ITERATED VANISHING CYCLES, CONVOLUTION, AND A MOTIVIC ANALOGUE OF A CONJECTURE OF STEENBRINK

GIL GUIBERT, FRANÇOIS LOESER, and MICHEL MERLE

Abstract
We prove a motivic analogue of Steenbrink's conjecture [25, Conjecture 2.2] on the Hodge spectrum (proved by M. Saito in [21]). To achieve this, we construct and compute motivic iterated vanishing cycles associated with two functions. We are also led to introduce a more general version of the convolution operator appearing in the motivic Thom-Sebastiani formula. Throughout the article we use the framework of relative equivariant Grothendieck rings of varieties endowed with an algebraic torus action.

Contents

1. Introduction

Let us start by recalling the statement of Steenbrink's conjecture [25, Conjecture 2.2]. Let $f : X \to \mathbf{A}^1$ be a function on a smooth complex algebraic variety. Let x be a closed point of $f^{-1}(0)$. Steenbrink introduced in [24] and [25] the spectrum $\mathrm{Sp}(f, x)$ of f at x. It is a fractional Laurent polynomial $\sum_{\alpha \in \mathbf{Q}} n_\alpha t^\alpha$, n_α in $\mathbf{Z}$, which is constructed using the action of the monodromy on the mixed Hodge structure on the cohomology of the Milnor fiber at x. When f has an isolated singularity at x, all n_α are in $\mathbf{N}$, and the exponents of f, counted with multiplicity n_α, are exactly the rational numbers α with n_α not equal to zero.

DUKE MATHEMATICAL JOURNAL
Vol. 132, No. 3, © 2006
Received 10 December 2003. Revision received 22 June 2005.
2000 *Mathematics Subject Classification*. Primary 14B05, 14B07, 14J17, 32S05, 32S25, 32S30, 32S35, 32S55.

Let us assume now that the singular locus of f is a curve Γ having r local components Γ_ℓ, $1 \leq \ell \leq r$ in a neighborhood of x. We denote by m_ℓ the multiplicity of Γ_ℓ. Let g be a generic linear form vanishing at x (that is, a function g vanishing at x whose differential at x is a generic linear form). For N large enough, the function $f+g^N$ has an isolated singularity at x. In a neighborhood of the complement Γ_ℓ° to $\{x\}$ in Γ_ℓ, we may view f as a family of isolated hypersurface singularities parametrized by Γ_ℓ°. The cohomology of the Milnor fiber of this hypersurface singularity is naturally endowed with the action of two commuting monodromies; the monodromy of the function and the monodromy of a generator of the local fundamental group of Γ_ℓ°. We denote by $\alpha_{\ell,j}$ the exponents of that isolated hypersurface singularity, and we denote by $\beta_{\ell,j}$ the corresponding rational numbers in [0, 1) such that the complex numbers $\exp(2\pi i\beta_{\ell,j})$ are the eigenvalues of the monodromy along Γ_ℓ°.

CONJECTURE 1.1 (Steenbrink [25, Conjecture 2.2])
For $N \gg 0$,

$$\mathrm{Sp}(f + g^N, x) - \mathrm{Sp}(f, x) = \sum_{\ell,j} t^{\alpha_{\ell,j}+(\beta_{\ell,j}/m_\ell N)} \frac{1-t}{1-t^{1/m_\ell N}}. \tag{1.1.1}$$

The conjecture of Steenbrink has been proved by M. Saito in [21], using his theory of mixed Hodge modules (see [18], [20]). Later, A. Némethi and J. H. M. Steenbrink [17] gave another proof, still relying on the theory of mixed Hodge modules. Also, forgetting the integer part of the exponents of the spectrum, (1.1.1) has been proved by D. Siersma [23] in terms of zeta functions of the monodromy. Notice that, taking ordinary Euler characteristics, (1.1.1) specializes to a result of I. Iomdin [14], who was the first to compare vanishing cohomologies of f and $f + g^N$. The convention we use here (see (6.6.2)) to define $\mathrm{Sp}(f, x)$ slightly differs from the original one and corresponds to what is denoted by $\mathrm{Sp}'(f, x)$ in [21].

Recently, using motivic integration, Denef and Loeser introduced the motivic Milnor fiber $\mathscr{S}_{f,x}$ (see [5], [8]). It is a virtual variety endowed with an action of the group scheme $\hat{\mu}$ of roots of unity, and the Hodge spectrum $\mathrm{Sp}(f, x)$ can be retrieved from $\mathscr{S}_{f,x}$ (see [8]). They also showed that an analogue of the Thom-Sebastiani theorem holds for the motivic Milnor fiber. This result was first stated in a (completed) Grothendieck ring (see [7]) of Chow motives and then extended to a Grothendieck ring of virtual varieties endowed with a $\hat{\mu}$-action in [16] and [8] using a convolution product $*$ introduced in [16]. It is also convenient to slightly modify the virtual varieties $\mathscr{S}_{f,x}$, which correspond to nearby cycles, into virtual varieties $\mathscr{S}^\phi_{f,x}$ corresponding to vanishing cycles.

It is then quite natural to ask for a motivic analogue of Steenbrink's conjecture in terms of motivic Milnor fibers. The present article is devoted to give a complete answer to that question. Our main result, Theorem 5.7, expresses (in its local version, Corollary 5.16) for x a closed point, where f and g both vanish, and for $N \gg 0$ the difference $\mathscr{S}^{\phi}_{f,x} - \mathscr{S}^{\phi}_{f+g^N,x}$ as $\Psi_\Sigma(\mathscr{S}_{g^N,x}(\mathscr{S}^{\phi}_f))$, where $\mathscr{S}_{g^N,x}(\mathscr{S}^{\phi}_f)$ corresponds to iterated motivic vanishing cycles and Ψ_Σ is a generalization of the convolution product $*$. In fact, in Theorem 5.7, we no longer assume any condition on the singular locus of f; also, g is no longer assumed to be a generic linear form and can be any function vanishing at x. Formula (1.1.1) may be deduced from Theorem 5.7 by considering the Hodge spectrum.

The plan of the article is the following. In Section 2 we introduce the basic Grothendieck rings that we use. Then, in Section 3, we recall the definition of the motivic Milnor fiber, and we extend it to the whole Grothendieck ring. Such an extension has also been done by F. Bittner in [3], using the weak factorization theorem, and in her work [2]; the construction we present here, based on motivic integration, is quite different. We then extend the construction to the equivariant setting in order to define iterated vanishing cycles in the motivic framework in Section 4. In Section 5 we first define our generalized convolution operator Ψ_Σ and explain its relation with the convolution product $*$. This gives us the opportunity to prove the associativity of the convolution product $*$, a fact already mentioned in [8]. Then comes the heart of the article, that is, the proof of Theorem 5.7. We conclude the section by explaining how one recovers the motivic Thom-Sebastiani theorem of [7], [16], and [8] from Theorem 5.7. The final section, Section 6, is devoted to applications to the Hodge-Steenbrink spectrum; in particular, we deduce Steenbrink's conjecture, Conjecture 1.1, from Theorem 5.7.

2. Grothendieck rings

2.1

By a variety over a field k, we mean a separated and reduced scheme of finite type over k. If X is a scheme, we denote by $|X|$ the corresponding reduced scheme. If an algebraic group G acts on a variety X, we say the action is good if every G-orbit is contained in an affine open subset of X. Let Y be a variety over k, and let $p : A \to Y$ be an affine bundle for the Zariski topology. (The fibers of p are affine spaces, and the transition morphisms between trivializing charts are affine.) In particular, the fibers of p have the structure of affine spaces. Let G be a linear algebraic group. A good action of G on A is said to be affine if it is a lifting of a good action on Y and its restriction to all fibers is affine. Note that affine actions on an affine bundle extend to its relative projective bundle compactification.

If G is finite and X and Y are two varieties with good G-action, we denote by $X \times^G Y$ the quotient of the product $X \times Y$ by the equivalence relation $(gx, y) \equiv (x, gy)$. The action of G on, say, the first factor of $X \times Y$ induces a good G-action on $X \times^G Y$.

For $n \geq 1$, we denote by μ_n the group scheme of nth roots of unity and by $\hat{\mu}$ the projective limit $\varprojlim \mu_n$ of the projective system with transition morphisms $\mu_{nd} \to \mu_n$

given by $x \mapsto x^d$. In this article all $\hat{\mu}$-actions, and more generally, all $\hat{\mu}^r$-actions, are assumed to factorize through a finite quotient.

2.2

Throughout the article k is a field of characteristic zero. For S, a variety over k, we denote by $K_0(\mathrm{Var}_S)$ the Grothendieck ring of varieties over S (see [8]). Let us recall that it is generated by classes of morphisms of varieties $X \to S$ and that it is also generated by classes of such morphisms with X smooth over k, and it suffices to consider relations for smooth varieties. We denote by $\mathbf{L} = \mathbf{L}_S$ the class of the trivial line bundle over S and set $\mathscr{M}_S$ for the localization $K_0(\mathrm{Var}_S)[\mathbf{L}^{-1}]$. As in [9], let us consider Grothendieck rings of varieties with $\hat{\mu}$-action. They are defined similarly, using the category $\mathrm{Var}_S^{\hat{\mu}}$ of varieties with good $\hat{\mu}$-action over S, but adding the additional relation

$$[Y \times \mathbf{A}_k^n, \sigma] = [Y \times \mathbf{A}_k^n, \sigma'] \tag{2.2.1}$$

if σ and σ' are two liftings of the same $\hat{\mu}$-action on Y to an affine action on $Y \times \mathbf{A}_k^n$. We denote them by $K_0(\mathrm{Var}_S^{\hat{\mu}})$ and $\mathscr{M}_S^{\hat{\mu}}$. One can more generally replace $\hat{\mu}$ by $\hat{\mu}^r$ in these definitions and define $K_0(\mathrm{Var}_S^{\hat{\mu}^r})$ and $\mathscr{M}_S^{\hat{\mu}^r}$. In [3] Bittner considers similar equivariant rings but with an additional relation that is, a priori, coarser than the one we use here.

2.3

In the present article, instead of varieties with $\hat{\mu}$-action over S, we choose to work in the equivalent setting of varieties with $\mathbf{G}_m$-action with some additional structure.

Let Y be a variety with good $\mathbf{G}_m^r$-action. We say that a morphism $\pi : Y \to \mathbf{G}_m^r$ is diagonally monomial of weight $\mathbf{n}$ in $\mathbf{N}_{>0}^r$ if $\pi(\lambda x) = \lambda^{\mathbf{n}} \pi(x)$ for all λ in $\mathbf{G}_m^r$ and x in Y. Fix $\mathbf{n}$ in $\mathbf{N}_{>0}^r$. We denote by $\mathrm{Var}_{S \times \mathbf{G}_m^r}^{\mathbf{G}_m^r, \mathbf{n}}$ the category of varieties $Y \to S \times \mathbf{G}_m^r$ over $S \times \mathbf{G}_m^r$ with good $\mathbf{G}_m^r$-action such that, furthermore, the fibers of the projection $\pi_1 : Y \to S$ are $\mathbf{G}_m^r$-invariant and the projection $\pi_2 : Y \to \mathbf{G}_m^r$ is diagonally monomial of weight $\mathbf{n}$. We define the Grothendieck group $K_0(\mathrm{Var}_{S \times \mathbf{G}_m^r}^{\mathbf{G}_m^r, \mathbf{n}})$ as the free abelian group on isomorphism classes of objects $Y \to S \times \mathbf{G}_m^r$ in $\mathrm{Var}_{S \times \mathbf{G}_m^r}^{\mathbf{G}_m^r, \mathbf{n}}$, modulo the relations

$$[Y \to S \times \mathbf{G}_m^r] = [Y' \to S \times \mathbf{G}_m^r] + [Y \backslash Y' \to S \times \mathbf{G}_m^r] \tag{2.3.1}$$

for Y' closed $\mathbf{G}_m^r$-invariant in Y and, for $f : Y \to S \times \mathbf{G}_m^r$ in $\mathrm{Var}_{S \times \mathbf{G}_m^r}^{\mathbf{G}_m^r, \mathbf{n}}$,

$$[Y \times \mathbf{A}_k^n \to S \times \mathbf{G}_m^r, \sigma] = [Y \times \mathbf{A}_k^n \to S \times \mathbf{G}_m^r, \sigma'] \tag{2.3.2}$$

if σ and σ' are two liftings of the same $\mathbf{G}_m^r$-action on Y to affine actions, the morphism $Y \times \mathbf{A}_k^n \to S \times \mathbf{G}_m^r$ being the composition of f with the projection on the first factor. Of course, in (2.3.2), instead of the trivial affine bundle, we could have considered any affine bundle over Y.

Fiber product over $S \times \mathbf{G}_m^r$ with diagonal action induces a product in the category $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}$, which allows us to endow $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m,\mathbf{n}})$ with a natural ring structure. Note that the unit $1_{S\times\mathbf{G}_m^r}$ for the product is the class of the identity morphism on $S \times \mathbf{G}_m^r$, the $\mathbf{G}_m^r$-action on $S \times \mathbf{G}_m^r$ being the trivial one on S and the standard multiplicative translation on $\mathbf{G}_m^r$.There is a natural structure of $K_0(\mathrm{Var}_k)$-module on $K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}})$. We denote by $\mathbf{L}_{S\times\mathbf{G}_m^r} = \mathbf{L}$ the element $\mathbf{L}\cdot 1_{S\times\mathbf{G}_m^r}$ in this module, and we set $\mathscr{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}} = K_0(\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}})[\mathbf{L}^{-1}]$.

If $f : S \to S'$ is a morphism of varieties, composition with f leads to a push-forward morphism $f_! : \mathscr{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}} \to \mathscr{M}_{S'\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}$, while fiber product leads to a pullback morphism $f^* : \mathscr{M}_{S'\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}} \to \mathscr{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}$. (These morphisms may already be defined at the K_0-level.)

2.4

For $\mathbf{n}$ in $\mathbf{N}_{>0}^r$, we denote by $\mu_{\mathbf{n}}$ the group $\mu_{n_1} \times \cdots \times \mu_{n_r}$. We consider the functor

$$G_{\mathbf{n}} : \mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}} \longrightarrow \mathrm{Var}_S^{\mu_{\mathbf{n}}}, \tag{2.4.1}$$

assigning to $p : Y \to S \times \mathbf{G}_m^r$ the fiber at 1 of the morphism $Y \to \mathbf{G}_m^r$ obtained by composition with projection on the second factor. Note that this fiber carries a natural $\mu_{\mathbf{n}}$-action by the monomiality assumption.

On the other side, if $f : X \to S$ is a variety over S with good $\mu_{\mathbf{n}}$-action, we may consider the variety $F_{\mathbf{n}}(X) := X \times^{\mu_{\mathbf{n}}} \mathbf{G}_m^r$ and view it as a variety over $S \times \mathbf{G}_m^r$ by sending the class of (x, λ) to $(f(x), \lambda^{\mathbf{n}})$. The standard $\mathbf{G}_m^r$-action by multiplicative translation on $\mathbf{G}_m^r$ induces a $\mathbf{G}_m^r$-action on $F_{\mathbf{n}}(X)$. Note that the second projection is diagonally monomial of weight $\mathbf{n}$; hence, $F_{\mathbf{n}}$ is in fact a functor

$$F_{\mathbf{n}} : \mathrm{Var}_S^{\mu_{\mathbf{n}}} \longrightarrow \mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}. \tag{2.4.2}$$

LEMMA 2.5

The functors $F_{\mathbf{n}}$ and $G_{\mathbf{n}}$ are mutually quasi-inverse, so that the categories $\mathrm{Var}_S^{\mu_{\mathbf{n}}}$ *and* $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}$ *are equivalent.*

Proof

It is quite clear that $G_{\mathbf{n}}(F_{\mathbf{n}}(X))$ is isomorphic to X for X in $\mathrm{Var}_S^{\mu_{\mathbf{n}}}$. For X in $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}$, set $Y := G_{\mathbf{n}}(X)$. We have a natural morphism $Y \times \mathbf{G}_m^r \to X$, sending (y, λ) to λy. Clearly, this morphism induces an isomorphism between $Y \times^{\mu_{\mathbf{n}}} \mathbf{G}_m^r$ and X in $\mathrm{Var}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r,\mathbf{n}}$. □

We consider the partial order $\mathbf{n} \prec \mathbf{m}$ on $\mathbf{N}^r_{>0}$ given by divisibility of each coordinate; that is, $\mathbf{n} \prec \mathbf{m}$ if $\mathbf{n} = \mathbf{km}$ for some $\mathbf{k}$ in $\mathbf{N}^r_{>0}$. If $\mathbf{n} = \mathbf{km}$, we have a natural functor

$$\theta^{\mathbf{m}}_{\mathbf{n}} : \mathrm{Var}^{\mathbf{G}^r_m,\mathbf{m}}_{S\times\mathbf{G}^r_m} \longrightarrow \mathrm{Var}^{\mathbf{G}^r_m,\mathbf{n}}_{S\times\mathbf{G}^r_m}, \tag{2.5.1}$$

sending $X \to S \times \mathbf{G}^r_m$ to the same object but with the action $\lambda \mapsto \lambda x$ on X replaced by $\lambda \mapsto \lambda^{\mathbf{k}} x$. We define the category $\mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m}$ as the colimit of the inductive system of categories $\mathrm{Var}^{\mathbf{G}^r_m,\mathbf{n}}_{S\times\mathbf{G}^r_m}$. We define $K_0(\mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m})$ and $\mathscr{M}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m}$ as in Section 2.3. Clearly, $K_0(\mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m})$ and $\mathscr{M}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m}$ are, respectively, the colimits of the rings $K_0(\mathrm{Var}^{\mathbf{G}^r_m,\mathbf{n}}_{S\times\mathbf{G}^r_m})$ and $\mathscr{M}^{\mathbf{G}^r_m,\mathbf{n}}_{S\times\mathbf{G}^r_m}$. Since the category $\mathrm{Var}^{\hat{\mu}^r}_S$ is the colimit of the categories $\mathrm{Var}^{\hat{\mu}_{\mathbf{n}}}_S$, we have the following statement.

PROPOSITION 2.6
There is a unique pair of functors

$$G : \mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m} \longrightarrow \mathrm{Var}^{\hat{\mu}^r}_S \tag{2.6.1}$$

and

$$F : \mathrm{Var}^{\hat{\mu}^r}_S \longrightarrow \mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m} \tag{2.6.2}$$

which restrict to $G_{\mathbf{n}}$ and $F_{\mathbf{n}}$ for every $\mathbf{n}$. They are mutually quasi-inverse. In particular, G induces canonical isomorphisms

$$K_0(\mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m}) \simeq K_0(\mathrm{Var}^{\hat{\mu}^r}_S) \qquad \textit{and} \qquad \mathscr{M}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m} \simeq \mathscr{M}^{\hat{\mu}^r}_S \tag{2.6.3}$$

compatible with the operations $f_!$ and f^*. □

2.7
Let Y be a variety with good $\mathbf{G}^r_m$-action. We say that a morphism $\pi : Y \to \mathbf{G}^r_m$ is monomial if it is equivariant with respect to some transitive $\mathbf{G}^r_m$-action on $\mathbf{G}^r_m$ (see Section 4.6 for monomial morphisms that are not diagonally monomial morphisms). More generally, consider a variety $(p, \pi) : Y \to S \times \mathbf{G}^r_m$ over $S \times \mathbf{G}^r_m$ with good $\mathbf{G}^r_m$-action such that, furthermore, the fibers of $p : Y \to S$ are $\mathbf{G}^r_m$-invariant and $\pi : Y \to \mathbf{G}^r_m$ is monomial. By elementary linear algebra, there exists a group morphism $\varrho : \mathbf{G}^r_m \to \mathbf{G}^r_m$ such that if we compose the original $\mathbf{G}^r_m$-action on Y with ϱ, the morphism π becomes diagonally monomial for that new action. Furthermore, the image of $(p, \pi) : Y \to S \times \mathbf{G}^r_m$ with the action twisted by ϱ in $\mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m}$, and hence, also its class in $K_0(\mathrm{Var}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m})$ and in $\mathscr{M}^{\mathbf{G}^r_m}_{S\times\mathbf{G}^r_m}$, does not depend on ϱ. We denote that class by $[(p, \pi) : Y \to S \times \mathbf{G}^r_m]$. Indeed, the first statement amounts to saying that for every matrix A in $\mathrm{M}_r(\mathbf{Z}) \cap \mathrm{GL}_r(\mathbf{Q})$, there exists B in $\mathrm{M}_r(\mathbf{Z}) \cap \mathrm{GL}_r(\mathbf{Q})$ such that BA is diagonal with coefficients in $\mathbf{N}_{>0}$,

and the second statement follows from the observation that if B' is another such matrix, there exist diagonal matrices C and C' with coefficients in $\mathbf{N}_{>0}$ such that $CB = C'B'$.

More generally, if W is a constructible subset of Y which is stable by the $\mathbf{G}_m^r$-action, we call a morphism $\pi : W \to \mathbf{G}_m^r$ piecewise monomial if there is a finite partition of W into locally closed $\mathbf{G}_m^r$-invariant subsets on which the restriction of π is a monomial morphism. To such a W, endowed with a morphism $(p,\pi) : W \to S\times\mathbf{G}_m^r$ such that the fibers of $p : W \to S$ are $\mathbf{G}_m^r$-invariant and $\pi : W \to \mathbf{G}_m^r$ is piecewise monomial, we assign by additivity a class $[(p,\pi) : W \to S \times \mathbf{G}_m^r]$ in $\mathscr{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r}$.

2.8. Rational series

Let A be one of the rings $\mathbf{Z}[\mathbf{L},\mathbf{L}^{-1}]$, $\mathbf{Z}[\mathbf{L},\mathbf{L}^{-1},(1/(1-\mathbf{L}^{-i}))_{i>0}]$, $\mathscr{M}_{S\times\mathbf{G}_m^r}^{\mathbf{G}_m^r}$. We denote by $A[[T]]_{\mathrm{sr}}$ the A-submodule of $A[[T]]$ generated by 1 and by finite products of terms $p_{e,i}(T) = \mathbf{L}^e T^i/(1-\mathbf{L}^e T^i)$ with e in $\mathbf{Z}$ and i in $\mathbf{N}_{>0}$. There is a unique A-linear morphism

$$\lim_{T\mapsto\infty} : A[[T]]_{\mathrm{sr}} \longrightarrow A \tag{2.8.1}$$

such that

$$\lim_{T\mapsto\infty}\Big(\prod_{i\in I} p_{e_i,j_i}(T)\Big) = (-1)^{|I|} \tag{2.8.2}$$

for every family $((e_i, j_i))_{i\in I}$ in $\mathbf{Z}\times\mathbf{N}_{>0}$ with I finite and maybe empty.

