The Annals of Statistics
1981, Vol. 9, No. 2, 408–417

ISOTROPY AND SPHERICITY: SOME CHARACTERISATIONS OF THE NORMAL DISTRIBUTION

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Main result: $X_1, X_2, \cdots, X_n$ are independent random variables valued in Euclidean spaces $E_1, E_2, \cdots, E_n$ such that $P[X_j = 0] = 0$ for all j. Denote $R = [\sum_{j=1}^n \|X_j\|^2]^{1/2}$. Suppose that $(R^{-1}X_1, R^{-1}X_2, \cdots, R^{-1}X_n)$ is uniformly distributed on the sphere of $\oplus_{j=1}^n E_j$. Then the X_j are normal if $n \geq 3$. The case $n = 2$ and the case of Hilbert spaces are also studied.

1. Definitions and statement of results on probability distributions. In a nonzero, finite-dimensional Euclidean space E with scalar product $\langle \cdot, \cdot \rangle_E$ and norm $\| \cdot \|_E$, $S(E)$ is the sphere with radius 1. We consider a random variable X valued in E with distribution μ. Recall that μ is completely determined by its characteristic function $\int_E \exp(i\langle t, x \rangle_E)\mu(dx) = \hat{\mu}(t)$ defined for t in E, and if X is valued in $(0, +\infty)$, μ is completely determined by the characteristic function of the distribution of log X, which is $\int_0^\infty x^{it}\mu(dx)$, defined for t in the real line $\mathbb{R}$.

Definition 1.1. The *normal distribution* $\nu_{E,a}$ on E with variance $a \geq 0$ is defined by:

$$\hat{\nu}_{E,a}(t) = \exp(-a\|t\|_E^2/2).$$

The *Cauchy distribution* γ_E on E is defined by $\hat{\gamma}_E(t) = \exp(-\|t\|_E)$. *The uniform distribution* σ_E on $S(E)$ is defined as the distribution of $X/\|X\|_E$, where the distribution of X is $\nu_{E,1}$.

Definition 1.2. The random variable X in E, or its distribution μ, will be said to be *spherical* in E if the distribution of $\langle \alpha, X \rangle_E$ does not depend on α, when α lies on the unit sphere $S(E)$. It will be said to be *isotropic* in E if $\mu(\{0\}) = 0$ and if the distribution of $X/\|X\|_E$ is σ_E. It will be said to be *infinitely-spherical* in E if there exists a probability distribution ρ on $[0, +\infty)$ such that

$$\hat{\mu}(t) = \int_0^\infty \exp(-a\|t\|_E^2/2)\rho(da).$$

The adjective "infinitely-spherical" alludes to the fact that such a distribution is, for any Euclidean space F bigger than E, the orthogonal projection onto E of some spherical distribution on F. We shall give in Proposition 4.1 an elementary proof of this, well known as "Schoenberg's theorem." Note that the sphericity implies isotropy if $P[X = 0] = 0$, and that in dimension 1, sphericity is symmetry, isotropy is $P[X < 0] = P[X > 0] = ½$.

When we consider several Euclidean spaces $E_1, E_2, \cdots, E_n$ then $\oplus_{j=1}^n E_j$ denotes the direct orthogonal sum, and is Euclidean. If all E_j are equal to the same E, we denote $\oplus_{j=1}^n E_j = E^n$: so, $\mathbb{R}^n$ has its natural Euclidean structure. We shall prove the following theorems:

Theorem 1.1. *Let X_1 and X_2 be two independent random variables valued in nonzero finite dimensional Euclidean spaces E_1 and E_2. Then $X = (X_1, X_2)$ is spherical in $E =$*

Received July 1978; revised January 1980.
AMS (1970) *subject classifications.* Primary 62E10; secondary 60B15.
Key words and phrases. Normal distribution, Cauchy distribution, cylindrical-distribution.

$E_1 \oplus E_2$ if and only if there exists $a \geq 0$ such that the distribution of X is the normal distribution $\nu_{E,a}$.

THEOREM 1.2. *Let X_1 and X_2 be two independent random variables valued in nonzero finite dimensional Euclidean spaces E_1 and E_2. Then the following properties are equivalent:*

(i) *$X = (X_1, X_2)$ is isotropic in $E = E_1 \oplus E_2$.*

(ii) *X_1 and X_2 are spherical in E_1 and E_2, $P[X_1 = 0] = P[X_2 = 0] = 0$, and the distribution of $X_1/\langle \alpha_2, X_2\rangle_{E_2}$ is γ_{E_1} for all α_2 in $S(E_2)$.*

(iii) *X_1 and X_2 are spherical in E_1 and E_2, $P[X_1 = 0] = P[X_2 = 0] = 0$, and if $d_1 = \dim E_1$ and $d_2 = \dim E_2$:*

$$\mathbb{E}[\|X_1\|_{E_1}^{it}]\mathbb{E}[\|X_2\|_{E_2}^{-it} = \prod_{k=0}^{\infty}\left[1 + \frac{it}{2k + d_1}\right]^{-1}\left[1 - \frac{it}{2k + d_2}\right]^{-1} \tag{1.1}$$

for all real t.

