Noname manuscript No.
(will be inserted by the editor)

ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws

Christophe Prieur · Frédéric Mazenc

Received: date / Accepted: date

Abstract A family of time-varying hyperbolic systems of balance laws is considered. The partial differential equations of this family can be stabilized by selecting suitable boundary conditions. For the stabilized systems, the classical technique of construction of Lyapunov functions provides a function which is a weak Lyapunov function in some cases, but is not in others. We transform this function through a strictification approach to obtain a time-varying strict Lyapunov function. It allows us to establish asymptotic stability in the general case and a robustness property with respect to additive disturbances of Input-to-State Stability (ISS) type. Two examples illustrate the results.

Keywords Strictification; Lyapunov function; hyperbolic PDE; system of balance laws

1 Introduction

Lyapunov function based techniques are central in the study of dynamical systems. This is especially true for those having an infinite number of dynamics. These systems are usually modelled by time-delay systems or partial differential equations (PDEs). For the latter family of systems, Lyapunov functions are useful for the analysis of many different types of problems, such as the existence of solutions for the heat equation [3], or the controllability of

Christophe Prieur
Department of Automatic Control, Gipsa-lab, 961 rue de la Houille Blanche, BP 46, 38402 Grenoble Cedex, France,
Tel.: +33-4-76-82-64-82 Fax: +33-4-76-82-63-88
E-mail: christophe.prieur@gipsa-lab.grenoble-inp.fr
Work supported by HYCON2 Network of Excellence “Highly-Complex and Networked Control Systems”, grant agreement 257462.

Frédéric Mazenc
Team INRIA DISCO, L2S, CNRS-Supelec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France
E-mail: Frederic.MAZENC@lss.supelec.fr

the semilinear wave equation [8]. The present paper focuses on a class of one-dimensional hyperbolic equations like those written as a system of conservation laws. The study of this class of PDEs is crucial when considering a wide range of physical networks having an engineering relevance. Among the potential applications we have in mind, there are hydraulic networks (for irrigation or navigation), electric line networks, road traffic networks [17] or gas flow in pipeline networks [1,11]. The importance of these applications motivates a lot of theoretical questions on hyperbolic systems which for instance pertain to optimal control and controllability as considered in [4,15,16].

The stabilizability of such systems is often proved by means of a Lyapunov function as illustrated by the contributions [11,21,32] where different control problems are solved for particular hyperbolic equations. For more general nonlinear hyperbolic equations, the knowledge of Lyapunov functions can be useful for the stability analysis of a system of conservation laws (see [7]), or even for the design for these systems of stabilizing boundary controls (see the recent work [6]).

To demonstrate asymptotic stability through the knowledge of a weak Lyapunov function i.e. a Lyapunov function whose derivative, along the trajectories of the system which is considered, is nonpositive, the celebrated LaSalle invariance principle has to be invoked (see e.g. [3,24,36]). It requires to state a precompactness property for the solutions, which may be difficult to prove (and is not even always satisfied, as illustrated by the hyperbolic systems considered in [7]). This technical step is not needed when is available a strict Lyapunov function i.e. a Lyapunov function whose derivative, along the trajectories of the system which is considered, is negative definite. Thus designing such a strict Lyapunov function is a way to overcome this technical difficulty, as done for example in [7]. These remarks motivate the present paper which is devoted to new Lyapunov techniques for the study of stability and robustness properties of Input-to-State-Stable (ISS) type for a family of time-varying linear hyperbolic PDEs with disturbances. By first applying the classical technique of construction of Lyapunov functions available in the literature, we will obtain a function which is a weak Lyapunov function for some of the systems we consider, but not for the others. Next, we will transform this function through a strictification approach, which owes a great deal to the one presented in [26], (see also [27,29]) and obtain that way a strict Lyapunov function that makes it possible to estimate the robustness of the stability of the systems with respect to the presence of uncertainties and/or external disturbances. This function is given by an explicit expression.

It is worth mentioning that although the ISS notion is very popular in the area of the dynamical systems of finite dimension (see e.g. the recent surveys [19,37]), or for systems with delay (see for instance [28]), the present work is, to the best of our knowledge, the first one which uses it to characterize a robustness property for hyperbolic PDEs. This work parallels what has been done in [30] where ISS-Lyapunov functions for semilinear time-invariant parabolic PDEs are derived using strictification techniques (see also [9] where ISS properties are compared for a reaction-diffusion system with its finite dimensional counterpart without diffusion). On the other hand, the construction we shall present is significantly different from the one in [30] because the family of systems we will study is very different. In particular the systems studied in [30] are time-invariant whereas the systems considered in the present work are time-varying. Moreover an existence result for the stability when the perturbations are vanishing is given in [33] by studying the Riemann coordinates. The present paper gives a constructive estimation even when the perturbations are not vanishing.

The main result of the present paper is applied to the problem of the regulation of the flow in a channel. This problem has been considered for a long time in the literature, as reported in the survey paper [25] which involves a comprehensive bibliography, and is still an active field of research, as illustrated for instance in [31] where a predictive control is computed using a Lyapunov function. We will consider the linearized Saint-Venant–Exner equations which model the dynamics of the flow in an open channel coupled with the moving sediment bed, see [13, 18] (see also [10]). For this hyperbolic system of balance laws, we will construct a strict Lyapunov function from which an ISS property can be derived.

The paper is organized as follows. Basic definitions and notions are introduced in Section 2. In Section 3 the analysis of the robustness of a linear time-varying hyperbolic PDE with uncertainties is carried out by means of the design of an ISS-Lyapunov function. In Section 4, the main result is illustrated by an academic example. Section 5 is devoted to an other application, that is the design of stabilizing boundary controls for the Saint-Venant–Exner equations. Concluding remarks in Section 6 end the work.

Notation. Throughout the paper, the argument of the functions will be omitted or simplified when no confusion can arise from the context. A continuous function $\alpha : [0,\infty) \to [0,\infty)$ belongs to class $\mathcal{K}$ provided it is increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if, in addition, $\alpha(r) \to \infty$ as $r \to \infty$. For any integer n, we let Id denote the identity matrix of dimension n. Given a vector ξ in $\mathbb{R}^n$, the components of ξ are denoted ξ_i for each $i = 1 \ldots, n$. Given a continuously differentiable function $A : \mathbb{R}^n \to \mathbb{R}$, $\frac{\partial A}{\partial \Xi}(\Xi)$ stands for the vector $(\frac{\partial A}{\partial \xi_1}(\Xi), \ldots, \frac{\partial A}{\partial \xi_n}(\Xi)) \in \mathbb{R}^n$. The norm induced from the Euclidean inner product of two vectors will be denoted by $|\cdot|$. Given a matrix A, its induced matrix norm will be denoted by $||A||$, and

$$\mathtt{Sym}(A) = \frac{1}{2}(A + A^\top)$$

stands for the symmetric part of A. Moreover, given a vector $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$, $\mathtt{Diag}(a_1, \ldots, a_n)$ is the diagonal matrix with the vector $(a_1, \ldots, a_n)$ on its diagonal. We denote the set of diagonal positive definite matrices in $\mathbb{R}^{n\times n}$ by $\mathcal{D}_{n,+}$. The norm $|\bullet|_{L^2(0,L)}$ is defined by: $|\phi|_{L^2(0,L)} = \sqrt{\int_0^L |\phi(z)|^2 dz}$, for any function $\phi : (0, L) \to \mathbb{R}^n$ such that $\int_0^L |\phi(z)|^2 dz < +\infty$. Finally, given two topological spaces X and Y, we denote by $C^0(X; Y)$ (resp. $C^1(X; Y)$) the set of the continuous (resp. continuously differentiable) functions from X to Y. Following [6], we introduce the notation, for all matrices $M \in \mathbb{R}^{n\times n}$,

$$\rho_1(M) = \inf\{\|\Delta M \Delta^{-1}\|,\ \Delta \in \mathcal{D}_{n,+}\}\ . \tag{1}$$

2 Basic definitions and notions

Throughout our work, we will consider linear partial differential equations of the form

$$\frac{\partial X}{\partial t}(z,t) + \Lambda(z,t)\frac{\partial X}{\partial z}(z,t) = F(z,t)X(z,t) + \delta(z,t)\ , \tag{2}$$

where $z \in [0, L]$, $t \in [0, +\infty)$, and $\Lambda(z,t) = \mathtt{Diag}(\lambda_1(z,t), \dots, \lambda_n(z,t))$ is a diagonal matrix in $\mathbb{R}^{n\times n}$ whose m first diagonal terms are nonnegative and the $n-m$ last diagonal terms are nonpositive (we will say that the hyperbolicity assumption holds, when additionally the λ_i's are never vanishing). We assume that the function δ is a disturbance of class C^1, F is a periodic function with respect to t of period T and of class C^1, Λ is a function of class C^1, periodic with respect to t of period T.