2.9

Let I be a finite set. We consider rational polyhedral convex cones in $\mathbf{R}_{>0}^I$. By this, we mean a convex subset of $\mathbf{R}_{>0}^I$ defined by a finite number of integral linear inequalities of type $a \geq 0$ or $b > 0$ and stable by multiplication by $\mathbf{R}_{>0}$. Let Δ be such a cone in $\mathbf{R}_{>0}^I$. We denote by $\bar\Delta$ its closure in $\mathbf{R}_{\geq 0}^I$.

Let ℓ and ν be integral linear forms on $\mathbf{Z}^I$ which are positive on $\bar\Delta\setminus\{0\}$. Let us consider the series

$$S_{\Delta,\ell,\nu}(T) := \sum_{k\in\Delta\cap\mathbf{N}_{>0}^I} T^{\ell(k)}\mathbf{L}^{-\nu(k)} \tag{2.9.1}$$

in $\mathbf{Z}[\mathbf{L},\mathbf{L}^{-1}][[T]]$.

In the special case when Δ is open in its linear span and $\bar\Delta$ is generated by vectors $(e_1,\dots,e_m)$ which are part of a $\mathbf{Z}$-basis of the $\mathbf{Z}$-module $\mathbf{Z}^I$, the series $S_{\Delta,\ell,\nu}$ lies in $\mathbf{Z}[\mathbf{L},\mathbf{L}^{-1}][[T]]_{\mathrm{sr}}$ and $\lim_{T\mapsto\infty} S_{\Delta,\ell,\nu}(T)$ is equal to $(-1)^{\dim(\Delta)}$. By additivity with respect to disjoint union of cones with the positivity assumption, one deduces that, in general, $S_{\Delta,\ell,\nu}$ lies in $\mathbf{Z}[\mathbf{L},\mathbf{L}^{-1}][[T]]_{\mathrm{sr}}$ and that $\lim_{T\mapsto\infty} S_{\Delta,\ell,\nu}(T)$ is equal to $\chi(\Delta)$, the Euler characteristic with compact supports of Δ.

In particular, we get the following lemma (cf. [13, Lemma 2.1.5] and [4, pages 1006–1007]).

LEMMA 2.10
Let Δ be a rational polyhedral convex cone in $\mathbf{R}^I_{>0}$ defined by

$$\sum_{i\in K} a_i x_i \leq \sum_{i\in I\setminus K} a_i x_i \tag{2.10.1}$$

with a_i in $\mathbf{N}$, $a_i > 0$ for i in K, and K and $I\setminus K$ nonempty. If ℓ and ν are integral linear forms positive on $\bar{\Delta}\setminus\{0\}$, then $\lim_{T\mapsto\infty} S_{\Delta,\ell,\nu}(T) = 0$.

3. Motivic vanishing cycles

3.1. Arc spaces
As usual, we denote by $\mathscr{L}_n(X)$ the space of arcs of order n, also known as the nth jet space on X. It is a k-scheme whose set of K-points, for K a field containing k, is the set of morphisms φ : Spec $K[t]/t^{n+1} \to X$. There are canonical morphisms $\mathscr{L}_{n+1}(X) \to \mathscr{L}_n(X)$ which are $\mathbf{A}^d_k$-bundles when X is smooth of pure dimension d. The arc space $\mathscr{L}(X)$ is defined as the projective limit of this system. We denote by $\pi_n : \mathscr{L}(X) \to \mathscr{L}_n(X)$ the canonical morphism. There is a canonical $\mathbf{G}_m$-action on $\mathscr{L}_n(X)$ and on $\mathscr{L}(X)$ given by $a\cdot\varphi(t) = \varphi(at)$.

For an element φ in $K[[t]]$ or in $K[t]/t^{n+1}$, we denote by $\mathrm{ord}_t(\varphi)$ the valuation of φ and by $\mathrm{ac}(\varphi)$ its first nonzero coefficient with the convention $\mathrm{ac}(0) = 0$.

3.2. Motivic zeta function and Motivic Milnor fiber
Let us start by recalling some basic constructions introduced by Denef and Loeser in [5], [9], and [8].

Let X be a smooth variety over k of pure dimension d, and let $g : X \to \mathbf{A}^1_k$ be a morphism. We set $X_0(g)$ for the zero locus of g and consider for $n \geq 1$ the variety

$$\mathscr{X}_n(g) := \bigl\{\varphi \in \mathscr{L}_n(X) \bigm| \mathrm{ord}_t\, g(\varphi) = n\bigr\}. \tag{3.2.1}$$

Note that $\mathscr{X}_n(g)$ is invariant by the $\mathbf{G}_m$-action on $\mathscr{L}_n(X)$. Furthermore, g induces a morphism $g_n : \mathscr{X}_n(g) \to \mathbf{G}_m$, assigning to a point φ in $\mathscr{L}_n(X)$ the coefficient $\mathrm{ac}(g(\varphi))$ of t^n in $g(\varphi)$, which we also denote by $\mathrm{ac}(g)(\varphi)$. This morphism is diagonally monomial of weight n with respect to the $\mathbf{G}_m$-action on $\mathscr{X}_n(g)$ since $g_n(a\cdot\varphi) = a^n g_n(\varphi)$, so we can consider the class $[\mathscr{X}_n(g)]$ of $\mathscr{X}_n(g)$ in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$.

We now consider the motivic zeta function

$$Z_g(T) := \sum_{n\geq 1} [\mathscr{X}_n(g)]\, \mathbf{L}^{-nd}\, T^n \tag{3.2.2}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times \mathbf{G}_m}[[T]]$. Note that $Z_g = 0$ if $g = 0$ on X.

Denef and Loeser showed in [5] and [8] (see also [9]) that $Z_g(T)$ is a rational series by giving a formula for $Z_g(T)$ in terms of a resolution of f.

3.3. Resolutions

Let us introduce some notation and terminology. Let X be a smooth variety of pure dimension d, and let Z be a closed subset of X of codimension everywhere not less than 1. By a log-resolution $h : Y \to X$ of (X, Z), we mean a proper morphism $h : Y \to X$ with Y smooth such that the restriction of $h : Y \backslash h^{-1}(Z) \to X\backslash Z$ is an isomorphism and such that $h^{-1}(Z)$ is a divisor with normal crossings. We denote by E_i, i in A, the set of irreducible components of the divisor $h^{-1}(Z)$. For $I \subset A$, we set

$$E_I := \bigcap_{i\in I} E_i \tag{3.3.1}$$

and

$$E_I^\circ := E_I \backslash \bigcup_{j\notin I} E_j. \tag{3.3.2}$$

We denote by ν_{E_i} the normal bundle of E_i in Y, by $\nu^J_{E_I}$ the fiber product for J contained in I of the restrictions to E_I of the bundles ν_{E_i}, i in J, and by $\pi_I^J : \nu^J_{E_I} \to E_I$ the canonical projections. For any of these vector bundles ν, we denote by $\overline{\nu}$ the projective bundle associated to the sum of ν with the trivial line bundle.

We denote by U_{E_i} the complement of the zero section in ν_{E_i} and by U_I^J (resp., $U^J_{E_I}$) the fiber product for J contained in I of the restrictions of the spaces U_{E_i}, i in J, to E_I° (resp., E_I). We still denote by π_I^J the induced projection from U_I^J (resp., $U^J_{E_I}$) onto E_I° (resp., E_I).

When $J = I$, we simply write ν_{E_I} (resp., $\overline{\nu}_{E_I}$, π_I, U_I, U_{E_I}) for $\nu^I_{E_I}$ (resp., $\overline{\nu}^I_{E_I}$, π_I^I, U_I^I, $U^I_{E_I}$).

If $\mathscr{I}$ is a sheaf of ideals defining a closed subscheme Z and $h^{-1}(\mathscr{I})\mathscr{O}_Y$ is locally principal, we define $N_i(\mathscr{I})$, the multiplicity of $\mathscr{I}$ along E_i, by the equality of divisors

$$h^{-1}(Z) = \sum_{i\in A} N_i(\mathscr{I}) E_i. \tag{3.3.3}$$

If $\mathscr{I}$ is a sheaf of principal ideals generated by a function g, we write $N_i(g)$ for $N_i(\mathscr{I})$. Similarly, we define integers ν_i by the equality of divisors

$$K_Y = h^* K_X + \sum_{i\in A} (\nu_i - 1) E_i. \tag{3.3.4}$$

Let $\mathscr{I}_1$ and $\mathscr{I}_2$ be two sheaves of ideals on X whose associated reduced closed subschemes Z_1 and Z_2 have codimension of at least one. Let $h : Y \to X$ be a log-resolution of $(X, Z_1 \cup Z_2)$ such that $h^*(\mathscr{I}_1)$ and $h^*(\mathscr{I}_2)$ are locally principal. Then we set

$$\gamma_h(\mathscr{I}_1, \mathscr{I}_2) := \sup_{\{i \in A \mid N_i(\mathscr{I}_2) > 0\}} \frac{N_i(\mathscr{I}_1)}{N_i(\mathscr{I}_2)}. \tag{3.3.5}$$

If x is a closed point of Z_2, we set

$$\gamma_{h,x}(\mathscr{I}_1, \mathscr{I}_2) = \sup_{\{i \in A_x \mid N_i(\mathscr{I}_2) > 0\}} \frac{N_i(\mathscr{I}_1)}{N_i(\mathscr{I}_2)} \tag{3.3.6}$$

with A_x the set of i in A such that $|h^{-1}(x)| \cap E_i \neq \emptyset$. Finally, we define $\gamma(\mathscr{I}_1, \mathscr{I}_2)$, respectively, $\gamma_x(\mathscr{I}_1, \mathscr{I}_2)$, as the infimum of all $\gamma_h(\mathscr{I}_1, \mathscr{I}_2)$, respectively, $\gamma_{h,x}(\mathscr{I}_1, \mathscr{I}_2)$, for h, a log-resolution of $(X, Z_1 \cup Z_2)$ such that $h^*(\mathscr{I}_1)$ and $h^*(\mathscr{I}_2)$ are locally principal.

3.4

Let g be a function on a smooth variety X of pure dimension d. Assume that $X_0(g)$ is nowhere dense in X. Let F be a reduced divisor containing $X_0(g)$, and let $h : Y \to X$ be a log-resolution of (X, F). We fix I such that there exists i in I with $N_i(g) > 0$. Let us explain how g induces a morphism $g_I : U_I \to \mathbf{G}_m$. Note that the function $g \circ h$ induces a function

$$\bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(g)}|_{E_I} \longrightarrow \mathbf{A}_k^1. \tag{3.4.1}$$

We define $g_I : \nu_{E_I} \to \mathbf{A}_k^1$ as the composition of this last function with the natural morphism $\nu_{E_I} \to \bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(g)}|_{E_I}$, sending (y_i) to $\bigotimes y_i^{\otimes N_i(g)}$. We still denote by g_I the induced morphism from U_I (resp., U_{E_I}) to $\mathbf{G}_m$ (resp., $\mathbf{A}_k^1$).

We view U_I as a variety over $X_0(g) \times \mathbf{G}_m$ via the morphism $(h \circ \pi_I, g_I)$. The group $\mathbf{G}_m$ has a natural action on each U_{E_i}, so the diagonal action induces a $\mathbf{G}_m$-action on U_I. Furthermore, the morphism g_I is monomial, so $U_I \to X_0(g) \times \mathbf{G}_m$ has a class in $\mathscr{M}_{X_0(g) \times \mathbf{G}_m}^{\mathbf{G}_m}$ which we denote by $[U_I]$.

3.5

The morphism g_I may be described in terms of the following variant of the deformation to the normal cone to E_I in Y (see [12]). We consider the affine space $\mathbf{A}_k^I = \operatorname{Spec} k[u_i]_{i \in I}$ and the subsheaf

$$\mathscr{A}_I := \sum_{\mathbf{n} \in \mathbf{N}^I} \mathscr{O}_{Y \times \mathbf{A}_k^I}\Big(-\sum_{i \in I} n_i (E_i \times \mathbf{A}_k^I)\Big) \prod_{i \in I} u_i^{-n_i} \tag{3.5.1}$$

of $\mathscr{O}_{Y\times\mathbf{A}_k^I}[u_i^{-1}]_{i\in I}$. It is a sheaf of rings, and we set $CY_I := \operatorname{Spec}\mathscr{A}_I$. The natural inclusion $\mathscr{O}_{Y\times\mathbf{A}_k^I}\to\mathscr{A}_I$ induces a morphism $\pi: CY_I\to Y\times\mathbf{A}_k^I$; hence, a morphism $p: CY_I\to\mathbf{A}_k^I$. With the ring $\mathscr{A}_I$ being a graded subring of the ring $\mathscr{O}_Y[u_i,u_i^{-1}]_{i\in I}$, we consider the corresponding $\mathbf{G}_m^I$-action σ_I on CY_I, leaving sections of $\mathscr{O}_Y$ invariant and acting by $(\lambda_i,u_i)\mapsto\lambda_i^{-1}u_i$ on u_i. We may then identify equivariantly ν_{E_I} with the fiber $p^{-1}(0)$. The image by the inclusion $\mathscr{O}_{Y\times\mathbf{A}_k^I}\to\mathscr{A}_I$ of the function $g\circ h$ is divisible by $\prod_{i\in I}u_i^{N_i(g)}$ in $\mathscr{A}_I$, so we may consider the quotient $\widetilde{g}_I$ in $\mathscr{A}_I$. The restriction of $\widetilde{g}_I$ to the fiber $p^{-1}(0)\simeq\nu_{E_I}$ is nothing else than g_I. As g may vanish only on the divisors E_i, i in A, the function g_I does not vanish on U_I and induces a monomial morphism $g_I: U_I\longrightarrow\mathbf{G}_m$.

Let us note the following "transitivity" property. If we write I as a disjoint union $K\sqcup J$, one notices that $p^{-1}(0\times\mathbf{G}_m^J)$ is equivariantly isomorphic to $\nu_{E_K}\times\mathbf{G}_m^J$. Hence, restricting $p: CY_I\to\mathbf{A}_k^I$ to $p^{-1}(0\times\mathbf{A}_k^J)$, the function $g_I: U_I\longmapsto\mathbf{G}_m$ can be obtained from g_K by the same process as we obtained it from g, replacing Y by ν_{E_K}, I by J, and g by $g_K:\nu_{E_K}\longrightarrow\mathbf{A}_k^1$.

3.6

We now assume that $F = X_0(g)$; that is, $h: Y\to X$ is a log-resolution of $(X, X_0(g))$. In this case, h induces a bijection between $\mathscr{L}(Y)\backslash\mathscr{L}\big(|h^{-1}(X_0(g))|\big)$ and $\mathscr{L}(X)\backslash\mathscr{L}(X_0(g))$. By using the change of variable formula in a way completely similar to [5] and [8], one deduces the equality

$$Z_g(T) = \sum_{\emptyset\neq I\subset A}[U_I]\prod_{i\in I}\frac{1}{T^{-N_i(g)}\mathbf{L}^{\nu_i}-1} \tag{3.6.1}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]$.

In particular, with the notation of Section 2.8, the function $Z_g(T)$ is rational and belongs to $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$; hence, we can consider $\lim_{T\mapsto\infty}Z_g(T)$ in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$.

We set

$$\mathscr{S}_g := -\lim_{T\mapsto\infty}Z_g(T), \tag{3.6.2}$$

which by (3.6.1) may be expressed on a resolution h as

$$\mathscr{S}_g = -\sum_{\emptyset\neq I\subset A}(-1)^{|I|}[U_I], \tag{3.6.3}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$.

We also consider in this article the motivic vanishing cycles defined as

$$\mathscr{S}_g^{\phi} := (-1)^{d-1}\big(\mathscr{S}_g - [\mathbf{G}_m\times X_0(g)]\big) \tag{3.6.4}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$. Here d denotes the dimension of X, and $\mathbf{G}_m \times X_0(g)$ is endowed with the standard $\mathbf{G}_m$-action on the first factor and with the trivial $\mathbf{G}_m$-action on the second factor.

3.7. A modified zeta function

We now explain how to extend $\mathscr{S}_g$ to the whole Grothendieck group $\mathscr{M}_X$ in such a way that $\mathscr{S}_g([X \to X])$ is equal to $\mathscr{S}_g$. A similar result has been obtained by F. Bittner in [3]. We present here a somewhat different approach that avoids the use of the weak factorization theorem by constructing directly $\mathscr{S}_g([Y \to X])$ for generators of $\mathscr{M}_X$ of the form $Y \to X$ with Y smooth.

Let X be a smooth variety of pure dimension d, and let U be a dense open subset in X. Consider again a function $g : X \to \mathbf{A}^1_k$. We denote by F the closed subset $X\backslash U$ and by $\mathscr{I}_F$ the ideal of functions vanishing on F. We start by defining $\mathscr{S}_g([U \to X])$.

Fix $\gamma \geq 1$ a positive integer. We consider the modified zeta function $Z^{\gamma}_{g,U}(T)$, defined as follows. For $n \geq 1$, we consider the constructible set

$$\mathscr{X}^{\gamma n}_n(g, U) := \bigl\{\varphi \in \mathscr{L}_{\gamma n}(X) \bigm| \mathrm{ord}_t g(\varphi) = n, \mathrm{ord}_t \varphi^*(\mathscr{I}_F) \leq \gamma n\bigr\}. \tag{3.7.1}$$

As in Section 3.2, we consider the morphism $\mathscr{X}^{\gamma n}_n(g, U) \to \mathbf{G}_m$ induced by $\varphi \mapsto \mathrm{ac}(g(\varphi))$. It is piecewise monomial, so we can consider the class $[\mathscr{X}^{\gamma n}_n(g, U)]$ in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$ by Section 2.7. We set

$$Z^{\gamma}_{g,U}(T) := \sum_{n\geq 1} [\mathscr{X}^{\gamma n}_n(g, U)]\, \mathbf{L}^{-\gamma n d}\, T^n \tag{3.7.2}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]$. Note that for $U = X$, $Z^{\gamma}_{g,U}(T)$ is equal to $Z_g(T)$ for every γ since, in this case, $[\mathscr{X}^{\gamma n}_n(g, U)]\mathbf{L}^{-\gamma n d} = [\mathscr{X}_n(g)]\mathbf{L}^{-nd}$. Note also that $Z^{\gamma}_{g,U}(T) = 0$ if g is identically zero on X.

If $X_0(g)$ is nowhere dense in X and $h : Y \to X$ is a log-resolution of $(X, F \cup X_0(g))$, we denote by C the set $\{i \in A \mid N_i(g) \neq 0\}$.

PROPOSITION 3.8

Let U be a dense open subset in the smooth variety X of pure dimension d with a function $g : X \to \mathbf{A}^1_k$. There exists γ_0 such that for every $\gamma > \gamma_0$, the series $Z^{\gamma}_{g,U}(T)$ lies in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$ and $\lim_{T\mapsto\infty} Z^{\gamma}_{g,U}(T)$ is independent of $\gamma > \gamma_0$. We set $\mathscr{S}_{g,U} = -\lim_{T\mapsto\infty} Z^{\gamma}_{g,U}(T)$. Furthermore, if $X_0(g)$ is nowhere dense in X and $h : Y \to X$ is a log-resolution of $(X, F \cup X_0(g))$,

$$\mathscr{S}_{g,U} = -\sum_{\substack{I\neq\emptyset\\ I\subset C}} (-1)^{|I|}\, [U_I] = h_!(\mathscr{S}_{g\circ h, h^{-1}(U)}) \tag{3.8.1}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$.

Proof
We may assume that $X_0(g)$ is nowhere dense in X. Let $h : Y \to X$ be a log-resolution of $(X, F \cup X_0(g))$. As in the proof of [9, Theorem 2.4], we deduce from the change of variable formula, or more precisely from [6, Lemma 3.4], that

$$Z^{\gamma}_{g,U}(T) = \sum_{I \cap C \neq \emptyset} [U_I]\, S_I(T) \tag{3.8.2}$$

with

$$S_I(T) = \sum_{\substack{k_i \geq 1 \\ \sum_i k_i N_i(\mathscr{I}_F) \leq \gamma \sum_i k_i N_i(g)}} \prod_{i \in I} (T^{N_i(g)} \mathbf{L}^{-\nu_i})^{k_i}. \tag{3.8.3}$$

Assume first that $I \subset C$. For $\gamma \geq \sup_{i \in I} \big(N_i(\mathscr{I}_F)/(N_i(g))\big)$, we have $\sum_i k_i N_i(\mathscr{I}_F) \leq \gamma \sum_i k_i N_i(g)$ for all $k_i \geq 1$, $i \in I$. It follows that $S_I(T)$ lies in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}[[T]]_{\mathrm{sr}}$ and $\lim_{T \mapsto \infty} S_I(T) = (-1)^{|I|}$, as soon as $\gamma \geq \sup_{i \in I} \big(N_i(\mathscr{I}_F)/(N_i(g))\big)$.

