Zhikun Hou
Shiyu Chen
Department of Mechanical
Engineering
Worcester Polytechnic Institute
Worcester, MA 01609

Iterated Dynamic Condensation Technique and Its Applications in Modal Testing

This article addresses an application of the dynamic condensation in modal testing, which requires measurements of natural frequencies and a limited number of components of modal shapes only. The measured frequencies can be employed as the initial estimates in the iterated dynamic condensation to effectively find an exact solution for the original eigenvalue problem and determine the transformation matrix between the testing and nontesting coordinates. Complete modal shapes including the nontesting coordinates can then be estimated based on the limited measurements. Numerical results are presented to illustrate the feasibility and efficiency of the iterated dynamic condensation. A comparison with the experimental results of a multiple disk system is also provided. © 1997 John Wiley & Sons, Inc.

INTRODUCTION

Modal testing has been employed in many engineering disciplines to determine the vibration characteristics of a system including its natural frequencies and the corresponding modal shapes. With development of hardware and software, the modal testing technique has become more and more efficient, reliable, and accurate and has been widely adopted in the industry. However, for large complex structures, the testing is usually costly and time consuming, especially when mode shapes are concerned. On the other hand, the finite element method has become a powerful tool for engineering structural analysis. However, applications of the method depends, in many cases, on how the mathematical model appropriately represents the real physical system; therefore, an experimental

verification is often required. Many efforts have been devoted to bridge the gap between these two approaches including using modal testing to identify the boundary conditions of real physical systems (Chen and Wang, 1991), using the finite element model to estimate the complete modal shapes based on the limited measurements (Chen et al., 1995; Stephan, 1986), and some applications in structural dynamic modification (Baruch, 1978; Chen et al., 1995).

This article addresses an application of the iterated dynamic condensation technique in the modal testing. Generally speaking, the measurement of natural frequencies of a system is relatively easier to implemented and less costly. Engineers can easily select the testing coordinates for which the testing facilities are available and implementation of the test is convenient. Using the measured fre-

quencies as the initial estimates in the iterated dynamic condensation, exact solutions for the natural frequencies of the finite element formulation can be efficiently obtained. And, in turn, the relationship between the testing and nontesting coordinates for the modes of interest can be found. Therefore, the nontesting coordinates of the modal shapes can be estimated based on the limited measurements of the testing coordinates, and the full or complete modal vectors can be constructed.

A brief summary of the iterated dynamic condensation is provided in the following section. Its applications in the modal testing are then discussed. Numerical results for vibration analysis of a cantilever beam are presented to illustrate the accuracy and efficiency of the dynamic condensation iteration. A comparison with the experimental results for a multiple disk system is provided to demonstrate the application of the dynamic condensation in the modal testing.

DYNAMIC CONDENSATION TECHNIQUES

In structural dynamics, the natural frequencies and eigenmodes of a structure can be obtained by solving the following generalized eigenvalue problem:

$$KX = \omega^2 M X \tag{1}$$

where K and M are the stiffness and mass matrices of the structure, respectively, ω is a natural frequency and X is the corresponding eigenvector, or mode shape of the system. It is assumed that K is semipositive definite and M is positive definite. Condensation techniques for large-scale generalized eigenvalue problems have been proposed to reduce the size of the problem and improve the computational efficiency including Guyan (1965), Irons (1965), Kuhar and Stahle (1974), Leung (1978), Paz (1978), Singh and Suarez (1992), Suarez and Singh (1991), Vysloukh et al. (1973), and Zhang and Hou (1980, 1981). Using these approaches, the components of the unknown eigenvector, X, are classified as the primary (or master) coordinates and the secondary (or slave) coordinates, i.e.,

$$X = \left\{ \frac{X_p}{X_s} \right\}, \tag{2}$$

where X_p and X_s are, respectively, the subvectors of the primary and secondary coordinates. In a partitioned form, Eq. (1) can be rewritten as

$$\begin{bmatrix} K_{pp} - \omega^2 M_{pp} & K_{ps} - \omega^2 M_{ps} \\ K_{sp} - \omega^2 M_{sp} & K_{ss} - \omega^2 M_{ss} \end{bmatrix} \left\{ \frac{X_p}{X_s} \right\} = \left\{ \frac{0}{0} \right\}, \tag{3}$$

