# Iterated line graphs with only negative eigenvalues $-2$, their complements and energy

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**Abstract**

The graphs with all equal negative or positive eigenvalues are special kind in the spectral graph theory. In this article, several iterated line graphs $\mathcal{L}^k(G)$ with all equal negative eigenvalues $-2$ are characterized for $k \geq 1$ and their energy consequences are presented. Also, the spectra and the energy of complement of these graphs are obtained, interestingly they have exactly two positive eigenvalues with different multiplicities. Moreover, we characterize a large class of equienergetic graphs which generalize some of the existing results. There are two different quotient matrices defined for an equitable partition of $H$-join (generalized composition) of regular graphs to find the spectrum (partial) of adjacency matrix, Laplacian matrix and signless Laplacian matrix, it has been proved that these two quotient matrices give the same respective spectrum of graphs.

**Keywords:** Eigenvalues of graphs, Energy of a graph, Equitable partition, Iterated line graphs, Complement of iterated line graphs.
**Mathematics Subject Classification:** 05C50, 05C76.

## 1 Introduction

Spectral graph theory is aimed at answering the questions related to the graph theory in terms of eigenvalues of matrices which are naturally associated with graphs. Graphs with all equal positive or negative eigenvalues are very special kind in the study of eigenvalues of graphs and its easy to find energy on these class of graphs. Also, the graphs with an

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exact number of positive or negative eigenvalues has applications in finding equienergetic graphs. The line graph of a graph has a special property that its least eigenvalue is not smaller than $-2$. In the last two decades graphs with least eigenvalues $-2$ have been well studied [9, 10, 39]. The problem of particular interest in this class of graphs is the graphs with all negative eigenvalues equal to $-2$. This problem can be restated in terms of the eigenvalues of signless Laplacian matrix Q for line graphs as the graphs with all its signless Laplacian eigenvalues belongs to the set $[2, \infty) \cup \{0\}$ with the help of the relation (1). In [33] Ramane et al. obtained the spectra and the energy of iterated regular line graphs $\mathcal{L}^k(G)$ for $k \geq 2$ and presented infinitely many pairs of non-trivial equienergetic graphs which belong to the above class of graphs. Let $\varrho$ be the property that a graph $G$ has all its negative eigenvalues equal to $-2$. In this paper, we are motivated to find several classes of iterated line graphs $\mathcal{L}^k(G)$ with the property $\varrho$ for $k \geq 1$, spectra of their complements and energy consequences. As a consequence of energy of these graphs we present a large class of equienergetic graphs which generalize some of the existing results. The energy relation between a regular graph $G$ and its complement were studied in [27, 28, 29, 30, 32], we extend the results pertaining $\mathcal{E}(\overline{G}) = \mathcal{E}(G)$ to non-regular iterated line graphs with the property $\rho$. The energy of line graphs is well studied in [11, 18, 24, 33, 34]. In this paper, we also present hyperenergetic iterated line graphs and their complements. The software sage math [37] is used to verify some of the results.

This paper is organized as follows. In section 2, basic definitions, equitable partition, known results on spectra and energy of graphs are presented. In section 3, it is proved that the two different quotient matrices defined for an equitable partition of $H$-join of regular graphs give the same spectrum(partial) for adjacency matrix, Laplacian matrix and signless Laplacian matrix. Section 4 provides the results on spectra and energy of iterated line graphs satisfying the property $\rho$. Section 5 deals with the spectra and the energy of complements of graphs which are presented in section 4. In section 6, upper bound is given for independence number of iterated line graphs and their complements. Also, a result on minimum order of the connected graph when its complement is connected and line graph satisfying the property $\rho$.

## 2 Preliminaries

In this paper, simple and undirected graphs $G$ are considered with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Let $e = v_iv_j$ be an edge of $G$ with its end vertices $v_i$ and $v_j$. The order and size of a graph $G$ is the number of vertices and number of edges respectively in it. The complement $\overline{G}$ of a graph $G$ has same vertices as in $G$ but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. Let the subgraph of a graph $G$ obtained by deleting vertices $v_1, v_2, \dots, v_k$, $k < n$ and the edges incident to them in $G$ be $G - \{v_1, v_2, \dots, v_k\}$ and simply $G - v$ if one vertex $v$ and edges incident to it are deleted. The **adjacency matrix** $A(G)$ or simply $A$ of a graph $G$ of order $n$ is the $n \times n$ matrix indexed by $V(G)$ whose $(i, j)$-th entry is defined as $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. The **Laplacian matrix** $L$ and **signless Laplacian matrix** $Q$ of a graph is defined as $L = D - A$ and $Q = D + A$ respectively, where $D = [d_i]$ is the diagonal degree matrix of suitable order whose $i$-th diagonal entry $d_i$ is the degree of the vertex $v_i$. The matrix $Q$ is positive semi-definite and has real eigenvalues. Given any square matrix $M$, the multiset of its eigenvalues is called the **spectrum** of $M$. Denote the characteristic polynomial and spectrum of the matrix $M$, respectively by $\varphi(M; x)$ and $Sp(M)$. Let $aSp(M) \pm b = \{a\lambda \pm b : \lambda \in Sp(M)\}$ for any two real numbers $a$ and $b$. The spectrum of a graph is the spectrum of its adjacency matrix $A$. The positive (negative) inertia of a graph $G$ is the number of positive (negative) eigenvalues of $G$ denoted by $n^+(n^-)$. Given a graph $G$ we denote the spectrum of adjacency matrix and signless Laplacian matrix respectively by $Sp_A(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_k^{m_k}\}$ and $Sp_Q(G) = \{q_1^{m_1}, q_2^{m_2}, \dots, q_k^{m_k}\}$, where $\lambda_i$'s and $q_i$'s are indexed in descending order, and $m_i$ is the multiplicity of the respective eigenvalue for $1 \le i \le k$. Denote the least eigenvalue of signless Laplacian by $q_{\min}$. The **energy** [16] of a graph $G$ is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| = 2\sum_{i=1}^{n^+} \lambda_i = -2\sum_{i=1}^{n^-} \lambda_{n-i+1}.$$

Two graphs $G_1$ and $G_2$ of same order are said to be **equienergetic** if their energies are equal. A set of vertices in a graph $G$ is independent if no two vertices in the set are adjacent. The **independence number** $\alpha(G)$ of a graph $G$ is the maximum cardinality of an independent set of vertices in $G$. As usual the graphs $C_n, K_n$ and $P_n$ denote the cycle, complete graph and path on $n$ vertices respectively.

**Definition 2.1.** [26] *The* **Turán graph** $T_r(n)$, $r \geq 1$, *is the complete* $r$*-partite graph of order* $n$ *with all parts of size either* $\lfloor n/r \rfloor$ *or* $\lceil n/r \rceil$.

**Definition 2.2.** [20] *The* **line graph** $\mathcal{L}(G)$ *of a graph* $G$ *is the graph with vertex set same as the edge set of* $G$*, two vertices in* $\mathcal{L}(G)$ *are adjacent if the corresponding edges in* $G$ *have a vertex in common. The* $k$*-th* **iterated line graph** *of* $G$ *for* $k \in \{0, 1, 2, \ldots\}$ *is defined as* $\mathcal{L}^k(G) = \mathcal{L}(\mathcal{L}^{k-1}(G))$*, where* $\mathcal{L}^0(G) = G$ *and* $\mathcal{L}^1(G) = \mathcal{L}(G)$.

Let $n_k$ and $m_k$ denote the order and the size of $\mathcal{L}^k(G)$ for $k \in \{0, 1, 2, \ldots\}$. It is noted that $n_k = m_{k-1}$ for $k \in \{1, 2, \ldots\}$. Let us denote the eigenvalues of $\mathcal{L}^k(G)$ and its complement $\overline{\mathcal{L}^k(G)}$ respectively by $\lambda_{k(j)}, \overline{\lambda}_{k(j)}$ for $k \in \{1, 2, \ldots\}$ and $1 \leq j \leq n_k$. The complement of a line graph is also called jump graph [6]. The important spectral property of line graph is that its least eigenvalue is not less than $-2$ [22].

**Theorem 2.3.** [8] *If* $G$ *is a* $r$*-regular graph of order* $n$ *with spectrum* $Sp_A(G) = \{r, \lambda_2, \ldots, \lambda_n\}$*, then the complement graph of* $G$ *is a* $(n-r-1)$*-regular graph with spectrum* $Sp_A(\overline{G}) = \{n-r-1, -1-\lambda_n, \ldots, -1-\lambda_2\}$.