Now assume that $\emptyset \neq I \backslash C = K$. For $\gamma \geq \sup_{i \in I \backslash K} \big(N_i(\mathscr{I}_F)/(N_i(g))\big)$, the sum runs over the points with coordinates in $\mathbf{N}_{>0}$ of the cone Δ_I in $\mathbf{R}^I_{>0}$ defined by the single inequality

$$\sum_{i \in K} a_i x_i \leq \sum_{i \in I \backslash K} a_i x_i \tag{3.8.4}$$

with a_i in $\mathbf{N}$ and $a_i > 0$ for i in K. Note that both K and $I \backslash K$ are nonempty. It follows from Lemma 2.10 that in this case, $S_I(T)$ lies in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}[[T]]_{\mathrm{sr}}$ and $\lim_{T \mapsto \infty} S_I(T) = 0$. The statement we have to prove then holds if we set $\gamma_0 = \sup_{i \in C} \big(N_i(\mathscr{I}_F)/(N_i(g))\big) = \gamma_h(\mathscr{I}_F, (g))$. Note that since this holds for any h, we could also take $\gamma_0 = \gamma(\mathscr{I}_F, (g))$. □

THEOREM 3.9 (Extension to the Grothendieck group)
Let X be a variety with a function $g : X \to \mathbf{A}^1_k$. There exists a unique $\mathscr{M}_k$-linear group morphism

$$\mathscr{S}_g : \mathscr{M}_X \longrightarrow \mathscr{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m} \tag{3.9.1}$$

such that for every proper morphism $p : Z \to X$ with Z smooth and for every dense open subset U in Z,

$$\mathscr{S}_g([U \to X]) = p_!(\mathscr{S}_{g \circ p, U}). \tag{3.9.2}$$

Proof
Since $K_0(\mathrm{Var}_X)$ is generated by classes $[U \to X]$ with U smooth connected, and since every such $U \to X$ may be embedded in a proper morphism $Z \to X$ with Z smooth

and U dense in Z, uniqueness is clear. For existence, let us first note that if we define $\mathscr{S}_g([U \to X]) = \mathscr{S}_g([U])$ by the right-hand side of (3.9.2), the result is independent from the choice of the embedding in a proper morphism $p : Z \to X$. Indeed, this is clear if $g \circ p$ vanishes identically on U, so we may assume that $(g \circ p)^{-1}(0)$ is of codimension > 0. In this case, if we have another such morphism $p' : Z' \to X$, there exists a smooth variety W with proper morphisms $h : W \to Z$ and $h' : W \to Z'$ such that $p \circ h = p' \circ h'$ and such that h and h' are, respectively, log-resolutions of $(Z, (Z \backslash U) \cup (g \circ p)^{-1}(0))$ and $(Z', (Z' \backslash U) \cup (g \circ p')^{-1}(0))$, so the statement follows from (3.8.1).

Let us now prove the following additivity statement. If $\kappa : U \to X$ is a morphism with U smooth and W is a smooth closed subset of U, then

$$\mathscr{S}_g([U \to X]) = \mathscr{S}_g([W \to X]) + \mathscr{S}_g([U \backslash W \to X]). \tag{3.9.3}$$

We may assume that U and W are connected and that $U \backslash W$ is dense in U. The result being trivial if $g \circ \kappa$ vanishes identically, we may assume that this is not the case. By Hironaka's strong resolution of singularities, we may embed U in a smooth variety Z with $p : Z \to X$ a proper morphism extending κ such that $Z \backslash U$ is a normal crossings divisor and the closure $\overline{W}$ of W in Z is smooth. Again by Hironaka's strong resolution of singularities, there exists a log-resolution $h : \widetilde{Z} \to Z$ of $(Z, (Z \backslash U) \cup (g \circ p)^{-1}(0))$ such that the closure $\widetilde{W}$ of $h^{-1}(W)$ in $\widetilde{Z}$ is smooth and intersects the divisor $D := h^{-1}((Z \backslash U) \cup (g \circ p)^{-1}(0))$ transversally. We denote by E_i, i in A, the irreducible components of the divisor D and use the notation of Section 3.3. It follows from the definition and (3.8.1) that

$$\mathscr{S}_g([U \to X]) = -\sum_{\substack{I \neq \emptyset \\ I \subset C}} (-1)^{|I|} \, [U_I] \tag{3.9.4}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g) \times \mathbf{G}_m}$. Note that if W is contained in $(g \circ p)^{-1}(0)$, the previous discussion still holds for U replaced by $U \backslash W$, so we have $\mathscr{S}_g([U \to X]) = \mathscr{S}_g([(U \backslash W) \to X])$, and (3.9.3) follows since $\mathscr{S}_g([W \to X]) = 0$ in this case.

Now we assume that W is not contained in $(g \circ p)^{-1}(0)$. Note that the morphism $h_0 : \widetilde{W} \to \overline{W}$ induced by h is a log-resolution of $(\overline{W}, (\overline{W} \backslash W) \cup (g \circ p)^{-1}_{|\overline{W}}(0))$. Furthermore, the irreducible components of the normal crossings divisor $h_0^{-1}((\overline{W} \backslash W) \cup (g \circ p)^{-1}_{|\overline{W}}(0))$ are exactly those among the $E_i \cap \widetilde{W}$ which are nonempty. Hence, denoting by $U_I|_{E_I^\circ \cap \widetilde{W}}$ the restriction of the bundle U_I to $E_I^\circ \cap \widetilde{W}$, it follows from the definition and (3.8.1) that

$$\mathscr{S}_g([W \to X]) = -\sum_{\substack{I \neq \emptyset \\ I \subset C}} (-1)^{|I|} \, [U_I|_{E_I^\circ \cap \widetilde{W}}] \tag{3.9.5}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$. Let us now consider the blowup $h' : Z' \to \widetilde{Z}$ of $\widetilde{Z}$ along $\widetilde{W}$. The exceptional divisor W' of $\widetilde{W}$ is smooth. Furthermore, $h \circ h' : Z' \to Z$ is a log-resolution of $\big(Z, (Z\backslash(U\backslash W)) \cup (g \circ p)^{-1}(0)\big)$, and $D' := (h \circ h')^{-1}\big((Z\backslash(U\backslash W)) \cup (g \circ p)^{-1}(0)\big)$ is a normal crossings divisor whose irreducible components are the strict transforms E'_i of E_i in Z', i in A together with W'. We set $A' := A \sqcup \{0\}$ and $E'_0 := W'$ in order to use the notation of Section 3.3 in this setting, adding everywhere $'$ as an exponent. Again, it follows from the definition and (3.8.1) that

$$\mathscr{S}_g\big([(U\backslash W) \to X]\big) = -\sum_{\substack{I\neq\emptyset\\ I\subset C'}} (-1)^{|I|}\,[U'_I] \tag{3.9.6}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$, where $C' = \{i \in A' | N_i(g \circ p \circ h \circ h') \neq 0\}$. The hypothesis made on W ensures that $C' = C$. So it is enough to prove that for I nonempty and contained in C,

$$[U_I] = [U_I|_{E_I^\circ\cap\widetilde{W}}] + [U'_I] \tag{3.9.7}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$, which follows from the fact that the restriction $U_I|_{E_I^\circ\backslash(E_I^\circ\cap\widetilde{W})}$ of the bundle U_I to $E_I^\circ\backslash(E_I^\circ \cap \widetilde{W})$ and the bundle U'_I have the same class in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$ since h' is an isomorphism outside W'. This concludes the proof of (3.9.3).

Let $U \to X$ again be in Var_X with U smooth and connected. Let W be a smooth proper variety over k. Note that

$$\mathscr{S}_g([W \times U \to X]) = [W]\,\mathscr{S}_g([U \to X]) \tag{3.9.8}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$. Indeed, let us embed $U \to X$ in $p : Z \to X$ with Z smooth and proper and U dense in Z. We may assume that $g \circ p$ is not identically zero. If $h : Y \to Z$ is a log-resolution of $(Z, (Z\backslash U) \cup (g \circ p)^{-1}(0))$, then $W \times U \to X$ may be embedded in $W \times Z \to X$ and $\mathrm{id} \times h : W \times Y \to W \times Z$ is a log-resolution of $\big(W \times Z, ((W \times Z)\backslash(W \times U)) \cup (W \times g \circ p)^{-1}(0)\big)$; hence, (3.9.8) follows from (3.8.1) and (3.9.2). By the additivity statement that we already proved, relation (3.9.8) in fact holds for every variety W over k, so our construction of $\mathscr{S}_g$ may be extended uniquely by $\mathscr{M}_k$-linearity to a $\mathscr{M}_k$-linear group morphism $\mathscr{M}_X \to \mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}$, which finishes the proof. □

3.10. The equivariant setting

Let X be a variety with a function $g : X \to \mathbf{A}^1_k$. By Theorem 3.9, there is a canonical morphism

$$\mathscr{S}_g : \mathscr{M}_X \longrightarrow \mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}. \tag{3.10.1}$$

We want to lift this morphism to a morphism still denoted by $\mathscr{S}_g$,

$$\mathscr{S}_g : \mathscr{M}^{\mathbf{G}_m^r}_{X\times\mathbf{G}_m^r} \longrightarrow \mathscr{M}^{\mathbf{G}_m^r\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}, \tag{3.10.2}$$

such that the diagram

$$\begin{array}{ccc} \mathscr{M}^{\mathbf{G}_m^r}_{X\times\mathbf{G}_m^r} & \xrightarrow{\mathscr{S}_g} & \mathscr{M}^{\mathbf{G}_m^r\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m} \\ \downarrow & & \downarrow \\ \mathscr{M}_X & \xrightarrow{\mathscr{S}_g} & \mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m} \end{array} \tag{3.10.3}$$

is commutative, the vertical arrows being given by forgetting the $\mathbf{G}_m^r$-action and taking the fiber over 1 in $\mathbf{G}_m^r$.

Let us start with some basic facts that we use without further mention. We fix the variety X, which we consider as endowed with the trivial $\mathbf{G}_m^r$-action. Let Z be a smooth variety of pure dimension d endowed with a good $\mathbf{G}_m^r$-action and an equivariant morphism $p : Z \to X$. The induced action on the affine bundles $\mathscr{L}_{n+1}(Z) \to \mathscr{L}_n(Z)$ is affine. In particular, by relation (2.3.2), $[\mathscr{L}_{n+1}(Z) \to X] = \mathbf{L}^d[\mathscr{L}_n(Z) \to X]$ in $\mathscr{M}^{\mathbf{G}_m^r}_{X\times\mathbf{G}_m^r}$. Similarly, if $h : Y \to Z$ is a proper birational $\mathbf{G}_m^r$-equivariant morphism with Y smooth with a good $\mathbf{G}_m^r$-action, the fibrations occurring in [6, Lemma 3.4] are (piecewise) affine bundles and the induced $\mathbf{G}_m^r$-action is affine; hence, by relation (2.3.2), one does not see the action on the fibers in the Grothendieck ring $\mathscr{M}^{\mathbf{G}_m^r}_{X\times\mathbf{G}_m^r}$.

We now assume that X is endowed with a morphism $g : X \to \mathbf{A}^1_k$ and that Z is endowed with a monomial morphism $\mathbf{f} = (f_1, \ldots, f_r) : Z \to \mathbf{G}_m^r$ such that $(p, \mathbf{f}) : Z \to X \times \mathbf{G}_m^r$ is proper. We consider an open dense subset U of Z, stable under the $\mathbf{G}_m^r$-action. Similarly, as in (3.7.1), we set

$$\mathscr{X}_n^{\gamma n}(g\circ p, U) := \bigl\{\varphi \in \mathscr{L}_{\gamma n}(Z) \bigm| \mathrm{ord}_t(g\circ p)(\varphi) = n, \mathrm{ord}_t\varphi^*(\mathscr{I}_F) \le \gamma n\bigr\} \tag{3.10.4}$$

with $F := Z\backslash U$. The $\mathbf{G}_m^r$-action on Z induces a $\mathbf{G}_m^r$-action on $\mathscr{X}_n^{\gamma n}(g\circ p, U)$ via its induced action on the arc space. On the other side, the standard $\mathbf{G}_m$-action on arcs

$$(\lambda\cdot\varphi)(t) := \varphi(\lambda t) \tag{3.10.5}$$

induces a $\mathbf{G}_m$-action on $\mathscr{X}_n^{\gamma n}(g\circ p, U)$. In this way, we get a $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action on $\mathscr{X}_n^{\gamma n}(g\circ p, U)$. The morphism

$$\bigl(\mathbf{f}\circ\pi_0, \mathrm{ac}(g\circ p)\bigr) : \mathscr{X}_n^{\gamma n}(g\circ p, U) \to \mathbf{G}_m^r \times \mathbf{G}_m \tag{3.10.6}$$

is piecewise monomial; hence, proceeding as in Section 2.7, we may assign to

$$\mathscr{X}_n^{\gamma n}(g \circ p, U) \longrightarrow X_0(g) \times \mathbf{G}_m^r \times \mathbf{G}_m \tag{3.10.7}$$

a class $[\mathscr{X}_n^{\gamma n}(g \circ p, U)]$ in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$.

Similarly, as in (3.7.2), we consider the corresponding series

$$Z_{g\circ p,U}^{\gamma}(T) := \sum_{n\geq 1} [\mathscr{X}_n^{\gamma n}(g \circ p, U)]\, \mathbf{L}^{-\gamma n d}\, T^n \tag{3.10.8}$$

in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}[[T]]$.

Proceeding as in the proof of Proposition 3.8, one proves that there exists a γ_0 such that for every $\gamma > \gamma_0$, the series $Z_{g\circ p,U}^{\gamma}(T)$ belongs to $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$ and $\lim_{T\mapsto\infty} Z_{g\circ p,U}^{\gamma}(T)$ is independent of $\gamma > \gamma_0$. Indeed, we may assume that the zero locus $Z_0(g \circ p)$ of $g \circ p$ is nowhere dense in Z, and in this case, we now use a $\mathbf{G}_m^r$-equivariant log-resolution of $(Z, (Z\backslash U) \cup Z_0(g \circ p))$. (For the existence of equivariant resolutions, see [1], [10], [11], [28], [29].) We now define $\mathscr{S}_{g\circ p,U}$ in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$ as $-\lim_{T\mapsto\infty} Z_{g\circ p,U}^{\gamma}(T)$ for $\gamma > \gamma_0$.

Still assuming that $Z_0(g \circ p)$ is nowhere dense in Z, let $h : Y \to Z$ be such a $\mathbf{G}_m^r$-equivariant log-resolution. We again use the notation introduced in Section 3.3. By connectedness of $\mathbf{G}_m^r$, the $\mathbf{G}_m^r$-action on Z induces the trivial action on the set of strata E_I° for I subset of A. The $\mathbf{G}_m^r$-action on Y induces an action on the normal bundles to the divisors E_i for i in A, and hence, on U_I for I subset of A. We also consider the $\mathbf{G}_m$-action on U_I which is the diagonal action induced by the canonical $\mathbf{G}_m^I$-action on U_I. In this way, we get a $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action on U_I. Furthermore, with the notation of Section 3.4, the morphisms $\mathbf{f}$ and g induce morphisms $\mathbf{f}_I : U_I \to \mathbf{G}_m^r$ and $\mathbf{g}_I : U_I \to \mathbf{G}_m$. Note that the morphism $(\mathbf{f}_I, g_I) : U_I \to \mathbf{G}_m^r \times \mathbf{G}_m$ is monomial with respect to the $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action since g_I is invariant by the $\mathbf{G}_m^r$-action and monomial with respect to the $\mathbf{G}_m$-action and the morphism $\mathbf{f}_I : U_I \to \mathbf{G}_m^r$ is induced from $\mathbf{f}$ via the projection $U_I \to Z$. We can then consider the class $[U_I]$ in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$ of the morphism

$$(p \circ h \circ \pi_I, \mathbf{f}_I, g_I) : U_I \longrightarrow X_0(g) \times \mathbf{G}_m^r \times \mathbf{G}_m. \tag{3.10.9}$$

Similarly, as in Proposition 3.8, we get that the equality

$$\mathscr{S}_{g\circ p,U} = \sum_{I\neq\emptyset, I\subset C} (-1)^{|I|}[U_I] \tag{3.10.10}$$

holds in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$.

Remark 3.11
When $r = 0$, what is denoted here by $[\mathscr{X}_n^{\gamma n}(g \circ p, U)]$, $Z^{\gamma}_{g\circ p,U}(T)$, and $\mathscr{S}_{g\circ p,U}$ corresponds to what was denoted by $p_!([\mathscr{X}_n^{\gamma n}(g \circ p, U)])$, $p_!(Z^{\gamma}_{g\circ p,U}(T))$, and $p_!(\mathscr{S}_{g\circ p,U})$ in the nonequivariant setting. This slight conflict of notation probably does not lead to confusion.

We can now state the following equivariant analogue of Theorem 3.9.

THEOREM 3.12
Let X be a variety with a function $g : X \to \mathbf{A}^1_k$. We consider X endowed with the trivial $\mathbf{G}^r_m$-action. There exists a unique $\mathscr{M}_k$-linear group morphism

$$\mathscr{S}_g : \mathscr{M}^{\mathbf{G}^r_m}_{X\times\mathbf{G}^r_m} \longrightarrow \mathscr{M}^{\mathbf{G}^r_m\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}^r_m\times\mathbf{G}_m} \tag{3.12.1}$$

such that for every smooth variety Z with good $\mathbf{G}^r_m$-action which is endowed with an equivariant morphism $p : Z \to X$ and a monomial morphism $\mathbf{f} : Z \to \mathbf{G}^r_m$ such that the morphism $(p, \mathbf{f}) : Z \to X \times \mathbf{G}^r_m$ is proper, and for every open dense subset U of Z which is stable under the $\mathbf{G}^r_m$-action,

$$\mathscr{S}_g([U \to X \times \mathbf{G}^r_m]) = \mathscr{S}_{g\circ p,U} \tag{3.12.2}$$

in $\mathscr{M}^{\mathbf{G}^r_m\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}^r_m\times\mathbf{G}_m}$.

Proof
Let us denote by $K'_0(\mathrm{Var}^{\mathbf{G}^r_m}_{X\times\mathbf{G}^r_m})$ the Grothendieck ring defined similarly as $K_0(\mathrm{Var}^{\mathbf{G}^r_m}_{X\times\mathbf{G}^r_m})$ but without relation (2.3.2). Let U be a smooth variety over k with a good $\mathbf{G}^r_m$-action endowed with an equivariant morphism $\kappa : U \to X$ and with a monomial morphism $\mathbf{f}_U : U \to \mathbf{G}^r_m$. Note that U may be embedded equivariantly as an open dense subset of a smooth variety Z with good $\mathbf{G}^r_m$-action, endowed with an equivariant morphism $p : Z \to X$ extending κ and a monomial morphism $\mathbf{f} : Z \to \mathbf{G}^r_m$ extending $\mathbf{f}_U$, such that $(p, \mathbf{f}) : Z \to X \times \mathbf{G}^r_m$ is proper. Indeed, using the equivalence of categories of Proposition 2.6 and Section 2.7, it is enough to know that every smooth variety U_0 endowed with a good $\hat{\mu}^r$-action and with an equivariant morphism $\kappa_0 : U_0 \to X$ with X endowed with the trivial $\hat{\mu}^r$-action may be embedded equivariantly as an open dense subset in a smooth variety Z_0 with good $\hat{\mu}^r$-action, endowed with a proper equivariant morphism $Z_0 \to X$ extending κ_0, which follows from [5, Appendix] and also from Sumihiro's equivariant completion result in [26]. Hence, we can proceed exactly as in the proof of Theorem 3.9 in an equivariant way, getting existence and unicity of a $K_0(\mathrm{Var}_k)$-linear morphism

$$\mathscr{S}_g : K'_0(\mathrm{Var}^{\mathbf{G}^r_m}_{X\times\mathbf{G}^r_m}) \longrightarrow \mathscr{M}^{\mathbf{G}^r_m\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}^r_m\times\mathbf{G}_m} \tag{3.12.3}$$

such that for every smooth variety Z with good $\mathbf{G}_m^r$-action, endowed with an equivariant morphism $p : Z \to X$ and a monomial morphism $\mathbf{f} : Z \to \mathbf{G}_m^r$ such that $(p, \mathbf{f}) : Z \to X \times \mathbf{G}_m^r$ is proper, and for every open dense subset U of Z which is stable under the $\mathbf{G}_m^r$-action,

$$\mathscr{S}_g([U \to X \times \mathbf{G}_m^r]) = \mathscr{S}_{g \circ p, U} \tag{3.12.4}$$

in $\mathscr{M}_{X_0(g) \times \mathbf{G}_m^r \times \mathbf{G}_m}^{\mathbf{G}_m^r \times \mathbf{G}_m}$.

Let us now prove the compatibility of the morphism $\mathscr{S}_g$ with the additional relation (2.3.2). Let U be a smooth variety over k endowed with a good $\mathbf{G}_m^r$-action, with an equivariant morphism $\kappa : U \to X$, and with a monomial morphism $\mathbf{f}_U : U \to \mathbf{G}_m^r$. Let $q : B \to U$ be an affine bundle everywhere of rank s with a good affine $\mathbf{G}_m^r$-action over the action on U. We claim that U may be embedded equivariantly as an open dense subset in a smooth variety Z with good $\mathbf{G}_m^r$-action, endowed with an equivariant morphism $p : Z \to X$ extending κ and with a monomial morphism $\mathbf{f} : Z \to \mathbf{G}_m^r$ extending $\mathbf{f}_U$, such that $(p, \mathbf{f}) : Z \to X \times \mathbf{G}_m^r$ is proper and such that, furthermore, the affine bundle B with its affine $\mathbf{G}_m^r$-action extends to an affine bundle $\widetilde{B} \to Z$ with an affine $\mathbf{G}_m^r$-action over the action on Z extending the previous one. Indeed, this follows, using again the equivalence of categories of Proposition 2.6 and Section 2.7, from Lemma 3.14. To prove that $\mathscr{S}_g([B \to X \times \mathbf{G}_m^r])$ does not depend on the affine $\mathbf{G}_m^r$-action on B over the action on U, it is enough to check that

$$\mathscr{S}_g([B \to X \times \mathbf{G}_m^r]) = \mathbf{L}^s\, \mathscr{S}_g([U \to X \times \mathbf{G}_m^r]). \tag{3.12.5}$$

We may assume that $(g \circ p)^{-1}(0)$ is nowhere dense in Z. Let $h : Y \to Z$ be a $\mathbf{G}_m^r$-log-resolution of $(Z, (Z \backslash U) \cup (g \circ p)^{-1}(0))$. We denote by E_i, i in A, the irreducible components of $h^{-1}((Z \backslash U) \cup (g \circ p)^{-1}(0))$, and it follows from (3.10.10) that with the notation of Sections 3.3 and 3.10,

$$\mathscr{S}_g([U \to X \times \mathbf{G}_m^r]) = -\sum_{\emptyset \neq I \subset C} (-1)^{|I|} [U_I] \tag{3.12.6}$$

in $\mathscr{M}_{X_0(g) \times \mathbf{G}_m^r \times \mathbf{G}_m}^{\mathbf{G}_m^r \times \mathbf{G}_m}$. Let us consider the projective bundle $\lambda : Z' \to Z$ on Z, which is the relative projective completion of the bundle $\widetilde{B}$. In particular, Z' is endowed with a (projective) $\mathbf{G}_m^r$-action. We consider the pullback $Y' \to Y$ of the bundle Z' along the morphism h. We get a proper morphism $h' : Y' \to Z'$, which is an equivariant log-resolution of $(Z', (Z' \backslash B) \cup (g \circ p \circ \lambda)^{-1}(0))$. The set of irreducible components of $h'^{-1}((Z' \backslash B) \cup (g \circ p \circ \lambda)^{-1}(0))$ consists of the restriction E_i' of Y' to E_i for i in A together with H_∞, the divisor at infinity of the projective bundle Y'. We set $A' := A \sqcup \{0\}$ and $E_0' := H_\infty$ in order to use the notation of Sections 3.3 and 3.10 in this setting, adding everywhere $'$ as an exponent. In particular, for every nonempty subset I of C', we denote by U'_I the corresponding variety with $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action and with a monomial morphism $(\mathbf{f}'_I, g'_I) : U'_I \to \mathbf{G}_m^r \times \mathbf{G}_m$. Since $g \circ p \circ \lambda$ is not identically zero on H_∞, we have $C' = C$. It follows again from (3.10.10) that

$$\mathscr{S}_g([B \to X \times \mathbf{G}_m^r]) = - \sum_{\emptyset \neq I \subset C} (-1)^{|I|} [U'_I] \tag{3.12.7}$$

in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$. Now remark that the natural morphism $p_I : U'_I \to U_I$ is an affine bundle of rank s with an affine $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action over the one on U_I. Furthermore, the monomial morphism $U'_I \to \mathbf{G}_m^r \times \mathbf{G}_m$ is the composition of the monomial morphism $U_I \to \mathbf{G}_m^r \times \mathbf{G}_m$ with p_I. One deduces that $[U'_I] = \mathbf{L}^s [U_I]$ in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$, and (3.12.5) follows.

