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ITERATED PROBABILITY DISTRIBUTIONS AND EXTREMES WITH RANDOM SAMPLE SIZE

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Abstract. In this paper the possible nondegenerated limit distributions for the n-fold mapping of a given probability distribution are considered. If the mapping used for the iteration procedure is a probability generating function of a positive integer-valued random variable then the results can be applied to the max-stability of distributions of random variables with random sample size.

Key words and phrases: Distribution-preserving mappings, stable distributions, extremes, random sample size.

1. Introduction

It is well known that for many real-world situations stable distributions are a useful tool for modeling (e.g. Kruglov and Korolev (1990), Rachev (1991)). For example, sum-stable and especially geometric-sum-stable distributions are used for financial modeling like asset or stock returns (Mittnik and Rachev (1991, 1993), Rachev and Sen Gupta (1992) and references therein). For geometric-sum-stability some results on limit distributions, rate of convergence and domains of attraction can be found for example in Gnedenko (1982, 1983), Rachev and Samorodnitsky (1992), Rachev and Sen Gupta (1992).

In many cases, however, not sum-stability but max-stability is the proper concept. For example, if $X_1, X_2, \dots$ are insurance claims and N the (random) number of claims, then the distribution of

$$X_{N,N} = \max_{1 \le k \le N} X_k$$

must be known in order to calculate the insurance premium for reinsurance treaties based on ordered claims (like in the ECOMOR or LCR case, e.g. Kremer (1983)). For modeling possible reinsurance situations therefore those probability distributions are of special interest which remain unchanged (up to some linear transformation) by taking the maximum with respect to a given class of integer-valued random variables.

To be precise, let $X_1, X_2, \ldots$ be i.i.d. random variables with cumulative distribution function (cdf) $F(x)$. Moreover, let $\phi(t) = E(t^N)$ be the probability generating function (pgf) of a nonnegative integer-valued random variable N. Then the cdf $G(x)$ of $X_{N,N}$ is

$$G(x) = \phi(F(x)), \qquad x \in \mathbb{R}. \tag{1.1}$$

Now let $\mathcal{N} = \{\phi_\theta \mid \theta \in \Theta\}$ be a parameterized set of pgf's of nonnegative integer-valued random variables N_θ, $\theta \in \Theta \subset \mathbb{R}$. A cdf G is called $\mathcal{N}$-max-stable, if there exist measurable functions $a(\theta)$, $b(\theta)$ such that

$$\phi_\theta(G(a(\theta)x + b(\theta))) = G(x), \qquad \forall x \in \mathbb{R}, \quad \forall \theta \in \Theta. \tag{1.2}$$

In the literature, several authors investigated the possible $\mathcal{N}$-max-stable distributions for a given set $\mathcal{N}$ (Baringhaus (1980), Gnedenko (1982), Voorn (1987, 1989), Rachev and Resnick (1991), Bunge (1993), Brücks (1993)).

Let, for example, ϕ_θ be the pgf of a geometric distribution with parameter $\theta \in (0,1)$. The only max-stable-distributions with respect to $\mathcal{N} = \{\phi_\theta \mid \theta \in (0,1)\}$ are the logistic, the loglogistic and the negative loglogistic distribution.

In order to extend the known results we embed the above mentioned problem in the following framework, replacing the set $\mathcal{N}$ of pgf's by the class $\mathcal{G}$ of cdf-preserving mappings:

(1.3) Let $\mathcal{G} = \{g \mid g : [0,1] \to [0,1],\ g(0) = 0,\ g(1) = 1,\ g$ strictly increasing and continuous$\}$.

(i) Given $g \in \mathcal{G}$ and nondegenerate cdf's F and G, are there constants $a_n > 0$, $b_n \in \mathbb{R}$ such that
$$\lim_{n\to\infty} g^n(F(a_n x + b_n)) = G(x), \qquad \forall x \in C_G$$
(with $C_G = \{x \mid x \in \mathbb{R}, G(x) \text{ is continuous in } x\}$ and $g^n = g \circ g \circ \cdots \circ g$ n-times)?