**Theorem 2.4.** [35] *If* $G$ *is a* $r$*-regular graph of order* $n$ *and size* $m$ *with spectrum* $Sp_A(G) = \{r, \lambda_2, \ldots, \lambda_n\}$*, then line graph of* $G$ *is a* $(2r-2)$*-regular graph with spectrum* $Sp_A(\mathcal{L}(G)) = \{2r-2, \lambda_2 + r - 2, \ldots, \lambda_n + r - 2, -2^{m-n}\}$.

Let $n$ and $m$ be the order and the size of a graph $G$. The following is the relation between the eigenvalues of line graph and signless Laplacian of a graph $G$ [7]

$$Sp_A\big(\mathcal{L}(G)\big) = \{-2^{m-n}\} \cup \big(Sp_Q(G) - 2\big). \tag{1}$$

the multiplicity of the eigenvalue $-2$ in $\mathcal{L}(G)$ is equal to $m - n + 1$ if $G$ is bipartite and equal to $m - n$ if $G$ is not bipartite [9].

**Theorem 2.5.** [11] *Let* $G$ *be a graph of order* $n > 2$ *with* $m$ *edges and minimum degree* $\delta$*. If* $\delta \geq \frac{n}{2} + 1$*. Then the line graph* $\mathcal{L}(G)$ *satisfies the property* $\rho$ *with energy* $4(m-n)$*. Thus, the line graphs of all such graphs of order* $n$ *and size* $m$ *with property* $\delta \geq \frac{n}{2} + 1$ *are mutually equienergetic.*

**Theorem 2.6.** [24] *Let* $G$ *be a graph of order* $n \geq 5$ *and size* $m$*. If* $m \geq 2n$*, the line graph* $\mathcal{L}(G)$ *is hyperenergetic.*

The following is an elegant relation even though not sharp between the smallest eigenvalue $\lambda_{min}$ of $A$ and the smallest eigenvalue $q_{min}$ of $Q$.

**Proposition 2.7.** [14] *If $G$ is a graph with minimum degree $\delta$ and maximum degree $\Delta$, then*

$$q_{min} - \Delta \leq \lambda_{min} \leq q_{min} - \delta.$$

**Proposition 2.8.** [13] *If $G$ is a spanning subgraph of a graph $G'$, then $q_{min}(G) \leq q_{min}(G')$.*

**Theorem 2.9.** [21] *If $G$ is a graph with vertex $v$, then $q_{min}(G) - 1 \leq q_{min}(G - v)$.*

Let $\mathbf{j}$ denotes all one's column vector.

**Theorem 2.10.** [19] *If $\lambda \in Sp_A(G)$, then $-1-\lambda \in Sp_A(\overline{G})$ if and only if $\mathbf{j}^T Y = 0$ for some eigenvector $Y$ of $G$ corresponding to the eigenvalue $\lambda$.*

**Proposition 2.11.** [8] *In the line graph $\mathcal{L}(G)$ of a graph $G$ the eigenspace of the eigenvalue $-2$ is orthogonal to the vector $\mathbf{j}$.*

The Weyl's eigenvalue inequality [23] $\lambda_j(M_1) + \lambda_k(M_2) \leq \lambda_i(M_1 + M_2)$, $j + k - n \geq i$ for sum of two Hermitian matrices $M_1$ and $M_2$ of order $n$ gives the following useful eigenvalues inequality on a graph $G$ of order $n$ and its complement $\overline{G}$ [27].

$$\lambda_j + \overline{\lambda}_{n-j+2} \leq -1 \text{ for } j \in \{2, 3, \ldots, n\}. \tag{2}$$

**Proposition 2.12.** [15] *Let $M$ be any square matrix of order $n$ with the characteristic polynomial $\varphi(M; x) = \sum\limits_{r=0}^{n} (-1)^n m_r x^{n-r}$. Then $m_r$ is equal to the sum of the principal minors of $M$ of order $r$.*

**Definition 2.13.** [2] *Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The* **extended bipartite double** *$Ebd(G)$ of a graph $G$ is the bipartite graph with its partite sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ in which two vertices $x_i$ and $y_j$ are adjacent iff $i = j$ or $v_i$ and $v_j$ are adjacent in $G$. The $Ebd(G)$ is also called the extended double cover* [1].

**Theorem 2.14.** [2] *Let $Ebd(G)$ is the extended bipartite double of a graph $G$.*

1. *If $G$ is connected, then $Ebd(G)$ is connected.*

*2. The spectrum of $Ebd(G)$ is $Sp_A(Ebd(G)) = (-Sp_A(G) - 1) \cup (Sp_A(G) + 1)$.*

**Definition 2.15.** [36] *If $G$ is a graph of order $n$ with labeled vertices, then the graph $G[H_1, H_2, \ldots, H_n]$ called* **generalized composition** *or $H$-***join** [4] *which is obtained from the graphs $H_1, H_2, \ldots, H_n$ by joining every vertex of $H_i$ to every vertex of $H_j$ if $v_i$ is adjacent to $v_j$ in $G$.*

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_p)$ be a partition of a vertex set of a graph $G$. The partition $\pi$ is called **equitable partition** [5, 36] if for each $i, j = 1, 2, \ldots, p$ there exists a number $c_{ij}$ such that for every vertex $v \in \pi_i$ there are exactly $c_{ij}$ edges between $v$ and the vertices in $\pi_j$. If $\pi$ is an equitable partition, then the associated $p \times p$ matrix with rows and columns corresponding to the partite sets $\pi_1, \pi_2, \ldots, \pi_p$ is called **quotient matrix**. Let $A_\pi$ is the $p \times p$ matrix with $(ij)$-th element $a^\pi_{ij}$ equal to $c_{ij}$, and the $p \times p$ diagonal matrix $D_\pi$ whose $i$-th diagonal element equal to $\sum_{k=1}^{p} c_{ik}$. If $\pi$ is an equitable partition, we denote the quotient matrix for adjacency, Laplacian and signless Laplacian matrices of a graph, respectively by $A_\pi, L_\pi$ and $Q_\pi$. The matrices $A_\pi, L_\pi$ and $Q_\pi$ are given by $A_\pi = [a^\pi_{ij}]$ [36], $L_\pi = [l^\pi_{ij}] = D_\pi - A_\pi$ [3] and $Q_\pi = [q^\pi_{ij}] = D_\pi + A_\pi$ [5, 40]. It is noted that these are non-symmetric matrices.

Let $H_i$ be $r_i$-regular graphs of order $n_i$ for $i = 1, 2, \ldots, p$. In case of $H$-join $G[H_1, H_2, \ldots, H_p]$, where $G$ is a graph of order $p$, let us denote the quotient matrix for adjacency, Laplacian and signless Laplacian of a graph, respectively by $A^H_\pi = [a^H_{ij}]$, $L^H_\pi = [l^H_{ij}]$ and $Q^H_\pi = [q^H_{ij}]$. These matrices are defined as [4, 41]

$$a^H_{ij} = \begin{cases} c_{ij} & \text{if } i = j \\ \sqrt{n_i n_j} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases},$$

$$l^H_{ij} = \begin{cases} l^\pi_{ij} & \text{if } i = j \\ -\sqrt{n_i n_j} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad q^H_{ij} = \begin{cases} q^\pi_{ij} & \text{if } i = j \\ \sqrt{n_i n_j} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

It is noted that these are symmetric matrices.

If $\pi = (\pi_1, \pi_2, \ldots, \pi_p)$ is an equitable partition of a graph $G$ with cardinality of $\pi_i$, $|\pi_i| = m_i$ for $i = 1, \ldots, p$, then $\pi$ also equitable partition to its complement $\overline{G}$. The quotient matrix $\overline{A}_\pi$ of $\overline{G}$ is given by $\overline{A}_\pi = \mathbf{J}_\pi - I - A_\pi$ [38], where $\mathbf{J}_\pi$ is the matrix of order $p$ whose $(i,j)$-th element equal to $m_j$ and $I$ is the identity matrix of order $p$.