The boundary conditions are written as

$$\begin{pmatrix} X_+(0,t) \\ X_-(L,t) \end{pmatrix} = K \begin{pmatrix} X_+(L,t) \\ X_-(0,t) \end{pmatrix}\ , \tag{3}$$

where $X = \begin{pmatrix} X_+ \\ X_- \end{pmatrix}$, $X_+ \in \mathbb{R}^m$, $X_- \in \mathbb{R}^{n-m}$, and $K \in \mathbb{R}^{n\times n}$ is a constant matrix.

The initial condition is

$$X(z,0) = X^0(z)\ ,\ \forall z \in (0,L)\ , \tag{4}$$

where X^0 is a function $[0,L] :\to \mathbb{R}^n$ of class C^1 satisfying the following zero-order compatibility condition:

$$\begin{pmatrix} X_+^0(0) \\ X_-^0(L) \end{pmatrix} = K \begin{pmatrix} X_+^0(L) \\ X_-^0(0) \end{pmatrix} \tag{5}$$

and the following first-order compatibility condition:

$$\begin{pmatrix} -\lambda_1(0,0)\frac{dX_1^0}{dz}(0)+(F(0,0)X^0(0)+\delta(0,0))_1 \\ \vdots \\ -\lambda_m(0,0)\frac{dX_m^0}{dz}(0)+(F(0,0)X^0(0)+\delta(0,0))_m \\ -\lambda_{m+1}(L,0)\frac{dX_{m+1}^0}{dz}(L)+(F(L,0)X^0(L)+\delta(L,0))_{m+1} \\ \vdots \\ -\lambda_n(L,0)\frac{dX_n^0}{dz}(L)+(F(L,0)X^0(L)+\delta(L,0))_n \end{pmatrix} = K \begin{pmatrix} -\lambda_1(L,0)\frac{dX_1^0}{dz}(L)+(F(L,0)X^0(L)+\delta(L,0))_1 \\ \vdots \\ -\lambda_m(L,0)\frac{dX_m^0}{dz}(L)+(F(L,0)X^0(L)+\delta(L,0))_m \\ -\lambda_{m+1}(0,0)\frac{dX_{m+1}^0}{dz}(0)+(F(0,0)X^0(0)+\delta(0,0))_{m+1} \\ \vdots \\ -\lambda_n(0,0)\frac{dX_n^0}{dz}(0)+(F(0,0)X^0(0)+\delta(0,0))_n \end{pmatrix} . \tag{6}$$

As proved in [20], if the function Λ is of class C^1 and satisfies the hyperbolicity assumption, if the function δ is of class C^1, and if the initial condition is of class C^1 and satisfies the compatibility conditions (5) and (6), there exists an unique classical solution of the system (2), with the boundary conditions (3) and the initial condition (4), defined for all $t \geq 0$.

Now we introduce the notions of weak, strict and ISS-Lyapunov functions that we consider in this paper (see for instance [24, Def. 3.62] for the notion of Lyapunov function and [30] for the notion of ISS-Lyapunov function in an infinite dimensional context).

Definition 1 *Let $\nu : L^2(0,L) \times \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, periodic with respect to its second argument. The function ν is said to be a* weak Lyapunov function *for (2) with (3), if there are two functions κ_S and κ_M of class $\mathcal{K}_\infty$ such that, for all functions $\phi \in L^2(0,L)$ and for all $t \in [0,+\infty)$,*

$$\kappa_S\left(|\phi|_{L^2(0,L)}\right) \leq \nu(\phi,t) \leq \int_0^L \kappa_M\left(|\phi(z)|\right) dz \tag{7}$$

and, in the absence of δ, for all solutions of (2) satisfying (3), and all $t \geq 0$,

$$\frac{d\nu(X(.,t),t)}{dt} \leq 0 \ .$$

The function ν is said to be a strict Lyapunov function *for (2) with (3) if, in the absence of δ, there exists a real number $\lambda_1 > 0$ such that, for all solutions of (2) satisfying (3), and for all $t \geq 0$,*

$$\frac{d\nu(X(.,t),t)}{dt} \leq -\lambda_1 \nu(X(.,t),t) \ .$$

The function ν is said to be an ISS-Lyapunov function *for (2) with (3) if there exist a positive real number $\lambda_1 > 0$ and a function λ_2 of class $\mathcal{K}$ such that, for all continuous functions δ, for all solutions of (2) satisfying (3), and for all $t \geq 0$,*

$$\frac{d\nu(X(.,t),t)}{dt} \leq -\lambda_1 \nu(X(.,t),t) + \int_0^L \lambda_2(|\delta(z,t)|)dz \ .$$

Remark 1 1. *For conciseness, we will often use the notation $\dot{\nu}$ instead of $\frac{d\nu(X(.,t),t)}{dt}$.*

2. *Let us recall that, when is known a weak Lyapunov function, asymptotic stability can be often established via the celebrated LaSalle invariance principle (see [24, Theorem 3.64] among other references).*

3. *When a strict Lyapunov function ν exists for (2) with (3) and δ is not present, then the value of ν along the solutions of (2) satisfying (3) exponentially decays to zero and therefore each solution $X(z,t)$ satisfies $\lim_{t\to+\infty} |X(.,t)|_{L^2(0,L)} = 0$. When in addition, there exists a function κ_L of class $\mathcal{K}_\infty$, such that, for all functions $\phi \in L^2(0,L)$ and for all $t \geq 0$,*

$$\nu(\phi,t) \leq \kappa_L \left(|\phi|_{L^2(0,L)}\right) \ , \tag{8}$$

then the system (2) is globally asymptotically stable for the topology of the norm L^2.

4. *When the system (2) with (3) admits an ISS-Lyapunov function ν, then, one can check through elementary calculations*[1] *that, for all solutions of (2) satisfying (3) and for all instants $t \geq t_0$, the inequality*

$$|X(.,t)|_{L^2(0,L)} \leq \kappa_S^{-1}\left(\varphi_1(t,t_0,X)\right) + \kappa_S^{-1}\left(\varphi_2(t,t_0)\right)$$

with

$$\varphi_1(t,t_0,X) = 2e^{-\lambda_1(t-t_0)} \int_0^L \kappa_M(|X(z,t_0)|)dz$$

and

$$\varphi_2(t,t_0) = \frac{2}{\lambda_1} \sup_{\tau\in[t_0,t]} \left(\int_0^L \lambda_2(|\delta(z,\tau)|)dz \right)$$

holds. This inequality is the analogue for the PDEs (2) with (3) of the ISS inequalities for ordinary differential equations. It gives an estimation of the influence of the disturbance δ on the solutions of the system (2) with (3). ◦

[1] For instance this inequality follows from the fact that we have, for all κ of class $\mathcal{K}$ (that is for all continuous, zero at zero, and increasing functions $\kappa : [0,+\infty) \to [0,+\infty)$), and for all positive values a and b,

$$\kappa(a+b) \leq \kappa(2a) + \kappa(2b) \ ,$$

and from the fact that the function κ_S^{-1} is zero at zero and increasing.