(ii) Given $g \in \mathcal{G}$, are there constants $a_n > 0$, $b_n \in \mathbb{R}$ and a nondegenerate cdf G such that
$$g^n(G(a_n x + b_n)) = G(x), \qquad \forall x \in \mathbb{R}, \quad n \in \mathbb{N}?$$

As in extreme value theory with non-random sample size it turns out that the problems (1.3)(i) and (ii) are linked in the following way:

Let $\mathcal{F}$ be the set of all nondegenerate cdf's F on $\mathbb{R}^1$. We say that $F \in \mathcal{F}$ belongs to the g-domain of attraction of G ($F \in DA_g(G)$) if (1.3)(i) is fulfilled and G is called g-stable if (1.3)(ii) holds. Then it can be shown that G is g-stable iff the g-domain of attraction $DA_g(G)$ is nonempty; clearly $G \in DA_g(G)$. For a given $g \in \mathcal{G}$ therefore the set of all g-stable cdf's is of main interest.

In this paper we first deal with the existence and uniqueness of g-stable distributions for a given $g \in \mathcal{G}$. Taking into account only $g \in \mathcal{G}$ without any fixed point in $(0,1)$ we get a result on the support of stable distributions. It is easy to see that for every g there is a noncountable set of types of g-stable distributions.

By restricting attention to proper subsets $\mathcal{G}^* \subset \mathcal{G}$ of functions g_s with a group structure in s it turns out that there exist exactly three types of stable distributions depending on the set $\mathcal{G}^*$. Thus, if $\mathcal{G}^*$ is a set of pgf's it uniquely characterizes three different types of $\mathcal{G}^*$-max-stable distributions where $\mathcal{G}^*$-max-stability means g-stability for all $g \in \mathcal{G}^*$.

The paper is organized as follows: in Section 2 some results on iterating cdf's by a given $g \in \mathcal{G}$ are obtained. Section 3 deals with the stability of nondegenerated cdf's with respect to some subset $\mathcal{G}^* \subset \mathcal{G}$. Finally, Section 4 shows the application of the results of Section 3 to the max-stability of distributions for samples with random sample size.

2. Iterated distributions and stability

Let $\mathcal{G}$ be defined as in (1.3) and let $\mathcal{F}$ be the set of all nondegenerate cdf's F on $\mathbb{R}^1$. If $G \in \mathcal{F}$ is g-stable for $g \in \mathcal{G}$, then only two cases occur:

(2.1) (i) These exists $b \in \mathbb{R}$, $b \neq 0$, such that

$$g^n(G(x)) = G(x + nb), \qquad x \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

(ii) There exist $a > 0$, $a \neq 1$ and $b \in \mathbb{R}$, such that

$$g^n(G(x)) = G(a^n(x-b)+b), \qquad x \in \mathbb{R}, \quad n \in \mathbb{Z}.$$

This follows from the fact that the stability condition means

$$g(G(\alpha_1 x + \beta_1)) = G(x) \qquad \text{for all } x \in \mathbb{R}, \quad \text{for some } \alpha_1 > 0, \ \beta_1 \in \mathbb{R},$$

which is equivalent to

$$g(G(x)) = G(ax+d) \qquad \text{with } a = \alpha_1^{-1}, \ d = -\beta_1\alpha_1^{-1}.$$

Hence, induction leads to

$$g^n(G(x)) = G(h^n(x)), \qquad x \in \mathbb{R}, \ n \in \mathbb{Z}, \quad \text{with } h(x) = ax + d.$$

(For negative integers, the mapping g^{-n}, $n \in \mathbb{N}$, denotes the n-fold mapping $g^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}$ (n times) where g^{-1} always exists due to the definition of $\mathcal{G}$.) The distinction between $a = 1$ and $a \neq 1$ gives the desired result.