**Proposition 2.16.** [38] *Let the graphs $G_1$ and $G_2$ are co-spectral with equitable partitions ${}^1\pi$ and ${}^2\pi$ respectively. If $A_{{}^1\pi} = A_{{}^2\pi}$, then the graphs $\overline{G_1}$ and $\overline{G_2}$ are co-spectral.*

**Theorem 2.17.** [41] *Let $G$ be a labeled graph of order $n$ and $H_i$ be $r_i$-regular graph of order $n_i$ for $i = 1, 2, \ldots, n$. If $\mathsf{G} = G[H_1, H_2, \ldots, H_n]$, then*

$$Sp_Q(\mathsf{G}) = \Big(\cup_{i=1}^{n}\Big((q_{ii}^{\pi} - 2r_i) + (Sp_Q(H_i) \setminus \{2r_i\})\Big)\Big) \cup \Big(Sp(Q_{\pi}^{H})\Big).$$

Suppose that $n \geq 2$ and $M = [m_{ij}] \in \mathbb{C}^{n\times n}$. The Geršgorin discs $D_i, i = 1, 2, \ldots, n$ of the matrix $M$ are defined as the closed circular regions $D_i = \{z \in \mathbb{C} : |z - m_{ii}| \leq R_i\}$ in the complex plane, where

$$R_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |m_{ij}| \quad \text{is the radius of } D_i.$$

**Theorem 2.18** (Geršgorin)**.** [23] *Let $n \geq 2$ and $M \in \mathbb{C}^{n\times n}$. All eigenvalues of the matrix $M$ lie in the region $D = \cup_{i=1}^{n} D_i$, where $D_i, i = 1, 2, \ldots, n$ are the Geršgorin discs of $M$.*

# 3 Spectra of Quotient Matrices

**Theorem 3.1.** *Let $\mathbf{G} = G[H_1, H_2, \ldots, H_n]$, where $H_i$ is $r_i$-regular graph of order $n_i$ for $i = 1, 2, \ldots, n$. Then the spectrum of the quotient matrices $A_\pi, L_\pi$ and $Q_\pi$ equal to the spectrum of the quotient matrices $A_\pi^H, L_\pi^H$ and $Q_\pi^H$ respectively.*

*Proof.* It can be easily seen that $\pi = (V(H_1), V(H_2), \ldots, V(H_n))$ is an equitable partition of $\mathbf{G}$. Let us first prove the result for the matrices $Q_\pi$ and $Q_\pi^H$ with the help of Proposition 2.12 and similar can be applied for the remaining matrices. The entries of the matrices $Q_\pi$ and $Q_\pi^H$ can be written as $q_{ij}^{\pi} = a_{ij}n_j$, $q_{ij}^{H} = a_{ij}\sqrt{n_i n_j}$ for $i \neq j$ and $q_{ii}^{\pi} = q_{ii}^{H}$, where $a_{ij}$ is the entry of the adjacency matrix of $G$. Let $S_n$ be the set of all permutations over the set $\{1, 2, \ldots, n\}$ and if $\sigma \in S_n$ denote its sign by $sgn(\sigma)$. Let $M_r^{'}$ be the principal minor of order $r$ which is obtained by deleting any $n - r$ rows and corresponding columns of the matrix $Q_\pi$ and the respective principal minor of the matrix $Q_\pi^H$ be $M_r^{''}$ for $1 \leq r \leq n$.

$$
\begin{aligned}
M_r^{'} &= \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r} q^{\pi}_{k\sigma(k)} = \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r} a_{k\sigma(k)} n_{\sigma(k)} \\
&= \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r} a_{k\sigma(k)} n_k \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
M_r^{''} &= \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r} q^{H}_{k\sigma(k)} = \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r} a_{k\sigma(k)} \sqrt{n_k n_{\sigma(k)}} \\
&= \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r} a_{k\sigma(k)} n_k \quad \text{if } k \neq \sigma(k) \text{ for any } k = 1, 2, \ldots, r.
\end{aligned}
$$

If $k = \sigma(k)$ for $k = 1, 2, \ldots, p$ and $1 \leq p \leq r$, then we have

$$
M_r^{'} = M_r^{''} = \sum_{\sigma \in S_r} sgn(\sigma) \prod_{k=1}^{r-p} a_{k\sigma(k)} n_k (q^{\pi}_{kk})^p.
$$

Which shows that $M_r^{'} = M_r^{''}$ to each $r$ for $1 \leq r \leq n$. Hence, the sum of principal minors of order $r$ of the matrices $Q_\pi$ and $Q_\pi^H$ are equal. It shows that the matrices $Q_\pi$ and $Q_\pi^H$ have the same characteristic polynomials by Proposition 2.12. Thus, the matrices $Q_\pi$ and $Q_\pi^H$ have the same spectrum. □

Now Theorem 2.17, with the help of above Theorem can be stated in the following way.

**Proposition 3.2.** *Let $G$ be a labeled graph of order $n$ and $H_i$ be $r_i$-regular graph of order $n_i$ for $i = 1, 2, \ldots, n$. If $\mathsf{G} = G[H_1, H_2, \ldots, H_n]$, then*

$$
Sp_Q(\mathsf{G}) = \Big( \cup_{i=1}^{n} \Big( R_i + (Sp_Q(H_i) \setminus \{2r_i\}) \Big) \Big) \cup \Big( Sp(Q_\pi) \Big)
$$

*where $R_i$ is the $i$-th row sum excluding the diagonal entry of $Q_\pi$ for $i = 1, 2, \ldots, n$.*

*Proof.* Proof follows by Theorem 3.1 and an observation that $q^{\pi}_{ii} = 2r_i + R_i$ for $i = 1, 2, \ldots, n$. □

# 4 Spectra and Energy of iterated line graphs with the property $\rho$

**Theorem 4.1.** *Let $G$ be a graph of order $n_0$ and size $m_0$ with $d_u + d_v \geq 6$ to each edge $e = uv$ in $G$. Then the graphs $\mathcal{L}^k(G)$ satisfy the property $\varrho$ for $k \geq 2$. And all the iterated line graphs $\mathcal{L}^k(G)$ of such graphs $G$ are mutually equienergetic with energy $4(n_k - n_{k-1})$ for $k \geq 2$.*

*Proof.* If $G$ is a graph of order $n_0$ and size $m_0$ with an edge $e = uv$, then the degree of a vertex corresponding to an edge $e$ in $\mathcal{L}(G)$ is $d_u + d_v - 2$. But the condition $d_u + d_v \geq 6$ to each edge $e = uv$ in $G$ implies $d_u + d_v - 2 \geq 4$ which shows that the minimum degree of each vertex in $\mathcal{L}(G)$ is at least four. It is well known that the least eigenvalue of the line graph $\mathcal{L}(G)$ is not smaller than $-2$. Hence, the least eigenvalues $\lambda_{min}, q_{min}$ and the minimum degree $\delta$ of $\mathcal{L}(G)$ by Proposition 2.7 satisfy $q_{min} \geq \lambda_{min} + \delta \geq -2 + 4 = 2$. Now, by relation (1) $\mathcal{L}(\mathcal{L}(G)) = \mathcal{L}^2(G)$ satisfies the property $\rho$ with the multiplicity of $-2$ equal to $m_1 - n_1$. One can find easily that the minimum degree $\delta$ in the line graphs $\mathcal{L}^k(G)$ increases as $k$ increases, which shows that $q_{min}$ of $\mathcal{L}^k(G)$ increases as $k$ increases and which is at least 2. Hence, by relation (1) the iterated line graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ to each $k \geq 2$ with their energy equal to $4(m_{k-1} - n_{k-1}) = 4(n_k - n_{k-1})$. □

A tree graph is called caterpillar if the removal of all pendant vertices in it makes a path graph. The spectra and energy of line graph of caterpillars is studied in [34]

**Example 4.2.** The graph in Figure 1 is the caterpillar $C(4, 3, 4)$. This graph satisfies the conditions in Theorem 4.1, therefore all the graphs $\mathcal{L}^k\big(C(4, 3, 4)\big)$ satisfy the property $\rho$ for $k \geq 2$.

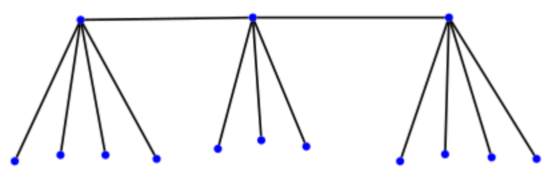

Figure 1

**Corollary 4.3.** *Let $G$ be a graph of order $n_0$ and size $m_0$ with minimum degree $\delta \geq 3$. Then the graphs $\mathcal{L}^k(G)$ satisfy the property $\varrho$ for $k \geq 2$. And all the iterated line graphs $\mathcal{L}^k(G)$ of such graphs $G$ are mutually equienergetic with energy $4(n_k - n_{k-1})$ for $k \geq 2$.*

**Remark 4.4.** In [33] Ramane et al. obtained the spectra and energy of iterated line graphs of regular graphs of degree $r \geq 3$ and thereby characterized large class of pairs of non-trivial equienergetic regular graphs. It is noted that the results of [33] become particular case of Corollary 4.3.

Now, Theorem 4.1 naturally motivates to discuss when $\mathcal{L}(G)$ satisfies the property $\rho$ as $\mathcal{L}^k(G)$ satisfy the property $\rho$ for $k \geq 2$ in a graph $G$ with $d_u + d_v \geq 6$ to each edge $e = uv$. In this direction we have the following results.