3 ISS-Lyapunov functions for hyperbolic systems

Before stating the main theoretical result of the work, some comments are needed. Since, in the case where the system (2) is such that $m < n$, we can replace $X(z,t)$ by $\begin{pmatrix} X_+(z,t) \\ X_-(L-z,t) \end{pmatrix}$ and we may assume without loss of generality that Λ is diagonal and positive semidefinite. Then the boundary conditions (3) become

$$X(0,t) = KX(L,t) \ . \tag{9}$$

Next, we recall an important result given in [6] because it sheds light on the problem we consider and, more precisely, on the assumptions we introduce below. If Λ is constant, diagonal and positive definite and if $\rho_1(K) < 1$, where the ρ_1 is the function defined in (1), then the system (2), when $F(z,t) = 0$ and $\delta(z,t) = 0$ for all $z \in [0,L], t \geq 0$, with the boundary conditions (9) is exponentially stable in H^2-norm. Moreover, there exist a diagonal positive definite matrix[2] $Q \in \mathbb{R}^{n\times n}$ and a positive constant $\varepsilon > 0$ such that

$$\mathtt{Sym}(Q\Lambda - K^\top Q\Lambda K) \geq \varepsilon Id \ . \tag{10}$$

Furthermore, following what has been assumed for parabolic equations in [30], it might seem natural to consider the case where $F(z,t)$ possesses some stability properties. On the other hand, there is no reason to believe that this property is always needed.

These remarks lead us to introduce the following assumption:

Assumption 1 *For all $t \geq 0$ and for all $z \in [0,L]$, all the entries of the diagonal matrix $\Lambda(z,t)$ are nonnegative. There exist a symmetric positive definite matrix Q, a real number $\alpha \in (0,1)$, a continuous real-valued function r, periodic of period $T > 0$ such that the constant*

$$B = \int_0^T \left[\frac{\max\{r(m),0\}}{||Q||} + \frac{\min\{r(m),0\}}{\lambda_Q} \right] dm \ , \tag{11}$$

where λ_Q is the smallest eigenvalue of Q, is positive. Moreover, for all $t \geq 0$ and for all $z \in [0,L]$, the following inequalities:

$$\mathtt{Sym}\left(\alpha Q\Lambda(L,t) - K^\top Q\Lambda(0,t)K\right) \geq 0 \ , \tag{12}$$

$$\mathtt{Sym}\left(Q\Lambda(z,t)\right) \geq r(t)Id \ , \tag{13}$$

$$\mathtt{Sym}\left(Q\frac{\partial\Lambda}{\partial z}(z,t) + 2QF(z,t)\right) \leq 0 \tag{14}$$

are satisfied.

[2] such a matrix Q may be obtained by selecting a diagonal positive definite matrix Δ such that $\Delta K \Delta^{-1} < Id$. Then selecting $Q = \Delta^2 \Lambda^{-1}$ and $\varepsilon > 0$ sufficiently small we have (10).

Let us introduce the function

$$q(t) = \mu \left[\frac{\max\{r(t), 0\}}{||Q||} + \frac{\min\{r(t), 0\}}{\lambda_Q} \right] - \frac{\mu B}{2T} \ , \tag{15}$$

where Q and r are the matrix and the function in Assumption 1

We are ready to state and prove the main result of the paper.

Theorem 1 *Assume the system (2) with the boundary conditions (9) satisfies Assumption 1. Let μ be any real number such that*

$$0 < \mu \leq -\frac{1}{L} \ln(\alpha) \ . \tag{16}$$

Then the function $U : L^2(0,L) \times \mathbb{R} \to \mathbb{R}$ defined, for all $\phi \in L^2(0,L)$ and $t \in \mathbb{R}$, by

$$U(\phi, t) = \exp\left(\frac{1}{T} \int_{t-T}^{t} \int_{\ell}^{t} q(m) dm d\ell \right) \int_0^L \phi(z)^\top Q \phi(z) e^{-\mu z} dz \ , \tag{17}$$

where q is the function defined in (15), is an ISS-Lyapunov function for the system (2) with the boundary conditions (9).

Remark 2 1. *Assumption 1 does not imply that for all fixed $z \in [0,L]$, the ordinary differential equation $\dot{X} = F(z,t)X$ is stable. In Section 4, we will study an example where this system is unstable.*

2. *The fact that Q is symmetric positive definite and all the entries of $\Lambda(z,t)$ are nonnegative does not imply that $\mathtt{Sym}\,(Q\Lambda(z,t))$ is positive definite. That is the reason why we do not assume that r is a nonnegative function.*

3. *Assumption 1 holds when, in the system (2), $\Lambda(z,t)$ is constant, $F(z,t)$ is constant, $\delta(z,t) = 0$ for all $z \in [0,L]$ and $t \geq 0$ and the boundary condition (3) satisfies*

$$\mathtt{Sym}\left(Q\Lambda - K^\top Q \Lambda K\right) \geq 0 \ , \quad \mathtt{Sym}\,(QF) \leq 0$$

for a suitable symmetric positive definite matrix Q. Therefore Theorem 1 generalizes the sufficient conditions of [10] for the exponential stability of linear hyperbolic systems of balance laws (when F is diagonally marginally stable), to the time-varying case and to the semilinear perturbed case (without assuming that F is diagonally marginally stable).

4. *The Lyapunov function U defined in (17) is a time-varying function, periodic of period T. In the case where the system is time-invariant, one can choose a constant function q, and then it is obtained a time-invariant function (17) which is a quite usual Lyapunov function candidate in the context of the stability analysis of PDEs (see e.g., [5, 6, 39]).*

$\circ$

Proof. We begin the proof by showing that the function V defined by, for all $\phi \in L^2(0,L)$,

$$V(\phi) = \int_0^L \phi(z)^\top Q \phi(z) e^{-\mu z} dz$$

is a weak Lyapunov function for the system (2) with the boundary conditions (9) when Assumption 1 and (16) are satisfied, r is a nonnegative function, and δ is identically equal to zero.

We note for later use that, for all $\phi \in L^2(0, L)$,

$$\lambda_Q \int_0^L |\phi(z)|^2 e^{-\mu z}\ dz \leq V(\phi) \leq ||Q|| \int_0^L |\phi(z)|^2 e^{-\mu z}\ dz\ . \tag{18}$$

Now, we compute the time-derivative of V along the solutions of (2) with (9):

$$\begin{aligned}\dot{V} &= 2 \int_0^L X(z,t)^\top Q \frac{\partial X}{\partial t}(z,t) e^{-\mu z} dz \\ &= 2 \int_0^L X(z,t)^\top Q \left[-\Lambda(z,t) \frac{\partial X}{\partial z}(z,t) + F(z,t) X(z,t) + \delta(z,t) \right] e^{-\mu z} dz \\ &= -R_1(X(.,t),t) + R_2(X(.,t),t) + R_3(X(.,t),t)\ ,\end{aligned} \tag{19}$$

with

$$R_1(\phi,t) = 2 \int_0^L \phi(z)^\top Q \Lambda(z,t) \frac{\partial \phi}{\partial z}(z) e^{-\mu z} dz\ ,$$

$$R_2(\phi,t) = 2 \int_0^L \phi(z)^\top Q F(z,t) \phi(z) e^{-\mu z} dz\ , \tag{20}$$

$$R_3(\phi,t) = 2 \int_0^L \phi(z)^\top Q \delta(z,t) e^{-\mu z} dz\ . \tag{21}$$

Now, observe that

$$\begin{aligned}R_1(\phi,t) = &\int_0^L \frac{\partial(\phi(z)^\top Q \Lambda(z,t) \phi(z))}{\partial z} e^{-\mu z} dz \\ &- \int_0^L \phi(z)^\top Q \frac{\partial \Lambda}{\partial z}(z,t) \phi(z) e^{-\mu z} dz\ .\end{aligned}$$

Performing an integration by parts on the first integral, we obtain

$$\begin{aligned}R_1(\phi,t) = &\phi(L)^\top Q \Lambda(L,t) \phi(L) e^{-\mu L} - \phi(0)^\top Q \Lambda(0,t) \phi(0) \\ &+\mu \int_0^L \phi(z)^\top Q \Lambda(z,t) \phi(z) e^{-\mu z} dz \\ &- \int_0^L \phi(z)^\top Q \frac{\partial \Lambda}{\partial z}(z,t) \phi(z) e^{-\mu z} dz\ .\end{aligned} \tag{22}$$