Furthermore, if X_1 and X_2 are infinitely spherical and (i) *is true, there exists $a > 0$ such that the distribution of X is the normal distribution $\nu_{E,a}$.*

THEOREM 1.3. *Let X_1, X_2 and X_3 be three independent random variables valued in nonzero finite dimensional Euclidean spaces E_1, E_2 and E_3. Then $X = (X_1, X_2, X_3)$ is isotropic in $E = E_1 \oplus E_2 \oplus E_3$ if and only if there exists $a > 0$ such that the distribution of X is the normal distribution $\nu_{E,a}$.*

THEOREM 1.4. *Let X_1 and X_2 be two independent random variables valued in a nonzero finite dimensional Euclidean space E, with the same distribution μ. Suppose that*

$$\mu\{x; \langle \alpha, x\rangle_E = 0\} = 0 \text{ for all } \alpha \text{ in } S(E).$$

Then the following properties are equivalent:

(i) *$X_1/\langle \alpha, X_2\rangle_E$ is spherical for all α in $S(E)$.*

(ii) *$X_1\langle \alpha, X_2\rangle_E$ is spherical for all α in $S(E)$.*

(iii) *μ is spherical.*

Furthermore, (X_1, X_2) is isotropic in E^2 if and only if the distribution of $X_1/\langle \alpha, X_2\rangle_E$ is γ_E for all α in $S(E)$.

THEOREM 1.5. *Let X_1', X_1'', X_2', X_2'' be four independent random variables, where X_1' and X_2' are valued in E' with the same distribution, X_1'' and X_2'' are valued in E'' with the same distribution, and E' and E'' are nonzero-finite dimensional Euclidean spaces. Denote $E = E' \oplus E''$; suppose that*

$$P[\langle \alpha', X_1'\rangle_{E'} + \langle \alpha'', X_1''\rangle_{E''} = 0] = 0$$

for all (α', α'') in $S(E)$, and let $X_1 = (X_1', X_1'')$, $X_2 = (X_2', X_2'')$. Then the following properties are equivalent:

(i) *$X_1/\langle \alpha, X_2\rangle_E$ is spherical in E for all α in $S(E)$.*

(ii) *$X_1\langle \alpha, X_2\rangle_E$ is spherical in E for all α in $S(E)$.*

(iii) *There exists $a > 0$ such that the distribution of X_1 and X_2 is the normal distribution $\nu_{E,a}$.*

2. Definitions and statement of the result on cylindrical-distributions. For an infinite-dimensional Hilbert space E with scalar product $\langle \cdot, \cdot\rangle_E$ and norm $\|\cdot\|_E$, denote by $\mathscr{F}(E)$ the set of finite-dimensional linear subspaces of E. If $V \supset W$ and if V and W are in $\mathscr{F}(E)$, let p_{VW} be the orthogonal projection from V to W.

DEFINITION 2.1. A cylindrical-distribution μ on E is a set $\mu = (\mu_V;\ V \in \mathscr{F}(E))$ of probability distributions μ_V on V such that the image of μ_V by p_{VW} is μ_W when $V \supset W$.

DEFINITION 2.2. The normal cylindrical-distribution on E with variance $a \geq 0$ is defined by $(\nu_{V,a};\ V \in \mathscr{F}(E))$.

DEFINITION 2.3. The cylindrical-distribution μ on E will be said to be *spherical* if μ_V is spherical on V for all V in $\mathscr{F}(E)$. It will be said to be isotropic if μ_V is isotropic on V for all V in $\mathscr{F}(E)$.

Here is a characterisation of normal cylindrical-distributions:

THEOREM 2.1. *Let μ_1 and μ_2 be two cylindrical-distributions on two infinite-dimensional Hilbert spaces E_1 and E_2. Then $\mu = \mu_1 \otimes \mu_2$ is isotropic on the direct orthogonal sum $E_1 \oplus E_2$ if and only if there exists $a > 0$ such that μ is normal with variance a.*

3. Comments. This paper arises from a question raised by Professors J. L. Philoche and M. Keane (Rennes), which was: "Is The Theorem 1.3 true for $E_1 = E_2 = E_3 = \mathbb{R}$ and X_1, X_2, X_3 with the same distribution?" Professor J. L. Philoche wrote an interesting paper (mainly expository) [10] on isotropy and sphericity: the proofs of Propositions 3.1, 3.2 and 3.3 below can be found in [10].

PROPOSITION 3.1. *Let E be a finite dimensional Euclidean space, V a nonzero linear subspace of V, and p_V the orthogonal projection from E to V. If X is a spherical* (resp. *isotropic*) *random variable on E, $p_V(X)$ is spherical* (resp. *isotropic*) *on V. In particular $P[\langle \alpha, X \rangle_E = 0] = 0$ if X is isotropic in E and α is in $S(E)$.*

This proposition enables us to amplify in a trivial manner our theorems: for instance, Theorem 1.3 remains true if we use n random variables ($n \geq 3$) instead of three.