The assertions in (2.1)(i) and (ii) correspond to $a = 1$ and $a \neq 1$ and we call a cdf G $g[1,b]$-stable or $g[a,b]$-stable ($a \neq 1$) respectively. Let

$$\text{type}(G) = \{F \mid F \in \mathcal{F}, \exists a > 0, b \in \mathbb{R} : F(ax+b) = G(x), x \in \mathbb{R}\}$$

then it is easy to see that all members of type(G) for a given $g \in \mathcal{G}$ have the same stability property. That means that in every type-equivalence class of $g[1,b]$-stable distribution there is a $g[1,-1]$-stable one and likewise a $g[a,0]$-stable distribution in every $g[a,b]$-stable class for a given $a > 0$, $a \neq 1$.

When considering the existence of g-stable distributions as well as their structure it is necessary to have a closer look at $g \in \mathcal{G}$. If, for example, g has a fixed point in the interior of $[0,1]$, then there does not exist a continuous $g[1,b]$-stable distribution:

Let $y_0 \in (0,1)$ with $g(y_0) = y_0$, G be $g[1,b]$-stable and $x \in \mathbb{R}$ with $G(x) = y_0$. The stability condition yields

$$y_0 = g^n(y_0) = g^n(G(x)) = G(x+nb) \qquad \text{for all} \quad n \in \mathbb{Z},$$

which is a contradiction to $G \in \mathcal{F}$.

Therefore we restrict ourselves to $g \in \mathcal{G}$ without any fixed point in $(0,1)$. That is, instead of $\mathcal{G}$ we consider the set

$$\underline{\mathcal{G}} \cup \overline{\mathcal{G}} = \{g \mid g \in \mathcal{G}, g\,|_{(0,1)} < id\,|_{(0,1)}\} \cup \{g \mid g \in \mathcal{G}, g\,|_{(0,1)} > id\,|_{(0,1)}\}.$$

For $g \in \underline{\mathcal{G}}$ we have $g^{-1} \in \overline{\mathcal{G}}$ and vice versa. It suffices to consider $g \in \underline{\mathcal{G}}$, since g and g^{-1} belong to the same stability class. (In this case we have $b < 0$ for all $g[1,b]$-stable distributions.)

The following theorem shows that the structures of g-stable distributions and max-stable distributions with non-random sample size coincide (Gensler (1992)).

THEOREM 2.1. *Let $g \in \underline{\mathcal{G}}$, $G \in \mathcal{F}$, $b \neq 0$, $a > 0$, $a \neq 1$, and $c \in \mathbb{R}$.*
(i) *If G is $g[1,b]$-stable, then $0 < G(x) < 1$ for all $x \in \mathbb{R}$.*
(ii) *If G is $g[a,c]$-stable, then*

$$G(x) \begin{cases} = 0 & \text{for } x \le c \\ \in (0,1) & \text{for } x > c \end{cases} \qquad \text{if } a < 1,$$

$$G(x) \begin{cases} \in (0,1) & \text{for } x < c \\ = 1 & \text{for } x \ge c \end{cases} \qquad \text{if } a > 1.$$

Moreover, G is continuous in c.

PROOF.
(i) $G(x) \in (0,1)$ follows from the fact that 0 and 1 are the only fixed points of g.
(ii) The $g[a,c]$-stability of G is equivalent to

$$g^n(G(x)) = G(a^n(x-c)+c) \qquad \text{for all} \quad x \in \mathbb{R}, \ n \in \mathbb{Z}.$$

Thus,

$$g^n(G(c)) = G(c) \in \{0,1\} \qquad \text{because of } g \in \underline{\mathcal{G}}.$$

For the same reason it is

$$G(x) > g^n(G(x)) = G(a^n(x-c)+c)$$

since $g(y) < y$ for $g \in \underline{\mathcal{G}}$, $y \in (0,1)$.

Thus, we find $x > a^n(x-c)+c$ which leads to either

$$(a < 1 \text{ and } x > c) \quad \text{or} \quad (a > 1 \text{ and } x < c).$$

The result is obtained by considering the two possible cases $G(c) = 0$ and $G(c) = 1$. □

There are more similarities between max-stable and g-stable distributions: e.g., the different types of g-stable distributions can be transformed into each other by some bijective function as shown in the following lemma.