where K_{ij} and M_{ij} $(I, j = p, s)$ are the submatrices of K and M and 0 represents a zero vector. The primary and secondary coordinates are related by

$$X_s = R X_p, \tag{4}$$

where R is given by

$$R = -(K_{ss} - \omega^2 M_{ss})^{-1}(K_{sp} - \omega^2 M_{sp}). \tag{5}$$

Define a transformation matrix, T, as follows:

$$X = T X_p. \quad T = \begin{bmatrix} I \\ R \end{bmatrix}, \tag{6}$$

in which I is a $p \times p$ identity matrix. Using the transformation matrix defined, an equivalent eigenvalue problem in terms of the primary coordinates, X_p, can be derived as

$$K^* X_p = \omega^2 M^* X_p, \tag{7}$$

where

$$\begin{aligned} K^* &= T^T K T, \\ M^* &= T^T M T, \end{aligned} \tag{8}$$

are the equivalent stiffness and mass matrices, respectively. It can be shown that K^* remains semipositive definite and M^* remains positive definite, provided the rank of the transformation matrix is p.

Generally speaking, the transformation matrix, T, is a function of the unknown natural frequency, ω. Therefore, the new formulation in terms of X_p seems not to bring any computational advantage into the solution. To implement this technique, Guyan (1965) neglected the inertia effect in Eq. (5), i.e.,

$$R = -K_{ss}^{-1} K_{sp}. \tag{9}$$

Guyan's condensation, referred to as the static condensation, may lead to an exact solution for

the static problem but only an approximate solution for its dynamic counterpart. The error caused by the static condensation increases for higher order modes due to the increasing inertia effects. To improve the accuracy, Kuhar and Stahle (1974) used

$$R = -(K_{ss} - \bar{\omega}^2 M_{ss})^{-1}(K_{sp} - \bar{\omega}^2 M_{sp}) \quad (10)$$

where $\bar{\omega}$ is an estimated natural frequency. Their approximation, referred to as the dynamic condensation, took the inertia effect into account. The results are an approximate solution but are expected to be improved for the mode(s) whose corresponding natural frequency(s) is close to the estimated frequency.

Leung (1978), Singh and Suarez (1992), Suarez and Singh (1991) and Zhang and Hou (1980, 1981), proposed various versions of the iterated dynamic condensation technique. For example, in Zhang and Hou (1980), to obtain the eigensolution for the ith mode, a sequence solution of $\omega_i^{(0)}$, $\omega_i^{(1)}$, ... $\omega_i^{(j)}$, $\omega_i^{(j+1)}$, ... is constructed where $\omega_i^{(0)}$ is an initial estimate for the ith natural frequency by using the static condensation and $\omega_i^{(j+1)}$ is the $(j + 1)$th iterated estimation by using

$$R^{(j+1)} = -(K_{ss} - \omega_i^{(j)2} M_{ss})^{-1}(K_{sp} - \omega_i^{(j)2} M_{sp}). \quad (11)$$

Zhang and Hou (1980, 1981) showed that the iterated dynamic condensation may lead to an exact solution for the eigenvalue problem provided the primary coordinates are properly selected. A criterion for selecting the principal coordinates was given. Numerical results were presented for its application in vibration analysis of engine blades with the superparametric shell element. Singh and Suarez (Singh and Suarez, 1992; Suarez and Singh, 1991) used the iterated dynamic condensation in the synthesis of substructure eigenproperties and numerical examples are presented to demonstrate the implementation of the procedure.

MODAL TESTING BASED ON DYNAMIC CONDENSATION

The governing differential equation of a linear multiple degree of freedom system is written as

$$M \frac{d^2}{dt^2} \underline{x}(t) + C \frac{d}{dt} \underline{x}(t) + K\underline{x}(t) = \underline{F}(t), \quad (12)$$

where M, C, and K are, respectively, the mass, damping, and stiffness matrices; $\underline{x}(t)$ is the response vector; and $\underline{F}(t)$ is the external excitation vector. Equation (12) may be obtained by the finite element formulation or some other approximation techniques. The associated generalized eigenvalue problem is expressed by Eq. (1).