PROPOSITION 3.2. *Let X be a random variable valued in a finite dimensional Euclidean space E such that $P[X = 0] = 0$. Then X is spherical if and only if X is isotropic and $X/\|X\|_E$ and $\|X\|_E$ are independent.*

The next proposition is classical and is one of the simplest characterisations of the normal distribution:

PROPOSITION 3.3. *Let X be a real random variable such that for all real θ and t:*

$$\mathbb{E}[\exp(itX)] = \mathbb{E}[\exp(itX\cos\theta)\mathbb{E}[\exp(itX\sin\theta)].$$

There then exists $a \geq 0$ such that X is normal with variance a.

Let us make some comments on theorems of Section 1. Theorem 1.1 is well known as "Maxwell's theorem" (see [4] page 187, Section 3b). We state it here for reference; its proof is typical of our methods of proof. Theorem 1.2 is the main theorem of the paper: compared with Theorem 1.1 it shows that isotropy contrasts strongly with sphericity for two independent random variables. The last part of Theorem 1.4 for $E = \mathbb{R}$ is well known and there exists numerous explicit examples of nonnormal distributions μ on the real line such that if X_1 and X_2 are independent with the same distribution μ, then X_1/X_2 is Cauchy distributed; a nice one is $\mu(dx) = \sqrt{2}[\pi(1 + x^4)]^{-1}\,dx$. A more obvious example is the distribution μ of $1/X$ where the real random variable X has a normal distribution.

Bibliographical data on this subject can be found in the monograph by E. Lukacs and R. G. Laha [9]. More generally, if we consider part (iii) of Theorem 1.2, we see that there are a lot of ways to write the second member of (1.1) as the product of two characteristic functions, hence to find independent random variables X_1 and X_2 such that (X_1, X_2) is isotropic; it would be difficult to classify them even with the further restriction of Theorem 1.4 that X_1 and X_2 have the same distribution.

A nice application of Theorem 1.3 to functions of real variables is the following: suppose that f_1, f_2 and f_3 are positive integrable functions on $\mathbb{R}$ such that the function

$$F(x_1, x_2, x_3) = \int_0^\infty f_1(\rho x_1) f_2(\rho x_2) f_3(\rho x_3) \rho^2 \, d\rho$$

is a constant on $S(\mathbb{R}^3)$, then there exists four positive constants A_1, A_2, A_3 and B such that $f_j(x) = A_j \exp(-Bx^2)$, for $j = 1, 2, 3$.

Theorem 1.3 is actually a simple corollary of Theorem 1.2. It is not completely new: for $E_1 = E_2 = E_3 = \mathbb{R}$, an equivalent result is proved in [6] and [7] with the further hypothesis of symmetry for X_1, X_2, X_3. Let us quote also a companion result, found by A. A. Zinger [14] if $X_1, X_2, \cdots, X_n$ are independent and identically distributed real random variables, denote $\bar{X} = (X_1 + \cdots + X_n)/n$; consider the subspace E of $\mathbb{R}^n$ defined by $E = \{(x_1, \cdots, x_n); x_1 + \cdots + x_n = 0\}$. Then if $n \geq 6$, $(X_1 - \bar{X}, \cdots, X_n - \bar{X})$ isotropic in E implies that X_1 is normal (with mean not necessarily zero). I am indebted to Professor E. Lukacs for the reference [14].

A cylindrical-distribution (called in French: "promesure de masse 1") is not necessarily the set of projections of some probability distribution on the Hilbert space. For a discussion of this problem, a motivation of the definition and a historical perspective, Bourbaki [3] can be consulted. He uses them to give a short and beautiful introduction to Brownian motion. Note that Bourbaki calls $\mu_{V^\perp}$ what we call μ_V: we took advantage of the fact that we restricted ourselves to Hilbert spaces. For an application of Theorem 2.1, we consider the Hilbert space $E = L^2[0, 1]$ of real functions which are square-integrable with respect to Lebesgue measure on [0, 1], the space $\mathscr{C}$ of real continuous functions f on [0, 1] such that $f(0) = 0$, with sup-norm, and the continuous linear $P: E \to \mathscr{C}$ defined by:

$$(Pf)(t) = \int_0^t f(x) \, dx.$$

The Wiener theorem (see [3], page 83) says that if μ is the normal cylindrical distribution on E with variance 1, the image of μ by P on $\mathscr{C}$ is the Wiener probability distribution on $\mathscr{C}$. Using that theorem, Theorem 2.1 implies that if μ_1 and μ_2 are cylindrical-distributions on E such that $\mu_1 \otimes \mu_2$ is isotropic in E^2, the image of $\mu_1 \otimes \mu_2$ by the map: $P_2: E^2 \to \mathscr{C}^2$ defined by

$$(f_1, f_2) \mapsto \left(\int_0^t f_1(x) \, dx, \int_0^t f_2(x) \, dx \right)$$

is the Wiener probability distribution for the two dimensional Brownian motion on [0, 1] (with some normalisation, since the variance is not necessarily 1).