LEMMA 2.1. *Under the assumptions of Theorem* 2.1 *let*

$$T_{(a,b,c)} : \mathbb{R} \to I = \begin{cases} (c,\infty) & \text{for } a < 1 \\ (-\infty, c) & \text{for } a > 1 \end{cases} \quad \text{defined by}$$

$$T_{(a,b,c)}(x) = c + \operatorname{sign}(1-a)a^{b^{-1}x}$$

$$\text{for } x \in \mathbb{R} \quad \text{with } \operatorname{sign}(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases}$$

Then the following holds:
(i) *If G is $g[a,c]$-stable, then $G^*_b(x) = G(T_{(a,b,c)}(x))$ is $g[1,b]$-stable.*
(ii) *If G is $g[a,b]$-stable, then $G^*_{a,c}(x)$ is $g[a,c]$-stable where*

$$G^*_{a,c}(x) = \begin{cases} 0 & \text{for } x < c \\ G(T^{-1}_{(a,b,c)}(x)) & \text{for } x \geq c \end{cases} \quad \text{for } a < 1 \quad \text{and}$$

$$G^*_{a,c}(x) = \begin{cases} G(T^{-1}_{(a,b,c)}(x)) & \text{for } x < c \\ 1 & \text{for } x \geq c \end{cases} \quad \text{for } a > 1.$$

PROOF. Since $T_{(a,b,c)}(x)$ is continuous, strictly increasing and bijective, (i) and (ii) follow by straight forward calculations. □

We may use Lemma 2.2 to conclude that G is continuous in c (cf. Theorem 2.1).

Besides the structure of g-stable distributions, the existence and uniqueness (in the sense of type uniqueness) is of main interest. Due to the simple conditions for the stability we obtain the following result.

LEMMA 2.2. *For $g \in \underline{\mathcal{G}}$ the set of type-equivalence classes of $g[1,b]$-stable distributions is noncountable.*

PROOF. Let $g \in \underline{\mathcal{G}}$, $\alpha \in (0,1)$, $\beta = g^{-1}(\alpha)$ and $T_\lambda : [0,1) \to (0,1)$ for $\lambda \in (\alpha, \beta]$ be a family of functions which are strictly increasing, right-continuous and for which $T_\lambda(0) = \alpha$ and $\lim_{x\to 1} T_\lambda(x) = \lambda$ are satisfied.

Then

$$G_\lambda(x) = g^{-n}(T_\lambda(x-n)) \qquad \text{for} \quad x \in [n, n+1), \; n \in \mathbb{Z},$$

leads to a family of $g[1,1]$-stable distributions for $\lambda \in (\alpha, \beta]$ with

$$\text{type}(G_\lambda) \neq \text{type}(G_{\lambda'}) \qquad \text{for} \quad \lambda \neq \lambda', \; \lambda, \lambda' \in (\alpha, \beta].$$

(A similar construction is used by Voorn (1987), p. 841.) □

3. Continuous stability

In comparison with Lemma 2.3, the situation changes if the stability condition (1.3)(i) is required to be fulfilled for proper subsets $\mathcal{G}^* \subset \mathcal{G}$ as in the $\mathcal{N}$-max-stable case (1.2). Let

$$\mathcal{G}^* = \{g_s \mid g_s \in \mathcal{G}, s > 0, g_s \circ g_t = g_{s \cdot t}, s, t > 0, g_s \text{ pointwise continuous in } s\}.$$

Such a class is called M-class. Obviously, we have $g_1 = id$, $g_s^{-1} = g_{s^{-1}}$ and $g_s^n = g_{s^n}$ for all $n \in \mathbb{N}$.