Distinguishing the components of the modal vectors as the testing coordinates and nontesting coordinates, the modal vector may be written as

$$\underline{X} = \left\{ \begin{array}{c} \underline{X}_t \\ \underline{X}_{nt} \end{array} \right\}, \quad (13)$$

where $\underline{X}_t$ is an $N_t \times 1$ subvector for all testing coordinates and $\underline{X}_{nt}$ is an $N_{nt} \times 1$ subvector for all nontesting coordinates. The testing coordinates should be carefully chosen based on the availability of the testing facilities, feasibility of implementation, and understanding of the physical nature of the modes of interest. The number of the testing coordinates should be larger than the number of modes of interest and may be much less than the total number of degrees of freedom of the system.

Assume that the natural frequencies and incomplete modal shape of the first n ($n \leq N_t$) modes are measured and denoted by ω_i^* and $\underline{X}_i^*$ ($i = 1$, $2, \ldots, n$). These measured natural frequencies may be used as initial estimates of frequencies of the modes of interest in the iterated dynamic condensation. In detail, ω_i and $\underline{X}_i$ ($i = 1, 2, \ldots, n$) are obtained by the following iteration:

$$K^{(j+1)} \underline{X}_t^{(j+1)} = \omega^2 M_t^{(j+1)}, \quad j = 0, 1, 2, \ldots, \quad (14)$$

where
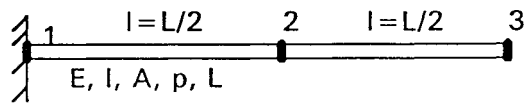
$$K^{(j+1)} = T^{(j+1)T} K T^{(j+1)}, \quad M^{(j+1)} = T^{(j+1)T} M T^{(j+1)}, \quad (15)$$

in which

$$T^{(j+1)} = \left\{ \begin{array}{c} I \\ -(K_{ss} - \omega_i^{(j)2} M_{ss})^{-1}(K_{sp} - \omega_i^{(j)2} M_{sp}) \end{array} \right\}, \quad (16)$$

and $(\cdot)^T$ stands for the transpose of a matrix or vector. In Eq. (16) $\omega_i^{(j)}$ is the jth estimated natural frequency of the ith mode and $\omega_i^{(0)} = \omega_i^*$, i.e., the corresponding experimental result. It is expected that under certain conditions the iteration con-

FIGURE 1 A simple finite element model for a cantilever beam.

verges (Zhang and Hou, 1980, 1981). Assume that the limiting values of the ith natural frequency, the corresponding mode shape, and the transformation matrix are represented by $\omega_i^{(\infty)}$, $X_{l,i}^{(\infty)}$, and $T^{(\infty)}$, respectively. The complete modal vector including both testing and nontesting coordinates can be estimated by

$$X_i^{(\infty)} = T^{(\infty)} X_{l,i}^*. \tag{17}$$

Computational experience shows that the process converges after only a few of iterations if the testing coordinates are appropriately selected, as demonstrated later by numerical examples.

NUMERICAL EXAMPLES

Numerical results for three example problems are provided to demonstrate concepts of iterated dynamic condensation techniques. The first example illustrates a step by step procedure of the approach for a two-element model of a cantilever beam. The second example demonstrates how to use experimental results for the natural frequencies of a cantilever beam as the initial estimates to find more accurate values for the natural frequencies using the iterated dynamic condensation for a relatively large finite element model. Results for the beam problem with general boundary conditions are also listed. The third example shows an application for modal testing for a multiple-disk shaft system. A comparison is given for both natural frequencies and model shapes.

EXAMPLE 1: Vibration analysis of a cantilever beam using a two-element model.

To demonstrate the feasibility and efficiency of the iterated dynamic condensation technique, transversal vibration of a cantilever beam, as shown in Fig. 1, is considered. E is the Young's modulus of the beam material, I and A are, respectively, the moment of inertia and the area of the beam cross section, L is the total length of the

beam, and ρ is the mass density per unit volume of the beam material.

To demonstrate the basic concepts and procedure of the proposed approach, a simple finite element model with three nodes and two elements is employed. There are 2 degree of freedoms for each node, i.e., the lateral displacement, y_i, and rotation (or slope), θ_i ($i = 1, 2, 3$).