4. A further look to sphericity. Let us comment now on the notion of infinite sphericity as used in Definition 1.2. Actually, there are three related concepts:

(i) The infinite sphericity in a finite dimensional space.

(ii) The sphericity of a cylindrical-distribution in infinite Hilbert space.

(iii) The sphericity of a distribution on a sequence space.

We characterise these situations in the next three propositions: all the results of this section are more or less known.

Concerning the first concept, denote by $\mathscr{S}_n$ the set of spherical distributions on the Euclidean space $\mathbb{R}^n$ and by $\mathscr{S}_{n,k}$ the set of images of distributions of $\mathscr{S}_n$ on $\mathbb{R}^k$ by the natural projection $\mathbb{R}^n \to \mathbb{R}^k$ if $k \le n$. Obviously $\mathscr{S}_{n,k} \supset \mathscr{S}_{n+1,k}$. The following proposition explains the term "infinitely-spherical". Its proof is due to I. J. Schoenberg [13] and can also be found in N. I. Achieser ([1], page 200).

PROPOSITION 4.1. *Let k be a positive integer. The distribution μ belongs to $\bigcap_{n\ge k} \mathscr{S}_{n,k}$ if and only if μ is infinitely-spherical.*

Proofs of this proposition in [1] and [13] use Bessel functions. Let us give an elementary proof using only Levy's theorem on continuity of characteristic functions and the weak law of large numbers.

PROOF OF PROPOSITION 4.1. The "if" part being obvious, we concentrate on the converse. We consider the pre-Hilbertian space E of sequences of real numbers $x = (x_1, x_2, \ldots, x_n, \ldots)$ such that $x_j \ne 0$ only for a finite number of j, with the scalar product $\langle x, y\rangle = \sum_{j=1}^{\infty} x_j y_j$. The subspace $\{x \in E;\ x_j = 0 \text{ for } j > n\}$ is simply denoted by $\mathbb{R}^n$ and for $k \le n$, $p_{n,k}$ is the canonical projection $\mathbb{R}^n \to \mathbb{R}^k$; $\|\cdot\|_n$ is the norm in $\mathbb{R}^n$ and $\nu_{n,1}$ is the normal distribution in $\mathbb{R}^n$ with

$$\hat{\nu}_{n,1}(t) = \exp(-\|t\|_n^2/2) \quad \text{for} \quad t \quad \text{in } \mathbb{R}^n.$$

Let us denote by $\mathscr{L}(X)$ the distribution of a random variable X; if μ and μ_n are probability distributions on a finite dimensional vector space V, $\mu_n \to \mu$ as $n \to \infty$, means weak convergence (that is $\int_V f\, d\mu_n \to \int_V f\, d\mu$, as $n \to \infty$, for all bounded continuous functions on V).

The hypothesis is the following: for each $n \ge k$ there exists a spherical random variable X_n on $\mathbb{R}^n$ such that

$$\mathscr{L}[p_{nk}X_n] = \mathscr{L}[X_k] = \mu \quad \text{for} \quad n \ge k. \tag{4.1}$$

Note that the X_n are *not* defined on the same probability space and we have not $p_{nk}X_n = X_k$. Without lost of generality we may suppose $P[X_k = 0] = 0$; it is easy to come back afterward to the case where this is not true. Define $\theta_n = X_n/\|X_n\|_n$. Now:

$$\mathscr{L}[\sqrt{n}\,p_{nk}\theta_n] \to \nu_{k,1} \quad \text{as} \quad n \to \infty. \tag{4.2}$$

This fact is known as "Poincaré's lemma (see [11]); its proof is easy: consider a sequence $(Y_j)_{j=1}^{\infty}$ of independent real random variables, with normal distribution $\nu_{\mathbb{R},1}$, and $R_n = [Y_1^2 + \cdots + Y_n^2]^{1/2}$. So

$$\mathscr{L}[\sqrt{n}\,p_{nk}\theta_n] = \mathscr{L}\left[\frac{\sqrt{n}}{R_n}(Y_1, Y_2, \ldots, Y_k)\right].$$

But $R_n^2/n \to 1$ as $\to \infty$, in probability from the law of large numbers and this proves (4.2). Denote now for real t and for $n \ge k$:

$$\alpha_n(t) = \mathbb{E}[(n^{-1/2}\|X_n\|_n)^{it}],\ \beta_n(t) = \mathbb{E}[(n^{1/2}\|p_{nk}\theta_n\|_k)^{it}]$$

and $\gamma(t) = \mathbb{E}\,[\|X_k\|_k^{it}]$. Then (4.1) gives $\alpha_n(t)\beta_n(t) = \gamma(t)$. But:

$$\beta_n(t) \to 2^{it/2}\Gamma\left(\frac{it}{2} + \frac{k}{2}\right)\Big/\Gamma\left(\frac{k}{2}\right) = \beta(t), \quad \text{as } n \to \infty.$$

Hence, from Levy's theorem (see, for instance, [4], Th.2, page 481), γ/β is the characteristic function of some real random variable ξ. The distribution of $\exp \xi$ being denoted by ρ on $(0, +\infty)$ we get:

$$\alpha_n(t) \to \alpha(t) = \int_0^{\infty} a^{it/2}\rho(da), \quad \text{as } n \to \infty.$$

The fact that $\gamma(t) = \alpha(t)\beta(t)$ implies now that X_k is infinitely spherical, which is the desired result.