The following stability concept is defined for M-classes:

Definition. Let $\mathcal{G}^*$ be a M-class. A cdf $G \in \mathcal{F}$ is called G^*-stable if there exist measurable functions $a(s) > 0$, $b(s)$ for $s > 0$ such that

$$g_s[G(a(s)x + b(s))] = G(x), \qquad \forall x \in \mathbb{R}, \; s > 0.$$

Remark. In suffices to require the condition in Definition for all $s = n \in \mathbb{N}$ as in the case of max-stability with non-random sample size.

By analogy with (2.1), two different cases are possible:

Lemma 3.1. *If $G \in \mathcal{F}$ is $\mathcal{G}^*$-stable then one of the following cases occurs.*
(i) *There exists $b \in \mathbb{R}$, $b \neq 0$, such that*

$$g_s(G(x)) = G(x + b \ln s), \qquad \forall x \in \mathbb{R}, \; s > 0.$$

(ii) *There exist $d \in \mathbb{R}$, $d \neq 0$ and $c \in \mathbb{R}$ such that*

$$g_s(G(x)) = G(s^d(x-c)+c), \qquad \forall x \in \mathbb{R}, \; s > 0.$$

Proof. For $s, t > 0$ the property of M-classes and the stability condition yield

$$G(x) = g_{st}(G(h_{st}(x))) \quad \text{and}$$
$$G(x) = g_{st}(G((h_s \circ h_t)(x)))$$

with

$$h_s(x) = a(s)x + b(s).$$

Hence,

$$G(x) = G[(h_s \circ h_t \circ h_{st}^{-1}(x)] \quad \text{for all} \quad x \in \mathbb{R}.$$

Using the stability condition it turns out that

$$h_s \circ h_t \circ h_{st}^{-1} = id \qquad \text{or equivalently}$$
$$a(s)[a(t)x + b(t)] + b(s) = a(st)x + b(st).$$

The distinction between $a(s) = 1$ and $a(s) \neq 1$ gives the desired result. □

According to these two different cases, G is called $\mathcal{G}^*[1,b]$-stable or $\mathcal{G}^*[d,c]$-stable, respectively.

A comparison with (2.1) shows the connection between the two stability concepts:

(3.1) For $\mathcal{G}^*$ M-class, $G \in \mathcal{F}$, $b, c, d \in \mathbb{R}$ with $b, c \neq 0$ we have

(i) G is $\mathcal{G}^*[1,b]$-stable iff G is $g_s[1, b \ln s]$-stable for all $s > 0$, $s \neq 1$.

(ii) G is $\mathcal{G}^*[d,c]$-stable iff G is $g_s[s^d, c]$-stable for all $s > 0$, $s \neq 1$.

Therefore, the results of Section 2 can be applied to characterize $\mathcal{G}^*$-stable distributions. For example, cdf's of the same type possess the same stability property. Moreover, restricting to

$$\underline{\mathcal{G}}^* = \{g \mid g \in \mathcal{G}^*, g\mid_{(0,1)} < id\mid_{(0,1)} \text{ for all } s > 1 \text{ or all } s < 1\}$$

it turns out that Theorem 2.1 is valid here, too. Referring to the support, there are three types of $\mathcal{G}^*$-stable distributions which can be transformed to each other by the transformations $T_{(a,b,c)}$ used in Lemma 2.1.

Putting $b_s = b \ln s$, $a_s = s^d$ we obtain

$$T_{(a_s,b_s,c)}(x) = c + \operatorname{sign}(1 - a_s) a_s^{b_s^{-1}x} = c - \operatorname{sign}(d) e^{db^{-1}x} = T_{(e^d,b,c)}(x)$$

which does not depend on s.