The corresponding generalized eigenvalue problem can be obtained as

$$K\underline{X} = \omega^2 M \underline{X},$$

where

$$K = \frac{EI}{l^3} \begin{pmatrix} 24 & 0 & -12 & 6 \\ 0 & 8 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{pmatrix},$$

$$M = \frac{\rho A l}{420} \begin{pmatrix} 312 & 0 & 54 & -13 \\ 0 & 8 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{pmatrix}, \tag{18}$$

$$X = \begin{Bmatrix} y_2 \\ \phi_2 \\ y_3 \\ \varphi_3 \end{Bmatrix}.$$

For convenience, $\varphi_i = l\theta_i$, $i = 1, 2, 3$ are introduced. The exact solutions for the first two modes of the original complete problem without condensation are

$$\lambda_1 = \frac{\omega_1^2}{\dfrac{EI}{\rho A l^4}} = 3.5177, \quad X_1 = \begin{Bmatrix} 0.246646 \\ 0.844903 \\ 0.726460 \\ 1.000000 \end{Bmatrix}, \tag{19}$$

and

$$\lambda_2 = \frac{\omega_2^2}{\dfrac{EI}{\rho A l^4}} = 22.221, \quad X_1 = \begin{Bmatrix} 0.149924 \\ -0.09023 \\ -0.20771 \\ -1.00000 \end{Bmatrix}. \tag{20}$$

Choose the testing coordinates as

$$X_t = \begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix}, \qquad (21)$$

i.e., the deflections at nodes 2 and 3. Using the static condensation, an approximate solution can be obtained as

$$\lambda_1^{(0)} = \frac{\omega_1^{(0)2}}{\frac{EI}{\rho A l^4}} = 3.5219, \quad X_1^{(0)} = \begin{Bmatrix} 0.241260 \\ 0.815853 \\ 0.710569 \\ 1.000000 \end{Bmatrix}, \quad (22)$$

and

$$\lambda_2^{(0)} = \frac{\omega_2^{(0)2}}{\frac{EI}{\rho A l^4}} = 22.279, \quad X_2^{(0)} = \begin{Bmatrix} 0.140709 \\ -0.051915 \\ -0.201277 \\ -1.00000 \end{Bmatrix}. \quad (23)$$

Quite accurate results are obtained. For the first mode, the error for the frequency is 0.12% and the maximum error for the mode shape is 3.4%. For the second mode, the error reaches 0.26% for the frequency but 42% for the mode shape.

Using the dynamic condensation with an estimated frequency of $\lambda_1^{(0)} = 3.5219$, one obtains

$$\lambda_1^{(1)} = \frac{\omega_1^{(1)2}}{\frac{EI}{\rho A l^4}} = 3.5177, \quad X_1^{(1)} = \begin{Bmatrix} 0.246658 \\ 0.844974 \\ 0.726499 \\ 1.000000 \end{Bmatrix}, \quad (24)$$

for the first mode, which is almost the corresponding exact solution, and

$$\lambda_2^{(1)} = \frac{\omega_2^{(1)2}}{\frac{EI}{\rho A l^4}} = 22.276, \quad X_2^{(1)} = \begin{Bmatrix} 0.140965 \\ -0.052637 \\ -0.201350 \\ -1.000000 \end{Bmatrix}, \quad (25)$$

for the second mode, which does not show any significant improvement.

The solution for the second mode can be improved by using $\omega_2^{(1)} = 22.276$, or directly using the second eigenvalue from the static condensation, as the new estimated frequency in the dynamic condensation, which leads to the following results:

$$\lambda_2^{(2)} = \frac{\omega_2^{(2)2}}{\frac{EI}{\rho A l^4}} = 22.221, \quad X_2^{(2)} = \begin{Bmatrix} 0.149968 \\ -0.090504 \\ -0.207766 \\ -1.000000 \end{Bmatrix}. \quad (26)$$

The error in frequency has been reduced to almost zero and for the mode shape, the error reduces to 0.3%.

The efficiency of implementing the iterated dynamic condensation depends, to a large extent, on the choice of the testing coordinates. As a demonstration, the above example was repeated with a different choice of the testing coordinates as

$$X_t = \begin{Bmatrix} y_3 \\ \varphi_3 \end{Bmatrix} \qquad (27)$$

i.e., the deflection and rotation at node 3. The static condensation leads to the following approximate solutions:

$$\lambda_1^{(0)} = \frac{\omega_1^{(0)2}}{\frac{EI}{\rho A l^4}} = 3.5327, \quad X_1^{(0)} = \begin{Bmatrix} 0.237976 \\ 0.838928 \\ 0.725952 \\ 1.000000 \end{Bmatrix}, \quad (28)$$

and

$$\lambda_2^{(0)} = \frac{\omega_2^{(0)2}}{\frac{EI}{\rho A l^4}} = 34.807, \quad X_2^{(0)} = \begin{Bmatrix} 0.059405 \\ -0.053214 \\ -0.131191 \\ -1.00000 \end{Bmatrix}, \quad (29)$$

For the first mode, the error for the frequency is 0.43% and the maximum error for the mode shape is 3.4%. For the second mode, the error reaches 56% for the frequency and 60% for the mode shape.