The above proposition implies now strong restrictions on spherical cylindrical-distributions:

PROPOSITION 4.2. *Let $\mu = (\mu_V\colon V \in \mathscr{F}(E))$ a spherical cylindrical-distribution on the infinite dimensional Hilbert space E. Then there exists a distribution ρ on $[0, +\infty)$ such that*

$$\hat{\mu}_V(t) = \int_0^\infty \exp(-a\|t\|_V^2/2)\rho(da) \text{ for } t \text{ in } V.$$

Furthermore μ is not *a distribution E (except in the trivial case $\rho(\{0\}) = 1$).*

PROOF. An obvious consequence from Proposition 4.1. is that μ_V is infinitely spherical for any V in $\mathscr{F}(E)$. To verify that the corresponding measure ρV actually does not depend on V, consider V_1 and V_2 in $\mathscr{F}(E)$; we get for t in V_1:

$$\int_0^{+\infty} \exp\left(-a\frac{\|t\|_{V_1}^2}{2}\right)\rho_{V_1}(da) = \hat{\mu}_{V_1}(t) = \hat{\mu}_{V_1+V_2}(t) = \int_0^\infty \exp\left(-a\frac{\|t\|_{V_1}^2}{2}\right)\rho_{V_1+V_2}(da)$$

since $\|t\|_{V_1}^2 = \|t\|_{V_1+V_2}^2$. Hence ρ_{V_1} and $\rho_{V_1+V_2}$ have the same Laplace transform and are equal. Symmetry gives $\rho_{V_2} = \rho_{V_1+V_2}$ and proves the first part.

To see that μ is not a probability distribution, suppose that there exists a random variable X valued in E such that the orthogonal projection X_V on the finite dimensional V of X is μ_V distributed. Since $\|X_V\|_V \le \|X\|_E$, we get for positive x:

$$P[\|X\|^2 < x] \le P[\|X_V\|_V^2 < x] \quad \text{for all } V \text{ in } \mathscr{F}(E)).$$

But clearly, $\|X\|_V^2$ is the product of two independent random variables: the first one is ρ distributed, the second is χ^2 distributed with parameter $n = \dim V$. So, we get $P[\|X\|^2 < x] = 0$ for all $x > 0$, a contradiction.

For simplicity, we state the last result on the space $\mathbb{R}^N$ of sequence of real numbers, and not on E^N where E is an Euclidean space:

PROPOSITION 4.3. *Let μ be a probability distribution on the space $\mathbb{R}^N$ of real sequences $X = (X_0, X_1, \ldots, X_n), \ldots)$ equipped with the usual σ-field. Suppose that $(X_0, X_1, \ldots, X_n)$ is spherical for each integer n. Then there exists a probability measure ρ on $[0, +\infty)$ such that μ is the distribution of $\sqrt{V}Y_0, \sqrt{V}Y_1, \sqrt{V}Y_2, \ldots)$ where $V, Y_0, Y_1, \ldots$ are independent random variables V, being ρ distributed and Y_n with normal distribution $\nu_{\mathbb{R},1}$.*

A proof of this is given in [5]. Generalizations, replacing sphericity by isotropy *and* stationarity can be found in [2] and [8].

5. Proof of Theorem 1.1. We prove it first for $E_1 = E_2 = \mathbb{R}$. Let $\varphi_j(t) = \mathbb{E}[\exp(itX_j)]$ $j = 1, 2$. Since the distribution of $X_1 \cos\theta + X_2 \sin\theta$ does not depend on θ in $\mathbb{R}$, then $\varphi_1(t\cos\theta)\varphi_2(t\sin\theta)$ does not depend on θ. Taking $\theta = 0$ and $\theta = \pi/2$, we get $\varphi_1 = \varphi_2$, and then $\varphi_1(t\cos\theta)\varphi_1(t\sin\theta) = \varphi_1(t)$ for all real t and θ. Proposition 3.3 gives the result.

For the general case, we take α_j in $S(E_j)$ $j = 1, 2$; Then for real θ, $(\alpha_1\cos\theta, \alpha_2\sin\theta)$ is in $S(E)$. So from the one dimensional case $\langle\alpha_1, X_1\rangle_{E_1}$ and $\langle\alpha_2, X_2\rangle_{E_2}$ are normal with the same variance and the result follows.