Combining (19) and (22), we obtain

$$\begin{aligned}\dot{V} = &-X(L,t)^\top Q \Lambda(L,t) X(L,t) e^{-\mu L} + X(0,t)^\top Q \Lambda(0,t) X(0,t) \\ &+R_4(X(.,t),t) + R_3(X(.,t),t)\ ,\end{aligned}$$

with

$$\begin{aligned}R_4(\phi,t) = &-\mu \int_0^L \phi(z)^\top Q \Lambda(z,t) \phi(z) e^{-\mu z} dz \\ &+ \int_0^L \phi(z)^\top Q \frac{\partial \Lambda}{\partial z}(z,t) \phi(z) e^{-\mu z} dz + R_2(\phi,t)\ ,\end{aligned}$$

where R_2 is the function defined in (20). Using (9), we obtain

$$\begin{aligned}\dot V &= -X(L,t)^\top Q\Lambda(L,t)X(L,t)e^{-\mu L} + X(L,t)^\top K^\top Q\Lambda(0,t)KX(L,t)\\ &\quad +R_4(X(.,t),t) + R_3(X(.,t),t)\ .\end{aligned}$$

By grouping the terms and using the notation

$$N(t) = K^\top Q\Lambda(0,t)K$$

we obtain

$$\dot V = X(L,t)^\top \left[N(t) - e^{-\mu L}Q\Lambda(L,t)\right] X(L,t) + R_4(X(.,t),t) + R_3(X(.,t),t)\ .$$

Grouping the terms in $R_4(X(.,t),t)$, we obtain

$$\begin{aligned}\dot V &= X(L,t)^\top \left[N(t) - e^{-\mu L}Q\Lambda(L,t)\right] X(L,t)\\ &\quad -\int_0^L X(z,t)^\top QM(z,t)X(z,t)e^{-\mu z}dz + R_3(X(.,t),t)\ ,\end{aligned}$$

with

$$M(z,t) = \mu\Lambda(z,t) - \frac{\partial \Lambda}{\partial z}(z,t) - 2F(z,t)\ .$$

The inequalities (12) and (16) imply that

$$\dot V \leq -\int_0^L X(z,t)^\top QM(z,t)X(z,t)e^{-\mu z}dz + R_3(X(.,t),t)\ .$$

It follows from (14) that

$$\dot V \leq -\mu\int_0^L X(z,t)^\top Q\Lambda(z,t)X(z,t)e^{-\mu z}dz + R_3(X(.,t),t)\ .$$

Using (13), we deduce that

$$\dot V \leq -\mu r(t)\int_0^L |X(z,t)|^2 e^{-\mu z}dz + R_3(X(.,t),t)\ .$$

From (18) and the definition of R_3 in (21), we deduce that

$$\dot V \leq -\mu\left[\tfrac{\max\{r(t),0\}}{||Q||} + \tfrac{\min\{r(t),0\}}{\lambda_Q}\right] V(X(.,t)) + 2||Q||\int_0^L |X(z,t)||\delta(z,t)|e^{-\mu z}dz\ .$$

From Cauchy-Schwarz inequality, it follows that, for all $\kappa > 0$,

$$\begin{aligned}\dot V &\leq -\mu\left[\frac{\max\{r(t),0\}}{||Q||} + \frac{\min\{r(t),0\}}{\lambda_Q}\right] V(X(.,t))\\ &\quad +2||Q||\kappa\int_0^L |X(z,t)|^2 e^{-\mu z}dz + \frac{||Q||}{2\kappa}\int_0^L |\delta(z,t)|^2 e^{-\mu z}dz \qquad (23)\\ &\leq -q_\kappa(t)V(X(.,t)) + \frac{||Q||}{2\kappa}\int_0^L |\delta(z,t)|^2 dz\ , \qquad (24)\end{aligned}$$

with $q_\kappa(t) = \mu\left[\frac{\max\{r(t),0\}}{||Q||} + \frac{\min\{r(t),0\}}{\lambda_Q}\right] - \frac{2||Q||\kappa}{\lambda_Q}$.

The inequality (23) implies that when $r(t)$ is nonnegative, δ is identically equal to zero and κ small enough, the function V is a weak Lyapunov function for the system (2) with the initial conditions (9). However, we did not assume that the function r is nonnegative and we aim at establishing that the system is ISS with respect to δ. This leads us to apply a strictification technique which transforms V into a strict Lyapunov function. The technique of [26, Chapter 11] leads us to consider the time-varying candidate Lyapunov function

$$U_\kappa(\phi, t) = e^{s_\kappa(t)} V(\phi) \ ,$$

with $s_\kappa(t) = \frac{1}{T}\displaystyle\int_{t-T}^{t}\int_{\ell}^{t} q_\kappa(m)dmd\ell$. Through elementary calculations, one can prove the following:

Claim 2 *For all t in $\mathbb{R}$, we have $\frac{d}{dt}s_\kappa(t) = q_\kappa(t) - \frac{1}{T}\displaystyle\int_{t-T}^{t} q_\kappa(m)dm$.*

With (24) and Claim 2, we get that the time-derivative of U_κ along the solutions of (2) with the initial conditions (9) satisfies:

$$\begin{aligned}
\dot{U}_\kappa \le& -e^{s_\kappa(t)} q_\kappa(t) V(X(.,t)) + \frac{||Q||}{2\kappa} e^{s_\kappa(t)} \int_0^L |\delta(z,t)|^2 dz \\
&+ e^{s_\kappa(t)} \left[q_\kappa(t) - \tfrac{1}{T}\int_{t-T}^{t} q_\kappa(m)dm\right] V(X(.,t)) \\
\le& \frac{||Q||}{2\kappa} e^{s_\kappa(t)} \int_0^L |\delta(z,t)|^2 dz - e^{s_\kappa(t)} \frac{1}{T}\int_{t-T}^{t} q_\kappa(m)dm \ V(X(.,t)) \ .
\end{aligned}$$

Since r is periodic of period T, we have

$$\begin{aligned}
\int_{t-T}^{t} q_\kappa(m)dm &= \mu \int_{t-T}^{t}\left[\frac{\max\{r(m),0\}}{||Q||} + \frac{\min\{r(m),0\}}{\lambda_Q}\right] dm - \frac{2T||Q||\kappa}{\lambda_Q} \\
&= \mu B - \tfrac{2T||Q||\kappa}{\lambda_Q} \ ,
\end{aligned}$$

where B is the constant defined in (11). We deduce that the value

$$\kappa = \frac{\mu B \lambda_Q}{4T||Q||} \ , \tag{25}$$

which is positive because B is positive, gives

$$\begin{aligned}
\dot{U} &\le -\tfrac{\mu}{2T} B U(X(.,t),t) + \tfrac{||Q||}{2\kappa} e^{s_\kappa(t)} \int_0^L |\delta(z,t)|^2 dz \\
&\le -\tfrac{\mu}{2T} B U(X(.,t),t) + c_1 \int_0^L |\delta(z,t)|^2 dz \ ,
\end{aligned}$$

with $c_1 = 2T\frac{||Q||^2}{\mu B \lambda_Q} e^{T\frac{\mu r_M}{||Q||}}$, $r_M = \sup_{\{m \in [0,T]\}} \{r(m)\}$ and $U = U_\kappa$ for κ defined in (25). Moreover, (18) ensures that there are two positive constants c_2 and c_3 such that, for all $t \in \mathbb{R}$ and $\phi \in L^2(0, L)$,

$$c_2 \int_0^L |\phi(z)|^2 \, dz \le U(\phi, t) \le c_3 \int_0^L |\phi(z)|^2 \, dz \ .$$

Therefore inequalities of the type (7) are satisfied. We deduce that U is an ISS-Lyapunov function, as introduced in Definition 1. This concludes the proof of Theorem 1. •

4 Benchmark example

In this section we consider the system (2) and the boundary conditions (3) with the following data, for all z in $[0, L]$ and for all $t \ge 0$,

$$\begin{array}{l} X(z,t) \in \mathbb{R} \ , \ L = 1 \ , \\ \Lambda(z,t) = \sin^2(t) \left[\cos^2(t) + 1 - \frac{1}{2}z\right] , F(z,t) = \frac{\sin^2(t)}{5} \ , \\ K = \frac{1}{2} \ . \end{array}$$