**Lemma 4.5.** *Let $G$ be a labeled connected graph of order $n \geq 2$ and $H_i$ be $r_i$-regular graph of order $n_i$, where $r_i \geq 1$, for $i = 1, 2, \dots, n$. Then the least eigenvalue of the quotient matrix $Q_\pi$ of the graph $G[H_1, H_2, \dots, H_n]$ is at least 2.*

*Proof.* The signless Laplacian $Q$ of a graph $G$ is positive semi-definite and has real eigenvalues which implies that the eigenvalues of quotient matrix $Q_\pi$ are real by Proposition 3.2. Observe that in $i$-th row of $Q_\pi$ the diagonal entry $q_{ii}^\pi$ satisfies the equality $q_{ii}^\pi = 2r_i + R_i$, where $R_i$ is the $i$-th row sum excluding $q_{ii}^\pi$ of $Q_\pi$ for $i = 1, 2, \dots, n$. By using Geršgorin Theorem, all the eigenvalues of $Q_\pi$ belong to the union of the closed intervals $[2r_i, 2(r_i + R_i)]$ for $i = 1, 2, \dots, n$. Now, the result follows from the condition that $r_i \geq 1$. □

**Theorem 4.6.** *Let $G$ be a labeled connected graph of order $n \geq 2$ and $H_i$ be $r_i$-regular graph of order $p_i$, where $r_i \geq 1$, for $i = 1, 2, \dots, n$. If $\mathsf{G} = G[H_1, H_2, \dots, H_n]$ is a graph of order $n_0$ and size $m_0$, then the graphs $\mathcal{L}^k(\mathsf{G})$ satisfy the property $\rho$ for $k \geq 1$. And the line graphs of graphs $G'$ of order $n_k$ and size $m'_k$ for which $\mathcal{L}^k(\mathsf{G})$ is a spanning subgraph are mutually equienergetic with energy $4(m'_k - n_k)$ for $k \geq 0$.*

*Proof.* As $G$ is connected graph of order $n \geq 2$, for every vertex $v_i$ in $G$ there exists at least one adjacent vertex $v_j$ which shows that each vertex of the graph $H_i$ is adjacent to every vertex of $H_j$ for at least one $j$ in $\mathsf{G} = G[H_1, H_2, \dots, H_n]$. But $H_i$ are $r_i$-regular graph of order $p_i$ with $r_i \geq 1$ implies $p_i \geq 2$. Thus, the quotient matrix $Q_\pi$ corresponding to an equitable partition $\pi = (V(H_1), V(H_2), \dots, V(H_n))$ has non-diagonal entry $q_{ij}^\pi \geq 2$ in its $i$-th row for at least one $j$. Since, the signless Laplacian matrix $Q$ of any graph is positive semi-definite, it has all non-negative eigenvalues. By using Proposition 3.2 and Lemma 4.5, we get that every signless Laplacian eigenvalue of $\mathsf{G}$ is at least 2. Now by relation (1), the line graph $\mathcal{L}(\mathsf{G})$ satisfies the property $\rho$. It can be observed that the minimum order graph $\mathsf{G}$ with possible least degrees is $\mathsf{G} = K_2[K_2, K_2]$, which shows that minimum degree $\delta$ of $\mathsf{G}$ is at least 3. Thus, by Corollary 4.3 all the iterated line graphs $\mathcal{L}^k(\mathsf{G})$ for $k \geq 1$ satisfy the property $\rho$. Since, $\mathsf{G}$ is a spanning subgraph of $G'$, from Proposition 2.8 $G'$ also has $q_{min}$ at least 2 which implies that the graph $\mathcal{L}(G')$ also satisfies the property $\rho$ with its energy equal to $4(m'_k - n_k)$ for $k \geq 0$ by relation (1). This completes the proof. □

**Remark 4.7.** Let $G$ be a labeled connected graph of order $n \geq 2$ and $H_i$ be $r_i$-regular graph of order $p_i$ with $r_i \geq 1$, for $i = 1, 2, \ldots, n$. Let $\mathsf{G} = G[H_1, \ldots, H_j, \ldots, H_n]$ for $1 \leq j \leq i$ and $H_{s_1}$, $H_{s_2}$ be two $r$-regular graphs of same order with $r \geq 1$. If $\mathsf{G}_1 = G[H_1, \ldots, H_{s_1}, \ldots, H_n]$ and $\mathsf{G}_2 = G[H_1, \ldots, H_{s_2}, \ldots, H_n]$, then the graphs $\mathcal{L}^k(\mathsf{G}_1)$ and $\mathcal{L}^k(\mathsf{G}_2)$ are equienergetic for $k \geq 1$. And also if $H_{s_1}$ and $H_{s_2}$ are non co-spectral (co-spectral) graphs, then we get non co-spectral (co-spectral) graphs $\mathcal{L}^k(\mathsf{G}_1)$ and $\mathcal{L}^k(\mathsf{G}_2)$ respectively for $k \geq 0$ as they have same quotient matrices.

**Remark 4.8.** In Theorem 2.5, K. C. Das et al. characterized large class of equienergetic line graphs of graphs of order $n$ with the condition on minimum degree $\delta \geq \frac{n}{2}+1$. But one can construct equienergetic line graphs of graphs of order $n$ with the condition $\delta \leq \frac{n}{2} + 1$ by using Theorem 4.6. For example if $\mathsf{G} = K_2[K_2, C_n]$, $n \geq 6$ we can always construct non-isomorphic equienergetic line graphs of graph $\mathsf{G}$ with energy equal to $4 \times$ (size of $\mathsf{G}-$order of $\mathsf{G}$) for $\delta = 4$ by adding the edges between non-adjacent vertices of $\mathsf{G}$.

**Theorem 4.9.** *Let $G$ be a labeled connected graph of order $n \geq 2$ and $H_i$ be $r_i$-regular graph of order $p_i$, where $r_i \geq 2$, for $i = 1, 2, \ldots, n$. If $\mathsf{G} = G[H_1, H_2, \ldots, H_n]$, then the graphs $\mathcal{L}^k(\mathsf{G} - v)$ satisfy the property $\rho$ for $k \geq 1$. Moreover, if $n_i \geq min\{2r_i : r_i \geq s, s \in \{2, 3, \ldots\}\}$, then the graphs $\mathcal{L}^k(\mathsf{G} - \{v_1, v_2, \ldots, v_{2(s-1)}\})$ satisfy the property $\rho$ for $k \geq 1$.*

*Proof.* All the eigenvalues of $Q_\pi$ of $\mathsf{G}$ belongs to the union of the closed intervals $[2r_i, 2(r_i+R_i)]$ for $i = 1, 2, \ldots, n$ by Lemma 4.5 which shows that each eigenvalue of $Q_\pi$ is at least 4 as $r_i \geq 2$. Each row of the matrix $Q_\pi$ has at least one non-diagonal entry at least 3 as $r_i \geq 2$ and $G$ is connected of order $n \geq 2$. By Proposition 3.2 the $q_{min}$ of $\mathsf{G}$ is at least 3. And by Theorem 2.9 the $q_{min}$ of $\mathsf{G} - v$ is at least 2. Hence, $\mathcal{L}(\mathsf{G} - v)$ satisfies the property $\rho$ by relation (1). If $n_i \geq min\{2r_i : r_i \geq s, s \in \{2, 3, \ldots\}\}$, then the $q_{min}$ of $Q_\pi$ is at least $2s$, and each row of $Q_\pi$ has at least one non-diagonal entry at least $2s$ with $s \in \{2, 3, \ldots\}$. By Proposition 3.2 the $q_{min}$ of $\mathsf{G}$ is at least $2s$. It is easy to observe by Theorem 2.9 that $q_{min}(\mathsf{G}) - 2(s-1) \leq q_{min}\big(\mathsf{G} - \{v_1, v_2, \ldots, v_{2(s-1)}\}\big)$. Hence,

$\mathcal{L}\big(\mathsf{G}-\{v_1,v_2,\ldots,v_{2(s-1)}\}\big)$ satisfies the property $\rho$ by relation (1). The graphs $\mathsf{G}-v$ for $r_i \geq 2$ and $\mathsf{G}-\{v_1,v_2,\ldots,v_{2(s-1)}\}$ for $n_i \geq min\{2r_i : r_i \geq s, s \in \{2,3,\ldots\}\}$ both are graphs with minimum degree $\delta \geq 3$. Thus by Corollary 4.3 all the iterated line graphs $\mathcal{L}^k(\mathsf{G}-v)$ and $\mathcal{L}^k(\mathsf{G}-\{v_1,v_2,\ldots,v_{2(s-1)}\})$ for $k \geq 1$ satisfy the property $\rho$. □

The following is an interesting result due to the minimum number of edges in join of two connected graphs.