6. Proofs of Theorems 1.2 and 1.3. Let us explain first how Theorem 1.3. is a simple corollary of Theorem 1.2: consider $X_1' = (X_2, X_3)$. Since (X_1, X_1') is isotropic, X_1' is spherical (Theorem 1.2.) with $P[X'_1 = 0] = 0$. Since X_2 and X_3 are independent, Theorem 1.1. shows that X'_1 is normal in $E_2 \oplus E_3$ with variance $a > 0$. The same reasoning shows that (X_1, X_3) is normal in $E_1 \oplus E_3$ with the *same* variance a, and the result is proved. The "only if" part is trivial. Now we embark upon a proof of Theorem 1.2.

i $\Rightarrow$ ii. We prove it first for $E_1 = E_2 = \mathbb{R}$. Denote by μ and ν the distributions of X_1 and X_2. The measures μ^+ and ν^+ (resp. μ^- and ν^-) are the restrictions of the distributions of X_1 and X_2 (resp. of $-X_1$ and $-X_2$) to $(0, +\infty)$. Hypothesis (i) and Proposition 3.1. imply $\mu(\{0\}) = \mu(\{0\}) = 0$ and

$$P[X_1 > 0] = P[X_1 < 0] = P[X_2 > 0] = P[X_2 < 0] = 1/2.$$

But for real t:

$$\frac{2}{\pi}\int_0^{\pi/2} (\tan\theta)^{it}\, d\theta = \frac{2}{\pi}\int_0^{\infty} q^{it}(1+q^2)^{-1}\, dq = \left(\cosh\frac{\pi t}{2}\right)^{-1}. \tag{6.1}$$

Thus for all ϵ and η in $\{-, +\}$ and for real t:

$$\int_0^\infty\int_0^\infty x_1^{it}x_2^{-it}\mu^\epsilon(dx_1)\nu^\eta(dx_2) = \left(4\cosh\frac{\pi t}{2}\right)^{-1}.$$

Since $(\cosh(\pi t/2)^{-1}$ is never zero, we get for all t:

$$\int_0^\infty x_2^{-it}\nu^\eta(dx_2) \neq 0 \quad\text{and}\quad \int_0^\infty x_1^{it}\mu^+(dx_1) = \int_0^\infty x_1^{it}\mu^-(dx_1).$$

This implies $\mu^+ = \mu^-$. In the same way $\nu^+ = \nu^-$ and symmetry (= sphericity) of μ and ν is proved. By (6.1) X_1/X_2 is Cauchy distributed.

Now we prove (i) $\Rightarrow$ (ii) for general E_1 and E_2. For α_1 in $S(E_1)$ and α_2 in $S(E_2)$, the random variable $(\langle\alpha_1, X_1\rangle_{E_1}, \langle\alpha_2, X_2\rangle_{E_2})$ is isotropic in $\mathbb{R}^2$. Denote for real t:

$$\varphi_{\alpha_1}(t) = \mathbb{E}\,[|\langle\alpha_1, X_1\rangle_{E_1}|^{it}] \quad\text{and}\quad \psi_{\alpha_2}(t) = \mathbb{E}\,[|\langle\alpha_2, X_2\rangle_{E_2}|^{-it}].$$

From the one-dimensional case and (6.1) we get for real t;

$$\varphi_{\alpha_1}(t)\psi_{\alpha_2}(-t) = \left(\cosh\frac{\pi t}{2}\right)^{-1}. \tag{6.2}$$

Hence $\varphi_{\alpha_1}(t)$ and $\psi_{\alpha_2}(t)$ are independent of α_1 and α_2 respectively. From the one-dimensional case again, $\langle\alpha_1, X_1\rangle_{E_1}$ and $\langle\alpha_2, X_2\rangle_{E_2}$ are symmetric, so their distributions are independent of α_1 in $S(E_1)$ and α_2 in $S(E_2)$. The remainder of (ii) follows from (6.2).

(ii) $\Rightarrow$ (iii). Let $\theta_j = X_j/\|X_j\|_{E_j}, j = 1, 2$. Proposition 3.2. implies that θ_1, θ_2, $\|X_1\|_{E_1}$ and $\|X_2\|_{E_2}$ are independent. Let $d_j = \dim E_j$, and for all α_j in $S(E_j)$ and real t:

$$\varphi_{d_j}(t) = \mathbb{E}\,[|\langle\alpha_j, \theta_j\rangle|^{it}] \qquad j = 1, 2.$$

Obviously $\varphi_{d_j}(t)$ does not depend on α_j. Using (6.2) we get for real t:

$$\varphi_{d_1}(t)\varphi_{d_2}(-t)\;\mathbb{E}\,[\|X_1\|_{E_1}^{it}]\;\mathbb{E}\,[|X_2\|_{E_2}^{-it}] = \left(\cosh\frac{\pi t}{2}\right)^{-1}. \tag{6.3}$$