A remarkable feature of this system if that the system $\dot{\xi} = F(z,t)\xi$, which rewrites as $\dot{\xi} = \frac{\sin^2(t)}{5}\xi$, is exponentially unstable. Now, we show that Theorem 1 applies to the PDE we consider. Let $Q = 1$ and $\alpha = \frac{1}{2}$. Then we have, for all $t \ge 0$,

$$\alpha Q \Lambda(1,t) - K^\top Q \Lambda(0,t) K = \frac{1}{4} \sin^2(t) \cos^2(t)$$

and thus (12) is satisfied. It is clear that (13) holds with the function $r(t) = \sin^2(t) \left[\cos^2(t) + \frac{1}{2}\right]$, which is periodic of period 2π. Then the corresponding value of B is positive, and we compute $B = \frac{3\pi}{4}$. Finally, for all $z \in [0,1]$, $t \ge 0$,

$$\begin{array}{rl} \mathtt{Sym}\left(Q\frac{\partial \Lambda}{\partial z}(z,t) + 2QF(z,t)\right) & = -\frac{\sin^2(t)}{2} + 2\frac{\sin^2(t)}{5} \\ & \le -\frac{\sin^2(t)}{10} \ . \end{array}$$

Therefore (14) holds and Assumption 1 is satisfied. We conclude that Theorem 1 applies. It follows that the system defined by

$$\frac{\partial X}{\partial t}(z,t) + \sin^2(t) \left[\cos^2(t) + 1 - \frac{1}{2}z\right] \frac{\partial X}{\partial z}(z,t) = \frac{\sin^2(t)}{5} X(z,t) + \delta(z,t) \tag{26}$$

for all z in $(0,1)$ and for all $t \ge 0$, with the boundary condition

$$X(0,t) = \frac{1}{2} X(1,t)$$

is asymptotically stable and, for this system, the function (17) is an ISS-Lyapunov function (with respect to δ). Let us compute this ISS-Lyapunov function. With (15) and the choice $Q = 1$ and $\mu = \frac{1}{2}$, we compute

$$\begin{aligned}
\tfrac{1}{2\pi}\int_{t-2\pi}^{t}\int_{\ell}^{t} q(m)dmd\ell &= \tfrac{1}{2\pi}\int_{t-2\pi}^{t}\int_{\ell}^{t}\left(\tfrac{1}{2}\sin^2(m)[\cos^2(m)+\tfrac{1}{2}] - \tfrac{3}{32}\right)dmd\ell\ , \\
&= \tfrac{1}{2\pi}\int_{t-2\pi}^{t}\int_{\ell}^{t}\left(\tfrac{1-\cos(2m)}{8} + \tfrac{1-\cos(4m)}{16} - \tfrac{3}{32}\right)dmd\ell\ , \\
&= \tfrac{1}{2\pi}\int_{t-2\pi}^{t}\left(\tfrac{3}{32}(t-l) - \tfrac{\sin(2t)}{16} - \tfrac{\sin(4t)}{64}\right)d\ell\ , \\
&= \tfrac{3\pi}{32} - \tfrac{\sin(2t)}{16} - \tfrac{\sin(4t)}{64}\ .
\end{aligned}$$

Therefore the ISS-Lyapunov function is given by the expression, for all ϕ in $L^2(0,1)$ and for all $t \geq 0$,

$$U(\phi, t) = \exp\left(\frac{3\pi}{32} - \frac{1}{16}\sin(2t) - \frac{1}{64}\sin(4t)\right)\int_0^1 |\phi(z)|^2 e^{-\frac{1}{2}z}dz\ . \tag{27}$$

To numerically check the stability and the ISS property of the system, let us discretize the hyperbolic equation (26) using a two-step variant of the Lax-Friedrichs (LxF) method [35] and the solver [34] on Matlab©[3]. We select the parameters of the numerical scheme so that the CFL condition for the stability holds. More precisely, we divide the space domain $[0, 1]$ into 100 intervals of identical length, and, choosing 10 as final time, we set a time discretization of 5×10^{-3}. For the initial condition, we select the function $X(z, 0) = z + 1$, for all $z \in [0, 1]$. For the perturbation, we choose, for all $z \in [0, 1]$,

$$\delta(z,t) = \sin^2(\pi t) \text{ when } t < 5 \text{ and } \delta(z,t) = 0 \text{ when } t \geq 5\ .$$

The time evolutions of the solution X and of the Lyapunov function U given by (27) are in Figures 1 and 2 respectively. We can observe that the solution converges as expected to the equilibrium.

5 Application on the design of boundary control for the Saint-Venant–Exner equations

5.1 Dynamics using the physical and Riemann variables

In this section, we apply the main result of Section 3 to the Saint-Venant–Exner model which is an example of nonlinear hyperbolic system of balance laws.

We consider a prismatic open channel with a rectangular cross-section, and a unit width. The dynamics of the height and of the velocity of the water in the pool are usually described by the shallow water equation (also called the Saint-Venant equation) as considered in [14]. To take into account the effect

[3] The simulation codes can be downloaded from
www.gipsa-lab.fr/∼christophe.prieur/Codes/2012-Prieur-Mazenc-Ex1.zip

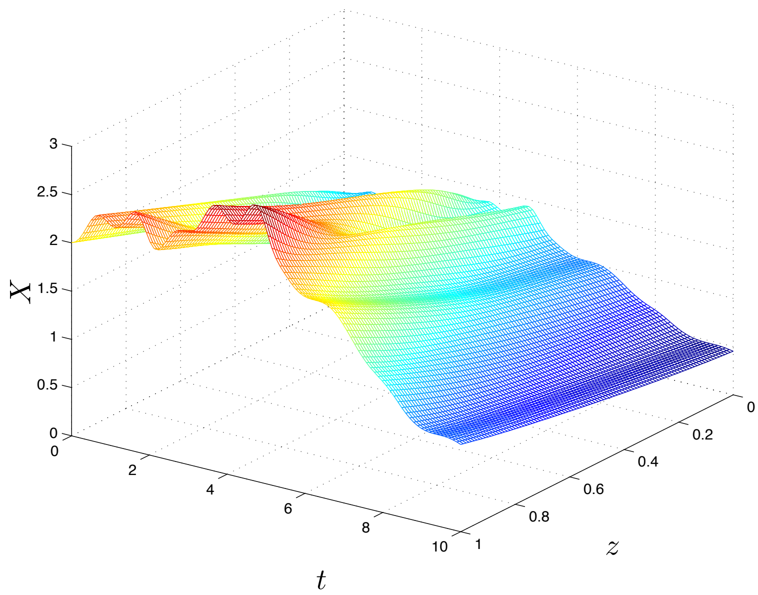

Fig. 1 Solution of (26) for $t \in [0, 10]$ and for $z \in [0, 1]$

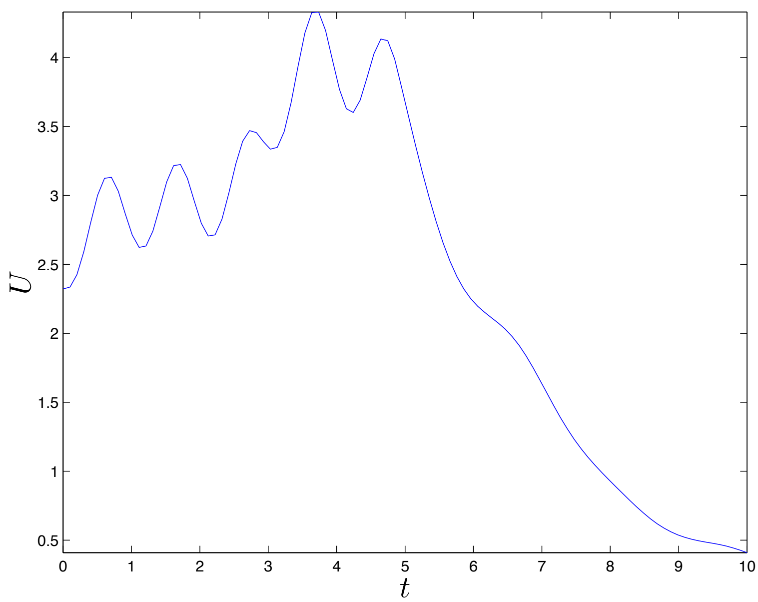

Fig. 2 Time-evolution of the Lyapunov function U given by (27) along the solution of (26) for $t \in [0, 10]$

of the sediment on the flow, this system of equations should be modified as described in [13,18] (see also [10]). This yields the following model