Table 1.   First Three Natural Frequencies of Beams with Different Boundary Conditions

Boundary Conditions	Mode No.	Zeroth Iteration	First Iteration	Second Iteration	Third Iteration	Fourth Iteration	Fifth Iteration	Exact (FEM)	Exact (BVP)
Fixed–free	1	3.2[a]	3.522	3.516				3.516	3.516
beam	2	19.5[a]	22.07	22.034	22.034			22.034	22.0345
	3	52.0[a]	61.90	61.697	61.697			61.697	61.6972
Fixed–Fixed	1	0	22.532	22.373				22.373	22.3733
beam	2	0	63.177	61.677	61.673			61.673	61.6728
	3	0	158.338	129.280	122.883	121.029	120.904	120.904	120.9034
Fixed–hinged	1	0	15.471	15.418				15.418	15.4182
beam	2	0	52.870	50.022	49.965			49.965	49.9645
	3	0	145.851	97.265	106.739	103.373	104.493	104.248	104.2477
Hinged–hinged	1	0	9.883	9.870				9.870	9.87
beam	2	0	40.444	39.481	39.478			39.478	39.5
	3	0	131.550	79.38	91.62	89.23	88.83	88.823	88.9

[a]Experimental measurements.

The first iteration yields the following improved results:

$$\lambda_1^{(1)} = \frac{\omega_1^{(1)2}}{\dfrac{EI}{\rho A l^4}} = 3.5177, \quad X_1^{(1)} = \begin{Bmatrix} 0.246719 \\ 0.844948 \\ 0.726460 \\ 1.000000 \end{Bmatrix}, \quad (30)$$

and

$$\lambda_2^{(1)} = \frac{\omega_2^{(1)2}}{\dfrac{EI}{\rho A l^4}} = 17.761, \quad X_2^{(1)} = \begin{Bmatrix} 0.056902 \\ -0.315803 \\ -0.276364 \\ -1.000000 \end{Bmatrix}, \quad (31)$$

While the solution for the first mode is very close to the exact solution, the solution for the second mode remains in a significant error. However, using the iterated dynamic condensation, a good result for the second mode may also be obtained after 13 iterations. For example, the solution in the 14th iteration is

$$\lambda_2^{(14)} = \frac{\omega_2^{(14)2}}{\dfrac{EI}{\rho A l^4}} = 22.221, \quad X_2^{(14)} = \begin{Bmatrix} 0.149923 \\ -0.090211 \\ -0.207693 \\ -1.000000 \end{Bmatrix}. \quad (32)$$

EXAMPLE 2: Determination of natural frequencies of beams with general boundary conditions using the iterated dynamic condensation

To demonstrate efficiency of the procedure for a large-scale problem, a finer finite element mesh with 40 nodes and 39 elements for beams with general boundary conditions is used for modeling. The finite element model has a total of 80 degrees of freedom. The nodes are numbered in an ascending order from one end. Three rotations at node numbers 5, 16, and 35 and two translations at node numbers 11 and 26 are randomly selected as the testing coordinates. The selection does not violate the convergence criteria. Boundary conditions of the beam includes four appropriate combinations of the boundary conditions for the fixed, hinged, or free end, i.e., hinged–hinged, fixed–hinged, fixed–fixed, and fixed–free. Results for the first three natural frequencies are listed in Table 1 where there are two sets of exact solutions: one is from the full finite element model and the other is from the corresponding boundary value problem (Thomson, 1993). The results for the
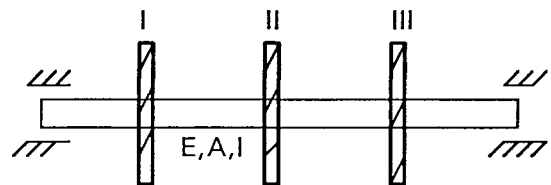
FIGURE 2   A three-disk shaft system.