**Corollary 4.10.** *If $m,n \geq 3$, then the graphs $\mathcal{L}^k\big(K_2[P_n,P_m]\big)$ satisfy the property $\rho$ for $k \geq 1$.*

*Proof.* Let $G = K_2, H_1 = C_{n+1}$ and $H_2 = C_{m+1}$ for $m,n \geq 3$ in the above Theorem. Let $v_1 \in H_1$ and $v_2 \in H_2$, then deleting the vertices $v_1$ and $v_2$ and the edges incident to them respectively in $H_1$ and $H_2$ we get $\mathsf{G}-\{v_1,v_2\} = K_2[P_n,P_m]$. Hence, the graphs $\mathcal{L}^k(K_2[P_n,P_m])$ satisfy the property $\rho$ for $k \geq 1$. □

**Definition 4.11.** [17] *Let $v$ be the vertex of the complete graph $K_n$, $n \geq 3$ and let $e_i$, $i = 1,\ldots,p$, $1 \leq p \leq n-1$ be its distinct edges all being incident to $v$. The graph $Ka_n(p)$ is obtained by deleting $e_i$, $i = 1,\ldots,p$ from $K_n$, where $Ka_n(0) = K_n$.*

With this notation we have the following result.

**Theorem 4.12.** *If $n \geq 6$, then the graphs $\mathcal{L}^k\big(Ka_n(p)\big)$, $1 \leq p \leq n-4$ satisfy the property $\rho$ for $k \geq 1$.*

*Proof.* All the graphs of order up to 5 with their line graphs satisfying the property $\rho$ are $C_4, K_4, K_{3,2}, K_5$ by Theorem 6.7. It is noted that none of these graphs are of the kind $Ka_n(p)$ for $p \geq 1$. If $n \geq 6$, the graph $Ka_n(n-4) = P_3[K_1,K_3,K_{n-4}]$. The quotient matrix $Q_\pi$ of $Ka_n(n-4)$ is

$$\begin{bmatrix} 3 & 3 & 0 \\ 1 & n+1 & n-4 \\ 0 & 3 & 2n-7 \end{bmatrix}$$

with its spectrum $Sp(Q_\pi) = \{n-\frac{1}{2}+\frac{1}{2}\sqrt{4n(n-7)+73}, n-2, n-\frac{1}{2}-\frac{1}{2}\sqrt{4n(n-7)+73}\}$. The signless Laplacian spectrum of $Ka_n(n-4)$ is $Sp_Q\big(Ka_n(n-4)\big) = \{(n-2)^2,(n-3)^{n-5}\} \cup Sp(Q_\pi)$. It is clear that all signless Laplacian eigenvalues of $Ka_n(n-4)$ are greater than or equal to 2 except the doubt about the third eigenvalue of $Sp(Q_\pi)$ for

$n \geq 6$. But this eigenvalue $n - \frac{1}{2} - \frac{1}{2}\sqrt{4n(n-7)+73} \geq 2$ if $n \geq 5$. Thus $\mathcal{L}\big(Ka_n(p)\big)$ for $1 \leq p \leq n-4$ satisfies the property $\rho$ by using Proposition 2.8 and the relation (1). It is easy to observe that the minimum degree of $Ka_n(p)$ for $1 \leq p \leq n-4$ is at least 3. Hence, by Corollary 4.3 all the iterated line graphs $\mathcal{L}^k(Ka_n(p))$ for $k \geq 1$ and $1 \leq p \leq n-4$ satisfy the property $\rho$. □

There are some class of graphs with least eigenvalue $-2$ like exceptional graphs and generalized line graphs [9]. If minimum degree $\delta \geq 4$ on these class of graphs, we have the following simple result.

**Theorem 4.13.** *Let $G$ be a graph with least eigenvalue $-2$ and minimum degree $\delta \geq 4$. Then the iterated line graphs $\mathcal{L}^k(G)$ satisfy the property $\varrho$ for $k \geq 1$.*

*Proof.* The least eigenvalues $\lambda_{min}, q_{min}$ and the minimum degree $\delta$ of $G$ by Proposition 2.7 satisfy $q_{min} \geq \lambda_{min} + \delta \geq -2+4 = 2$. Thus by using the relation (1) $\mathcal{L}(G)$ satisfies the property $\rho$. Now by using Corollary 4.3 all the iterated line graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ for $k \geq 1$. □

Theorem 2.5 of K. C. Das et al. can be generalized to the iterated line graphs.

**Theorem 4.14.** *Let $G$ be a graph of order $n_0 > 2$ and size $m_0$ with minimum degree $\delta \geq \frac{n_0}{2}+1$. Then the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ for $k \geq 1$.*

*Proof.* If $G$ be a graph of order $n_0 > 2$ and size $m_0$ with minimum degree $\delta \geq \frac{n_0}{2}+1$, then by Theorem 2.5 $\mathcal{L}(G)$ satisfies the property $\rho$. The existence of a graph $G$ with minimum degree $\delta \geq \frac{n_0}{2}+1$ implies $n_0 \geq 4$, that is the minimum degree of $G$ is at least 3. Hence by using Corollary 4.3 all the iterated line graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ for $k \geq 1$. □

The following inequality was given by Leonardo de Lima et al. in [12] for Turán graph.

$$(r-2)\left\lfloor \frac{n}{r} \right\rfloor < q_{min}\big(T_r(n)\big) \leq \Big(1-\frac{1}{r}\Big)n.$$

**Remark 4.15.** The above inequality is not valid when $r=3$, $n=6,7,8$ and $r=4$, $n=5$ as $Sp_Q\big(T_3(6)\big) = \{8, 4^3, 2^2\}$, $Sp_Q\big(T_3(7)\big) = \{9.2745, 5^2, 4^2, 3, 1.7251\}$, $Sp_Q\big(T_3(8)\big) = \{10.6056, 6, 5^4, 3.3944, 2\}$ and $Sp_Q\big(T_4(5)\big) = \{7.3723, 3^3, 1.6277\}$, which shows that $q_{min}\big(T_3(6)\big) = 2, q_{min}\big(T_3(7)\big) = 1.7251$, $q_{min}\big(T_3(8)\big) = 2$ and $q_{min}\big(T_4(5)\big) = 1.6277$ but the inequality gives strict lower bound 2 for $q_{min}\big(T_r(n)\big)$. However with the help of this inequality we have the following result.

**Proposition 4.16.** *If $r = 3$ and $n \geq 6$, $n \neq 7$ or $r = 4$ and $n \neq 5$ or $r \geq 5$, then the graphs $\mathcal{L}^k\big(T_r(n)\big)$ satisfy the property $\rho$ for $k \geq 1$.*

*Proof.* The condition on $r$, $n$ in the hypothesis and the above inequality ensure that $q_{min}\big(T_r(n)\big)$ is at least 2. Thus, $\mathcal{L}\big(T_r(n)\big)$ satisfies the property $\rho$ by using the relation (1). It is easy to observe that the minimum degree of $T_r(n)$ is at least 3. Hence by Corollary 4.3 all the iterated line graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ for $k \geq 1$. □

## 4.1 Iterated regular line graphs with property $\rho$

Most of the results so far discussed are non-regular iterated line graphs graphs satisfying the property $\rho$. One can get iterated regular line graphs $\mathcal{L}^k(G)$ satisfying property $\rho$ from Theorem 4.1 for $k \geq 2$ and from Theorem 4.6, Theorem 4.13 and Theorem 4.14 for $k \geq 1$. In the Proposition 4.16 if $r$ divides $n$, then we get iterated regular line graphs $\mathcal{L}^k(G)$ satisfying property $\rho$ for $k \geq 1$. Here, we present some more iterated regular line graphs $\mathcal{L}^k(G)$ for $k \geq 1$ by taking regular graph $G$.