In order to compute $\varphi_{d_j}(t), j = 1, 2$, we consider independent random variables Y_1 and Y_2 such that their distributions are $\nu_{E_1,1}$ and $\nu_{E_2,1}$; the distributions σ_{E_j} of θ_j are the same as $Y_j/\|Y_j\|_{E_j}$. Since $\|Y_j\|_{E_j}^2$ is χ^2 distributed with d_j degrees of freedom, if we replace (X_1, X_2)

by (Y_1, Y_2) in (6.3) we get:

$$\varphi_{d_1}(t)\varphi_{d_2}(t)\Gamma\left(\frac{d_1}{2}+\frac{it}{2}\right)\Gamma\left(\frac{d_2}{2}-\frac{it}{2}\right)\left[\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2}{2}\right)\right]^{-1} = \left(\cosh\frac{\pi t}{2}\right)^{-1}.$$

Comparing with (5.3):

$$\text{(6.4)}\qquad \mathbb{E}[\|X_1\|_{E_1}^{it}\ \mathbb{E}\,[\|X_2\|_{E_2}^{-it}] = \Gamma\left(\frac{d_1}{2}+\frac{it}{2}\right)\Gamma\left(\frac{d_2}{2}-\frac{it}{2}\right)\left[\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2}{2}\right)\right]^{-1}.$$

We can now use the product decomposition of gamma function (see, for instance, Sansone and Gerretsen [12], page 188):

$$[\Gamma(z)]^{-1} = e^{\gamma z}z\prod_{k=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$

This formula and (6.4) give (iii).

(iii) $\Rightarrow$ (i). The preceeding proof shows that $Q = \|X_1\|_{E_1}/\|X_2\|_{E_2}$ and $Q' = \|Y_1\|_{E_1}/\|Y_2\|_{E_2}$ have the same distribution when the distribution of (Y_1, Y_2) is $\nu_{E,1}$. Denote:

$$\|X\|_E = [\|X_1\|_{E_1}^2 + \|X_2\|_{E_2}^2]^{1/2} \quad\text{and}\quad \theta = X/\|X\|_E.$$

We get, keeping the notation θ_1 and θ_2 as above,:

$$\begin{aligned}\theta &= (\theta_1\|X_1\|_{E_1}/\|X\|_E,\ \theta_2\|X_2\|_{E_2}/\|X\|_E)\\ &= (\theta_1[Q^2+1]^{-1/2},\ \theta_2 Q[Q^2+1]^{-1/2}).\end{aligned}$$

Sphericity of X_1 and X_2 implies independence of θ_1, θ_2 and Q, and θ has the same distribution as

$$(\theta_1[Q'^2+1]^{-1/2},\ \theta_2 Q'[Q'^2+1]^{-1/2})$$

because we may suppose X_1, X_2, Y_1, Y_2 independent. Since the distributions of θ_1 and θ_2 are σ_{E_1} and σ_{E_2} the distribution of θ is σ_E.

Last part. We suppose now that (X_1, X_2) is isotropic in E and that there exist two distributions ρ_1 and ρ_2 on $(0, +\infty)$ such that for all α_j in $S(E_j)$ and real t:

$$\mathbb{E}\,[\exp(it\,\langle\alpha_j, X_j\rangle_{E_j})] = \int_0^\infty \exp\left(-t^2\frac{a}{2}\right)\rho_j(a). \qquad j=1,2.$$

This implies that for real t and $j = 1, 2$:

$$\text{(6.5)}\qquad \begin{aligned}\mathbb{E}\,[|\langle\alpha_j, X_j\rangle_{E_j}|^{it}] &= \int_0^{+\infty}\rho_j(da)\int_{-\infty}^{+\infty}|x|^{it}\exp\left(-\frac{x^2}{2a}\right)\frac{dx}{\sqrt{2\pi a}}\\ &= \frac{2^{it/2}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\frac{it}{2}\right)\int_0^\infty a^{it/2}\rho_j(da).\end{aligned}$$

Since (i) $\Leftrightarrow$ (ii), (6.2) implies for real t:

$$\begin{aligned}\mathbb{E}\,[|\langle\alpha_1, X_1\rangle_{E_1}|^{it}]\ \mathbb{E}\,[|\langle\alpha_2, X_2\rangle_{E_2}|^{-it}] &= \left(\operatorname{ch}\frac{\pi t}{2}\right)^{-1}\\ &= \frac{1}{\pi}\Gamma\left(\frac{1}{2}+\frac{it}{2}\right)\Gamma\left(\frac{1}{2}-\frac{it}{2}\right).\end{aligned}$$

This equality and (6.5) give for real t:

$$\int_0^\infty a^{it/2}\rho_1(da)\cdot\int_0^\infty a^{-(it/2)}\rho_2(da) = 1.$$

Standard reasoning shows that such equality implies that $\rho_1 = \rho_2 =$ a Dirac mass on some point $a > 0$, and this concludes the proof of Theorem 1.2.

7. Proofs of Theorems 1.4 and 1.5. Theorem 1.5 is a simple corollary of Theorem 1.4, which shows that $X_1 = (X_1', X_1'')$ is spherical under hypothesis (i) or (ii). Since X_1' and X_1'' are independent, we can use Theorem 1.1 and we get (iii). Converse part (iii) $\Rightarrow$ (i) and (ii) is trivial.