Table 2. Measured Natural Frequencies and Mode Shapes of Multidisk Rotor System

Mode	f_1 = 47.43 Hz	f_2 = 125.52 Hz
y_1	0.0	0.0
y_2	0.1245	0.3550
y_3	0.2150	0.0357
y_4	0.1315	−0.3534
y_5	0.0	0.0

fixed–free (cantilever) beam starts with a set of experimental measurements and the other three cases start with results from the static condensation.

Table 1 shows that accurate results for the natural frequencies can be iteratively obtained from the dynamically condensed system with a much smaller size. Very few iterations are needed for these investigated cases. Selection of the testing coordinates is rather arbitrary, provided certain mentioned requirements are satisfied. The numerical program is self-compensated, i.e., small errors introduced in each iteration will not affect the final results. All iterated results are obtained in just a few seconds using a 66-MHz 486 PC. Results for the mode shapes are not listed due to their size. In general, the convergence rate for the mode shapes is relatively slow and, therefore, more iteration steps are required for good results.

In the dynamic condensation iteration, only one particular mode is of interest in each run of the computer program, which is a disadvantage. To improve the efficiency of the proposed method,

an algorithm is developed for that purpose to calculate an arbitrarily assigned number of the eigenmodes in any selected order. The algorithm is proven to be efficient in the dynamic condensation iteration, even for the generate cases where duplicate eigenvalues occur.

EXAMPLE 3: An application of the dynamic condensation in model testing of a three-disk shaft system.

Consider a multiple-disk rotor system, as shown in Fig. 2. The system consists of three disks assembled on a steel shaft whose ends are fixed in two bearings with thin 502 glue. The diameter and length of the shaft are 6 and 290 mm, respectively. The three steel disks are identical, 78 mm in diameter and 15 mm in thickness. Disk II is in the middle of the shaft and disks I and II are 60 mm from the nearest end. Three transducers are used to measure the lateral displacement of the disks. A mass of 54 g is attached to each disk to account for the mass of the transducers. For disk I, an additional mass of 49.2 g is attached. Table 2 lists the measured natural frequencies and modal vector.

Note that the listed mode shapes are incomplete because they do not include the measurements of slopes at the nodes due to the lack of reliable sensors.

A simplified five-node finite element model is derived. Two rotational springs are introduced on both ends to model the imperfect boundary conditions, i.e., the shaft is neither simply supported

Table 3. Natural Frequency and Modal Shape of First Mode

	First Iteration	Second Iteration	Third Iteration	Full Model	Experiment
Frequency (Hz)	$f_1^{(1)}$ = 49.948	$f_1^{(2)}$ = 47.163	$f_1^{(3)}$ = 47.112	f_1 = 47.112	f_1^* = 47.43
Incomplete modal shape	$\begin{Bmatrix} 0.15551 \\ 0.12674 \end{Bmatrix}$	$\begin{Bmatrix} 0.12598 \\ 0.12685 \end{Bmatrix}$	$\begin{Bmatrix} 0.12913 \\ 0.12685 \end{Bmatrix}$	$\begin{Bmatrix} 0.12922 \\ 0.12918 \end{Bmatrix}$	$\begin{Bmatrix} 0.12456 \\ 0.13150 \end{Bmatrix}$
Complete modal vector	$\begin{Bmatrix} 0.0000 \\ 1.32411 \\ 0.15551 \\ 2.73960 \\ 0.27257 \\ -0.02737 \\ 0.12674 \\ -2.2254 \\ 0.00000 \\ -1.07347 \end{Bmatrix}$	$\begin{Bmatrix} 0.0000 \\ 0.95347 \\ 0.12598 \\ 2.67353 \\ 0.27257 \\ -0.03023 \\ 0.12685 \\ -2.7006 \\ 0.0000 \\ -0.95778 \end{Bmatrix}$	$\begin{Bmatrix} 0.00000 \\ 0.99306 \\ 0.12914 \\ 2.68060 \\ 0.27257 \\ -0.02949 \\ 0.12685 \\ -2.6432 \\ 0.00000 \\ -0.9729 \end{Bmatrix}$	$\begin{Bmatrix} 0.00000 \\ 0.99406 \\ 0.12922 \\ 2.68078 \\ 0.27257 \\ -0.03001 \\ 0.12918 \\ -2.6902 \\ 0.00000 \\ -1.09911 \end{Bmatrix}$	$\begin{Bmatrix} 0.00000 \\ 0.94344 \\ 0.12450 \\ 2.63747 \\ 0.27257 \\ 0.07370 \\ 0.13150 \\ -2.68615 \\ 0.00000 \\ -1.02268 \end{Bmatrix}$