**Theorem 4.17.** *If $G$ is a $r$-regular graph of order $n$ with $3 \leq r \leq \frac{n-1}{3}$, then the graphs $\mathcal{L}^k(\overline{G})$ satisfy the property $\rho$ for $k \geq 1$.*

*Proof.* Let $Sp_A(G) = \{r, \lambda_2^{m_2}, \ldots, \lambda_t^{m_t}\}$ such that $1 + \sum_{i=2}^{t} m_i = n$. Then by Theorem 2.3, $\overline{G}$ is also a regular graph with $Sp_A(\overline{G}) = \{n - r - 1, (-1 - \lambda_t)^{m_t}, \ldots, (-1 - \lambda_2)^{m_2}\}$ and by Theorem 2.4, $Sp_A(\mathcal{L}(\overline{G})) = \{2(n-1) - 2r - 2, (n - r - \lambda_t - 4)^{m_t}, \ldots, (n - r - \lambda_2 - 4)^{m_2}, -2^{n(n-r-3)/2}\}$. We shall prove that all eigenvalues of $\mathcal{L}(\overline{G})$ except $-2$ are non-negative. Since, $G$ is regular $\mathcal{L}(\overline{G})$ is also regular with degree $2(n-1) - 2r - 2$ which implies $2(n-1) - 2r - 2$ is positive. The condition $r \leq \frac{n-1}{3}$ gives that $n \geq 3r + 1$ which implies $n - r - \lambda_i - 4 \geq 2r - \lambda_i - 3 \geq 0$ as $r \geq 3$ to each $i = 2, \ldots, t$. Hence, $\mathcal{L}(\overline{G})$ satisfies the property $\rho$. Now, $n \geq 3r + 1$ implies $n - r - 1 \geq 2r \geq 6$ which shows that $\overline{G}$ is a regular graph with its least degree$\geq 6$. This implies the graphs $\mathcal{L}^k(\overline{G})$ satisfy the property $\rho$ to each $k \geq 1$ by Corollary 4.3. □

**Theorem 4.18.** *If $G$ is a $r$-regular graph of order $n \geq 8$ and $r \geq 1$, then the graphs $\mathcal{L}^k\big(\overline{\mathcal{L}(G)}\big)$, $k \geq 1$ satisfy the property $\rho$.*

*Proof.* Let $Sp_A(G) = \{r, \lambda_2^{m_2}, \ldots, \lambda_t^{m_t}\}$ such that $1 + \sum_{i=2}^{t} m_i = n$. Then by Theorem 2.4, $Sp_A(\mathcal{L}(G)) = \{2r - 2, (\lambda_2 + r - 2)^{m_2}, \ldots, (\lambda_t + r - 2)^{m_t}, -2^{n(r-2)/2}\}$ and since $\mathcal{L}(G)$

is regular by Theorem 2.3 $\overline{\mathcal{L}(G)}$ also a regular graph with $Sp_A\big(\overline{\mathcal{L}(G)}\big) = \{nr/2 - 2r + 1, 1^{n(r-2)/2}, (1-r-\lambda_t)^{m_t}, \ldots, (1-r-\lambda_2)^{m_2}\}$. Again, by Theorem 2.4, $Sp_A\big(\mathcal{L}\big(\overline{\mathcal{L}(G)}\big)\big) = \{r(n-4), ((n-4)r/2)^{n(r-2)/2}, ((n-6)r/2-\lambda_t)^{m_t}, \ldots, ((n-6)r/2-\lambda_2)^{m_2}, -2^{nr(nr-4r-2)/8}\}$. It is easy to see that all the eigenvalues of $\mathcal{L}\big(\overline{\mathcal{L}(G)}\big)$ are non-negative except $-2$ for $n \geq 8$. Hence $\mathcal{L}\big(\overline{\mathcal{L}(G)}\big)$ satisfies the property $\rho$. It is noted that the degree of $\overline{\mathcal{L}(G)}$ is $nr/2-2r+1$ which is at least 3 for $n \geq 8$ and $r \geq 1$. This implies the graphs $\mathcal{L}^k\big(\overline{\mathcal{L}(G)}\big)$ satisfy the property $\rho$ for $k \geq 1$ by Corollary 4.3. □

**Remark 4.19.** One can easily construct equienergetic graphs like in Theorem 4.6 for the results in Theorem 4.9 to Theorem 4.18.

**Theorem 4.20.** *Let $G$ be a graph with $d_u + d_v \geq 6$ to each edge $e = uv$ in $G$, then the graphs $\mathcal{L}^k(G)$ are hyperenergetic for $k \geq 2$.*

*Proof.* Since $G$ is a graph with $d_u + d_v \geq 6$ to each edge $e = uv$, $\mathcal{L}(G)$ is a graph with minimum degree $\delta \geq 4$. The number edges in $\mathcal{L}(G)$ is $m_1 = \frac{1}{2}\sum_{i=1}^{m_0} d_i \geq \frac{1}{2}(4m_0) = 2m_0 = 2n_1$, which implies that the graph $\mathcal{L}^2(G)$ is hyperenergetic by Theorem 2.6. Note that the minimum degree increases in the line graphs $\mathcal{L}^k(G)$ as $k$ increases for $k \geq 2$ and which is at least 4. Hence $m_k \geq 2n_k$, by Theorem 2.6, all the iterated line graphs $\mathcal{L}^k(G)$ are hyperenergetic for $k \geq 2$. □

# 5 Spectra and Energy of complement of iterated line graphs with the property $\rho$

**Lemma 5.1.** *Let $\mathcal{L}(G)$ be the line graph of a graph $G$ of order $n_0$ and size $m_0$. If $\mathcal{L}(G)$ has non-negative eigenvalue $\lambda_{1(j)}$, then its complement $\overline{\mathcal{L}(G)}$ has negative eigenvalue $\overline{\lambda}_{1(m_0-j+2)}$ for $j \in \{2, 3, \ldots, m_0\}$. In a particular case, if $\mathcal{L}(G)$ has eigenvalue $-2$, then its complement $\overline{\mathcal{L}(G)}$ has eigenvalue 1.*

*Proof.* If $G$ is a graph of order $n_0$ and size $m_0$, then its line graph $\mathcal{L}(G)$ is the graph of order $m_0$. If $\mathcal{L}(G)$ has non-negative eigenvalue $\lambda_{1(j)}$, then its complement $\overline{\mathcal{L}(G)}$ has eigenvalue $\overline{\lambda}_{1(m_0-j+2)} \leq -1-\lambda_{1(j)}$ by inequality 2, which is negative for $j \in \{2, 3, \ldots, m_0\}$. If $-2$ is the eigenvalue of $\mathcal{L}(G)$, then by Proposition 2.11 its eigenspace is orthogonal to all one's vector $\mathbf{j}$. Now by using the Theorem 2.10, the eigenvalue $-1-(-2) = 1$ is the eigenvalue of $\overline{\mathcal{L}(G)}$. This completes the proof. □

**Theorem 5.2.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 1$, then the graphs $\overline{\mathcal{L}^k(G)}$ for $k \geq 1$ have exactly two positive eigenvalues which are spectral radius and 1 with multiplicity $m_{k-1} - n_{k-1}$ and*

$$\mathcal{E}(\overline{\mathcal{L}^k(G)}) = 2\overline{\lambda}_{k(1)} + \frac{\mathcal{E}(\mathcal{L}^k(G))}{2}. \tag{3}$$

*Proof.* It is given that the iterated line graphs $\mathcal{L}^k(G)$ of $G$ for $k \geq 1$ have all negative eigenvalues equal to $-2$ with the multiplicity $m_{k-1} - n_{k-1} = n_k - n_{k-1}$. This implies the remaining $n_{k-1}$ eigenvalues of $\mathcal{L}^k(G)$ are non-negative. Now by Lemma 5.1, the complement of iterated line graphs $\overline{\mathcal{L}^k(G)}$ have negative eigenvalues $\overline{\lambda}_{k(n_k-j+2)}$ for $j \in \{2, 3, \ldots, n_{k-1}\}$ and positive eigenvalues $\overline{\lambda}_{k(n_k-j+2)} = 1$ for $j \in \{n_{k-1}+1, \ldots, n_k\}$. The only remaining eigenvalue of $\overline{\mathcal{L}^k(G)}$ is spectral radius $\overline{\lambda}_{k(1)}$ which must be greater than or equal to 1 and if $\overline{\mathcal{L}^k(G)}$ is connected, then $\overline{\lambda}_{k(1)} > 1$. Thus $\overline{\mathcal{L}^k(G)}$ have exactly two positive eigenvalues $\overline{\lambda}_{k(1)}$ and 1 with multiplicity $n_k - n_{k-1}$ for $k \geq 1$. Hence energy of $\overline{\mathcal{L}^k(G)}$ is $\mathcal{E}(\overline{\mathcal{L}^k(G)}) = 2(\overline{\lambda}_{k(1)} + n_k - n_{k-1})$. But $\mathcal{E}(\mathcal{L}^k(G)) = 4(n_k - n_{k-1})$ as $-2$ is only the negative eigenvalue of $\mathcal{L}^k(G)$ with the multiplicity $n_k - n_{k-1}$. Now with this we get the required energy relation between $\mathcal{E}(\overline{\mathcal{L}^k(G)})$ and $\mathcal{E}(\mathcal{L}^k(G))$ for $k \geq 1$. □

**Corollary 5.3.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 1$, then the graphs $\mathcal{L}^k(G)$ and $\overline{\mathcal{L}^k(G)}$ for $k \geq 1$ are equienergetic if and only if $n^-(\mathcal{L}^k(G)) = \overline{\lambda}_{k(1)}$.*