This leads to the main result of this section:

THEOREM 3.1. *For $\underline{\mathcal{G}}^*$ the set of all $\mathcal{G}^*$-stable distributions is either empty or consists of exactly three different types.*

PROOF. First it can be shown that every $\mathcal{G}^*[1,b]$-stable distribution is of the same type for all $b > 0$. Then the assertion follows from Theorem 2.1 and Lemma 2.1. □

Looking for possible $\mathcal{G}^*$-stable distributions for a given M-class $\mathcal{G}^*$ we first investigate the structure of $\mathcal{G}^*$ itself. Thus we have to regard a function

$$g : [0,1] \times (0,\infty) \to [0,1]$$

with the following properties

$$\text{(3.2)}\qquad \begin{aligned} &\text{(i)}\quad g(0,s)=0, g(1,s)=1, \qquad \forall s>0,\\ &\text{(ii)}\quad g(y,s) \text{ is strictly monotone in } y \text{ given } s \text{ and continuous}\\ &\qquad \text{in } s \text{ given } y,\\ &\text{(iii)}\quad g(g(y,s),t)=g(y,s\cdot t), \qquad \forall s,t>0, \quad y\in[0,1]. \end{aligned}$$

Thus, $g(y,s)$ is a slight modification of the translation equation in Aczél ((1961), Chapter 6), and we get the following

LEMMA 3.2. *The solutions of* (3.2) *are of the form*
(i) $g(y,s)=G(G^{-1}(y)+\ln s)$ *for* $y\in[0,1]$, $s>0$, *with* G *continuous, strictly increasing*, $0<G(x)<1$, $\forall x\in\mathbb{R}$ *and*

$$\lim_{x\to-\infty} G(x)=0, \qquad \lim_{x\to\infty} G(x)=1$$

or

(ii) $g(y,s)=G(c+s(G^{-1}(y)-c))$, $y\in[0,1]$, $s>0$ *for some* $c\in\mathbb{R}$, *with* G *used in* (i) *or those with the properties*

$$G(x)\begin{cases} =0 & \text{for } x<c\\ \in(0,1) & \text{for } x\geq c, \end{cases}$$

continuous, strictly increasing for $x\geq c$ *and* $\lim_{x\to\infty} G(x)=1$ *or*

$$G(x)\begin{cases} \in(0,1) & \text{for } x<c\\ =1 & \text{for } x\geq c, \end{cases}$$

continuous, strictly increasing for $x<c$ *and* $\lim_{x\to-\infty} G(x)=0$.

An immediate consequence of Lemma 3.2 is

COROLLARY 3.1. *Every continuous cdf* G *on* $\mathbb{R}$ *which is either strictly increasing for all* $x\in\mathbb{R}$ *or strictly increasing for* $x\geq c$ *(and equal to* 0 *for* $x<c$*) or strictly increasing for* $x<d$ *(and equal to* 1 *for* $x\geq d$*) for some* $c,d\in\mathbb{R}$ *is* $\mathcal{G}^*$*-stable with respect to some* M*-class* $\mathcal{G}^*=\mathcal{G}^*(G)$.

Example. Let $G(x)=\exp(-\exp(-x))$, $x\in\mathbb{R}$. Then G is $\mathcal{G}^*$-stable with respect to

$$\text{(3.3)}\qquad \begin{aligned} &G^{-1}(y)=-\ln(-\ln y) \quad \text{for} \quad y\in[0,1] \quad \text{and}\\ &\mathcal{G}^*(G)=\{g(\cdot,\cdot)\mid g(y,s)=y^s, s>0, y\in[0,1]\}. \end{aligned}$$

4. Max-stable distributions with random sample size

It is well known that a function

$$\phi : [0,1] \to \mathbb{R}$$

is a pgf of a nonnegative integer-valued random variable N iff
(i) $\phi([0,1]) \subset [0,1]$,
(ii) $\phi(1) = 1$ and
(iii) ϕ is absolutely monotone in $[0,1]$.
Therefore, the $\mathcal{N}$-max-stability (1.2) coincides with the $\mathcal{G}^*$-stability in Definition whenever $\mathcal{G}^*$ is a set of pgf's with $\phi(0) = 0$ for all $\phi \in \mathcal{G}^*$. Obviously, the absolute monotonicity of the pgf's yields $\phi \mid_{(0,1)} < id \mid_{(0,1)}$.