$$\begin{aligned}
&\frac{\partial \mathcal{H}}{\partial t} + \mathcal{V}\frac{\partial \mathcal{H}}{\partial z} + \mathcal{H}\frac{\partial \mathcal{V}}{\partial z} = \delta_1 \ , \\
&\frac{\partial \mathcal{V}}{\partial t} + \mathcal{V}\frac{\partial \mathcal{V}}{\partial z} + g\frac{\partial \mathcal{H}}{\partial z} + g\frac{\partial \mathcal{B}}{\partial z} = gS_b - C_f\frac{\mathcal{V}^2}{\mathcal{H}} + \delta_2 \ , \\
&\frac{\partial \mathcal{B}}{\partial t} + a\mathcal{V}^2\frac{\partial \mathcal{V}}{\partial z} = \delta_3 \ ,
\end{aligned} \tag{28}$$

where

- $\mathcal{H} = \mathcal{H}(z,t)$ is the water height at z in $[0,L]$ (L is the length of the pool), and at time $t \geq 0$;
- $\mathcal{V} = \mathcal{V}(z,t)$ is the water velocity;
- $\mathcal{B} = \mathcal{B}(z,t)$ is the bathymetry, i.e. the sediment layer above the channel bottom;
- g is the gravity constant;
- S_b is the slope (which is assumed to be constant);
- C_f is the friction coefficient (also assumed to be constant);
- a is a physical parameter to take into account the (constant) effects of the porosity and of the viscosity;
- $\delta = \delta(z,t) = (\delta_1(z,t), \delta_2(z,t), \delta_3(z,t))^\top$ is a disturbance, e.g. it can be a supply of water or an evaporation along the channel (see [14]).

In the previous model the disturbance may come from many different phenomena. The system (28) admits a steady-state $\mathcal{H}^\star$, $\mathcal{V}^\star$ and $\mathcal{B}^\star$ (i.e. a solution which does not depend on the time) which is constant with respect to the z-variable. Of course $\mathcal{H}^\star$, $\mathcal{V}^\star$ are such that the equality $gS_b\mathcal{H}^\star = C_f\mathcal{V}^{\star 2}$ is satisfied. The linearization of (28) at this equilibrium is carried out in [10] and is:

$$\begin{aligned}
&\frac{\partial h}{\partial t} + \mathcal{V}^\star\frac{\partial h}{\partial z} + \mathcal{H}^\star\frac{\partial v}{\partial z} = \delta_1 \ , \\
&\frac{\partial v}{\partial t} + \mathcal{V}^\star\frac{\partial v}{\partial z} + g\frac{\partial h}{\partial z} + g\frac{\partial b}{\partial z} = C_f\tfrac{\mathcal{V}^{\star 2}}{\mathcal{H}^{\star 2}}h - 2C_f\tfrac{\mathcal{V}^\star}{\mathcal{H}^\star}v + \delta_2 \ , \\
&\frac{\partial b}{\partial t} + a\mathcal{V}^{\star 2}\frac{\partial v}{\partial z} = \delta_3 \ .
\end{aligned} \tag{29}$$

Now we perform a change of coordinates for system (29). More precisely, we consider the classical characteristic Riemann coordinates (see e.g. [22]) defined for each $k = 1,\ 2,\ 3$ by [4]

$$X_k = \frac{1}{\theta_k(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\left[\left((\mathcal{V}^\star - \lambda_i)(\mathcal{V}^\star - \lambda_j) + g\mathcal{H}^\star\right)h + \mathcal{H}^\star\lambda_k v + g\mathcal{H}^\star b\right] \tag{30}$$

[4] In (30) (and similarly in other equations of this section), given $k = 1$, 2, or 3, the index i and j are in $\{1,2,3\}$ and such that i, j and k are different. In particular the product $(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)$ does not vanish.

for suitable constant characteristic velocities λ_k (which are quite complicated but explicitly computed in [18]), and denote

$$\theta_k = C_f \frac{\mathcal{V}^\star}{\mathcal{H}^\star} \frac{\lambda_k}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} \ . \tag{31}$$

Using this change of coordinates, we rewrite the system (29) as, for each $k = 1, 2, 3$,

$$\frac{\partial X_k}{\partial t} + \lambda_k \frac{\partial X_k}{\partial z} + \sum_{s=1}^{3} (2\lambda_s - 3\mathcal{V}^\star)\theta_s X_s = \delta_k \ . \tag{32}$$

This system belongs to the family of systems (2), where $X = (X_1,\ X_2,\ X_3)^\top$, $\Lambda(z,t) = \mathtt{Diag}(\lambda_1, \lambda_2, \lambda_3)$, and, for all $z \in [0, L]$, $t \geq 0$,

$$F(z,t) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \ , \tag{33}$$

with $\alpha_k = (3\mathcal{V}^\star - 2\lambda_k)\theta_k$ for all $k = 1,\ 2,\ 3$. It is proved in [18] that, for all $k = 1,\ 2,\ 3$, $\alpha_k < 0$ (and thus the matrix in (33) is stable) and that the three eigenvalues of Λ satisfy

$$\lambda_1 < 0 < \lambda_2 < \lambda_3 \ . \tag{34}$$

Let us explain how Theorem 1 can be applied to design a stabilizing boundary feedback control for the system (32).

5.2 Boundary conditions

The boundary conditions of (28) (and thus of (29)) are defined by hydraulic control devices such as pumps and valves. Here it is assumed that the water levels are measured at both ends of the open channel, and that the control action can be directly prescribed by the control devices. More precisely, in the present paper, we consider the following set of boundary conditions (see [2] for the two first boundary conditions and [10] for the last one):
1) the first boundary condition is made up of the equation that describes the operation of the gate at outflow of the reach:

$$\mathcal{H}(L,t)\mathcal{V}(L,t) = k_g \sqrt{[\mathcal{H}(L,t) - u_1(t)]^3} \ , \tag{35}$$

where k_g is a positive constant coefficient and, at each time instant t, $u_1(t)$ denotes the weir elevation which is a control input;
2) the second boundary condition imposes the value of the channel inflow rate that is controlled. It is denoted $u_2(t)$:

$$\mathcal{H}(0,t)\mathcal{V}(0,t) = u_2(t) \ ; \tag{36}$$

3) the last boundary condition is a physical constraint on the bathymetry:

$$\mathcal{B}(0,t) = B \ , \tag{37}$$

where B is a constant value.

By linearizing these boundary conditions, we derive the following boundary conditions for (29):

$$\mathcal{H}^\star v(L,t) + h(L,t)\mathcal{V}^\star = \frac{3}{2}k_g(h(L,t) - u_1(t) + u_1^\star)\sqrt{\mathcal{H}^\star - u_1^\star} \;, \tag{38}$$

$$\mathcal{H}^\star v(0,t) + h(0,t)\mathcal{V}^\star = u_2(t) - u_2^\star \;, \tag{39}$$

$$b(0,t) = 0 \;, \tag{40}$$

where $u_1^\star$ and $u_2^\star$ are the constant control actions at the equilibrium $(\mathcal{H}^\star, \mathcal{V}^\star, \mathcal{B}^\star)$.

Due to (30), for all constant values k_{12}, k_{13} and k_{21} in $\mathbb{R}$, the following conditions

$$X_1(L,t) = k_{12}X_2(L,t) + k_{13}X_3(L,t) \;, \tag{41}$$

$$X_2(0,t) = k_{21}X_1(0,t) \tag{42}$$

are equivalent to (38) and (39) for a suitable choice of the control actions $u_1(t)$ and $u_2(t)$. More precisely, assuming $\mathcal{H}^\star \neq u_1^\star$, (38) gives the following

$$u_1(t) = u_1^\star - \frac{2(\mathcal{H}^\star v(L,t) + h(L,t)\mathcal{V}^\star)}{3k_g\sqrt{\mathcal{H}^\star - u_1^\star}} + h(L,t) \;. \tag{43}$$

Before computing $v(L,t)$, note that, by letting

$$A = -C_f^2 \frac{\mathcal{V}^{\star 2}}{\mathcal{H}^{\star 2}} \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1 - \lambda_2)^2(\lambda_1 - \lambda_3)^2(\lambda_3 - \lambda_3)^2}$$

and, for each triplet of different index (i,j,k) in $\{1,2,3\}$, $\Psi_k = \frac{\theta_i\theta_j}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}$, due to (31), we have

$$\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3 = A(1 - k_{12} - k_{13}) \;.$$

Moreover, due to (34), A is positive, thus $\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3 \neq 0$, as soon as $k_{12} + k_{13} \neq 1$. Therefore, with a suitable choice of the tuning parameters k_{12} and k_{13}, the condition[5] $\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3 \neq 0$ holds.

