T}_t = T_t$.

To identify the domain of $T_t$, we recall that $\mathcal{D} = R_\lambda(C_0(T(M)))$ which is independent of $\lambda$. Clearly $C_0^1(T(M)) \subseteq \mathcal{D}$. Now if $u \in \mathcal{D}$, $u = R_\lambda f$ for some $f \in C_0(T(M))$. By 3.2a, $u = R^0_{1+\lambda} g$, $g = f + PR_\lambda f$. But for any $g \in C_0(T(M))$ the proof of Proposition 2.1 shows that $R^0_\lambda g$ is differentiable in the $Z$ direction, and that $D_Z(R^0_\lambda g) = \lambda(R^0_\lambda g) - g$. Hence $D_Z u$ exists and is an element of $C_0(T(M))$. Thus $\mathcal{D} = \{u \in C_0(M): D_Z u \in C_0(M)\}$. The proof is now complete.

**4. Convergence to Brownian motion.** We introduce a parameter $\epsilon > 0$ and consider a one-parameter family of isotropic transport processes corresponding to the backward equation

$$\partial u/\partial t = \epsilon D_Z u + (Pu - u). \tag{4.1}$$

This process can be constructed by replacing $\{\xi_n\}_{n \geqslant 1}$ by $\{\epsilon \xi_n\}_{n \geqslant 1}$ in the definition of $\widetilde{Y}(t)$. The solution of equation (4.1) with $u(0, x, \xi) = f(x, \xi)$ will be denoted by $T_t^{(\epsilon)} f$.

The Laplace-Beltrami operator $\Delta$ defines a Markov process on $M$, the Brownian notion $\{B^x(t), t < \zeta\}$ where $\zeta$ is the lifetime. We introduce the semigroup

$$U_t f(x) = E\{f(B^x(t)), t < \zeta\}$$

whose infinitesimal generator is an extension of $\Delta$. We shall assume the following:

$$U_t \textit{ maps } C_0(T(M)) \textit{ into } C_0(T(M)). \tag{4.2}$$

THEOREM 4.3. *Assume* (4.2). *Then for* $f \in C_0(M)$,

$$\lim_{\epsilon \to 0} T^{(\epsilon)}_{t/\epsilon^2} f = U_{t/n} f$$

*uniformly on M.*

If (4.2) is satisfied, then $P(\zeta = \infty) = 1$, but not conversely. (4.2) is satisfied if (a) $M$ is compact of (b) $M$ has bounded negative curvature (see below).

To prepare the proof, we set $B = P - I$, $A = D_Z$. Then

(4.4) $$PAPf = 0. \quad f \in C_0^1(T(M)).$$

(4.5) $$\{f: Bf = 0\} = C_0(M),$$

(4.6) $$\lim_{t\to\infty} e^{tB}f = Pf, \quad f \in C_0(T(M)).$$

The following proposition defines $B^{-1}$.

PROPOSITION 4.7. *Let $g \in C_0(T(M))$ with $\int_{T_x(M)} g(x, \xi)\mu_x(d\xi) = 0$. The unique solution of the equations*

$$Bf = g, \qquad \int_{T_x(M)} f(x, \xi)\mu_x(d\xi) = 0$$

*is given by $f(x, \xi) = -g(x, \xi)$.*

The proof is omitted.

We now define the operator

(4.7a) $$C = PAB^{-1}AP.$$

PROPOSITION 4.8. *Let $f \in C_0^2(M)$. Then $Cf = n^{-1}\Delta f$, where $\Delta$ is the Laplace-Beltrami operator on $M$.*

PROOF. We first verify that the components of any covariant vector $(\xi_1)$ satisfy

(4.9) $$\int_{T_x(M)} \xi_i\xi_j\mu_x(d\xi) = g_{ij}(x)/n \qquad (1 \leq i, j \leq n).$$