Thus, the characterization in Theorem 3.1 can be applied which means that for any set $\mathcal{N} = \{\phi_\theta \mid \theta \in \Theta\}$ of pgf's with $\phi_\theta(0) = 0$, $\forall \theta \in \Theta$ the set of all $\mathcal{N}$-max-stable distributions is either empty or consists of exactly three different types of distributions.

Moreover, Lemma 3.2 together with Corollary 3.1 yields uniqueness results concerning the max-stability when random sample sizes are considered:

Whenever some function $g(y,s) = g_s(y)$ which is formed by Lemma 3.2 for some proper cdf G turns out to be a pgf for $s < 1$ or $s > 1$ (or $s = n \in \mathbb{N}$) we know immediately all types of random-max-stable distributions with respect to $g_s(y)$ using the transformations $T_{(a,b,c)}$.

For example, we get the following results:

COROLLARY 4.1. *Let*

$$\mathcal{G}_1^* = \{g(\cdot,\cdot) \mid g(y,s) = 1 - (1-y)^s, s > 0, y \in [0,1]\}.$$

Then $g(y,s)$ is a pgf for $0 < s \leq 1$ and the only $\mathcal{G}_1^*$-max-stable distributions are the minimum-stable distributions in the non-random sample case.

PROOF. $\mathcal{G}_1^*$ is a M-class and $g(y,s)$ can be shown to be the pgf of the so-called Waring-distribution with the probability function

$$P(N_s = n) = \frac{s}{n} \prod_{i=1}^{n-1} \frac{i-s}{i} \quad \text{for} \quad n \in \mathbb{N},\ 0 < s \leq 1.$$

Since the exponential distribution with the cdf $F(x) = 1 - \exp(-x)$ for $x \geq 0$, is minimum-stable in the non-random case, we have

$$g(y,s) = F(sF^{-1}(y)) = 1 - (1-y)^s \quad \text{for} \quad s > 0.$$

(The result was already obtained by Brücks (1993) using the method of Voorn (1989).) □

Remark. Analogously, some more known results can be shown very easily:

(i) Let $\mathcal{G}_2^* = \{g(\cdot,\cdot) \mid g(y,s) = s \cdot y(1-(1-s)y)^{-1}, s > 0, y \in [0,1]\}$ be the set of pgf's of geometric distributions with the probability function

$$P(N_s = n) = s(1-s)^n, \quad n \in \mathbb{N}_0, \quad 0 < s \leq 1.$$

Then the only geometric-max-stable distributions are logistic, loglogistic and negative loglogistic distributions, because for the loglogistic distribution with cdf $F(x) = x(1+x)^{-1}$ for $x \geq 0$ we have

$$g(y,s) = F(sF^{-1}(y)) = sy(1-(1-s)y)^{-1}.$$

This result was obtained for example by Gnedenko (1982) and Rachev and Resnick (1991).

(ii) Example (3.3) also implies the uniqueness of the three known types of max-stable distributions in the non-random sample case, since $g(y,s) = y^s$ is for $s = n \in \mathbb{N}$ the pgf of the degenerate random variable N_n with $P(N_n = n) = 1$.

(iii) There is no unique correspondence between some M-class $\mathcal{G}^*$ and the cdf's F which are $\mathcal{G}^*$-stable.

For example, the logistic distribution with the cdf $F(x) = (1+\exp(-x))^{-1}$ for $x \in \mathbb{R}$ is

$$\mathcal{G}_3^*\left[\frac{1}{s}, 0\right]\text{-stable} \quad \text{for} \quad \mathcal{G}_3^* = \{g_1(\cdot,\cdot) \mid g(y,s) = y^s[y^s + (1-y)^s]^{-1}\} \quad \text{and}$$

$$\mathcal{G}_2^*[1, \ln s]\text{-stable} \quad \text{for} \quad \mathcal{G}_2^* \text{ as in (i).}$$

$g_1(y,s)$ however, is not a pgf since it has a fixed point at $y = \frac{1}{2}$ for all $s > 0$.

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