One then extends $\mathscr{S}_g$ by $\mathscr{M}_k$-linearity to a $\mathscr{M}_k$-linear group morphism

$$\mathscr{S}_g : \mathscr{M}_{X\times\mathbf{G}_m^r}^{\mathbf{G}_m^r} \longrightarrow \mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m} \tag{3.12.8}$$

similarly, as in the nonequivariant case. □

Remark 3.13

It follows from our constructions that the morphism $\mathscr{S}_g^{\hat{\mu}^r} : \mathscr{M}_X^{\hat{\mu}^r} \to \mathscr{M}_{X_0(g)}^{\hat{\mu}^{r+1}}$ deduced from (3.12.1) via the canonical isomorphism (2.6.3) is compatible with the one constructed by Bittner in [3] modulo the fact that our additional relation is finer than in that article. Indeed, they are easily checked to coincide on classes of $\hat{\mu}^r$-equivariant morphisms $Z \to X$ with Z smooth and proper. Note also that diagram (3.10.3) is indeed commutative by construction.

LEMMA 3.14

Let $\mathbf{n}$ be in $\mathbf{N}_{>0}^r$. Let X be a k-variety with trivial $\mu_{\mathbf{n}}$-action, and let U be a smooth variety with a good $\mu_{\mathbf{n}}$-action and an equivariant morphism $\kappa : U \to X$. Consider an affine bundle $B \to U$ with a good affine $\mu_{\mathbf{n}}$-action over the action on U. Then there exists an equivariant embedding of U as a dense open set in a smooth variety Z with good $\mu_{\mathbf{n}}$-action such that κ extends to a proper equivariant morphism $p : Z \to X$ and the affine bundle B with its affine $\mu_{\mathbf{n}}$-action extends to an affine bundle $\widetilde{B}$ on Z with an affine $\mu_{\mathbf{n}}$-action over the action on Z extending the previous one.

Proof

Set $G = \mu_{\mathbf{n}}$, and embed U equivariantly in V with a good G-action with $V \to X$ proper equivariant extending κ. The affine bundle $B \to U$ corresponds to an exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathscr{O}_U \longrightarrow 0 \tag{3.14.1}$$

on U such that the sheaf of local sections of the affine bundle is the preimage of 1 in F. The action of G on U gives a G-action on the exact sequence (3.14.1). (By a G-action on an $\mathscr{O}_U$-module F, we mean an isomorphism $a^*F \to p^*F$ satisfying the cocycle condition with $a : G \times U \to U$ the action and $p : G \times U \to U$ the projection on the second factor.) By blowing up the coherent ideal defining $V \setminus U$ with the reduced structure, we reduce to the case where the inclusion $j : U \to V$ is affine. By applying j_* to the exact sequence (3.14.1) and pulling back along $\mathscr{O}_V \to j_*\mathscr{O}_U$, we extend (3.14.1) to an exact sequence of quasi-coherent sheaves with G-action on V:

$$0 \longrightarrow E' \longrightarrow F' \longrightarrow \mathscr{O}_V \longrightarrow 0. \tag{3.14.2}$$

Let us note that F' is the direct limit of its G-invariant coherent subsheaves. Indeed, this follows from [15, Proposition 15.4] since (quasi-)coherent sheaves on the quotient stack $[V/G]$ correspond to (quasi-)coherent sheaves with G-action on V. It follows that we may assume that the sheaves in (3.14.2) are coherent. By restricting to a G-stable union of connected components of U, we may also assume that the vector bundle E is of constant rank s on U. Let $q : Z \to V$ be obtained by taking an equivariant resolution of the blowup of the sth Fitting ideal F_s of E', which is also the $(s+1)$th Fitting ideal F_{s+1} of F'. Applying q^* to (3.14.2) and modding out by torsion, we get an exact sequence of coherent sheaves with G-action

$$0 \longrightarrow \widetilde{E} \longrightarrow \widetilde{F} \longrightarrow \mathscr{O}_Z \longrightarrow 0 \tag{3.14.3}$$

on Z. Let us note that $\widetilde{E}$ and $\widetilde{F}$ are in fact locally free. Indeed, Z being normal, $\widetilde{E}$ and $\widetilde{F}$ are locally free outside a closed subvariety of codimension at least 2, but by construction, the Fitting ideals $F_s(\widetilde{E})$ and $F_{s+1}(\widetilde{F})$ are invertible; hence, they should be equal to $\mathscr{O}_Z$. The preimage of 1 in $\widetilde{F}$ is the sheaf of local sections of an affine bundle with G-action $\widetilde{B}$ on Z, satisfying the required properties. □

Remark 3.15
The proof of Lemma 3.14 was explained to us by Ofer Gabber and works in fact for any linear algebraic group G over k (see also [2, Lemma 7.4] for a similar, but different, extension lemma).

3.16. Compatibility with Hodge realization
We suppose here that $k = \mathbf{C}$. If X is a complex algebraic variety, we denote by MHM_X the category of mixed Hodge modules on X as defined in [20]. We denote by $K_0(\mathrm{MHM}_X)$ the corresponding Grothendieck ring. By additivity, there is a unique $\mathscr{M}_k$-linear morphism

$$H : \mathscr{M}_X \longrightarrow K_0(\mathrm{MHM}_X) \tag{3.16.1}$$

such that for any $p : Z \to X$ with Z smooth, $H([Z])$ is the class of the full direct image with compact supports $Rp_!(\mathbf{Q}_Z)$ in $K_0(\mathrm{MHM}_X)$ with $\mathbf{Q}_Z$ the trivial Hodge module on Z. Here we consider $K_0(\mathrm{MHM}_X)$ as an $\mathscr{M}_k$-module via its $K_0(\mathrm{MHM}_{\mathrm{Spec}\,\mathbf{C}})$-module structure and the Hodge realization map $H : \mathscr{M}_k \to K_0(\mathrm{MHM}_{\mathrm{Spec}\,\mathbf{C}})$. Note that $H(\mathbf{L}) = [\mathbf{Q}_X(-1)]$. If $\mu_{\mathbf{n}} = \mu_{n_1} \times \cdots \times \mu_{n_r}$ acts on Z, we may consider the automorphisms $T_1, \ldots, T_r$ on the cohomology objects $R^i p_!(\mathbf{Q}_Z)$ associated, respectively, to the action of the element with j-component $\exp(2\pi i/n_j)$ and other components 1. If we denote by $\mathrm{MHM}_X^{r-\mathrm{mon}}$ the category of mixed Hodge modules on X with r commuting automorphism of finite order, we get in this way a morphism

$$H : \mathscr{M}_X^{\hat{\mu}^r} \longrightarrow K_0(\mathrm{MHM}_X^{r-\mathrm{mon}}). \tag{3.16.2}$$

(That the morphism H is compatible with the additional relation (2.2.1) follows from the fact that for every affine bundle $p : A \to Y$ of rank s with an affine $\mu_{\mathbf{n}}$-action above a $\mu_{\mathbf{n}}$-action action on Y, there is a canonical equivariant isomorphism $Rp_!(\mathbf{Q}_A)[2s](s) \simeq \mathbf{Q}_Y$.) If $g : X \to \mathbf{A}^1$ is a function, there is a nearby cycle functor $\psi_g : \mathrm{MHM}_X \to \mathrm{MHM}_{X_0(g)}^{\mathrm{mon}}$ (see [20], [21]), which induces a morphism $\psi_g : K_0(\mathrm{MHM}_X) \to K_0(\mathrm{MHM}_{X_0(g)}^{\mathrm{mon}})$. By functoriality, the construction extends to morphisms $\psi_g : K_0(\mathrm{MHM}_X^{r-\mathrm{mon}}) \to K_0(\mathrm{MHM}_{X_0(g)}^{r+1-\mathrm{mon}})$.

PROPOSITION 3.17
For every $r \geq 0$ with the notation from Remark 3.13, the diagram

$$\begin{array}{ccc} \mathscr{M}_X^{\hat{\mu}^r} & \xrightarrow{\mathscr{S}_g^{\hat{\mu}^r}} & \mathscr{M}_{X_0(g)}^{\hat{\mu}^{r+1}} \\ \downarrow H & & \downarrow H \\ K_0(\mathrm{MHM}_X^{r-\mathrm{mon}}) & \xrightarrow{\psi_g} & K_0(\mathrm{MHM}_{X_0(g)}^{r+1-\mathrm{mon}}) \end{array} \tag{3.17.1}$$

is commutative.

Proof
It is enough to prove that $H(\mathscr{S}_g^{\hat{\mu}^r}([Z \to X])) = \psi_g(H([Z \to X]))$ for $p : Z \to X$ proper and Z smooth with $\hat{\mu}^r$-action. We can further reduce to the case where $(g \circ p)^{-1}(0)$ is a divisor with normal crossings which is stable by the $\hat{\mu}^r$-action. In that case, when $r = 0$, the statement is proved in [5, Theorem 4.2.1, Proposition 4.2.3] in a somewhat different language, when X is a point, but the proof carries over with no change to general X. Since the constructions in [5] may be performed in an equivariant way in the case of a $\hat{\mu}^r$-action, the proof extends directly to the case where $r > 0$. $\square$

4. Iterated vanishing cycles

4.1

Let X be a variety endowed with the trivial $\mathbf{G}_m^r$-action and with a function $g : X \to \mathbf{A}_k^1$. Let U be a smooth k-variety of pure dimension d with good $\mathbf{G}_m^r$-action endowed with an equivariant morphism $\kappa : U \to X$ and with a monomial morphism $\mathbf{f} = (f_1, \ldots, f_r) : U \to \mathbf{G}_m^r$. Let $U \to Y$ be an equivariant embedding as a dense open subset of a smooth variety Y with a good $\mathbf{G}_m^r$-action and with a proper equivariant morphism $p : Y \to X$. We assume that $(g \circ \kappa)^{-1}(0)$ is nowhere dense in U. Let $h : W \to Y$ be a $\mathbf{G}_m^r$-equivariant log-resolution of $(Y, (Y \backslash U) \cup (g \circ p)^{-1}(0))$. We now explain how to compute $\mathscr{S}_g([U \to X \times \mathbf{G}_m^r])$ in terms of W. Note that the present setup is different from the one in Theorem 3.12.

We denote by E_i, i in A, the irreducible components of $h^{-1}((Y \backslash U) \cup (g \circ p)^{-1}(0))$. We use again the notation from Sections 3.3 and 3.10, whenever possible. Let us assume that $I \cap C \neq \emptyset$. We can still consider the spaces U_I and the corresponding monomial morphism $g_I : U_I \to \mathbf{G}_m$. We denote by $h' : U' \to U$ the preimage of U in W, and we set $F := W \backslash U'$. The morphism $\mathbf{f} : U \to \mathbf{G}_m^r$ extends to a rational map $\widetilde{\mathbf{f}} : Y \dashrightarrow (\mathbf{P}_k^1)^r$. Furthermore, for i in A, there exist integers $N_i(f_j)$ in $\mathbf{Z}$ such that locally on W, each component $\widetilde{f}_j \circ h$ of $\widetilde{\mathbf{f}} \circ h$ may be written as $u \prod_{i \in A} x_i^{N_i(f_j)}$ with u a unit, x_i a local equation of E_i. Similarly to what we did for g_I, for every j, $1 \leq j \leq r$, we may define a rational map $f_{j,I} : \nu_{E_I} \dashrightarrow \mathbf{P}_k^1$, replacing $N_i(g)$ by $N_i(f_j)$, and we still denote by $f_{j,I}$ the induced morphism from U_I to $\mathbf{G}_m$. Finally, we get a morphism $\mathbf{f}_I : U_I \to \mathbf{G}_m^r$ which is monomial for the $\mathbf{G}_m^r$-action by Lemma 4.2. Similarly to what we observed in Section 3.10, this is enough to get that the morphism $(\mathbf{f}_I, g_I) : U_I \to \mathbf{G}_m^r \times \mathbf{G}_m$ is monomial for the $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action. We then denote by $[U_I]$ the corresponding class in $\mathscr{M}_{X_0(g) \times \mathbf{G}_m^r \times \mathbf{G}_m}^{\mathbf{G}_m^r \times \mathbf{G}_m}$.

LEMMA 4.2

Let W be a smooth variety with a good $\mathbf{G}_m^r$-action, and let U be a dense open stable by the $\mathbf{G}_m^r$-action. We assume that $F := W \backslash U$ is a divisor with normal crossings, and we denote by E_i, i in A, its irreducible components. We consider a monomial morphism $\mathbf{f} = (f_1, \ldots, f_r) : U \to \mathbf{G}_m^r$, and we denote by $\widetilde{\mathbf{f}} : W \dashrightarrow (\mathbf{P}_k^1)^r$ its extension as a rational map. For any nonempty subset I of A, the morphism $\mathbf{f}_I : U_I \to \mathbf{G}_m^r$ defined similarly as in Section 4.1 is monomial for the $\mathbf{G}_m^r$-action on U_I.

Proof

Consider the deformation CW_I to the normal cone to E_I in W described in Section 3.5. Hence, $CW_I := \operatorname{Spec} \mathscr{A}_I$ with

$$\mathscr{A}_I := \sum_{\mathbf{n} \in \mathbf{N}^I} \mathscr{O}_{W \times \mathbf{A}_k^I}\Big(- \sum_{i \in I} n_i (E_i \times \mathbf{A}_k^I)\Big) \prod_{i \in I} u_i^{-n_i}. \tag{4.2.1}$$

Letting $\mathbf{G}_m^r$ act trivially on each u_i, the $\mathbf{G}_m^r$-action on $\mathscr{O}_W$ induces a $\mathbf{G}_m^r$-action on $\mathscr{A}_I$ and on CW_I. For i in A, we denote by $\mathscr{J}_i$ the ideal of $\mathscr{A}_I$ generated by $u_i^{-1}\mathscr{O}_W(-E_i)$, respectively, $\mathscr{O}_W(-E_i)$ if $i \in I$, respectively, $i \notin I$, and we set $\mathscr{J} := \prod_{i\in A} \mathscr{J}_i$. We denote by CW_I° the complement in CW_I of the closed subset defined by $\mathscr{J}$. The fiber $CW_I^\circ \cap p^{-1}(0)$ may be identified equivariantly with U_I and CW_I° with $p^{-1}(\mathbf{G}_m^I) \simeq U \times \mathbf{G}_m^I$ (letting $\mathbf{G}_m^r$ act trivially on $\mathbf{G}_m^I$).

On $U \times \mathbf{G}_m^I$, we may consider the function $(x, u_i) \mapsto f_j(x)\prod_{i\in I} u_i^{-N_i(f_j)}$. As in Section 3.5, it extends to a morphism $F_j : CW_I^\circ \to \mathbf{G}_m$ whose restriction to U_I coincides with $f_{j,I}$. Let us consider the morphism $\mathbf{F} = (F_1, \ldots, F_r) : CW_I^\circ \to \mathbf{G}_m^r$. Since $\mathbf{f}$ is monomial and $\mathbf{G}_m^r$ acts trivially on u_i, the restriction of $\mathbf{F}$ to the dense open set $U \times \mathbf{G}_m^I$ is monomial; hence, $\mathbf{F}$ is monomial, and so is its restriction to U_I. □

4.3

Let γ and n be in $\mathbf{N}_{>0}$. We keep the notation from Section 4.1. In particular, $F = h^{-1}(Y\backslash U)$. Let φ be in $\mathscr{L}_{\gamma n}(W)$ with $\operatorname{ord}_t\varphi^*(\mathscr{I}_F) \leq \gamma n$ and $\operatorname{ord}_t g(\varphi) = n$. Let D denote the set consisting of all i in A such that $\varphi(0)$ lies in E_i, and consider a local equation $x_i = 0$ of E_i at $\varphi(0)$. By hypothesis, $x_i(\varphi)$ is nonzero in $\mathscr{L}_{\gamma n}(\mathbf{A}_k^1)$, so it has a well-defined order $\operatorname{ord}_t(x_i(\varphi))$ and an angular component $\operatorname{ac}(x_i(\varphi))$. Writing the component $\widetilde{f}_j \circ h$ of $\widetilde{\mathbf{f}} \circ h$ as $u\prod_{i\in D} x_i^{N_i(f_j)}$ with u a unit at $\varphi(0)$, we set

$$\operatorname{ord}_t(\widetilde{f}_j \circ h)(\varphi) := \sum_{i\in D} N_i(f_j)\operatorname{ord}_t\big(x_i(\varphi)\big) \tag{4.3.1}$$

and

$$\operatorname{ac}(\widetilde{f}_j \circ h)(\varphi) := u\big(\varphi(0)\big)\prod_{i\in D}\operatorname{ac}\big(x_i(\varphi)\big)^{N_i(f_j)}. \tag{4.3.2}$$

By abuse of notation, we write $(\widetilde{\mathbf{f}} \circ h)\varphi(0) \in \mathbf{G}_m^r$ to mean $\operatorname{ord}_t(\widetilde{f}_j \circ h)(\varphi) = 0$ for every $1 \leq j \leq r$.

Now we consider the constructible set

$$\mathscr{W}_n^{\gamma n} := \big\{\varphi \in \mathscr{L}_{\gamma n}(W) \,\big|\, \operatorname{ord}_t g(\varphi) = n, \operatorname{ord}_t\varphi^*(\mathscr{I}_F) \leq \gamma n, (\widetilde{\mathbf{f}} \circ h)(\varphi(0)) \in \mathbf{G}_m^r\big\}. \tag{4.3.3}$$

Similarly to the set in (3.10.4), $\mathscr{W}_n^{\gamma n}$ is endowed with a $(\mathbf{G}_m^r \times \mathbf{G}_m)$-action, and furthermore, the morphism $(\operatorname{ac}(\widetilde{f}_j \circ h), \operatorname{ac}(g)) : \mathscr{W}_n^{\gamma n} \to \mathbf{G}_m^r \times \mathbf{G}_m$ is piecewise monomial. We denote by $[\mathscr{W}_n^{\gamma n}]$ the corresponding class in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$. Let us consider the series

$$W^\gamma(T) := \sum_{n\geq 1}[\mathscr{W}_n^{\gamma n}]\,\mathbf{L}^{-\gamma n d}\,T^n \tag{4.3.4}$$

in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}[[T]]$.

For I a nonempty subset of A, we consider the cone

$$\Gamma(I) := \Big\{\mathbf{x} \in \mathbf{R}^I_{>0} \;\Big|\; \forall j \in \{1, \ldots, r\}, \sum_{i\in I} x_i N_i(f_j) = 0\Big\}, \tag{4.3.5}$$

and we denote by $d(I)$ its dimension. We also consider the cone

$$M_\gamma := \Big\{\mathbf{x} \in \mathbf{R}^I_{>0} \;\Big|\; \sum_{i\in I} x_i N_i(\mathscr{I}_F) \leq \gamma \sum_{i\in I\cap C} x_i N_i(g)\Big\}. \tag{4.3.6}$$

We denote by Δ the set of nonempty subsets I of A such that $\Gamma(I)$ is nonempty and contained in M_γ for $\gamma \gg 0$.

PROPOSITION 4.4
Let X be a variety with trivial $\mathbf{G}^r_m$-action and with a function $g : X \to \mathbf{A}^1_k$. Let U be a smooth k-variety of pure dimension d with good $\mathbf{G}^r_m$-action, endowed with an equivariant morphism $\kappa : U \to X$, and with a monomial morphism $\mathbf{f} = (f_1, \ldots, f_r) : U \to \mathbf{G}^r_m$. Let $U \to Y$ be an equivariant embedding as a dense open subvariety of a smooth variety Y with good $\mathbf{G}^r_m$-action and with a proper equivariant morphism $p : Y \to X$. We assume that $(g \circ \kappa)^{-1}(0)$ is nowhere dense in U. Let $h : W \to Y$ be a $\mathbf{G}^r_m$-equivariant log-resolution of $(Y, (Y \backslash U) \cup (g \circ p)^{-1}(0))$. There exists γ_0 such that for every $\gamma > \gamma_0$, the series $W^\gamma(T)$ lies in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$ and $\lim_{T\mapsto\infty} W^\gamma(T)$ is independent of $\gamma > \gamma_0$. Furthermore, if one sets $\mathscr{W} = -\lim_{T\mapsto\infty} W^\gamma(T)$, the following holds:

$$\mathscr{W} = -\sum_{I\in\Delta} (-1)^{d(I)}\, [U_I] \tag{4.4.1}$$

in $\mathscr{M}^{\mathbf{G}^r_m\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}^r_m\times\mathbf{G}_m}$.