Now we prove Theorem 1.4. (iii) $\Rightarrow$ (i) and (ii) is obvious. We show that (i) or (ii) implies (iii). We prove it first for dim $E = 1$, so we have to show that if X_1/X_2 or X_1X_2 is symmetric, then μ is symmetric. We consider for this the homomorphism h of the multiplicative group $\mathbb{R} \setminus \{0\}$ to the multiplicative group of complex numbers of modulus 1 defined by $h(x) = |x|^{it}$ sign x, for fixed real t. Let $\psi(t) = \mathbb{E}[h(X_1)]$. Note that μ is symmetric if and only if $\psi(t) = 0$ for all t. But

$$\mathbb{E}[h(X_1/X_2)] = |\psi(t)|^2$$
$$\mathbb{E}[h(X_1X_2)] = (\psi(t))^2.$$

So X_1/X_2 or X_1X_2 symmetric imply $\psi(t) = 0$ for all t.

We consider now the case dim $E > 1$. Let $\varphi_\alpha(t) = \mathbb{E}[|\langle \alpha, X_1\rangle_E|^{it}]$ if α is in $S(E)$ and t real. We separate the cases (i) and (ii). Suppose (i). Then $\varphi_{\alpha_1}(t)\varphi_\alpha(-t)$ is independent of α_1, so for all α_1 and α in $S(E)$ and real t:

$$\varphi_{\alpha_1}(t)\varphi_\alpha(-t) = \varphi_\alpha(t)\varphi_\alpha(-t). \tag{7.1}$$

This implies $\varphi_{\alpha_1}(t) = \varphi_\alpha(t)$ if $\varphi_\alpha(-t) \neq 0$. Suppose that $\varphi_{\alpha_1}(t) \neq 0$ and $\underline{\varphi_\alpha(t)} = 0$, we get a contradiction if we exchange (α, α_1) and $(t, -t)$ in (7.1), since $\varphi_\alpha(t) = \overline{\varphi_\alpha(-t)}$. Hence $\varphi_\alpha(t)$ does not depend on α in $S(E)$. From the one dimensional part of the proof applied to $\langle\alpha_1, X_1\rangle_E$ and $\langle\alpha, X_2\rangle_E$, we get $\langle\alpha_1, X_1\rangle_E$ symmetric. Since the distribution of $|\langle\alpha_1, X_1\rangle_E|$ does not depend on α_1, μ is spherical.

The proof of (ii) $\Rightarrow$ (iii) goes the same way and starts from

$$\varphi_{\alpha_1}(t)\varphi_\alpha(t) = \varphi_\alpha(t)\varphi_\alpha(t).$$

The proof of the last part of Theorem 1.4 is immediate, using the equivalence (i) $\Leftrightarrow$ (iii), the fact that γ_E is spherical and Theorem 1.2.

8. Proof of Theorem 2.1. Immediate, using Proposition 4.2. and the last part of Theorem 1.2.

REFERENCES

[1] ACHIESER, N. I. (1965). *The Classical Moment Problem.* Oliver and Boyd, Edinburgh, London.
[2] BERMAN, S. (1980). Stationarity, isotropy and sphericity in l_p. *Z. Wahrsbeinlichkeitstheorie und verw. Gebiete* **54** 21–23.
[3] BOURBAKI, N. (1969). *Eléments de Mathématique, Intégration.* Chapitre IX. Hermann, Paris.
[4] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications, Vol. II, 1st Ed.* Wiley, New York.
[5] KINGMAN, J. F. C. (1972). On random sequences with spherical symmetry. *Biometrika* **59** 492–494.
[6] KOTLARSKI, I. I. (1966). On characterizing the normal distribution by Student's law. *Biometrika* **53** 603–606.
[7] KOTLARSKI, I. I. (1967). On characterizing the gamma and the normal distribution. *Pacific J. Math.* **20** 69–76.

[8] LETAC, G. et MILHAUD, X. (1979). Une suite stationnaire et isotrope est sphérique. Z. *Wahrsheinlichkeitstheorie und verw. Gebiete* **49** 33–36.
[9] LUKACS, E. and LAHA, R. G. (1964). *Applications of Characteristic Functions.* Griffin's statistical monographs and courses, Griffin, London.
[10] PHILOCHE, J. L. (1977). Une condition de validité pour le test F. *Bull. Assoc. Statist. Univ* **1** 37–59.
[11] POINCARÉ, H. (1912). *Calcul des Probabilités.* Hermann, Paris.
[12] SANSONE, G. et GERRETSEN J. (1960). *Lectures on the Theory of Functions of a Complex Variable, Vol. 1.* P. Noordhoof, Grönigen.
[13] SCHOENBERG, I. J. (1938). Metric spaces and completely monotone functions. *Ann. Math.* **39** 811–841.
[14] ZINGER, A. A. (1956). On a problem of A. N. Kolmogorov. *Vestnik Leningrad. Univ.* **1** 53–56. (In Russian).

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