*Proof.* The energy relation (3) between $\mathcal{E}(\overline{\mathcal{L}^k(G)})$ and $\mathcal{E}(\mathcal{L}^k(G))$ by Theorem 5.2 can be written as $2\mathcal{E}(\overline{\mathcal{L}^k(G)}) - \mathcal{E}(\mathcal{L}^k(G)) = 4\overline{\lambda}_{k(1)}$ or $\mathcal{E}(\overline{\mathcal{L}^k(G)}) - \mathcal{E}(\mathcal{L}^k(G)) = 4\overline{\lambda}_{k(1)} - \mathcal{E}(\overline{\mathcal{L}^k(G)})$. Now the graphs $\mathcal{L}^k(G)$ and $\overline{\mathcal{L}^k(G)}$ for $k \geq 1$ are equienergetic if and only if $4\overline{\lambda}_{k(1)} - \mathcal{E}(\overline{\mathcal{L}^k(G)}) = 0$ or $4\overline{\lambda}_{k(1)} = \mathcal{E}(\overline{\mathcal{L}^k(G)})$. But $\mathcal{E}(\overline{\mathcal{L}^k(G)}) = 2(\overline{\lambda}_{k(1)} + n_k - n_{k-1})$, now with this we get $\overline{\lambda}_{k(1)} = n_k - n_{k-1}$. It is given that $n^-(\mathcal{L}^k(G)) = n_k - n_{k-1}$ which completes the proof. □

**Remark 5.4.** Recently Mojallal and Hansen in [27], the necessary and sufficient condition for a regular graph $G$ to be equienergetic with its complement $\overline{G}$ is given. In Corollary 5.3, we extended the same to the non-regular iterated line graphs $\mathcal{L}^k(G)$.

**Corollary 5.5.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 1$, then the complement of iterated* *line graphs $\overline{\mathcal{L}^k(G)}$ of such graphs $G$ are mutually equienergetic for $k \geq 1$ if and only if they have same spectral radius.*

*Proof.* If $G$ is the graph of order $n_0$ and size $m_0$, then the graphs $\mathcal{L}^k(G)$ have the order $n_k$ and size $m_k$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 1$, then all the iterated line graphs $\mathcal{L}^k(G)$ of such graphs $G$ of same order $n_0$ and same size $m_0$ are mutually equienergetic with energy $4(m_{k-1} - n_{k-1})$. Now using this fact in the energy relation (3) between $\mathcal{E}(\overline{\mathcal{L}^k(G)})$ and $\mathcal{E}(\mathcal{L}^k(G))$ by Theorem 5.2 completes the proof. $\square$

**Example 5.6.** If $k \geq 1$, the graphs $\overline{\mathcal{L}^k(\mathsf{G}_1)}$ and $\overline{\mathcal{L}^k(\mathsf{G}_2)}$ in the Remark 4.7 are equienergetic as they have same quotient matrices and the fact that the spectral radius of a quotient matrix coincides with the spectral radius of corresponding graph. Moreover, in the Remark 4.7 if $H_{s_1}$ and $H_{s_2}$ are non co-spectral (co-spectral) graphs, then we get non co-spectral (co-spectral) graphs $\overline{\mathcal{L}^k(\mathsf{G}_1)}$ and $\overline{\mathcal{L}^k(\mathsf{G}_2)}$ respectively for $k \geq 0$ by using the Proposition 2.16.

**Remark 5.7.** If $k \geq 1$ then the results in Theorem 5.2, Corollary 5.3 and Corollary 5.5 holds true for the iterated line graphs $\mathcal{L}^k(G)$ in Theorem 4.6 to Theorem 4.18. Similarly, if $k \geq 2$ these results holds true for the iterated line graphs $\mathcal{L}^k(G)$ in Theorem 4.1 and Corollary 4.3.

**Remark 5.8.** In [31] Ramane et al. obtained equienergetic regular graphs by means of complement of iterated regular line graphs $\overline{\mathcal{L}^k(G)}$ for $k \geq 2$ by taking regular graphs $G$ of same order and same degree $r \geq 3$ and thereby characterized large class of pairs of non-trivial equienergetic regular graphs. It is noted that all the results in this paper become particular case of Corollary 5.5 and Corollary 5.3.

**Theorem 5.9.** *Let $G$ be a graph with $d_u + d_v \geq 6$ to each edge $e = uv$ in $G$, then the graphs $\overline{\mathcal{L}^k(G)}$ are hyperenergetic for $k \geq 2$ if $\lambda_{k(1)} \leq \frac{n_k - 1}{2}$.*

*Proof.* If $G$ is the graph with $d_u + d_v \geq 6$ to each edge $e = uv$, then the iterated line graphs $\mathcal{L}^k(G)$ are hyperenergetic for $k \geq 2$ by Theorem 4.20, that is $\mathcal{E}(\mathcal{L}^k(G)) \geq 2(n_k - 1)$. And the energy relation (3) between $\mathcal{E}(\overline{\mathcal{L}^k(G)})$ and $\mathcal{E}(\mathcal{L}^k(G))$ by Theorem 5.2 is

$$\mathcal{E}(\overline{\mathcal{L}^k(G)}) = 2\overline{\lambda}_{k(1)} + \frac{\mathcal{E}(\mathcal{L}^k(G))}{2} \geq 2\overline{\lambda}_{k(1)} + \frac{1}{2}2(n_k - 1) = 2\overline{\lambda}_{k(1)} + (n_k - 1).$$

But it is well known that $\lambda_{k(1)}+\overline{\lambda}_{k(1)} \geq n_k-1$ which implies $\overline{\lambda}_{k(1)} \geq n_k-1-\lambda_{k(1)} \geq n_k-1-\frac{n_k-1}{2} = \frac{n_k-1}{2}$ if $\lambda_{k(1)} \leq \frac{n_k-1}{2}$. By using this we get $\mathcal{E}(\overline{\mathcal{L}^k(G)}) \geq 2\frac{n_k-1}{2}+(n_k-1) = 2(n_k-1)$ which completes the proof. □

# 6 Other results

All the results so far discussed are non-bipartite iterated line graphs. Here we study energy of some class of bipartite graphs by using iterated line graphs.

**Theorem 6.1.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k \geq 1$, then $\mathcal{E}\big(Ebd(\mathcal{L}^k(G))\big) = 2(3m_{k-1}-2n_{k-1})$. Thus, the graphs $Ebd(\mathcal{L}^k(G))$ of such graphs $G$ are mutually equienergetic for $k \geq 1$.*

*Proof.* If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k \geq 1$, then the graphs $\mathcal{L}^k(G)$ have non-negative eigenvalues $\lambda_{k(j)}$ for $j \in \{1,\ldots,n_{k-1}\}$. Now using Theorem 2.14, the graphs $Ebd(\mathcal{L}^k(G))$ have positive eigenvalues $\lambda_{k(j)}+1$ for $j \in \{1,\ldots,n_{k-1}\}$ and 1 with multiplicity $m_{k-1}-n_{k-1}$. Hence the energy of $Ebd(\mathcal{L}^k(G))$ is $\mathcal{E}\big(Ebd(\mathcal{L}^k(G))\big) = 2\big(\sum_{j=1}^{n_{k-1}}(\lambda_{k(j)}+1)+m_{k-1}-n_{k-1}\big) = 2\big(\sum_{j=1}^{n_{k-1}}\lambda_{k(j)}+m_{k-1}\big)$. But $\sum_{j=1}^{n_{k-1}}\lambda_{k(j)}-2(m_{k-1}-n_{k-1}) = 0$ for the graphs $\mathcal{L}^k(G)$, which implies $\mathcal{E}\big(Ebd(\mathcal{L}^k(G))\big) = 2(3m_{k-1}-2n_{k-1})$. If there are graphs $G$ of same order $n_0$ and same size $m_0$, then all the graphs $Ebd(\mathcal{L}^k(G))$ of such graphs $G$ are mutually equienergetic with energy $2(3m_{k-1}-2n_{k-1})$. □

**Theorem 6.2.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k \geq 1$, then $\mathcal{E}\big(Ebd(\overline{\mathcal{L}^k(G)})\big) = 2\big(2(\overline{\lambda}_{k(1)}+1)+3m_{k-1}-4n_{k-1}\big)$. Thus, all the graphs $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ of such graphs $G$ are mutually equienergetic for $k \geq 1$ if and only if the graphs $\overline{\mathcal{L}^k(G)}$ have same spectral radius.*