Proof
As in the proof of Proposition 3.8, we have

$$W^\gamma(T) = \sum_{I\cap C\neq\emptyset} [U_I]\, S_I(T) \tag{4.4.2}$$

with

$$S_I(T) = \sum_{\mathbf{k}\in\Gamma(I)\cap M_\gamma\cap\mathbf{N}^I_{>0}} \prod_{i\in I} (T^{N_i(g)}\mathbf{L}^{-1})^{k_i}. \tag{4.4.3}$$

The proof now goes on as the proof of Proposition 3.8 with $\mathbf{N}^I_{>0}$ replaced by $\Gamma(I)\cap\mathbf{N}^I_{>0}$. Indeed, note that the linear form $\sum_{i\in I\cap C} k_i N_i(g)$ is positive on $\overline{M_\gamma}\backslash\{0\}$ and that M_γ is empty if $I \cap C = \emptyset$. Assume first that I lies in Δ and $I \cap C \neq \emptyset$. Then it follows from Section 2.9 that $\lim_{T\mapsto\infty} S_I(T) = (-1)^{d(I)}$ for $\gamma \gg 0$. Assume now that $I \cap C \neq \emptyset$ and $I \notin \Delta$. In this case, necessarily, for $\gamma > 0$, the hyperplane $\sum_{i\in I} k_i N_i(\mathscr{I}_F) = \gamma \sum_{i\in I\cap C} k_i N_i(g)$ has a nonempty intersection with $\Gamma(I)$. It follows that the Euler characteristic $\chi(\Gamma(I) \cap M_\gamma)$ is equal to zero. □

PROPOSITION 4.5

Let X be a variety with trivial $\mathbf{G}_m^r$-action and with a function $g : X \to \mathbf{A}_k^1$. Let U be a smooth k-variety of pure dimension d with good $\mathbf{G}_m^r$-action, endowed with an equivariant morphism $\kappa : U \to X$, and with a monomial morphism $\mathbf{f} = (f_1, \dots, f_r) : U \to \mathbf{G}_m^r$. Let $U \to Y$ be an equivariant embedding as a dense open subvariety of a smooth variety Y with good $\mathbf{G}_m^r$-action and with a proper equivariant morphism $p : Y \to X$. We assume that $(g \circ \kappa)^{-1}(0)$ is nowhere dense in U. Let $h : W \to Y$ be a $\mathbf{G}_m^r$-equivariant log-resolution of $(Y, (Y\backslash U) \cup (g \circ p)^{-1}(0))$. Then with the previous notation, we have

$$\mathscr{S}_g([U \to X \times \mathbf{G}_m^r]) = -\sum_{I\in\Delta} (-1)^{d(I)}\, [U_I] \tag{4.5.1}$$

in $\mathscr{M}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}^{\mathbf{G}_m^r\times\mathbf{G}_m}$.

Proof

We may reduce to the case where the morphism $\mathbf{f} : U \to \mathbf{G}_m^r$ extends to a morphism $\widetilde{\mathbf{f}} : Y \to (\mathbf{P}_k^1)^r$. Indeed, there exists an equivariant embedding $U \to Y'$ of U as a dense open subvariety of a smooth variety Y' with a good $\mathbf{G}_m^r$-action and with a proper equivariant morphism $p' : Y' \to X$ such that $\mathbf{f}$ extends to a morphism $\widetilde{\mathbf{f}}' : Y' \to (\mathbf{P}_k^1)^r$. We may, furthermore, assume that there is a $\mathbf{G}_m^r$-equivariant proper morphism $Y' \to Y$ which restricts to the identity on U. Let $h' : W' \to Y'$ be a $\mathbf{G}_m^r$-equivariant log-resolution of $(Y', (Y'\backslash U) \cup (g \circ p')^{-1}(0))$. We may also assume that there is a $\mathbf{G}_m^r$-equivariant proper morphism $W' \to W$ such that the diagram

$$\begin{array}{ccc} W' & \xrightarrow{h'} & Y' \\ \downarrow & & \downarrow \\ W & \xrightarrow{h} & Y \end{array} \tag{4.5.2}$$

is commutative.

Consider $\mathscr{W}'$ defined as $\mathscr{W}$, but using W' instead of W. Since, temporarily, we work on W' and not on W, we denote by E_i, i in A, the irreducible components of $h'^{-1}((Y'\backslash U) \cup (g \circ p')^{-1}(0))$ and keep the previous notation, but for W' instead of W.

We have

$$\mathscr{W}' = -\lim_{T\mapsto\infty}\sum_{I\cap C\neq 0}[U_I]S_I(T), \tag{4.5.3}$$

while computing $W^\gamma(T)$ on W' using the change of variable formula or, more precisely, [6, Lemma 3.4], one gets

$$\mathscr{W} = -\lim_{T\mapsto\infty}\sum_{I\cap C\neq 0}[U_I]\widetilde{S}_I(T) \tag{4.5.4}$$

with

$$\widetilde{S}_I(T) = \sum_{\mathbf{k}\in\Gamma(I)\cap M_\gamma\cap\mathbf{N}^I_{>0}}\prod_{i\in I}(T^{N_i(g)}\mathbf{L}^{-m_i})^{k_i} \tag{4.5.5}$$

with $m_i \geq 1$. It follows that $\mathscr{W}' = \mathscr{W}$, and by Proposition 4.4, we can assume that $Y = Y'$ and $W = W'$.

Consider $Z := (\widetilde{\mathbf{f}}\circ h)^{-1}(\mathbf{G}^r_m)$ in W. Note that the image of the morphism $Z \to W\times\mathbf{G}^r_m$ given by the inclusion on the first factor and by the restriction of $\widetilde{\mathbf{f}}\circ h$ on the second factor is the closure of the image of the inclusion $U' \to W\times\mathbf{G}^r_m$ with U' the preimage of U in W. It follows that the morphism $(q, \widetilde{\mathbf{f}}\circ h_{|Z}) : Z \to X\times\mathbf{G}^r_m$ given by composition with $p\circ h$ on the first factor is proper. Since Z is smooth and the morphism $Z \to X\times\mathbf{G}^r_m$ is proper, it follows from (3.12.2) that

$$\mathscr{S}_g([U \to X\times\mathbf{G}^r_m]) = \mathscr{S}_{g\circ q, U'} \tag{4.5.6}$$

in $\mathscr{M}^{\mathbf{G}^r_m\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}^r_m\times\mathbf{G}_m}$. Note also that since $\mathbf{f} : U \to \mathbf{G}^r_m$ extends to a morphism $\widetilde{\mathbf{f}} : Y \to (\mathbf{P}^1_k)^r$, for a subset I of A with $I\cap C\neq\emptyset$, $\Gamma(I)$ is nonempty if and only if E°_I is contained in $(\widetilde{\mathbf{f}}\circ h)^{-1}(\mathbf{G}^r_m)$. Furthermore, if these conditions hold, $\Gamma(I) = \mathbf{R}^I_{>0}$. It follows that Δ consists exactly of those nonempty subsets of C for which E°_I is contained in $(\widetilde{\mathbf{f}}\circ h)^{-1}(\mathbf{G}^r_m)$; hence, the right-hand side of (4.5.1) may be rewritten as

$$-\sum_{\substack{\emptyset\neq I\subset C\\ E^\circ_I\subset(\widetilde{\mathbf{f}}\circ h)^{-1}(\mathbf{G}^r_m)}}(-1)^{|I|}[U_I], \tag{4.5.7}$$

and the required equation (4.5.1) follows now from (4.5.6) and (3.10.10). □

4.6. Iterated vanishing cycles

Now we consider a smooth variety X of pure dimension d with two functions $f : X\to\mathbf{A}^1_k$ and $g : X\to\mathbf{A}^1_k$. The motivic Milnor fiber $\mathscr{S}_f$ lies in $\mathscr{M}^{\mathbf{G}_m}_{X_0(f)\times\mathbf{G}_m}$. We still denote by g the function $X_0(f)\times\mathbf{G}_m\to\mathbf{A}^1_k$ obtained by composition of g with the projection $X_0(f) \times \mathbf{G}_m \to X$. Hence, thanks to Section 3.10, we may consider the image

$$\mathscr{S}_g(\mathscr{S}_f) = \mathscr{S}_g\big(\mathscr{S}_f([X \to X])\big) \tag{4.6.1}$$

of $\mathscr{S}_f = \mathscr{S}_f([X \to X])$ by the nearby cycles morphism

$$\mathscr{S}_g : \mathscr{M}^{\mathbf{G}_m}_{X_0(f)\times\mathbf{G}_m} \longrightarrow \mathscr{M}^{\mathbf{G}_m^2}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m^2}, \tag{4.6.2}$$

which lies in $\mathscr{M}^{\mathbf{G}_m^2}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m^2}$.

We now give an explicit description of $\mathscr{S}_g(\mathscr{S}_f)$ in terms of a log-resolution $h : Y \to X$ of $(X, X_0(f)\cup X_0(g))$. We denote by E_i, i in A, the irreducible components of $h^{-1}(X_0(f) \cup X_0(g))$, and we consider the sets

$$B = \big\{i \bigm| N_i(f) > 0\big\} \qquad \text{and} \qquad C = \big\{i \bigm| N_i(g) > 0\big\}. \tag{4.6.3}$$

Recall (see Section 3.3) that we denoted by U_I^J for $J \subset I$ the fiber product of the restrictions of the $\mathbf{G}_m$-bundles U_{E_i} for i in J to E_I°. Assume that $J := I \cap C$ and $K := I\backslash C$ are both nonempty. We now consider the fiber product $U_{K,J} := U_I^K \times_{E_I^\circ} U_I^J$, which has the same underlying variety as U_I. There is a natural $\mathbf{G}_m^2$-action on $U_{K,J}$, the first, respectively, second, $\mathbf{G}_m$-action being the diagonal action on U_I^K, respectively, U_I^J, and the trivial one on the other factor. The morphism $(f_I, g_I) : U_I = U_{K,J} \to \mathbf{G}_m^2$ being monomial, the morphism $(h \circ \pi_I, f_I, g_I) : U_I \to (X_0(f) \cap X_0(g)) \times \mathbf{G}_m^2$ has a class in $\mathscr{M}^{\mathbf{G}_m^2}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m^2}$ which we denote by $[U_{K,J}]$.

THEOREM 4.7
With the previous notation, we have

$$\mathscr{S}_g(\mathscr{S}_f) = \sum_{\substack{I\cap C=J\neq\emptyset\\ I\backslash C=K\neq\emptyset}} (-1)^{|I|}\,[U_{K,J}] \tag{4.7.1}$$

in $\mathscr{M}^{\mathbf{G}_m^2}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m^2}$.

Proof
Consider the inclusions $i : X_0(g) \times \mathbf{G}_m \hookrightarrow X \times \mathbf{G}_m$ and $j : (X\backslash X_0(g)) \times \mathbf{G}_m \hookrightarrow X \times \mathbf{G}_m$. Note that $\mathscr{S}_f - j_!(\mathscr{S}_{(f_{|X\backslash X_0(g)})})$ is supported by $X_0(g) \times \mathbf{G}_m$, that is, is of the form $i_!(\mathscr{A})$. Since $Y\backslash Y_0(g \circ h)$ is a log-resolution of $(X\backslash X_0(g), X_0(f)\backslash X_0(g))$, one deduces from the proof of (3.6.3) that

$$\mathscr{S}_f - \Big(- \sum_{\substack{K\cap C=\emptyset\\ K\neq\emptyset}} (-1)^{|K|}\,[U_K \to X_0(f) \times \mathbf{G}_m]\Big) \tag{4.7.2}$$

is supported by $X_0(g) \times \mathbf{G}_m$. Hence, since $\mathscr{S}_g$ is zero on objects of the form $i_!(\mathscr{A})$, we deduce that

$$\mathscr{S}_g(\mathscr{S}_f) = \mathscr{S}_g\Big(-\sum_{\substack{K\cap C=\emptyset\\ K\neq\emptyset}} (-1)^{|K|}\,[U_K \to X_0(f)\times \mathbf{G}_m]\Big). \tag{4.7.3}$$

To conclude, it is enough to check the following equality in $\mathscr{M}^{\mathbf{G}_m^2}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m^2}$ for every nonempty subset K of A such that $K\cap C=\emptyset$:

$$\mathscr{S}_g\big([U_K \to X_0(f)\times\mathbf{G}_m]\big) = -\sum_{\emptyset\neq J\subset C} (-1)^{|J|}[U_{K,J}]. \tag{4.7.4}$$

This follows from Proposition 4.5. Indeed, let us consider the projective bundle $\pi_K : \overline{\nu}_{E_K} \to E_K$ with the $\mathbf{G}_m$-action extending the diagonal one on ν_{E_K}. Let us set $A' := A \sqcup \{\infty\}$. The complement of U_K in $\overline{\nu}_{E_K}$ is a divisor with normal crossings whose irreducible components are

- the divisors $E'_j := \pi_K^{-1}(E_{K\cup\{j\}})$ for $j \notin K$ such that $E_{K\cup\{j\}} \neq \emptyset$;
- the divisor at infinity $E'_\infty := \overline{\nu}_{E_K}\backslash \nu_{E_K}$;
- the divisors E'_i for i in K defined as the closure of the fiber product, above E_K, of the zero section of ν_{E_i} with the ν_{E_ℓ}, ℓ in K, $\ell\neq i$.

Note that all these divisors are stable by the $\mathbf{G}_m$-action. We use the notation of Sections 3.3 and 3.10 with an exponent $'$.

We now determine the set Δ of nonempty subsets J' of A' such that $\Gamma(J')$ is nonempty and is contained in M_γ for $\gamma \gg 0$ with the notation of (4.3.5) and (4.3.6).

Note that for $\Gamma(J')$ to be nonempty, it is necessary that if $N_i(f_K) > 0$ (resp., $N_i(f_K) < 0$) for some i in J', then for some i' in J', $N_{i'}(f_K) < 0$ (resp., $N_{i'}(f_K) > 0$). This forces J' to be either of the form $J \sqcup \{\infty\}$ with $J\cap B \neq \emptyset$ or of the form J with $J \cap B = \emptyset$. In each case, the condition that $\Gamma(J')$ is contained in M_γ for $\gamma \gg 0$ implies that $J \subset C$ and, furthermore, that $d(J') = |J|$. We deduce that $J\sqcup\{\infty\}$ belongs to Δ if and only if $J \cap B \neq \emptyset$ and $J\subset C$, and we deduce that J belongs to Δ if and only if $J\cap B=\emptyset$, $J\subset C$, and $J\neq\emptyset$. It follows from Proposition 4.5 that

$$\mathscr{S}_g\big([U_K \to X_0(f)\times\mathbf{G}_m]\big) = -\sum_{\substack{\emptyset\neq J\subset C\\ J\cap B=\emptyset}} (-1)^{|J|}\,[U'_J] - \sum_{\substack{J\subset C\\ J\cap B\neq\emptyset}} (-1)^{|J|}\,[U'_{J\sqcup\{\infty\}}] \tag{4.7.5}$$

in $\mathscr{M}^{\mathbf{G}_m^r\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}$. To conclude, it is enough to note that if $\emptyset\neq J\subset C$ and $J\cap B=\emptyset$, then $[U'_J]=[U_{K,J}]$, and that if $J\subset C$ and $J\cap B\neq\emptyset$, we have $[U'_{J\sqcup\{\infty\}}]=[U_{K,J}]$.

Let us prove the second equality. We consider the image $\mathbf{P}(U_{E_K})$ of U_{E_K} in $\mathbf{P}(\nu_{E_K})$ and note that the canonical morphism $U_{E_K} \to \mathbf{P}(U_{E_K})$ is a $\mathbf{G}_m$-bundle, namely, the restriction to U_{E_K} of the tautological line bundle on $\mathbf{P}(\nu_{E_K})$. We identify $E''_\infty := E'_\infty\backslash\bigcup_{i\in K} E'_i$ with $\mathbf{P}(U_{E_K})$. The restriction of the tautological line bundle to $\mathbf{P}(U_{E_K})$ is dual to the restriction to E''_∞ of the normal bundle to E'_∞ in $\overline{\nu}_{E_K}$. We now have two $\mathbf{G}_m$-bundles on $E''_\infty = \mathbf{P}(U_{E_K})$, namely, U_{E_K} and the restriction, which we denote by U''_∞, to E''_∞ of the complement $U_{E'_\infty}$ of the zero section in the normal bundle $\nu_{E'_\infty}$. Let us denote by $U^a_{E_K}$ the $\mathbf{G}_m$-bundle U_{E_K} endowed with the inverse $\mathbf{G}_m$-action. The antipody $a : U_{E_K} \to U^a_{E_K}$, whose restriction to the fibers is given by $t \mapsto t^{-1}$, is an isomorphism of $\mathbf{G}_m$-bundles with $\mathbf{G}_m$-action. By the above description, U''_∞ may be identified, as a $\mathbf{G}_m$-bundle with $\mathbf{G}_m$-action, with the $\mathbf{G}_m$-bundle $U^a_{E_K}$.

Now consider the function f_K on ν_{E_K}. It induces a rational map $\widetilde{f}_K$ on $\overline{\nu}_{E_K}$. Let us check that under the above isomorphism, the restriction $f_K : U_{E_K} \to \mathbf{A}^1_k$ composed with the automorphism a corresponds to the morphism $f'_\infty : U''_\infty \to \mathbf{A}^1_k$ obtained from $\widetilde{f}_K$ by the construction of Section 4.1. Indeed, let U be an open subset of E_K above which the bundle ν_{E_K} is trivial, isomorphic to $U \times \mathbf{A}^K_k$. Denote by w_i for i in K the coordinates on $\mathbf{A}^K_k$. Fix ℓ in K. The restriction of U_{E_K} to U may be identified, equivariantly, with $U \times \mathbf{P}(\mathbf{G}^K_m) \times \mathbf{G}_m$ by $(u, (w_i)_{i\in K}) \mapsto (u, (x_i = w_i/w_\ell)_{i\in K\setminus\{\ell\}}, t = w_\ell)$, where $\mathbf{G}_m$ acts trivially on the first two factors and by multiplicative translation on the last one with $(x_i)_{i\in K\setminus\{\ell\}}$ the standard coordinates on $\mathbf{P}(\mathbf{G}^K_m) \simeq \mathbf{G}^{K\setminus\{\ell\}}_m$. If the restriction of f_K to $U \times \mathbf{G}^K_m$ is given by $v(u)\prod_{i\in K} w_i^{N_i}$, it may be rewritten, under the above identification, as $v(u)\prod_{i\in K\setminus\{\ell\}} x_i^{N_i} t^{\sum_{i\in K} N_i}$. Composing with the antipody a, we get the function $v(u)\prod_{i\in K\setminus\{\ell\}} x_i^{N_i} t^{-\sum_{i\in K} N_i}$, which corresponds to the restriction of the function f'_∞ on the corresponding open subset.

If J is a subset of C such that $E_{K\sqcup J} \neq \emptyset$, it follows from the "transitivity" property described in Section 3.5 that $f_{K\sqcup J}$ can be retrieved directly from $f_K : \nu_{E_K} \to \mathbf{A}^1_k$ and, similarly, that the rational map $f'_{J\sqcup\{\infty\}}$ can be retrieved directly from the rational map f'_∞ (obtained from $\widetilde{f}_K$ by the construction of Section 4.1) on $\nu_{E'_\infty}$. It follows that under the isomorphism between $U_{K\sqcup J}$ and $U'_{J\sqcup\{\infty\}}$ induced by φ, $f_{K\sqcup J}$ corresponds to $f'_{J\sqcup\{\infty\}}$.The same argument works for the functions induced by g on $U'_{K\sqcup J}$ and $U_{J\sqcup\{\infty\}}$. (Note that, in fact, $N_i(g) = 0$ for all i in K and $N_\infty(g) = 0$.)

The first equality, which is easier, is checked similarly using $E'_J = \pi_K^{-1}(E_{K\cup J})$ and the canonical isomorphism of bundles $\nu_{E'_J} \simeq (\pi_{K|E'_J})^*(\nu_{E_J|E_{K\cup J}})$ for $J \subset C$. □

5. Convolution and the main result

5.1. Convolution

Let us denote by a and b the coordinates on each factor of $\mathbf{G}^2_m$. Let X be a variety. We denote by $i : X \times (a+b)^{-1}(0) \to X \times \mathbf{G}^2_m$ the inclusion of the antidiagonal and by j the inclusion of its complement. We consider the morphism

$$a + b : X \times \mathbf{G}^2_m \setminus (a+b)^{-1}(0) \longrightarrow X \times \mathbf{G}_m, \qquad (5.1.1)$$

which is the identity on the X-factor and is equal to $a+b$ on the $(\mathbf{G}_m^2\backslash(a+b)^{-1}(0))$-factor. We denote by pr_1 and pr_2 the projection of $X\times\mathbf{G}_m\times(a+b)^{-1}(0)$ on $X\times\mathbf{G}_m$ and $X\times(a+b)^{-1}(0)$, respectively.

If A is an object in $\mathscr{M}_{X\times\mathbf{G}_m^2}$, the object

$$\Psi_\Sigma^0(A) := -(a+b)_! j^*(A) + \mathrm{pr}_{1!}\mathrm{pr}_2^* i^*(A) \tag{5.1.2}$$

lives in $\mathscr{M}_{X\times\mathbf{G}_m}$. We now explain how to lift Ψ_Σ^0 to an $\mathscr{M}_k$-linear group morphism $\Psi_\Sigma : \mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2} \to \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m}$.