Table 4. Natural Frequency and Modal Shape of Second Mode

	First Iteration	Second Iteration	Full Model	Experiment
Frequency (Hz)	$f_2^{(1)} = 123.36$	$f_2^{(2)} = 122.04$	$f_2 = 122.04$	$f_2^* = 123.52$
Incomplete modal vector	$\left\{ \begin{array}{c} 0.35500 \\ -0.37349 \end{array} \right\}$	$\left\{ \begin{array}{c} 0.35500 \\ -0.35772 \end{array} \right\}$	$\left\{ \begin{array}{c} 0.35500 \\ -0.35778 \end{array} \right\}$	$\left\{ \begin{array}{c} 0.3550 \\ -0.3534 \end{array} \right\}$
Complete modal vector	$\left\{ \begin{array}{c} 0.0000 \\ 3.84758 \\ 0.35500 \\ 3.11701 \\ -0.01636 \\ -7.92037 \\ -0.37349 \\ 3.44424 \\ 0.00000 \\ 3.81103 \end{array} \right\}$	$\left\{ \begin{array}{c} 0.00000 \\ 3.65317 \\ 0.35500 \\ 3.86886 \\ -0.02013 \\ -9.12539 \\ -0.35772 \\ 4.06675 \\ 0.00000 \\ 3.63686 \end{array} \right\}$	$\left\{ \begin{array}{c} 0.0000 \\ 3.66428 \\ 0.35500 \\ 3.82637 \\ -0.02257 \\ -9.09147 \\ -0.35340 \\ 4.07015 \\ 0.00000 \\ 3.63671 \end{array} \right\}$	$\left\{ \begin{array}{c} 0.00000 \\ 3.74653 \\ 0.35500 \\ 3.51471 \\ -0.56469 \\ -9.03183 \\ -0.35340 \\ 4.32105 \\ 0.00000 \\ 3.51307 \end{array} \right\}$

nor fixed. Chen and Wang (1991) estimated the rotational constants

$$k_1 = k_2 = 786.393 \text{ N} \cdot \text{m}.$$

The stiffness and mass matrices are derived by standard finite element formulations for the three-disk rotor system with diagonal modifications in the stiffness matrix due to the rotational springs and those in the mass matrix due additional masses. Choose y_2 and y_4, i.e., the lateral displacements of disks I and III, as the testing coordinates. The iterated dynamic condensation with the initial estimates from the static condensation leads to the results in Tables 3 and 4.

As compared with the results from the full analysis of the four-element model, the errors of frequency estimate for the first mode is 6.0% for the static iteration, 1.1% for the second iteration, and 0.00006% for the third iteration; errors of the frequency estimate for the second mode is 1.06% for the static iteration and 0.00024% for the fourth iteration. Using the measurements of deflection of disks I and III in a model testing, the corresponding full-mode shape including the deflection of disk II and rotations at different nodes can be recovered. The results are normalized in terms of the maximum deflection measurement for each mode. The mean square errors for the deflection components of the first mode is 3.1% for the static condensation, 0.77% for the second iteration, and 0.46% for the third iteration; those for the second mode are 4% for the static condensation and 3.6% for the fourth iteration. The iteration process can be accelerated

if the frequency measurements are directly employed as the initial estimates. This is especially useful for complex systems due to the difficulties in selecting the testing coordinates for the interesting modes and the expense in solving the associated large-scale generalized eigenvalue problems.

CONCLUSIONS

Dynamic condensation provides an efficient and accurate solution for large-scale generalized eigenvalue problems. A procedure using the dynamic iteration technique in modal testing is proposed to obtain complete modal shapes based on limited measurement data. The measured natural frequencies can be employed as the initial estimates in the iteration to effectively find the exact solution for the original eigenvalue problem and determine the transformation matrix between the testing and the nontesting coordinates. The complete modal shapes including the nontesting coordinates can then be estimated based on the limited measurements. The approach may be useful for model testing of complex structures for which the natural frequency measurements are feasible and the measurement of the complete modal shapes is not practical.

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