*Proof.* If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k \geq 1$, then the graphs $\overline{\mathcal{L}^k(G)}$ have negative eigenvalues $\overline{\lambda}_{k(n_k-j+2)} \leq -1$ for $j \in \{2,3,\ldots,n_{k-1}\}$ and two positive eigenvalues, spectral radius $\overline{\lambda}_{k(1)}$ and 1 with multiplicity $m_{k-1}-n_{k-1}$ by Theorem 5.2. Now using Theorem 2.14, the graphs $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ have non-negative eigenvalues $\overline{\lambda}_{k(1)}+1$, 2 with multiplicity $m_{k-1}-n_{k-1}$ and $-(\overline{\lambda}_{k(n_k-j+2)}+1)$ for $j \in \{2,3,\ldots,n_{k-1}\}$. Hence, the energy of $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ is $\mathcal{E}\big(Ebd\big(\overline{\mathcal{L}^k(G)}\big)\big) = 2\big(\overline{\lambda}_{k(1)}+$

$1+2(m_{k-1}-n_{k-1})+\sum_{j=2}^{n_{k-1}}(-\overline{\lambda}_{k(n_k-j+2)}-1)\big) = 2\big(\overline{\lambda}_{k(1)}+1+2(m_{k-1}-n_{k-1})-\sum_{j=2}^{n_{k-1}}\overline{\lambda}_{k(n_k-j+2)}-$ $n_{k-1}+1\big)$. But $\sum_{j=2}^{n_{k-1}}\overline{\lambda}_{k(n_k-j+2)}+\overline{\lambda}_{k(1)}+(m_{k-1}-n_{k-1})=0$ for the graphs $\overline{\mathcal{L}^k(G)}$, which implies $\mathcal{E}\big(Ebd\big(\overline{\mathcal{L}^k(G)}\big)\big)=2\big(2(\overline{\lambda}_{k(1)}+1)+3m_{k-1}-4n_{k-1}\big)$. If there are graphs $G$ of same order $n_0$ and same size $m_0$, then all the graphs $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ of such graphs $G$ are mutually equienergetic with energy $2\big(2(\overline{\lambda}_{k(1)}+1)+3m_{k-1}-4n_{k-1}\big)$ for $k\geq 1$ if and only if $\overline{\lambda}_{k(1)}$ is same for all the graphs $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ as $\mathcal{L}^k(G)$ have same order $n_k$ and same size $m_k$. □

**Remark 6.3.** Yaoping Hou and Lixin Xu in [25] studied spectra and energy of $Ebd\big(\mathcal{L}^2(G)\big)$ and $Ebd\big(\overline{\mathcal{L}^2(G)}\big)$, where $G$ is a $r$-regular graph of degree $r\geq 3$, thereby characterized large class of pairs of non-trivial bipartite equienergetic graphs $Ebd\big(\mathcal{L}^k(G)\big)$ and $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ for $k\geq 2$. It is noted that all these results become particular case of Theorem 6.1 and Theorem 6.2.

**Example 6.4.** The graphs $Ebd\big(\overline{\mathcal{L}^k(\mathsf{G}_1)}\big)$ and $Ebd\big(\overline{\mathcal{L}^k(\mathsf{G}_2)}\big)$ in the Remark 4.7 are equienergetic for $k\geq 1$ as the graphs $\overline{\mathcal{L}^k(\mathsf{G}_1)}$ and $\overline{\mathcal{L}^k(\mathsf{G}_2)}$ have same quotient matrices and the fact that the spectral radius of a quotient matrix coincides with the spectral radius of corresponding graph. Moreover, in the Remark 4.7 if $H_{s_1}$ and $H_{s_2}$ are non co-spectral (co-spectral) graphs, then we get non co-spectral (co-spectral) graphs $Ebd\big(\overline{\mathcal{L}^k(\mathsf{G}_1)}\big)$ and $Ebd\big(\overline{\mathcal{L}^k(\mathsf{G}_2)}\big)$ respectively for $k\geq 0$ by using Proposition 2.16.

**Theorem 6.5.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k\geq 1$, then the graphs $Ebd\big(\mathcal{L}^k(G)\big)$ and $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ are equienergetic for $k\geq 1$ if and only if $\overline{\lambda}_{k(1)}=n^-\big(\overline{\mathcal{L}^k(G)}\big)$.*

*Proof.* If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k\geq 1$, then by Theorem 6.1, we have $\mathcal{E}\big(Ebd(\mathcal{L}^k(G))\big)=2(3m_{k-1}-2n_{k-1})$ and $\mathcal{E}\big(Ebd(\overline{\mathcal{L}^k(G)})\big)=2\big(2(\overline{\lambda}_{k(1)}+1)+3m_{k-1}-4n_{k-1}\big)$ from Theorem 6.2. With these, we get $\mathcal{E}\big(Ebd(\overline{\mathcal{L}^k(G)})\big)=\mathcal{E}\big(Ebd(\mathcal{L}^k(G))\big)+2\big(2(\overline{\lambda}_{k(1)}+1)-2n_{k-1}\big)$ or $\mathcal{E}\big(Ebd(\overline{\mathcal{L}^k(G)})\big)-\mathcal{E}\big(Ebd(\mathcal{L}^k(G))\big)=4\big(\overline{\lambda}_{k(1)}+1-n_{k-1}\big)$. Therefore, the graphs $Ebd\big(\mathcal{L}^k(G)\big)$ and $Ebd\big(\overline{\mathcal{L}^k(G)}\big)$ are equienergetic for $k\geq 1$ if and only if $\overline{\lambda}_{k(1)}=n_{k-1}-1$. But $n_{k-1}-1=n^-\big(\overline{\mathcal{L}^k(G)}\big)$ by Theorem 5.2, which completes the proof. □

**Proposition 6.6.** *Let $G$ be a graph of order $n_0$ and size $m_0$. If the graphs $\mathcal{L}^k(G)$ satisfy the property $\rho$ with $-2$ multiplicity $m_{k-1}-n_{k-1}$ for $k\geq 1$, then $\alpha\big(\mathcal{L}^k(G)\big)\leq n_{k-1}$ and $\alpha\big(\overline{\mathcal{L}^k(G)}\big)\leq n^-\big(\mathcal{L}^k(G)\big)+1$.*

*Proof.* If $G$ is a graph of order $n_0$, it is well known that $n^-(G) \leq n_0 - \alpha(G)$. Using this fact for the graphs $\mathcal{L}^k(G)$, we get $m_{k-1} - n_{k-1} = n_k - n_{k-1} \leq n_k - \alpha\big(\mathcal{L}^k(G)\big)$ which implies $\alpha\big(\mathcal{L}^k(G)\big) \leq n_{k-1}$ for $k \geq 1$. Again using the fact for the graphs $\overline{\mathcal{L}^k(G)}$, we get $n_{k-1} - 1 \leq n_k - \alpha\big(\overline{\mathcal{L}^k(G)}\big)$ or $-(n_k - n_{k-1} + 1) \leq -\alpha\big(\overline{\mathcal{L}^k(G)}\big)$ which gives $\alpha\big(\overline{\mathcal{L}^k(G)}\big) \leq n^-\big(\mathcal{L}^k(G)\big) + 1$ for $k \geq 1$. □

The following is an interesting result on minimum order of a graph when the graph $G$ and its complement both connected with $\mathcal{L}(G)$ satisfying the property $\rho$ is considered.

**Theorem 6.7.** *The least order of a connected graph $G$ with its line graph satisfying the property $\varrho$ and connected $\overline{G}$ is 7.*

*Proof.* There are exactly 13 non-isomorphic connected graphs of order up to 6 with their line graphs satisfying the property $\varrho$. They are $C_4, K_4, K_{3,2}, K_5, K_{4,2}, K_{3,3}, K_6$ and the following graphs in Figure 2.

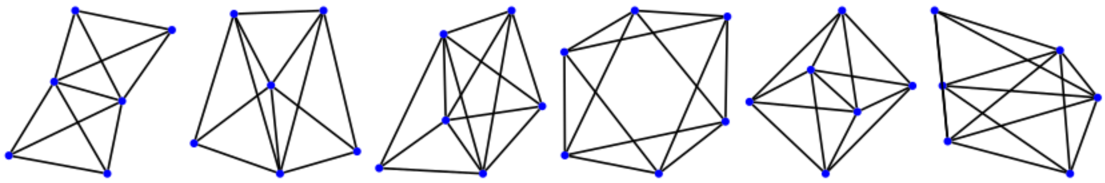

Figure 2

One can see that none of the above graphs have connected complement. The only connected graph of order 7 with its line graph satisfying the property $\varrho$ and having connected complement is given in Figure 3, which completes the proof.

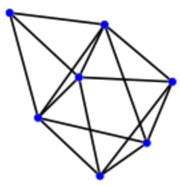

Figure 3

□

## Conclusion

Our study on eigenvalues of iterated line graphs reveals that the majority of these graphs have exact number of negative eigenvalues which are all equal to $-2$. Also, complements these graphs have exact number of positive eigenvalues one more than that of multiplicity of $-2$. Although we have presented many classes of iterated line graphs which have all negative eigenvalues equal to $-2$ but still the question remains to characterize all such graphs.

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