Let A be an object in $\mathrm{Var}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2,(n,m)}$ with class $[A]$ in $\mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2,(n,m)}$. It is endowed with a $\mathbf{G}_m^2$-action α for which the morphism to $\mathbf{G}_m^2$ is diagonally monomial of weight (n,m). We may consider the $\mathbf{G}_m$-action $\widetilde{\alpha}$ on A given by $\widetilde{\alpha}(\lambda)x = \alpha(\lambda^m,\lambda^n)x$. With some obvious abuse of notation, $(a+b)_! j^*([A])$ is the class of $a+b : A_{|a+b\neq 0} \to X\times\mathbf{G}_m$. If we endow $A_{|a+b\neq 0}$ with the $\mathbf{G}_m$-action induced by $\widetilde{\alpha}$, the morphism $(a+b) : A_{|a+b\neq 0} \to \mathbf{G}_m$ is diagonally monomial of weight nm. The term $\mathrm{pr}_{1!}\mathrm{pr}_2^* i^*([A])$ is the class of $A_{|a+b=0}\times\mathbf{G}_m \to X\times\mathbf{G}_m$, the morphism to $\mathbf{G}_m$ being the projection on the $\mathbf{G}_m$-factor. We endow $A_{|a+b=0}\times\mathbf{G}_m$ with the $\mathbf{G}_m$-action induced by $\widetilde{\alpha}$ on the first factor and the action $(\lambda, z)\mapsto \lambda^{nm}z$ on the second factor. Hence, we may set $\Psi_\Sigma^{n,m}([A]) = -[a+b : A_{|a+b\neq 0} \to X\times\mathbf{G}_m] + [A_{|a+b=0}\times\mathbf{G}_m \to X\times\mathbf{G}_m]$ in $\mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m,nm}$ and extend this construction in a unique way to an $\mathscr{M}_k$-linear group morphism

$$\Psi_\Sigma^{n,m} : \mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2,(n,m)} \longrightarrow \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m,nm}. \tag{5.1.3}$$

These morphisms being compatible with the morphisms induced by the transition morphisms of (2.5.1), after passing to the colimit, we get an $\mathscr{M}_k$-linear group morphism

$$\Psi_\Sigma : \mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2} \longrightarrow \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m}. \tag{5.1.4}$$

Let us now explain the relation of Ψ_Σ with the convolution product as considered in [7], [16], and [8]. There is a canonical morphism

$$\mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m} \times \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m} \longrightarrow \mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2} \tag{5.1.5}$$

sending (A,B) to $A\boxtimes B$, the fiber product over X of A and B; therefore, we may define

$$* : \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m} \times \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m} \longrightarrow \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m} \tag{5.1.6}$$

by

$$A * B = \Psi_\Sigma(A\boxtimes B). \tag{5.1.7}$$

If S is in $\mathrm{Var}_X^{\mu_n}$, respectively, in $\mathrm{Var}_X^{\mu_n \times \mu_n}$, we denote by $[S]$ the corresponding class in $\mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m}$, respectively, in $\mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2}$, via the isomorphism (2.6.3). Consider the Fermat curves F_1^n and F_0^n defined, respectively, by $x^n + y^n = 1$ and $x^n + y^n = 0$ in $\mathbf{G}_m^2$ with their standard $(\mu_n \times \mu_n)$-action. If A is a variety in $\mathrm{Var}_X^{\mu_n\times\mu_n}$, we have

$$\Psi_\Sigma([A]) = -[F_1^n \times^{\mu_n\times\mu_n} A] + [F_0^n \times^{\mu_n\times\mu_n} A], \tag{5.1.8}$$

the μ_n-action on each term in the right-hand side of (5.1.8) being the diagonal one. In particular, if A and B are two varieties in $\mathrm{Var}_X^{\mu_n}$, the convolution product $[A] * [B]$ is given by

$$[A] * [B] = -[F_1^n \times^{\mu_n\times\mu_n} (A \times_X B)] + [F_0^n \times^{\mu_n\times\mu_n} (A \times_X B)]. \tag{5.1.9}$$

The convolution product in [16] and [8] was defined when k contains all roots of unity. Since as soon as k contains an nth root of -1, we have $[F_0^n \times^{\mu_n\times\mu_n} (A \times_X B)] = (\mathbf{L} - 1)[(A \times_X B)/\mu_n]$, one gets that the convolution product in [16] and [8], when defined, coincides with the one in (5.1.9).

PROPOSITION 5.2
The convolution product on $\mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m}$ is commutative and associative. The unit element for the convolution product is 1, *the class of the identity $X \times \mathbf{G}_m \to X \times \mathbf{G}_m$ with the standard $\mathbf{G}_m$-action on the $\mathbf{G}_m$-factor.*

Proof
With commutativity being clear, let us prove the statement concerning associativity and unit element. For simplicity of notation, we assume that X is a point, and we first ignore the $\mathbf{G}_m$-actions; that is, we prove the corresponding statements for $\mathscr{M}_{\mathbf{G}_m}$. Consider $a : A \to \mathbf{G}_m$, $b : B \to \mathbf{G}_m$, $c : C \to \mathbf{G}_m$. By definition, the convolution product $A * B$ (with some abuse of notation, we denote by the same symbol varieties over $\mathbf{G}_m$ and their class in $\mathscr{M}_{\mathbf{G}_m}$) is equal to

$$-[a + b : (A \times B)_{|a+b\neq 0} \to \mathbf{G}_m] + [z : (A \times B \times \mathbf{G}_m)_{|a+b=0} \to \mathbf{G}_m] \tag{5.2.1}$$

with z the standard coordinate on $\mathbf{G}_m$.

Associativity follows from the following claim. $(A * B) * C$ is equal to

$$[a+b+c : (A \times B \times C)_{|a+b+c\neq 0} \to \mathbf{G}_m] - [z : (A \times B \times C \times \mathbf{G}_m)_{|a+b+c=0} \to \mathbf{G}_m], \tag{5.2.2}$$

Indeed, $(A * B) * C$ may be written as a sum of four terms. The first one,

$$[a + b + c : (A \times B \times C)_{|\substack{a+b+c\neq 0\\ a+b\neq 0}} \to \mathbf{G}_m], \tag{5.2.3}$$

may be rewritten as

$$[a+b+c:(A\times B\times C)_{|a+b+c\neq 0}\to \mathbf{G}_m]-[c:(A\times B\times C)_{|a+b=0}\to \mathbf{G}_m]. \tag{5.2.4}$$

The second one,

$$-[z:(A\times B\times C\times \mathbf{G}_m)_{|\substack{a+b+c=0\\ a+b\neq 0}}\to \mathbf{G}_m], \tag{5.2.5}$$

may be rewritten as

$$-[z:(A\times B\times C\times \mathbf{G}_m)_{|a+b+c=0}\to \mathbf{G}_m]. \tag{5.2.6}$$

The third one,

$$-[c+z:(A\times B\times C\times \mathbf{G}_m)_{|\substack{a+b=0\\ c+z\neq 0}}\to \mathbf{G}_m], \tag{5.2.7}$$

may be rewritten as

$$-[u:(A\times B\times C\times \mathbf{G}_m)_{|\substack{a+b=0\\ u\neq c}}\to \mathbf{G}_m] \tag{5.2.8}$$

since the corresponding spaces are isomorphic via $(\alpha,\beta,\gamma,z)\mapsto(\alpha,\beta,\gamma,u=c(\gamma)+z)$. Here u is a coordinate on some other copy of $\mathbf{G}_m$. The fourth term,

$$[u:(A\times B\times C\times \mathbf{G}_m\times \mathbf{G}_m)_{|\substack{a+b=0\\ c+z=0}}\to \mathbf{G}_m], \tag{5.2.9}$$

may be rewritten as

$$[u:(A\times B\times C\times \mathbf{G}_m)_{|a+b=0}\to \mathbf{G}_m]. \tag{5.2.10}$$

One deduces (5.2.2) by summing up (5.2.4), (5.2.6), (5.2.8), and (5.2.10).

For the statement concerning the unit element, one writes $A*\mathbf{G}_m$ as

$$-[a+z:(A\times \mathbf{G}_m)_{|a+z\neq 0}\to \mathbf{G}_m]+[u:(A\times \mathbf{G}_m\times \mathbf{G}_m)_{|a+z=0}\to \mathbf{G}_m]. \tag{5.2.11}$$

Since the first term may be rewritten as

$$-[u:(A\times \mathbf{G}_m)_{|a\neq u}\to \mathbf{G}_m] \tag{5.2.12}$$

and the second term as

$$[u:(A\times \mathbf{G}_m)\to \mathbf{G}_m], \tag{5.2.13}$$

it follows that $A*\mathbf{G}_m$ is equal to (the class of) A in $\mathscr{M}_{\mathbf{G}_m}$. The proofs for general X are just the same. As for $\mathbf{G}_m$-actions, since by the very constructions they are diagonally monomial of the same weight on each factor, all identifications we made are compatible with the $\mathbf{G}_m$-actions, and all statements still hold in $\mathscr{M}^{\mathbf{G}_m}_{X\times\mathbf{G}_m}$. □

Remark 5.3
Proposition 5.2, modulo the isomorphism (2.6.3), is already stated in [8, page 344].

5.4
In fact, associativity already holds at the Ψ_Σ-level. To formulate this, we need to introduce some more notation.

Let us denote by a, b, and c the coordinates on each factor of $\mathbf{G}_m^3$. For a variety X, we denote by i the inclusion $X \times (a+b+c)^{-1}(0) \hookrightarrow X \times \mathbf{G}_m^3$ and by j the inclusion of the complement. We consider the morphism

$$a+b+c : X \times \mathbf{G}_m^3 \backslash (a+b+c)^{-1}(0) \longrightarrow X \times \mathbf{G}_m, \tag{5.4.1}$$

which is the identity on the X-factor and is equal to $a+b+c$ on the $(\mathbf{G}_m^3 \backslash (a+b+c)^{-1}(0))$-factor. We denote by pr_1 and pr_2 the projection of $X \times \mathbf{G}_m \times (a+b+c)^{-1}(0)$ on $X \times \mathbf{G}_m$ and $X \times (a+b+c)^{-1}(0)$, respectively.

If A is an object in $\mathscr{M}_{X\times\mathbf{G}_m^3}^{\mathbf{G}_m^3}$, we consider the object

$$\Psi^0_{\Sigma_{123}}(A) := (a+b+c)_! j^*(A) - \mathrm{pr}_{1!}\mathrm{pr}_2^* i^*(A) \tag{5.4.2}$$

in $\mathscr{M}_{X\times\mathbf{G}_m}$. Similarly, as in Section 5.1, we extend $\Psi^0_{\Sigma_{123}}$ to an $\mathscr{M}_k$-linear group morphism $\Psi_{\Sigma_{123}} : \mathscr{M}_{X\times\mathbf{G}_m^3}^{\mathbf{G}_m^3} \to \mathscr{M}_{X\times\mathbf{G}_m}^{\mathbf{G}_m}$. We denote by A_{ij} the object A viewed as an element in $\mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2}$ by forgetting the projection and the action corresponding to the kth $\mathbf{G}_m$-factor with $\{i, j, k\} = \{1, 2, 3\}$. The object $\Psi_\Sigma(A_{ij})$ may now be endowed with a second projection to $\mathbf{G}_m$ and a second $\mathbf{G}_m$-action, namely, those corresponding to the kth $\mathbf{G}_m$-factor, so we get in fact an element in $\mathscr{M}_{X\times\mathbf{G}_m^2}^{\mathbf{G}_m^2}$ that we denote by $\Psi_{\Sigma_{ij}}(A)$.

PROPOSITION 5.5
Let A be an object in $\mathscr{M}_{X\times\mathbf{G}_m^3}^{\mathbf{G}_m^3}$. For every $1 \le i < j \le 3$, we have

$$\Psi_{\Sigma_{123}}(A) = \Psi_\Sigma\big(\Psi_{\Sigma_{ij}}(A)\big). \tag{5.5.1}$$

Proof
The proof is the same as the one for associativity in Proposition 5.2. Indeed, one just has to replace everywhere $A \times B \times C$ by A in the proof and to remark that (5.2.2) then becomes nothing else than $\Psi_{\Sigma_{123}}(A)$. □

5.6
Let us consider again a smooth variety X of pure dimension d with two functions f and g from X to $\mathbf{A}_k^1$. Let us denote by i_1 and i_2 the inclusion of $(X_0(f) \cap X_0(g)) \times \mathbf{G}_m$ in $X_0(f) \times \mathbf{G}_m$ and $X_0(f+g^N) \times \mathbf{G}_m$, respectively.

We can now state the main result of this article.

THEOREM 5.7

Let X be a smooth variety of pure dimension d, and let f and g be two functions from X to $\mathbf{A}^1_k$. For every $N > \gamma((f),(g))$, the equality

$$i_1^* \mathscr{S}_f^{\phi} - i_2^* \mathscr{S}_{f+g^N}^{\phi} = \Psi_\Sigma\big(\mathscr{S}_{g^N}(\mathscr{S}_f^{\phi})\big) \tag{5.7.1}$$

holds in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m}$.

Proof

Let φ be in $\mathscr{L}(X)$. A basic observation is that when the inequality $\operatorname{ord}_t f(\varphi) < N\operatorname{ord}_t g(\varphi)$ holds, $f(\varphi)$ and $(f+g^N)(\varphi)$ have the same order ord_t and the same angular coefficient ac. If A is a subset of $\mathscr{L}_n(X)$, we denote by A^+, respectively, A^0, the intersection of A with the set of arcs in $\mathscr{L}_n(X)$ such that $\operatorname{ord}_t f(\varphi) > N\operatorname{ord}_t g(\varphi)$, respectively, $\operatorname{ord}_t f(\varphi) = N\operatorname{ord}_t g(\varphi)$. In this way, one defines the series

$$Z_f^+(T) = \sum_{n\geq 1}[\mathscr{X}_n^+(f)]\,\mathbf{L}^{-nd}T^n \tag{5.7.2}$$

and

$$Z_f^0(T) = \sum_{n\geq 1}[\mathscr{X}_n^0(f)]\,\mathbf{L}^{-nd}T^n \tag{5.7.3}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0(f)\times\mathbf{G}_m}[[T]]$ and, similarly, the series $Z^+_{f+g^N}(T)$ and $Z^0_{f+g^N}(T)$ in $\mathscr{M}^{\mathbf{G}_m}_{X_0(f+g^N)\times\mathbf{G}_m}[[T]]$. It follows from the previous remark that

$$i_1^*Z_f(T) - i_2^*Z_{f+g^N}(T) = i_1^*\big(Z_f^+(T)+Z_f^0(T)\big) - i_2^*\big(Z^+_{f+g^N}(T)+Z^0_{f+g^N}(T)\big), \tag{5.7.4}$$

where we extend i_1^* and i_2^* to the series componentwise.

Let N be a positive integer. For any integer r, we denote by π_N the morphism $X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m \to X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m$ which is mapping (x,μ,λ) to (x,μ,λ^N). Then we have the following lemma.

LEMMA 5.8

Given a map $g : X \longrightarrow \mathbf{A}^1_k$ and the induced nearby cycles morphism $\mathscr{S}_g$ from $\mathscr{M}^{\mathbf{G}_m^r}_{X_0(g)\times\mathbf{G}_m^r}$ to $\mathscr{M}^{\mathbf{G}_m^r\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}$, then for any positive integer N, the following equality holds:

$$\mathscr{S}_{g^N} = \pi_{N!}\circ\mathscr{S}_g. \tag{5.8.1}$$

Proof

Let Z be a smooth variety with good $\mathbf{G}_m^r$-action, endowed with an equivariant morphism $p : Z \to X$ and a monomial morphism $\mathbf{f} : Z \to \mathbf{G}_m^r$, such that the morphism $(p,\mathbf{f}) : Z \to X\times\mathbf{G}_m^r$ is proper, and let $U = Z\setminus F$ be an open dense subset of Z

which is stable under the $\mathbf{G}_m^r$-action. For a positive integer γ with the notation of Section 3.10, let us consider the modified zeta function of g^N on U,

$$Z^{\gamma}_{g^N\circ p,U}(T) := \sum_{n\geq 1} [\mathscr{X}_n^{\gamma n}(g^N \circ p, U)]\,\mathbf{L}^{-\gamma n d}T^n, \tag{5.8.2}$$

in $\mathscr{M}^{\mathbf{G}_m^r\times\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m^r\times\mathbf{G}_m}[[T]]$. Since $\mathscr{X}_n^{\gamma n}(g^N \circ p, U)$ is empty unless N divides n and

$$[\mathscr{X}_{mN}^{\gamma m N}(g^N \circ p, U)] = \pi_{N!}\big([\mathscr{X}_m^{\gamma m N}(g \circ p, U)]\big), \tag{5.8.3}$$

we get that $Z^{\gamma}_{g^N\circ p,U}(T)$ is equal to $\pi_{N!}(Z^{\gamma N}_{g\circ p,U}(T^N))$, the limit of which, as T goes to infinity, is equal for γ big enough to $\pi_{N!}(\mathscr{S}_{g\circ p,U})$. The result follows from Theorem 3.12. □

LEMMA 5.9
Assume that $N > \gamma((f),(g))$.

Then the series $i_2^*(Z^+_{f+g^N}(T))$ lies in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$, and

$$\lim_{T\mapsto\infty} i_2^*\big(Z^+_{f+g^N}(T)\big) = -\mathscr{S}_{g^N}\big([X_0(f)]\big). \tag{5.9.1}$$

Proof
Note that $\mathscr{X}_n^+(f + g^N)$ is nonempty only if n is a multiple of N and that

$$[\mathscr{X}^+_{mN}(f+g^N)] = \pi_{N!}\Big(\Big[\Big\{\varphi \in \mathscr{L}_{mN}(X)\,\Big|\,\mathrm{ord}_t g(\varphi) = m,\ \mathrm{ord}_t f(\varphi) > Nm\Big\}\Big]\Big), \tag{5.9.2}$$

the variety on the right-hand side being endowed with the morphism to $\mathbf{G}_m$ induced by $\mathrm{ac}(g)$. Summing up, we may write by (3.7.2) and the proof of Lemma 5.8

$$Z^+_{f+g^N}(T) = \pi_{N!}\big(Z_g(T^N) - Z^N_{g,X\setminus X_0(f)}(T^N)\big). \tag{5.9.3}$$

By Proposition 3.8 and its proof, for $N > \gamma((f),(g))$, the series $Z^N_{g,X\setminus X_0(f)}(T)$ lies in $\mathscr{M}^{\mathbf{G}_m}_{X_0(g)\times\mathbf{G}_m}[[T]]_{\mathrm{sr}}$, and its $\lim_{T\mapsto\infty}$ is equal to $-\mathscr{S}_{g,X\setminus X_0(f)}$. The same holds for $Z^N_{g,X\setminus X_0(f)}(T^N)$, and the result follows since $\pi_{N!}(\mathscr{S}_g - \mathscr{S}_{g,X\setminus X_0(f)})$ is equal to $\mathscr{S}_{g^N}([X_0(f)])$ by Lemma 5.8. □

5.10
We fix an integer N such that $N > \gamma((f),(g))$, and we fix a log-resolution $h : Y \to X$ of $(X, X_0(f) \cup X_0(g))$ such that $N > \gamma_h((f),(g))$. We keep the notation used in Sections 3.3 and 4.6. In particular, $NN_i(g) > N_i(f)$ for $i \in C$. Note that the stratum E_I° is contained in $(g \circ h)^{-1}(0)$ if and only if $J = I \cap C$ is nonempty.

Fix a nonempty stratum E_I° in Y.

We consider the cones Δ_I^+ and Δ_I^0 in $\mathbf{R}_{>0}^I$ defined, respectively, by

$$\sum_{i\in I} N_i(f)x_i > N\sum_{j\in J} N_j(g)x_j \tag{5.10.1}$$

and

$$\sum_{i\in I} N_i(f)x_i = N\sum_{j\in J} N_j(g)x_j. \tag{5.10.2}$$

Note that when $K = I\backslash C$ is empty, $I = J$ and $N_i(f) < NN_i(g)$ for all i, and hence the cones Δ_I^+ and Δ_I^0 are both empty.

As in (3.8.2), we have

$$i_1^*\big(Z_f^+(T) + Z_f^0(T)\big) = \sum_{\substack{I\cap B\neq\emptyset\\ I\cap C\neq\emptyset}} [U_I]\,\Psi_I(T) \tag{5.10.3}$$

with

$$\Psi_I(T) = \sum_{\mathbf{k}\in(\Delta_I^+\cup\Delta_I^0)\cap\mathbf{N}_{>0}^I} \mathbf{L}^{-\sum_{i\in I}\nu_i k_i}\,T^{\sum_{i\in I}N_i(f)k_i}. \tag{5.10.4}$$

Since

$$\lim_{T\mapsto\infty}\Psi_I(T) = \chi(\Delta_I^+\cup\Delta_I^0) = 0, \tag{5.10.5}$$

we deduce that

$$\lim_{T\mapsto\infty} i_1^*\big(Z_f^+(T) + Z_f^0(T)\big) = 0. \tag{5.10.6}$$

5.11

We now want to compute the zeta function $Z^0_{f+g^N}(T)$. Fix $\mathbf{k}$ in $\Delta_I^0\cap\mathbf{N}_{>0}^I$, and denote by ϕ the finite morphism from $\mathbf{A}_k^1$ to $\mathbf{A}_k^I$ sending u to (u^{k_i}). We still denote by ϕ its restriction as a group morphism from $\mathbf{G}_m$ to $\mathbf{G}_m^I$. Taking the pullback along ϕ of the deformation to the normal cone to E_I in Y, $p_I : CY_I \to \mathbf{A}_k^I$, introduced in Section 3.5, one gets a morphism $p : CY_\mathbf{k} \to \mathbf{A}_k^1$ having the following description. The scheme $CY_\mathbf{k}$ may be identified with Spec $\mathscr{A}_\mathbf{k}$, where

$$\mathscr{A}_\mathbf{k} := \sum_{\mathbf{n}\in\mathbf{N}^I} \mathscr{O}_{Y\times\mathbf{A}_k^1}\Big(-\sum_{i\in I} n_i(E_i\times\mathbf{A}_k^1)\Big)u^{-\sum_{i\in I}k_i n_i} \tag{5.11.1}$$

is a subsheaf of $\mathscr{O}_{Y\times\mathbf{A}_k^1}[u^{-1}]$, and the natural inclusion $\mathscr{O}_{Y\times\mathbf{A}_k^1}\to\mathscr{A}_\mathbf{k}$ induces a morphism $\pi : CY_\mathbf{k}\to Y\times\mathbf{A}_k^1$, from which p is derived. Via the same inclusion, the functions $f \circ h$, $g^N \circ h$, and $(f + g^N) \circ h$ are divisible by $u^{\sum_{i \in I} k_i N_i(f)}$ in $\mathscr{A}_{\mathbf{k}}$. We denote the quotients by $\widetilde{f}_{\mathbf{k}}$, $\widetilde{g}^N_{\mathbf{k}}$, and $\widetilde{F}_{\mathbf{k}}$, respectively.

We denote by $\widetilde{E}_i$ the pullback of the divisor $E_i \times \mathbf{A}^1_k$ by π; by D the divisor globally defined on $CY_{\mathbf{k}}$ by $u = 0$; and by CE_i the divisors $\widetilde{E}_i - k_i D$, i in I (resp., $\widetilde{E}_i$, i not in I). We denote by $CY^{\circ}_{\mathbf{k}}$ the complement in $CY_{\mathbf{k}}$ of the union of the CE_i, i in A, and by Y° the complement in Y of the union of the E_i, i in A. We denote by F_I the function $f_I + g_I^N : U_I \to \mathbf{A}^1_k$.