Therefore, under the condition $k_{12} + k_{13} \neq 1$, we may compute $v(L,t)$ with (30) and (41), as a function of $h(L,t)$ and $b(L,t)$ as follows

$$\begin{aligned} v(L,t) = h(L,t)\Bigg[& \frac{g\mathcal{H}^\star(-\Psi_1 + k_{12}\Psi_2 + k_{13}\Psi_3) - \Psi_1(\mathcal{V}^\star - \lambda_2)(\mathcal{V}^\star - \lambda_3)}{\mathcal{H}^\star(\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3)} \\ & + \frac{k_{12}\Psi_2(\mathcal{V}^\star - \lambda_1)(\mathcal{V}^\star - \lambda_3) + k_{13}\Psi_3(\mathcal{V}^\star - \lambda_1)(\mathcal{V}^\star - \lambda_2)}{\mathcal{H}^\star(\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3)}\Bigg] \\ & + b(L,t)\frac{g(-\Psi_1 + k_{12}\Psi_2 + k_{13}\Psi_3)}{\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3} \;. \end{aligned} \tag{44}$$

[5] The condition $k_{12} + k_{13} \neq 1$ will hold with the numerical values that will be chosen in Section 5.3 below.

Combining (43) and (44), the control action depends only on $h(L,t)$ and $b(L,t)$, that is

$$\begin{aligned} u_1(t) = u_1^\star + h(L,t) \Bigg[1 - \frac{2\mathcal{H}^\star}{3k_g\sqrt{\mathcal{H}^\star - u_1^\star}} \\ \times \Bigg(\frac{g\mathcal{H}^\star(-\Psi_1 + k_{12}\Psi_2 + k_{13}\Psi_3) - \Psi_1(\mathcal{V}^\star - \lambda_2)(\mathcal{V}^\star - \lambda_3)}{\mathcal{H}^\star(\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3)} \\ + \frac{k_{12}\Psi_2(\mathcal{V}^\star - \lambda_1)(\mathcal{V}^\star - \lambda_3) + k_{13}\Psi_3(\mathcal{V}^\star - \lambda_1)(\mathcal{V}^\star - \lambda_2)}{\mathcal{H}^\star(\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3)} \Bigg) \\ - \frac{2\mathcal{V}^\star}{3k_g\sqrt{\mathcal{H}^\star - u_1^\star}} \Bigg] \\ - b(L,t)\frac{2\mathcal{H}^\star}{3k_g\sqrt{\mathcal{H}^\star - u_1^\star}} \frac{g\mathcal{H}^\star(-\Psi_1 + k_{12}\Psi_2 + k_{13}\Psi_3)}{\mathcal{H}^\star(\lambda_1\Psi_1 - k_{12}\lambda_2\Psi_2 - k_{13}\lambda_3\Psi_3)} \ . \end{aligned} \tag{45}$$

Similarly (30), (39), (40), and (42) give the following control

$$\begin{aligned} u_2(t) &= u_2^\star + h(0,t)\mathcal{V}^\star + \mathcal{H}^\star v(0,t) \\ &= u_2^\star + h(0,t)\, [\mathcal{V}^\star \\ &+ \frac{g\mathcal{H}^\star(-\Psi_2 + k_{21}\Psi_1) - \Psi_2(\mathcal{V}^\star - \lambda_1)(\mathcal{V}^\star - \lambda_3) + k_{21}\Psi_1(\mathcal{V}^\star - \lambda_2)(\mathcal{V}^\star - \lambda_3)}{\mathcal{H}^\star(\lambda_2\Psi_2 - k_{21}\lambda_1\Psi_1)} \Bigg] \ . \end{aligned} \tag{46}$$

By defining the output of the system (28) as the height at both ends of the channel and the bathymetry of the water at the outflow, the control (u_1, u_2) defined in (45) and (46) is an output feedback law.

The last boundary condition (40) is equivalent, for all $t \geq 0$, to

$$\sum_i [(\lambda_i - \mathcal{V}^\star)^2 - g\mathcal{H}^\star] X_i(0,t) = 0$$

and thus, assuming[6] $(\lambda_3 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star \neq 0$, and using (42), we have

$$\begin{aligned} X_3(0,t) &= -\frac{[(\lambda_1 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star]X_1(0,t) + [(\lambda_2 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star]X_2(0,t)}{(\lambda_3 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star} \\ &= -\frac{[(\lambda_1 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star] + k_{21}[(\lambda_2 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star]}{(\lambda_3 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star} X_1(0,t) \ . \end{aligned}$$

By performing the same change of spatial variables as in the beginning of Section 3, we may assume that, for all $i = 1, 2, 3$, $\lambda_i > 0$.

To summarize the boundary conditions in the characteristic Riemann coordinates are (9) with

$$K = \begin{pmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & 0 \\ \eta(k_{21}) & 0 & 0 \end{pmatrix} ,$$

where

$$\eta(k_{21}) = -\frac{[(\lambda_1 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star] + k_{21}[(\lambda_2 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star]}{(\lambda_3 - \mathcal{V}^\star)^2 - g\mathcal{H}^\star} \ .$$

[6] This will be the case with the numerical values that will be chosen in Section 5.3 below.

5.3 Numerical computation of a stabilizing boundary control and of an ISS-Lyapunov function

In this section, we show that computing a symmetric positive definite matrix Q such that Assumption 1 holds can be done by solving a convex optimization problem. This allows us to compute an output feedback law that is a stabilizing boundary control for (29), and to make explicit an ISS-Lyapunov for the linearized Saint-Venant–Exner equations (29). Note that [6] cannot be directly used since (29) is a quasilinear hyperbolic system (due to the presence of the non-zero left-hand side), and that [2,10] cannot be applied since the effect of the perturbations δ_1, δ_2 and δ_3 is additionally considered in this paper, and since the boundary conditions are different. Finally, in the paper [12], only the perturbed Saint-Venant equation is considered (without the dynamics of the sediment).

With the change of variables explained at the beginning of Section 3, we obtain a diagonal positive definite matrix Λ. Therefore, Assumption 1 holds as soon as there exists a symmetric positive definite matrix Q such that

$$\begin{array}{c}\mathtt{Sym}(Q\Lambda - K^\top Q\Lambda K) \geq 0 \ , \\ \mathtt{Sym}(QF) \leq 0 \ .\end{array} \tag{47}$$

Note that conditions (47) are Linear Matrix Inequalities (LMIs) with respect to the unknown symmetric positive definite matrix Q. Computing this matrix can be done using standard optimization softwares (see e.g. [38]). Let us solve this optimization problem and apply Theorem 1 using numerical values of [18] and of [12] [7]. The equilibrium is selected as $\mathcal{H}^\star = 0.1365$ [m], $\mathcal{V}^\star = 14.65$ [ms^{-1}], and $\mathcal{B}^\star = 0$ [m]. Next, consider system (32) with $\lambda_1 = -10$ [ms^{-1}], $\lambda_2 = 7.72 \times 10^{-4}$ [ms^{-1}], $\lambda_3 = 13$ [ms^{-1}] and the matrix F defined in (33). We take $g = 9.81$ [ms^{-1}], and we compute k_{21} such that

$$K = \begin{pmatrix} 0 & 0 & 0 \\ k_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{48}$$

This holds with the following tuning parameters

$$k_{12} = 0 \ , \quad k_{13} = 0 \ , \quad k_{21} = -0.09517 \ . \tag{49}$$

Now employing parser YALMIP [23] on Matlab©, we may check that[8]

$$Q = \begin{pmatrix} 8.111 \times 10^7 & -2.668 \times 10^3 & -7.215 \times 10^7 \\ \star & 2.927 \times 10^2 & 2.083 \times 10^3 \\ \star & \star & 6.544 \times 10^7 \end{pmatrix}$$

[7] The simulation codes can be downloaded from
www.gipsa-lab.fr/∼christophe.prieur/Codes/2012-Prieur-Mazenc-Ex2.zip

[8] In the following equation, the symmetric terms are denoted by $\star$.

ensures that Assumption 1 is satisfied. Thus selecting $\mu = 1.5 \times 10^{-2}$, we compute the following ISS-Lyapunov function, defined by, for all ϕ in $L^2(0,1)$,

$$U(\phi) = \int_0^1 \phi(z) Q \phi(z) e^{-\mu z} \, dz \tag{50}$$

for the system (32) with the boundary conditions (9) with K in (48). Therefore we have proved the following result:

Proposition 3 *The boundary control (45) and (46), with the tuning parameters k_{12}, k_{13} and k_{21} given by (49), is an asymptotically stabilizing boundary control for (29) when the δ_i are not present. With the height at both ends of the channel and the bathymetry of the water at the outflow as outputs of the system (28), it is an output feedback law.*

Moreover, performing the inverse of the change of variables (30), the function U given by (50) is an ISS-Lyapunov function for the linearized Saint-Venant–Exner equations (29) relative to $\delta(z,t)$.