Indeed, by the rotational invariance of $\mu_x$, we must have $\int\langle\xi, \mu\rangle\langle\xi, \nu\rangle\mu_x(d\xi) = \langle\mu, \nu\rangle/n$ for any two vectors $\mu, \nu \in T_x(M)$. The left-hand member of this equation is $\mu_k\nu_s g^{ik}(x)g^{js}(x)\int \xi_i\xi_j\mu_x(d\xi)$ and the right-hand side is $g^{ij}(x)\mu_i\nu_j$. Clearly the only possible choice is (4.9). Now we write the second covariant derivative

$$D_Z D_Z f = g^{il}\xi_l g^{js}\xi_s\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial f}{\partial x^k}\right).$$

Using (4.7a), we have

$$nPD_Z D_Z f = g_{ls}(x)g^{il}(x)g^{js}(x)\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma_{ij}^k\frac{\partial f}{\partial x^k}\right)$$
$$= g^{ij}(x)\left(\frac{\partial^2 f}{\partial x^i\partial x^i} - \Gamma_{ij}^k\frac{\partial f}{\partial x^2}\right)$$
$$= g^{ij}(x)D_iD_jf,$$

which is the usual expression for the Laplace-Beltrami operator [5].

PROOF OF THEOREM 4.3. We apply the method of T. G. Kurtz [8, Theorem 2.2]. $u = T_{t/\epsilon^2}^{(\epsilon)}f$ satisfies the equation

$$du/dt = A/\epsilon + Bu/\epsilon^2$$

where

$$PAP = 0, \qquad \lim_{\lambda\to\infty} \int_0^\infty \lambda e^{-\lambda t} e^{tB} f\, dt = Pf.$$

We must show that $C_0^1(M) \subseteq$ the closure of the range of $\lambda - \Delta$, $\lambda > 0$. But (4.2) implies [1] that the resolvent operator maps $C_0(M) \longrightarrow C_0(M)$. Therefore, the equation $(\lambda - \Delta)f = g \in C_0^1(M)$ is solved by $f(x) = E\{\int_0^\infty e^{-\lambda t} g(B_t^x)\, dt\} \in C_0(M)$. Hence by the theorem of Kurtz, we have for $f \in C_0^1(M)$,

$$\lim_{\epsilon\to 0} e^{t(A/\epsilon + B/\epsilon^2)} f = e^{tPAB^{-1}AP} Pf = e^{t\Delta/n} Pf$$

in the norm of $C_0(T(M))$, which was to be shown.

5. **Manifolds of negative curvature.** In order to verify (4.2) in some noncompact cases, we assume in addition that $M$ is an analytic simply connected manifold of negative curvature. In this case, the problem was considered by Azencott [1], who used a method based on Hasminskii's test. We will show below that by using a simple observation of Itô [7], (4.2) may be proved by a direct examination of the stochastic equations.

Indeed, in this case $M$ is homeomorphic to its tangent space at some $x_0 \in M$. Taking geodesic polar coordinates at $x$, we have [3]

$$r^2(B_t) = \int_0^t 2r(B_s)\, dW_s + \int_0^t \left[ 2 + 2\frac{\Theta'}{\Theta}(B_s) r(B_s) \right] ds$$

where $\Theta$ is the volume element in these coordinates $\Theta'/\Theta \leqslant (n-1)b \coth br$, where $K \geqslant -b^2$. Now we apply Itô's formula to $f = \log(1 + r^2)$:

$$\log[1 + r^2(B_t)] = \int_0^t \frac{2r(B_s)\,dW_s}{1 + r^2(B_s)} + \int_0^t \frac{2 + 2(\Theta'/\Theta)(B_s)r(B_s)}{1 + r^2(B_s)}\, ds - \int_0^t \frac{4r^2(B_s)}{[1 + r^2(B_s)]^2}\, ds.$$

To estimate the expectation, note that the first term above is an $L^2$-martingale and hence has mean 0. The third term is negative. For the second term, note that $|x \coth x| \leqslant 2 + 2x$. Therefore

$$E\{\log[1 + r^2(B_t)]\} \leqslant \int_0^t \frac{2 + 4(n-1)[1 + br(B_s)]}{1 + r^2(B_s)}\, ds \leqslant [4n - 2 + 2b(n-1)]\, t$$

where we have used the inequality $r/(1 + r^2) \leqslant \frac{1}{2}$.