LEMMA 5.12
The scheme $CY_{\mathbf{k}}$ is smooth, the morphism π induces an isomorphism above $\mathbf{A}^1_k \setminus \{0\}$, the morphism p is a smooth morphism, and its fiber $p^{-1}(0)$ may be naturally identified with the bundle ν_{E_I}. When restricted to $CY^{\circ}_{\mathbf{k}}$, the fiber of p above 0 is naturally identified with U_I, and π induces an isomorphism between $CY^{\circ}_{\mathbf{k}} \setminus p^{-1}(0)$ and $Y^{\circ} \times \mathbf{A}^1_k \setminus \{0\}$. The restriction of $\widetilde{f}_{\mathbf{k}}$ (resp., $\widetilde{g}_{\mathbf{k}}$, $\widetilde{F}_{\mathbf{k}}$) to the fiber $U_I \subset p^{-1}(0)$ is equal to f_I (resp., g_I, F_I).

Proof
Since $CY_{\mathbf{k}}$ is covered by open subsets of the form $\operatorname{Spec} \mathscr{O}_U[y_i, u]/(z_i - u^{k_i} y_i)$ with U an open subset on which the divisors E_i are defined by equations $z_i = 0$, the smoothness of $CY_{\mathbf{k}}$ is clear. The remaining properties are checked directly. □

The $\mathbf{G}^I_m$-action σ_I on CY_I induces via ϕ a $\mathbf{G}_m$-action on $CY_{\mathbf{k}}$ that we denote by σ, leaving sections of $\mathscr{O}_Y$ invariant and acting on u by $\sigma(\lambda) : u \mapsto \lambda^{-1} u$. Note that in coordinate charts such as in the proof of Lemma 5.12, σ leaves z_i invariant, and $\sigma(\lambda)$ maps y_i to $\lambda^{k_i} y_i$. We now have two different $\mathbf{G}_m$-actions on $\mathscr{L}_n(CY^{\circ}_{\mathbf{k}})$; the one induced by the standard $\mathbf{G}_m$-action on arc spaces, and the one induced by σ. We denote by $\widetilde{\sigma}$ the action given by the composition of these two (commuting) actions.

For φ in $\mathscr{L}_n(Y)$ with $\varphi(0)$ in E_i, we set $\operatorname{ord}_{E_i} \varphi := \operatorname{ord}_t z_i(\varphi)$, where z_i is any local equation of E_i at $\varphi(0)$.

Let us denote by $\widetilde{\mathscr{L}}_n(CY^{\circ}_{\mathbf{k}})$ the set of arcs φ in $\mathscr{L}_n(CY^{\circ}_{\mathbf{k}})$ such that $p(\varphi(t)) = t$. (In particular, $\varphi(0)$ is in U_I.) For such an arc φ, composition with π sends φ to an arc in $\mathscr{L}_n(Y \times \mathbf{A}^1_k)$, which is the graph of an arc in $\mathscr{L}_n(Y)$ not contained in the union of the divisors E_i, i in I. Note that $\widetilde{\mathscr{L}}_n(CY^{\circ}_{\mathbf{k}})$ is stable by $\widetilde{\sigma}$.

We consider $\mathscr{X}_{n,\mathbf{k}}$, the set of arcs φ in $\mathscr{L}_n(Y)$ such that $\varphi(0)$ is in E°_I and $\operatorname{ord}_{E_i} \varphi = k_i$ for $i \in I$.

LEMMA 5.13
Assume that $n \geq k_i$ for i in I. The morphism $\widetilde{\pi} : \widetilde{\mathscr{L}}_n(CY^{\circ}_{\mathbf{k}}) \to \mathscr{X}_{n,\mathbf{k}}$ induced by the projection $CY^{\circ}_{\mathbf{k}} \to Y$ is an affine bundle with fiber $\mathbf{A}_k^{\sum_I k_i}$. Furthermore, if $\widetilde{\mathscr{L}}_n(CY^{\circ}_{\mathbf{k}})$ is endowed with the $\mathbf{G}_m$-action induced by $\widetilde{\sigma}$ and $\mathscr{X}_{n,\mathbf{k}}$ with the standard $\mathbf{G}_m$-action, $\widetilde{\pi}$ is $\mathbf{G}_m$-equivariant, and the action of $\mathbf{G}_m$ on the affine bundle is affine. If $n \geq \sum_I k_i N_i(f)$,

the composed maps $\mathrm{ac}(f \circ h)(\widetilde{\pi}(\varphi))$ *and* $\mathrm{ac}(g \circ h)(\widetilde{\pi}(\varphi))$ *are equal, respectively, to* $f_I(\varphi(0))$ *and* $g_I(\varphi(0))$*, whereas*

$$\mathrm{ac}\big((f + g^N) \circ h\big)\big(\widetilde{\pi}(\varphi)\big) = \mathrm{ac}\big(\widetilde{F}_{\mathbf{k}}(\varphi)\big). \tag{5.13.1}$$

Furthermore, when $F_I(\varphi(0)) \neq 0$ *(hence,* $(\mathrm{ord}_t(f + g^N) \circ h)(\widetilde{\pi}(\varphi)) = \sum_I k_i N_i(f)$*), we have*

$$\mathrm{ac}\big((f + g^N) \circ h\big)\big(\widetilde{\pi}(\varphi)\big) = F_I\big(\varphi(0)\big). \tag{5.13.2}$$

Proof
Every point in E_I° is contained in an open subset U of Y such that the divisors E_i, $i \in I$, are defined by equations $z_i = 0$ in U and such that there exists, furthermore, $d - |I|$ functions w_j on U such that the family (z_i, w_j) gives rise to an étale morphism $U \to \mathbf{A}_k^d$. This morphism induces an isomorphism $\mathscr{L}_n(U) \simeq U \times_{\mathbf{A}_k^d} \mathscr{L}_n(\mathbf{A}_k^d)$ (see [6, Lemma 4.2]). Adding further the coordinate u gives an isomorphism $\mathscr{L}_n(U \times \mathbf{A}_k^1) \simeq (U \times \mathbf{A}_k^1) \times_{\mathbf{A}_k^{d+1}} \mathscr{L}_n(\mathbf{A}_k^{d+1})$. The family (y_i, w_j, u) with $z_i = y_i u^{k_i}$ induces an étale morphism $\pi^{-1}(U \times \mathbf{A}_k^1) \to \mathbf{A}_k^{d+1}$, and hence, an isomorphism $\mathscr{L}_n(\pi^{-1}(U \times \mathbf{A}_k^1)) \simeq (\pi^{-1}(U \times \mathbf{A}_k^1)) \times_{\mathbf{A}_k^{d+1}} \mathscr{L}_n(\mathbf{A}_k^{d+1})$. Under these isomorphisms, $\widetilde{\pi}$ just corresponds to multiplicating each y_i-component of an arc by t^{k_i}. Note in particular that in that description, the action of $\widetilde{\sigma}(\lambda)$ on a component $y_i(t)$ is given by $y_i(t) \mapsto \lambda^{k_i} y_i(\lambda t)$; hence, $\widetilde{\pi}$ is $\mathbf{G}_m$-equivariant. The rest of the statement follows also directly from that description. □

We define $\mathscr{Y}_{n,\mathbf{k}}$ as the subset of $\mathscr{X}_{n,\mathbf{k}}$ consisting of those arcs φ such that $\mathrm{ord}_t((f + g^N) \circ h)(\varphi) = n$. The constructible set $\mathscr{Y}_{n,\mathbf{k}}$ is stable by the usual $\mathbf{G}_m$-action on $\mathscr{L}_n(Y)$, and the morphism $\mathrm{ac}(f + g^N)$ defines a class $[\mathscr{Y}_{n,\mathbf{k}}]$ in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\cap X_0(g))\times\mathbf{G}_m}$. By definition, $\mathscr{Y}_{n,\mathbf{k}} = \emptyset$ if $n < \sum_I k_i N_i(f)$.

We then define $\widetilde{\mathscr{Y}}_{n,\mathbf{k}}$ as the preimage of $\mathscr{Y}_{n,\mathbf{k}}$ by the fibration $\widetilde{\pi}$ of Lemma 5.13. It consists of arcs φ in $\widetilde{\mathscr{L}}_n(CY_{\mathbf{k}}^\circ)$ such that $\mathrm{ord}_t \widetilde{F}_{\mathbf{k}}(\varphi) = n - \sum_I k_i N_i(f)$. We denote by $[\widetilde{\mathscr{Y}}_{n,\mathbf{k}}]$ the class of $\widetilde{\mathscr{Y}}_{n,\mathbf{k}}$ in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\times X_0(g))\times\mathbf{G}_m}$, the morphism $\widetilde{\mathscr{Y}}_{n,\mathbf{k}} \to \mathbf{G}_m$ being $\mathrm{ac}(\widetilde{F}_{\mathbf{k}})$ and the $\mathbf{G}_m$-action being induced by $\widetilde{\sigma}$. We denote by $[U_I \backslash (F_I^{-1}(0))]$ the class of $U_I \backslash (F_I^{-1}(0))$ in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\times X_0(g))\times\mathbf{G}_m}$, the $\mathbf{G}_m$-action being the natural diagonal action of weight $\mathbf{k}$ on $U_I \backslash (F_I^{-1}(0))$ and the morphism to $\mathbf{G}_m$ being the restriction of F_I. We also consider the class $[\mathbf{G}_m \times F_I^{-1}(0)]$ of $\mathbf{G}_m \times F_I^{-1}(0)$ in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\times X_0(g))\times\mathbf{G}_m}$, the $\mathbf{G}_m$-action on the second factor being the diagonal one and the morphism to $\mathbf{G}_m$ being the first projection.

LEMMA 5.14
The following equalities hold in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\times X_0(g))\times\mathbf{G}_m}$*:*
(1) $[\widetilde{\mathscr{Y}}_{n,\mathbf{k}}] = \mathbf{L}^{nd}\, [U_I \backslash (F_I^{-1}(0))]$ *if* $n = \sum_I k_i N_i(f)$*;*
(2) $[\widetilde{\mathscr{Y}}_{n,\mathbf{k}}] = \mathbf{L}^{nd-m}\, [\mathbf{G}_m \times F_I^{-1}(0)]$ *if* $n - \sum_I k_i N_i(f) = m > 0$.

Proof
If $n = \sum_I k_i N_i(f)$, $\widetilde{\mathscr{Y}}_{n,\mathbf{k}}$ is the set of arcs $\varphi(t)$ in $\mathscr{L}_n(CY_{\mathbf{k}}^\circ)$ such that $\varphi(0)$ lies in $U_I \backslash (F_I^{-1}(0))$ and $u(\varphi(t)) = t$, and (1) follows.

If $n - \sum_I k_i N_i(f) = m > 0$, $\widetilde{\mathscr{Y}}_{n,\mathbf{k}}$ is the set of arcs φ in $\mathscr{L}_n(CY_{\mathbf{k}}^\circ)$ such that $\mathrm{ord}_t(\widetilde{F}_{\mathbf{k}})(\varphi) = m$ and $u(\varphi(t)) = t$. Now let us observe that the morphism $(\widetilde{F}_{\mathbf{k}}, u) : CY_{\mathbf{k}}^\circ \to \mathbf{A}_k^2$ is smooth on a neighborhood of U_I in $CY_{\mathbf{k}}^\circ$ since u is a smooth function on $CY_{\mathbf{k}}^\circ$ and the restriction of $\widetilde{F}_{\mathbf{k}}$ to the divisor $u = 0$, identified with U_I, is $F_I = f_I + g_I^N$, which is a smooth function on U_I. The fact that $F_I = f_I + g_I^N$ is a smooth function on U_I is checked locally as follows. For i in $I \backslash C$ (recall that $I \backslash C$ is nonempty), and with local coordinates as in the proof of Lemma 5.13,

$$y_i \frac{\partial}{\partial y_i}(f_I + g_I^N) = y_i \frac{\partial f_I}{\partial y_i} = N_i(f) f_I \tag{5.14.1}$$

does not vanish on U_I. □

LEMMA 5.15
We have

$$i_2^*[\mathscr{X}_n^0(f + g^N)] = \sum_{\substack{I \cap C = J \neq \emptyset \\ I \backslash C = K \neq \emptyset}} \sum_{\mathbf{k} \in \Delta_I^0 \cap \mathbf{N}_{>0}^I} [\mathscr{Y}_{n,\mathbf{k}}] \, \mathbf{L}^{-\sum_{i \in I}(\nu_i - 1)k_i}. \tag{5.15.1}$$

Proof
This is a standard application of the change of variable formula, or more precisely, of [6, Lemma 3.4]. The proof is completely similar to the proof of [9, Theorem 2.4]. (Recall that Δ_I^0 is empty if K is empty.) □

It follows from Lemmas 5.15 and 5.13 that

$$i_2^* Z_{f+g^N}^0(T) = \sum_{n>0} \sum_{\substack{I \cap C = J \neq \emptyset \\ I \backslash C = K \neq \emptyset}} \sum_{\mathbf{k} \in \Delta_I^0 \cap \mathbf{N}_{>0}^I} [\widetilde{\mathscr{Y}}_{n,\mathbf{k}}] \, \mathbf{L}^{-\sum_{i \in I} \nu_i k_i} \mathbf{L}^{-nd} \, T^n. \tag{5.15.2}$$

Using Lemma 5.14, we deduce

$$i_2^* Z_{f+g^N}^0(T) = \sum_{\substack{I \cap C = J \neq \emptyset \\ I \backslash C = K \neq \emptyset}} \Big(\big[U_I \backslash \big(F_I^{-1}(0) \big) \big] + [\mathbf{G}_m \times F_I^{-1}(0)] \frac{\mathbf{L}^{-1} T}{1 - \mathbf{L}^{-1} T} \Big) \Phi_I(T) \tag{5.15.3}$$

with

$$\Phi_I(T) = \sum_{\mathbf{k} \in \Delta_I^0 \cap \mathbf{N}_{>0}^I} \mathbf{L}^{-\sum_{i \in I} \nu_i k_i} \, T^{\sum_{i \in I} N_i(f) k_i}. \tag{5.15.4}$$

Since $\Phi_I(T)$ lies in $\mathbf{Z}[\mathbf{L}, \mathbf{L}^{-1}][[T]]_{\mathrm{sr}}$ and

$$\lim_{T \mapsto \infty} \Phi_I(T) = \chi(\Delta_I^0) = (-1)^{|I|-1}, \tag{5.15.5}$$

$i_2^* Z^0_{f+g^N}(T)$ lies in $\mathscr{M}^{\mathbf{G}_m}_{(X_0(f)\cap X_0(g))\times \mathbf{G}_m}[[T]]_{\mathrm{sr}}$. Using Theorem 4.7, we deduce from (5.15.3) and (5.15.5) that

$$\lim_{T \mapsto \infty} i_2^* Z^0_{f+g^N}(T) = \Psi_\Sigma\big(\mathscr{S}_{g^N}(\mathscr{S}_f)\big). \tag{5.15.6}$$

We deduce from (5.7.4), (5.9.1), (5.15.6), and (5.10.6) that

$$i_1^* \mathscr{S}_f - i_2^* \mathscr{S}_{f+g^N} = \Psi_\Sigma\big(\mathscr{S}_{g^N}(\mathscr{S}_f)\big) - \mathscr{S}_{g^N}\big([X_0(f)]\big). \tag{5.15.7}$$

By Proposition 5.2, $\mathscr{S}_{g^N}([X_0(f)]) = \Psi_\Sigma\big(\mathscr{S}_{g^N}([\mathbf{G}_m \times X_0(f)])\big)$. Since $i_1^* \mathscr{S}^\phi_f - i_2^* \mathscr{S}^\phi_{f+g^N} = (-1)^{d-1}(i_1^* \mathscr{S}_f - i_2^* \mathscr{S}_{f+g^N})$, it follows from (5.15.7) that (5.7.1) holds for $N > \gamma((f), (g))$, which finishes the proof of Theorem 5.7. □

If f is a function on the smooth variety X of pure dimension d and x is a closed point of $X_0(f)$, we write $\mathscr{S}_{f,x}$ for $i_x^* \mathscr{S}_f$ and $\mathscr{S}^\phi_{f,x}$ for $i_x^* \mathscr{S}^\phi_f$, where i_x stands for the inclusion of x in $X_0(f)$. Note that $\mathscr{S}^\phi_{f,x} = (-1)^{d-1}(\mathscr{S}_{f,x} - [\mathbf{G}_m \times \{x\}])$.

Theorem 5.7 has the following local corollary.

COROLLARY 5.16

Let X be a smooth variety of pure dimension d, and let f and g be two functions from X to $\mathbf{A}^1_k$. Let x be a closed point of $X_0(f) \cap X_0(g)$. For every $N > \gamma_x((f), (g))$, the equality

$$\mathscr{S}^\phi_{f,x} - \mathscr{S}^\phi_{f+g^N,x} = \Psi_\Sigma\big(\mathscr{S}_{g^N,x}(\mathscr{S}^\phi_f)\big) \tag{5.16.1}$$

holds in $\mathscr{M}^{\mathbf{G}_m}_{\mathbf{G}_m}$.

Proof

The only point to be checked is that $\gamma((f), (g))$ may be replaced by the local invariant $\gamma_x((f), (g))$, which is clear from the proof of Theorem 5.7. □

5.17

Let us now explain how to deduce from Theorem 5.7 the motivic Thom-Sebastiani theorem of [7], [16], and [8].

Let X and Y be two varieties over k. For r and s in $\mathbf{N}$, cartesian product gives rise to an external product

$$\boxtimes : \mathscr{M}^{\mathbf{G}_m^r}_{X \times \mathbf{G}_m^r} \times \mathscr{M}^{\mathbf{G}_m^s}_{Y \times \mathbf{G}_m^s} \longrightarrow \mathscr{M}^{\mathbf{G}_m^{r+s}}_{X \times Y \times \mathbf{G}_m^{r+s}}. \tag{5.17.1}$$

(This is not to be confused with the one in (5.1.5).)

Let X_1 and X_2 be smooth varieties, and consider functions $f_1 : X_1 \to \mathbf{A}^1_k$ and $f_2 : X_2 \to \mathbf{A}^1_k$. We set $X_0 = f_1^{-1}(0) \times f_2^{-1}(0)$, and for any $Y \subset X_1 \times X_2$ containing X_0, we denote by i the inclusion of $X_0 \times \mathbf{G}_m$ in $Y \times \mathbf{G}_m$.

THEOREM 5.18
Let X_1 and X_2 be smooth varieties of pure dimension d_1 and d_2, and consider functions $f_1 : X_1 \to \mathbf{A}^1_k$ and $f_2 : X_2 \to \mathbf{A}^1_k$. Denote by $f_1 \oplus f_2$ the function on $X_1 \times X_2$ sending (x_1, x_2) to $f_1(x_1) + f_2(x_2)$. Then

$$i^* \mathscr{S}^{\phi}_{f_1 \oplus f_2} = \Psi_\Sigma(\mathscr{S}^{\phi}_{f_1} \boxtimes \mathscr{S}^{\phi}_{f_2}) \tag{5.18.1}$$

in $\mathscr{M}^{\mathbf{G}_m}_{X_0 \times \mathbf{G}_m}$.

Proof
We set $X = X_1 \times X_2$, and we denote by f and g the functions on X induced by f_1 and f_2, respectively. In particular, $f_1 \oplus f_2 = f + g$. If $Y_1 \to X_1$ is a log-resolution of $(X_1, f_1^{-1}(0))$ and $Y_2 \to X_2$ is a log-resolution of $(X_2, f_2^{-1}(0))$, $h : Y := Y_1 \times Y_2 \to X$ is a log-resolution of $(X, f^{-1}(0) \cup g^{-1}(0))$. Using such a log-resolution, it is easily checked that $\gamma((f), (g)) = 0$. By (5.15.7),

$$i^* \mathscr{S}_f - i^* \mathscr{S}_{f+g} = \Psi_\Sigma\big(\mathscr{S}_g(\mathscr{S}_f)\big) - i^* \mathscr{S}_g\big([X_0(f)]\big). \tag{5.18.2}$$

Using the log-resolution h, one checks that $i^* \mathscr{S}_f = \mathscr{S}_{f_1} \boxtimes [f_2^{-1}(0)]$, $\mathscr{S}_g(\mathscr{S}_f) = \mathscr{S}_{f_1} \boxtimes \mathscr{S}_{f_2}$, and $i^* \mathscr{S}_g([X_0(f)]) = [f_1^{-1}(0)] \boxtimes \mathscr{S}_{f_2}$. Hence, (5.18.2) may be rewritten as

$$\Psi_\Sigma(\mathscr{S}_{f_1} \boxtimes \mathscr{S}_{f_2}) = \mathscr{S}_{f_1} \boxtimes [f_2^{-1}(0)] + [f_1^{-1}(0)] \boxtimes \mathscr{S}_{f_2} - i^* \mathscr{S}_{f_1 \oplus f_2}. \tag{5.18.3}$$

Since $\mathscr{S}_{f_1} \boxtimes [f_2^{-1}(0)] = \Psi_\Sigma(\mathscr{S}_{f_1} \boxtimes [f_2^{-1}(0) \times \mathbf{G}_m])$ and $[f_1^{-1}(0)] \boxtimes \mathscr{S}_{f_2} = \Psi_\Sigma([f_1^{-1}(0) \times \mathbf{G}_m] \boxtimes \mathscr{S}_{f_2})$ (see the proof of the statement concerning the unit element in Proposition 5.2), (5.18.1) directly follows, by definition of $\mathscr{S}^{\phi}$. □

6. Spectrum and the Steenbrink conjecture

6.1
We now assume that $k = \mathbf{C}$. We denote by HS the abelian category of Hodge structures, and we denote by $K_0(\mathrm{HS})$ the corresponding Grothendieck ring (see, e.g., [8] for definitions). Note that any mixed Hodge structure has a canonical class in $K_0(\mathrm{HS})$. Recall that there is a canonical morphism

$$\chi_h : \mathscr{M}_{\mathbf{C}} \longrightarrow K_0(\mathrm{HS}), \tag{6.1.1}$$

which assigns to the class of a variety X the element $\sum_i (-1)^i [H_c^i(X, \mathbf{Q})]$ in $K_0(\mathrm{HS})$, where $[H_c^i(X, \mathbf{Q})]$ stands for the class of the mixed Hodge structure on $H_c^i(X, \mathbf{Q})$. Let us denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structures endowed with an automorphism of finite order and by $K_0(\mathrm{HS}^{\mathrm{mon}})$ the corresponding Grothendieck ring. Let us consider the ring morphism

$$\chi_h : \mathscr{M}_{\mathbf{G}_m}^{\mathbf{G}_m} \longrightarrow K_0(\mathrm{HS}^{\mathrm{mon}}), \tag{6.1.2}$$

deduced from (3.16.2) via (2.6.3) and composition with $K_0(\mathrm{MHM}^{\mathrm{mon}}_{\mathrm{Spec}\,\mathbf{C}}) \to K_0(\mathrm{HS}^{\mathrm{mon}})$. It is described as follows. If $[X]$ is the class of $f : X \to \mathbf{G}_m$ in $\mathscr{M}_{\mathbf{G}_m}^{\mathbf{G}_m}$ with X connected since f is monomial with respect to the $\mathbf{G}_m$-action, f is a locally trivial fibration for the complex topology. Furthermore, if the weight is, say, n, $x \mapsto \exp(2\pi i t/n)x$ is a geometric monodromy of finite order along the origin. It follows that X_1, the fiber of f at 1, is endowed with an automorphism of finite order T_f, and we have

$$\chi_h([f : X \to \mathbf{G}_m]) = \Big(\sum_i (-1)^i [H_c^i(X_1, \mathbf{Q})], T_f \Big). \tag{6.1.3}$$

There is a natural linear map called the Hodge spectrum,

$$\mathrm{hsp} : K_0(\mathrm{HS}^{\mathrm{mon}}) \longrightarrow \mathbf{Z}[\mathbf{Q}], \tag{6.1.4}$$

such that

$$\mathrm{hsp}([H]) := \sum_{\alpha \in \mathbf{Q} \cap [0,1)} t^{\alpha} \Big(\sum_{p,q \in \mathbf{Z}} \dim(H_{\alpha}^{p,q}) t^p \Big) \tag{6.1.5}$$

for any Hodge structure H with an automorphism of finite order, where $H_{\alpha}^{p,q}$ is the eigenspace of $H^{p,q}$ with respect to the eigenvalue $\exp(2\pi i \alpha)$. We identify here $\mathbf{Z}[\mathbf{Q}]$ with $\bigcup_{n \geq 1} \mathbf{Z}[t^{1/n}, t^{-1/n}]$.