6 Conclusions

For time-varying hyperbolic PDEs, we designed ISS-Lyapunov functions. These functions are time-varying and periodic. They make it possible to derive robustness properties of ISS type. We applied this result to two problems. The first one pertains to a benchmark hyperbolic equation, for which some simulations were performed to check the computation of an ISS-Lyapunov function.

The second one is the explicit computation of an ISS-Lyapunov function for the linear approximation around a steady state of the Saint-Venant–Exner equations that model the dynamics of the flow and of the sediment in an open channel.

This work leaves many questions open. The problem of designing ISS-Lyapunov functions for nonlinear hyperbolic equations will be considered in future works, possibly with the help of Lyapunov functions considered in [6]. In addition, it would be of interest to use an experimental channel to validate experimentally the prediction of the offset that is inferred from the proposed ISS-Lyapunov function, in a similar way as what is done in [12].

References

1. G. Bastin, J.-M. Coron, and B. d'Andréa Novel. Using hyperbolic systems of balance laws for modeling, control and stability analysis of physical networks. In *Lecture notes for the Pre-Congress Workshop on Complex Embedded and Networked Control Systems*, Seoul, Korea, 2008. 17th IFAC World Congress.
2. G. Bastin, J.-M. Coron, and B. d'Andréa Novel. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel. *Networks and Heterogeneous Media*, 4(2):177–187, 2009.
3. T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations.* Oxford University Press, 1998.

4. R.M. Colombo, G. Guerra, M. Herty, and V. Schleper. Optimal control in networks of pipes and canals. *SIAM Journal on Control and Optimization*, 48:2032–2050, 2009.
5. J.-M. Coron. On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain. *SIAM Journal on Control and Optimization*, 37(6):1874–1896, 1999.
6. J.-M. Coron, G. Bastin, and B. d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM Journal on Control and Optimization*, 47(3):1460–1498, 2008.
7. J.-M. Coron, B. d'Andréa Novel, and G. Bastin. A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws. *IEEE Transactions on Automatic Control*, 52(1):2–11, 2007.
8. J.-M. Coron and E. Trélat. Global steady-state stabilization and controllability of 1D semilinear wave equations. *Commun. Contemp. Math.*, 8(4):535–567, 2006.
9. S. Dashkovskiy and A. Mironchenko. On the uniform input-to-state stability of reaction-diffusion systems. In *49th IEEE Conference on Decision and Control (CDC)*, pages 6547–6552, Atlanta, GA, USA, 2010.
10. A. Diagne, G. Bastin, and J.-M. Coron. Lyapunov exponential stability of linear hyperbolic systems of balance laws. *Automatica*, to appear.
11. M. Dick, M. Gugat, and G. Leugering. Classical solutions and feedback stabilization for the gas flow in a sequence of pipes. *Networks and Heterogeneous Media*, 5(4):691–709, 2010.
12. V. Dos Santos and C. Prieur. Boundary control of open channels with numerical and experimental validations. *IEEE Transactions on Control Systems Technology*, 16(6):1252–1264, 2008.
13. W.H. Graf. *Hydraulics of sediment transport.* Water Resources Publications, Highlands Ranch, Colorado, USA, 1984.
14. W.H. Graf. *Fluvial hydraulics.* Wiley, 1998.
15. M. Gugat, M. Herty, A. Klar, and G. Leugering. Optimal control for traffic flow networks. *Journal of Optimization Theory and Applications*, 126(3):589–616, 2005.
16. M. Gugat and G. Leugering. Global boundary controllability of the Saint-Venant system for sloped canals with friction. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, 26(1):257–270, 2009.
17. B. Haut and G. Bastin. A second order model of road junctions in fluid models of traffic networks. *Networks and Heterogeneous Media*, 2(2):227–253, june 2007.
18. J. Hudson and P.K. Sweby. Formulations for numerically approximating hyperbolic systems governing sediment transport. *Journal of Scientific Computing*, 19:225–252, 2003.
19. B. Jayawardhana, H. Logemann, and E.P. Ryan. The circle criterion and input-to-state stability. *IEEE Control Systems*, 31(4):32–67, 2011.
20. I. Kmit. Classical solvability of nonlinear initial-boundary problems for first-order hyperbolic systems. *International Journal of Dynamical Systems and Differential Equations*, 1(3):191–195, 2008.
21. M. Krstic and A. Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57:750–758, 2008.
22. T.-T. Li. *Global classical solutions for quasilinear hyperbolic systems*, volume 32 of *RAM: Research in Applied Mathematics.* Masson, Paris, 1994.
23. J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proc. of the CACSD Conference*, Taipei, Taiwan, 2004.
24. Z.-H. Luo, B.-Z. Guo, and O. Morgul. *Stability and stabilization of infinite dimensional systems and applications.* Communications and Control Engineering. Springer-Verlag, New York, 1999.
25. P.-O. Malaterre, D.C. Rogers, and J. Schuurmans. Classification of canal control algorithms. *ASCE Journal of Irrigation and Drainage Engineering*, 124(1):3–10, 1998.
26. M. Malisoff and F. Mazenc. *Constructions of Strict Lyapunov Functions.* Communications and Control Engineering Series. Springer, London, UK, 2009.
27. F. Mazenc, M. Malisoff, and O. Bernard. A simplified design for strict Lyapunov functions under Matrosov conditions. *IEEE Transactions on Automatic Control*, 54(1):177–183, jan. 2009.

28. F. Mazenc, M. Malisoff, and Z. Lin. Further results on input-to-state stability for nonlinear systems with delayed feedbacks. *Automatica*, 44(9):2415–2421, 2008.
29. F. Mazenc and D. Nesic. Lyapunov functions for time-varying systems satisfying generalized conditions of Matrosov theorem. *Mathematics of Control, Signals, and Systems*, 19:151–182, 2007.
30. F. Mazenc and C. Prieur. Strict Lyapunov functions for semilinear parabolic partial differential equations. *Mathematical Control and Related Fields*, 1(2):231–250, 2011.
31. V.T. Pham, D. Georges, and G. Besancon. Predictive control with guaranteed stability for hyperbolic systems of conservation laws. In *49th IEEE Conference on Decision and Control (CDC)*, pages 6932–6937, Atlanta, GA, USA, 2010.
32. C. Prieur and J. de Halleux. Stabilization of a 1-D tank containing a fluid modeled by the shallow water equations. *Systems & Control Letters*, 52(3-4):167–178, 2004.
33. C. Prieur, J. Winkin, and G. Bastin. Robust boundary control of systems of conservation laws. *Mathematics of Control, Signals, and Systems*, 20(2):173–197, 2008.
34. L.F. Shampine. Solving hyperbolic PDEs in Matlab. *Appl. Numer. Anal. & Comput. Math.*, 2:346–358, 2005.
35. L.F. Shampine. Two-step Lax–Friedrichs method. *Applied Mathematics Letters*, 18:1134–1136, 2005.
36. M. Slemrod. A note on complete controllability and stabilizability for linear control systems in Hilbert space. *SIAM Journal on Control*, 12:500–508, 1974.
37. E.D. Sontag. Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, pages 163–220. Springer-Verlag, Berlin, 2007.
38. J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999. `http://sedumi.mcmaster.ca/`.
39. C.-Z. Xu and G. Sallet. Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM Control, Optimisation and Calculus of Variations*, 7:421–442, 2002.