Proposition 5.1. *Let $M$ be an analytic simply connected Riemann mani-*

*fold with $K \geqslant -b^2$. Then the diffusion semigroup $U_t$ maps $C_0(M)$ into $C_0(M)$.*

PROOF. Let $f \in C_0(M)$: given $\epsilon > 0$, let $K_R$ be a geodesic ball such that $|f(x)| \leqslant \epsilon$ if $x \notin K_R$. Now

$$U_t f(x) = Ef(B_t^x) = \left\{ \int_{K_R^c} + \int_{K_R} \right\} f(y) P_t^x(dy).$$

The first integral is bounded by $\epsilon$, by definition of $K_R$. To estimate the second integral, notice that if $B_t^x \in K_R$, then $d(B_t^x, x) \geqslant d(x, x_0) - R$. Therefore if $d(x, x_0) > R$,

$$\begin{aligned} \int_{K_R^c} P_t^x(dy) &\leqslant P\{d(B_t^x, x) \geqslant d(x, x_0) - R\} \\ &\leqslant P\{\log[1 + d^2(B_t^x, x)] \geqslant \log[1 + (d(x, x_0) - R)^2]\} \\ &\leqslant t(4n - 2 + 2b(n-1))/\log(1 + (d(x, x_0) - R)^2). \end{aligned}$$

Now let $d(x, x_0) \longrightarrow \infty$. Thus $\overline{\lim}\ |T_t f(x)| \leqslant \epsilon$ for each $\epsilon > 0$. Hence $U_t f \in C_0(M)$ which was to be proved.

ADDED IN PROOF. We have just learned of similar approximations of Brownian motion by geodesics in the works of Jorgensen [12] and Malliavin [13]. The latter construction generalizes the classical balayage method of Poincaré.

## REFERENCES


1. R. Azencott, *Behavior of diffusion semigroups at infinity*, Bull. Soc. Math. France 102 (1974), 193–240.
2. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Pure and Appl. Math., vol. 29, Academic Press, New York and London, 1968. MR 41 #9348.
3. A. Debiard, B. Gaveau and E. Mazet, *Temps d'arrêt des diffusions riemanniennes*, C. R. Acad. Sci. Paris Sér. A.-B 278 (1974), A723–A725. MR 49 #6381a.
4. R. Gangolli, *On the construction of certain diffusions on a differentiable manifold*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 (1964), 406–419. MR 29 #2870.
5. R. Goldberg, *Curvature and homology*, Academic Press, New York, 1968.
6. N. J. Hicks, *Notes on differential geometry*, Van Nostrand Math. Studies, no. 3, Van Nostrand, Princeton, N. J., 1965. MR 31 #3936.
7. K. Itô, *Stability of stochastic dynamical systems*, Lecture Notes in Math., vol. 294, Springer-Verlag, New York, 1973, pp. 1–7.
8. T. G. Kurtz, *A limit theorem for perturbed operator semigroups with applications to random evolutions*, J. Functional Analysis 12 (1973), 55–67.
9. ———, *Semigroups of conditioned shifts and approximation of Markov processes*, Ann. Probability 3 (1975), 618–642.
10. D. W. Stroock, *On the growth of stochastic integrals*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 18 (1971), 340–344. MR 44 #4825.
11. S. Watanabe and T. Watanabe, *Convergence of isotropic scattering transport process to Brownian motion*, Nagoya Math. J. 40 (1970), 161–171. MR 43 #5606.
12. E. Jorgensen, *The central limit problem for geodesic random walk*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 (1975), 1–64.
13. P. Malliavin, *Diffusions et géométrie différentielle globale*, Lecture Notes, August 1975, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, Paris 5.


DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201