We consider the composite morphism

$$\mathrm{Sp} := (\mathrm{hsp} \circ \chi_h) : \mathscr{M}_{\mathbf{G}_m}^{\mathbf{G}_m} \longrightarrow \mathbf{Z}[\mathbf{Q}]. \tag{6.1.6}$$

Note that Sp is a ring morphism for the convolution product $*$ on $\mathscr{M}_{\mathbf{G}_m}^{\mathbf{G}_m}$, by Lemma 6.3.

Denoting by $\mathrm{HS}^{2-\mathrm{mon}}$ the abelian category of Hodge structures endowed with two commuting automorphisms of finite order and by $K_0(\mathrm{HS}^{2-\mathrm{mon}})$ the corresponding Grothendieck ring, one deduces from (3.16.2) via (2.6.3) a ring morphism

$$\chi_h : \mathscr{M}_{\mathbf{G}_m^2}^{\mathbf{G}_m^2} \longrightarrow K_0(\mathrm{HS}^{2-\mathrm{mon}}) \tag{6.1.7}$$

having a description similar to (6.1.3).

Also, we can define a Hodge spectrum on $K_0(\mathrm{HS}^{2-\mathrm{mon}})$ as follows. Denote by $\pi : [0,1) \cap \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ the restriction of the projection $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$, and denote by $s : \mathbf{Q}/\mathbf{Z} \to [0,1) \cap \mathbf{Q}$ its inverse. The bijection $\mathbf{Q}/\mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}$ sending (a,b) to $s(a)+b$ induces an isomorphism of abelian groups between $\mathbf{Z}[\mathbf{Q}/\mathbf{Z} \times \mathbf{Z}]$ and $\mathbf{Z}[\mathbf{Q}]$. We define the spectrum

$$\mathrm{hsp} : K_0(\mathrm{HS}^{2-\mathrm{mon}}) \longrightarrow \mathbf{Z}[(\mathbf{Q}/\mathbf{Z})^2 \times \mathbf{Z}] \tag{6.1.8}$$

by

$$\mathrm{hsp}([H]) = \sum_{\alpha \in \mathbf{Q} \cap [0,1)} \sum_{\beta \in \mathbf{Q} \cap [0,1)} \sum_{p,q \in \mathbf{Z}} (\dim H^{p,q}_{\alpha,\beta}) t^{\pi(\alpha)} u^{\pi(\beta)} v^p \tag{6.1.9}$$

with $H^{p,q}_{\alpha,\beta}$ the eigenspace of $H^{p,q}$ with respect to the eigenvalue $\exp(2\pi i \alpha)$ for the first automorphism and $\exp(2\pi i \beta)$ for the the second automorphism. We denote by Sp the morphism of abelian groups

$$\mathrm{Sp} := (\mathrm{hsp} \circ \chi_h) : \mathscr{M}^{\mathbf{G}_m^2}_{\mathbf{G}_m^2} \longrightarrow \mathbf{Z}[(\mathbf{Q}/\mathbf{Z})^2 \times \mathbf{Z}]. \tag{6.1.10}$$

We denote by δ the morphism of abelian groups

$$\mathbf{Z}[(\mathbf{Q}/\mathbf{Z})^2 \times \mathbf{Z}] \longrightarrow \mathbf{Z}[\mathbf{Q}] \tag{6.1.11}$$

sending $t^a u^b v^c$ to $t^{s(a)+s(b)+c}$.

Let A be an element of $\mathscr{M}^{\mathbf{G}_m^2}_{\mathbf{G}_m^2}$. The relation between the spectrum of A and the spectrum of $\Psi_\Sigma(A)$ is given by the following proposition.

PROPOSITION 6.2
Let A be an element of $\mathscr{M}^{\mathbf{G}_m^2}_{\mathbf{G}_m^2}$. We have

$$\mathrm{Sp}\big(\Psi_\Sigma(A)\big) = \delta\big(\mathrm{Sp}(A)\big). \tag{6.2.1}$$

Proof
Let A be a smooth variety with a good $\mathbf{G}_m^2$-action and with a morphism to $\mathbf{G}_m^2$ which is diagonally monomial of weight (n,n), n in $\mathbf{N}_{>0}$. Let us denote by A_1 the fiber of A above $(1,1)$. By (5.1.8), and using the notation therein, we have

$$\Psi_\Sigma([A]) = -[F_1^n \times^{\mu_n \times \mu_n} A_1] + [F_0^n \times^{\mu_n \times \mu_n} A_1]. \tag{6.2.2}$$

The result follows from the following well-known computation of the cohomology of Fermat varieties (see [22] and [16, Lemma 7.1]). □

LEMMA 6.3
Let (α, β) be in $(\mathbf{Q}/\mathbf{Z})^2$. For every common denominator n of α and β, the Hodge type of the eigenspaces $H^i_c(F^n_1, \mathbf{C})(\alpha, \beta)$ and $H^i_c(F^n_0, \mathbf{C})(\alpha, \beta)$ of $\mu_n \times \mu_n$ in $H^i_c(F^n_1, \mathbf{C})$ and $H^i_c(F^n_0, \mathbf{C})$, respectively, with character $(\alpha, \beta) \in (n^{-1}\mathbf{Z}/\mathbf{Z})^2$ is independent of n and is computed as follows.

(1) *$H^1_c(F^n_1, \mathbf{C})(\alpha, \beta)$ is of rank 1 for $(\alpha, \beta) \neq (0, 0)$ and of rank 2 for $(\alpha, \beta) = (0, 0)$. $H^1_c(F^n_1, \mathbf{C})(\alpha, \beta)$ is of Hodge type $(0, 1)$ if $\alpha \neq 0 \neq \beta$, and $0 < s(\alpha) + s(\beta) < 1$, $(1, 0)$ if $1 < s(\alpha) + s(\beta) < 2$, and $(0, 0)$ otherwise, that is, if $\alpha = 0$ or $\beta = 0$ or $\alpha + \beta = 0$. $H^2_c(F^n_1)(0, 0)$ is of rank 1 and Hodge type $(1, 1)$. All other cohomology groups are zero.*

(2) *$H^1_c(F^n_0, \mathbf{C})(\alpha, -\alpha)$, respectively, $H^2_c(F^n_0, \mathbf{C})(\alpha, -\alpha)$, is of rank 1 and Hodge type $(0, 0)$, respectively, $(1, 1)$, for any α in $\mathbf{Q}/\mathbf{Z}$, and all other cohomology groups are zero.*

We also need the following obvious statement.

LEMMA 6.4
For $N \geq 1$, consider the morphism $\pi_N : \mathbf{G}^2_m \to \mathbf{G}^2_m$, given by $(a, b) \mapsto (a, b^N)$. For every A in $\mathscr{M}^{\mathbf{G}^2_m}_{\mathbf{G}^2_m}$,

$$\mathrm{Sp}\big(\pi_{N!}(A)\big) = \frac{1-u}{1-u^{1/N}} \mathrm{Sp}(A)(t, u^{1/N}, v). \tag{6.4.1}$$

6.5
Let X be a smooth complex algebraic variety of dimension d, and let f be a function $X \to \mathbf{A}^1$. Fix a closed point x of X at which f vanishes. Denote by F_x the Milnor fiber of f at x. The cohomology groups $H^i(F_x, \mathbf{Q})$ carry a natural mixed Hodge structure (see [24], [27], [18], [20]), which is compatible with the semisimplification of the monodromy operator $T_{f,x}$. Hence, we can define the Hodge characteristic $\chi_h(F_x)$ of F_x in $K_0(\mathrm{HS}^{\mathrm{mon}})$. The following statement follows from [5] and [8]. (It is also a consequence of Proposition 3.17.)

THEOREM 6.6
Assuming the previous notation, the following equality holds in $K_0(\mathrm{HS}^{\mathrm{mon}})$:

$$\chi_h(F_x) = \chi_h(\mathscr{S}_{f,x}). \tag{6.6.1}$$

In particular, if we define the Hodge spectrum of f at x as

$$\mathrm{Sp}(f, x) := (-1)^{d-1} \mathrm{hsp}\big(\chi_h(F_x) - 1\big), \tag{6.6.2}$$

it follows from Theorem 6.6 that

$$\mathrm{Sp}(f, x) = \mathrm{Sp}(\mathscr{S}^{\phi}_{f,x}). \tag{6.6.3}$$

Now if $g : X \to \mathbf{A}^1$ is another function vanishing at x, we set, by analogy with (6.6.3),

$$\mathrm{Sp}(f, g, x) := \mathrm{Sp}\big(\mathscr{S}_{g,x}(\mathscr{S}^{\phi}_{f})\big). \tag{6.6.4}$$

Let us denote by δ_N the morphism of abelian groups $\mathbf{Z}[(\mathbf{Q}/\mathbf{Z})^2 \times \mathbf{Z}] \to \mathbf{Z}[\mathbf{Q}]$ sending $t^a u^b v^c$ to $t^{s(a)+s(b)/N+c}$.

PROPOSITION 6.7
For every positive integer N, the spectrum of $\Psi_\Sigma(\mathscr{S}_{g^N,x}(\mathscr{S}^{\phi}_{f}))$ is equal to

$$\mathrm{Sp}\big(\Psi_\Sigma(\mathscr{S}_{g^N,x}(\mathscr{S}^{\phi}_{f}))\big) = \frac{1-t}{1-t^{1/N}}\delta_N\big(\mathrm{Sp}(f, g, x)\big). \tag{6.7.1}$$

Proof
The proof follows directly from Proposition 6.2 and Lemma 6.4. □

Hence, from Corollary 5.16, we deduce immediately the following statement.

THEOREM 6.8
Let X be a smooth variety of pure dimension d, and let f and g be two functions from X to $\mathbf{A}^1$. Let x be a closed point of $X_0(f) \cap X_0(g)$. Then, for $N > \gamma_x((f), (g))$,

$$\mathrm{Sp}(f, x) - \mathrm{Sp}(f + g^N, x) = \frac{1-t}{1-t^{1/N}}\, \delta_N\big(\mathrm{Sp}(f, g, x)\big). \tag{6.8.1}$$

6.9. Application to Steenbrink's conjecture
Let us now assume that the function g vanishes on all local components at x of the singular locus of f but a finite number of locally irreducible curves Γ_ℓ, $1 \le \ell \le r$. We denote by e_ℓ the order of g on Γ_ℓ.

As in the introduction, along the complement Γ°_ℓ to $\{x\}$ in Γ_ℓ, we may view f as a family of isolated hypersurface singularities parametrized by Γ°_ℓ. We denote by $\alpha_{\ell,j}$ the exponents of that isolated hypersurface singularity, and we note that there are two commuting monodromy actions on the cohomology of its Milnor fiber. The first one, denoted by T_f, is induced transversally by the monodromy action of f; the second one, denoted by T_τ, is the monodromy around x in Γ°_ℓ. Since the semisimplifications of T_f and T_τ can be diagonalized simultaneously, we may define rational numbers $\beta_{\ell,j}$ in $[0, 1)$, so that each $\exp(2\pi i \beta_{\ell,j})$ is the eigenvalue of the semisimplification of T_τ on the eigenspace of the semisimplification of T_f associated with $\alpha_{\ell,j}$.

We may now deduce from Theorem 5.7 the following statement, first proved by M. Saito in [21] and later given another proof by A. Némethi and J. H. M. Steenbrink in [17].

THEOREM 6.10
For $N > \gamma_x((f),(g))$, we have

$$\mathrm{Sp}(f+g^N,x)-\mathrm{Sp}(f,x)=\sum_{\ell,j} t^{\alpha_{\ell,j}+(\beta_{\ell,j}/e_\ell N)}\frac{1-t}{1-t^{1/e_\ell N}}. \tag{6.10.1}$$

Proof
For every ℓ, we set $\mathscr{S}^{\phi}_{f,\ell} := i_\ell^*(\mathscr{S}^{\phi}_f)$ with i_ℓ the inclusion of Γ°_ℓ in $X_0(f)$. Since $\mathscr{S}^{\phi}_f - \sum_\ell i_{\ell!}(\mathscr{S}^{\phi}_{f,\ell})$ has support in $X_0(g)\times \mathbf{G}_m$,

$$\mathscr{S}_{g^N,x}(\mathscr{S}^{\phi}_f)=\sum_\ell \mathscr{S}_{g^N,x}\big(i_{\ell!}(\mathscr{S}^{\phi}_{f,\ell})\big). \tag{6.10.2}$$

Now consider the normalization $n_\ell : \widetilde{\Gamma}_\ell \to \Gamma_\ell$. Let us choose a uniformizing parameter τ_ℓ at the preimage x_ℓ of x in $\widetilde{\Gamma}_\ell$. We may write $g\circ n_\ell = \eta\tau_\ell^{e_\ell}$ with η a local unit. We have

$$\mathscr{S}_{g^N,x}\big(i_{\ell!}(\mathscr{S}^{\phi}_{f,\ell})\big)=\mathscr{S}_{g^N\,|\,\Gamma_\ell,x}(\mathscr{S}^{\phi}_{f,\ell})=\mathscr{S}_{\eta\tau_\ell^{e_\ell N},x_\ell}(\mathscr{S}^{\phi}_{f,\ell}), \tag{6.10.3}$$

where in the last term, we view $\mathscr{S}^{\phi}_{f,\ell}$ as lying in $\mathscr{M}^{\mathbf{G}_m}_{\widetilde{\Gamma}_\ell\times\mathbf{G}_m}$. By Proposition 3.17,

$$\mathrm{Sp}\big(\mathscr{S}_{\eta\tau_\ell^{e_\ell N},x_\ell}(\mathscr{S}^{\phi}_{f,\ell})\big)=\mathrm{Sp}\big(\mathscr{S}_{\tau_\ell^{e_\ell N},x_\ell}(\mathscr{S}^{\phi}_{f,\ell})\big) \tag{6.10.4}$$

and

$$\mathrm{Sp}\big(\mathscr{S}_{\tau_\ell,x_\ell}(\mathscr{S}^{\phi}_{f,\ell})\big)=-\sum_j t^{\pi(\alpha_{\ell,j})}u^{\pi(\beta_{\ell,j})}v^{[\alpha_{\ell,j}]}, \tag{6.10.5}$$

where $[\alpha]$ denotes the integer part of α. Indeed, note that if H is the mixed Hodge module corresponding to a variation of mixed Hodge structure on a neighborhood of x_ℓ, the fiber at x_ℓ of $\psi_{\tau_\ell}(H)$ is nothing but the generic fiber of the variation endowed with the monodromy around x_ℓ. The sign in (6.10.5) results from the fact that the numbers $\alpha_{\ell,j}$ occurring in its right-hand side are the exponents of an isolated hypersurface singularity in an ambient space of dimension $d-1$ and not d. The result follows now from (6.10.2), (6.10.3), (6.10.4), and (6.10.5) by plugging together Corollary 5.16 and Proposition 6.7. □

Acknowledgments. We offer heartfelt thanks to the referee for reading several versions of the article with meticulous care and for providing us with comments that were most helpful in improving the exposition and eliminating many inaccuracies. We also thank J. Fernández de Bobadilla and A. Melle for their comments.

References

[1] E. BIERSTONE and P. D. MILMAN, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207 – 302. MR 1440306

[2] F. BITTNER, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), 1011 – 1032. MR 2059227

[3] ———, *On motivic zeta functions and the motivic nearby fiber*, Math. Z. **249** (2005), 63 – 83. MR 2106970

[4] J. DENEF, *On the degree of Igusa's local zeta function*, Amer. J. Math. **109** (1987), 991 – 1008. MR 0919001

[5] J. DENEF and F. LOESER, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), 505 – 537. MR 1618144

[6] ———, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), 201 – 232. MR 1664700

[7] ———, *Motivic exponential integrals and a motivic Thom-Sebastiani theorem*, Duke Math. J. **99** (1999), 285 – 309. MR 1708026

[8] ———, "Geometry on arc spaces of algebraic varieties" in *European Congress of Mathematics, Vol. 1 (Barcelona, 2000)*, Progr. Math. **201**, Birkhaüser, Basel, 2001, 327 – 348. MR 1905328

[9] ———, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology **41** (2002), 1031 – 1040. MR 1923998

[10] S. ENCINAS and H. HAUSER, *Strong resolution of singularities in characteristic zero*, Comment. Math. Helv. **77** (2002), 821 – 845. MR 1949115

[11] S. ENCINAS and O. VILLAMAYOR, *Good points and constructive resolution of singularities*, Acta Math. **181** (1998), 109 – 158. MR 1654779

[12] W. FULTON, *Intersection Theory*, Ergeb. Math. Grenzgeb. (3) **2**, Springer, Berlin, 1984. MR 0732620

[13] G. GUIBERT, *Espaces d'arcs et invariants d'Alexander*, Comment. Math. Helv. **77** (2002), 783 – 820. MR 1949114

[14] I. N. IOMDIN [YOMDIN], *Complex surfaces with a one-dimensional set of singularities* (in Russian), Sibirsk. Mat. Ž. **15** (1974), 1061 – 1082, 1181; English translation in Siberian Math. J. **15** (1974), 748 – 762. MR 0447621

[15] G. LAUMON and L. MORET-BAILLY, *Champs algébriques*, Ergeb. Math. Grenzgeb. (3) **39**, Springer, Berlin, 2000. MR 1771927

[16] E. LOOIJENGA, *Motivic measures*, Astérisque **276** (2002), 267 – 297, Séminaire Bourbaki 1999/2000, no. 874. MR 1886763

[17] A. NÉMETHI and J. H. M. STEENBRINK, *Spectral pairs, mixed Hodge modules, and series of plane curve singularities*, New York J. Math. **1** (1994/95), 149 – 177. MR 1343394

[18] M. SAITO, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), 849 – 995. MR 1000123

[19] ———, *Duality for vanishing cycle functors*, Publ. Res. Inst. Math. Sci. **25** (1989), 889 – 921. MR 1045997

[20] ———, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221 – 333. MR 1047415

[21] ———, *On Steenbrink's conjecture*, Math. Ann. **289** (1991), 703 – 716. MR 1103044

[22] T. SHIODA and T. KATSURA, *On Fermat varieties*, Tôhoku Math. J. (2) **31** (1979), 97 – 115. MR 0526513

[23] D. SIERSMA, *The monodromy of a series of hypersurface singularities*, Comment. Math. Helv. **65** (1990), 181 – 197. MR 1057239

[24] J. H. M. STEENBRINK, "Mixed Hodge structure on the vanishing cohomology" in *Real and Complex Singularities (Oslo, 1976)*, Sijthoff and Noordhoff, Alphen aan den Rijn, Netherlands, 1977, 525 – 563. MR 0485870

[25] ———, "The spectrum of hypersurface singularities" in *Actes du Colloque de Théorie de Hodge (Luminy, France, 1987)*, Astérisque **179 – 180**, Soc. Math. France, Montrouge, 1989, 11, 163 – 184. MR 1042806

[26] H. SUMIHIRO, *Equivariant completion, II*, J. Math. Kyoto Univ. **15** (1975), 573 – 605. MR 0387294

[27] A. N. VARCHENKO [VARČENKO], *Asymptotic Hodge structure on vanishing cohomology* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 540 – 591, 688; English translation in Math. USSR-Izv. **18** (1982), 469 – 512. MR 0623350

[28] O. E. VILLAMAYOR [VILLAMAYOR U.], *Constructiveness of Hironaka's resolution*, Ann. Sci. École Norm. Sup. (4) **22** (1989), 1 – 32. MR 0985852

[29] ———, *Patching local uniformizations*, Ann. Sci. École Norm. Sup. (4) **25** (1992), 629 – 677. MR 1198092

Guibert
15 rue du Cap, 94000 Créteil, France; guibert9@wanadoo.fr

Loeser
École Normale Supérieure, Département de mathématiques et applications, 45 rue d'Ulm, 75230 Paris CEDEX 05, France (UMR 8553 du CNRS); francois.loeser@ens.fr

Merle
Laboratoire J.-A. Dieudonné, Université de Nice–Sophia Antipolis, Parc Valrose, 06108 Nice CEDEX 02, France (UMR 6621 du CNRS); merle@math.unice.fr