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ITERATING THE BASIC CONSTRUCTION

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ABSTRACT. Let $N \subset M$ be a pair of type II_1 factors with finite Jones' index and $N \subset M \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots \subset M_{2n+1}$ be the associated tower of type II_1 factors obtained by iterating Jones' basic construction. We give an explicit formula of a projection in M_{2n+1} which implements the conditional expectation of M_n onto N, thus showing that M_{2n+1} comes naturally from the basic construction associated to the pair $N \subset M_n$. From this we deduce several properties of the relative commutant $N' \cap M_n$.

Introduction. Let $N \subset M$ be a pair of finite factors. Jones defined in [**1**] the index $[M : N]$ of N in M to be the coupling constant of N in its representation on $L^2(M)$. If this index is finite, then the trace preserving conditional expectation of M onto N, regarded as an operator on $L^2(M)$, generates together with M a finite factor M_1. This factor is called in Jones' terminology the extension of M by N and the construction of M_1 from M and N, the basic construction. The pair $M \subset M_1$ has the remarkable property that $[M_1 : M] = [M : N]$, so this procedure may be iterated to get an increasing sequence of finite factors $N \subset M \subset M_1 \subset M_2 \subset \cdots$ and together with it a sequence of projections $e_i \in M_{i+1}$, $i \geq 0$, implementing the conditional expectations at consecutive steps.

We prove in this paper that in this sequence of factors the basic construction arises periodically from n to n steps, for any n. In fact we give a formula for a projection f_n in M_{2n+1} that implements the conditional expectation of M_n onto N: f_n is a scalar multiple of the word of maximal length in $\{e_i\}_{0 \leq i \leq 2n}$, namely

$$f_n = [M : N]^{n(n+1)/2}(e_n e_{n-1} \cdots e_0)(e_{n+1} e_n \cdots e_1) \cdots (e_{2n} \cdots e_n).$$

We mention that this result was independently obtained by A. Ocneanu [**2**]. We apply this theorem to show that if the logarithm of the index $[M : N]$ equals the relative entropy $H(M|N)$ considered in [**3**], then one also has

$$H(M_n|N) = \ln[M_n : N] \quad \text{for every } n.$$

Since this equality characterizes an extremal case for an inclusion of factors, from the analysis of a similar situation in [**3**] we deduce several properties of the inclusion $N \subset M_n$ and of the relative commutant $N' \cap M_n$.

1. Preliminaries. Throughout this paper M will be a finite factor with normalized trace τ, $\tau(1) = 1$. We denote by $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$, the Hilbert norm given by τ and by $L^2(M, \tau)$ the Hilbert space completion of M in this norm. The canonical conjugation of $L^2(M, \tau)$ is denoted by J. It acts on $M \subset L^2(M, \tau)$

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by $Jx = x^*$ and satisfies $JMJ = M'$. In fact, if we regard M as acting by left multiplication on $L^2(M,\tau)$ then for $x \in M$, JxJ is the operator of right multiplication by x^*.

$N \subset M$ will denote a subfactor of M with $1_N = 1_M$ and E_N will be the unique normal trace preserving conditional expectation of M onto N. Note that E_N is just the restriction to $M \subset L^2(M,\tau)$ of the orthogonal projection e_N of $L^2(M,\tau)$ onto $L^2(N,\tau)$ (the closure of N in $L^2(M,\tau)$). The conditional expectation E_N, the projection e_N and the conjugation J are related by the properties

(i) If $x \in M$ then $x \in N$ iff $e_N x = x e_N$.

(ii) $e_N x e_N = E_N(x) e_N$, $x \in M$.

(iii) J commutes with e_N.

If the index of N in M is finite then from the pair $N \subset M$ one can construct a new pair of finite factors $M \subset M_1$ with the same index $[M_1 : M] = [M : N]$. The construction of M_1 is called the basic construction and the factor M_1 is called the extension of M by N.

We recall from [**1**] the definition and main properties of M_1:

1.1 PROPOSITION. *Define $M_1 = JN'J$. Then we have*

1° $M_1 = (M \cup \{e_N\})''$,

2° $[M_1 : M] = [M : N]$ *and if τ denotes the unique normalized trace on M_1 and E_M the τ preserving conditional expectation of M_1 onto M, then $E_M(e_N) = [M : N]^{-1} 1_M$ or equivalently $\tau(e_N x) = [M : N]^{-1}\tau(x)$ for every $x \in M$.*

Part 1° of this proposition can be made more precise: by [**3**], if $n + 1 \geq [M : N]$ then any element in M_1 is a sum of at most $(n+1)^2$ monomials of the form $x e_N y$, $x, y \in M$. Note that M_1 can also be described abstractly as the unique (up to isomorphism) finite factor M_1 which contains M and a projection e so that $[M_1 : M] = [M : N]$, $[e, y] = 0$ for $y \in N$, $exe = E_N(x)e$ for $x \in M$, and with the trace τ satisfying $\tau(ex) = [M_1 : M]^{-1}\tau(x)$, $x \in M$. In fact one of the conditions is redundant: the next proposition gives two equivalent ways of characterizing M_1.

1.2 PROPOSITION. *Let $N \subset M$ be a pair of finite factors with finite index and M_1 the extension of M by N. Let $\tilde{M}$ be a finite factor that contains M and with normalized trace $\tilde{\tau}$, E_M the $\tilde{\tau}$-preserving conditional expectation of $\tilde{M}$ onto M and $e \in \tilde{M}$ an orthogonal projection. Then the following conditions are equivalent:*

1° *There exists an isomorphism ϕ of M_1 onto $\tilde{M}$ such that $\phi(x) = x$ for $x \in M$ and $\phi(e_N) = e$.*

2° (i) $[e, y] = 0$, $y \in N$;

(ii) $E_M(e) = [\tilde{M} : M]^{-1} 1_M = [M : N]^{-1} 1_M$.

3° (i) *$exe = E_N(x)e$, $x \in M$, and $e \neq 0$;*

(ii) *e and M generate $\tilde{M}$ as a von Neumann algebra.*

PROOF. 1° implies 2° by the known properties of e_N.

Suppose 2° holds. Then by 1.8 of [**3**] we get that $\tilde{M}$ is the extension of M by P where $P = \{e\}' \cap M$. But (i) implies that $N \subset P$ and since $[M : P] = [\tilde{M} : M] = [M : N]$ we conclude that $N = P$. Thus e and M generate $\tilde{M}$ as a von Neumann algebra and again by 1.8 of [**3**] we get $E_N(x)e = exe$, for every $x \in M$.

Assume that 3° holds. Using the "orthonormal basis" of [3] it is easy to see that the map $\phi: M_1 \to \tilde{M}$ that sends $\sum x_i e_N y_i$ to $\sum x_i e y_i$ is a well-defined $*$-homomorphism. Moreover ϕ satisfies $m\phi(x) = \phi(mx)$ for every $m \in M$ and $x \in M_1$. This shows that $\phi(1)$ is a projection that commutes with e and with every $m \in M$. By (ii) we conclude that $\phi(1)$ is central and since $e \neq 0$ and M is a factor $\phi(1) = 1$. This implies now that $\phi(m) = \phi(m1) = m\phi(1) = m$ and since obviously $\phi(e_N) = e$ we get 1°. Q.E.D.

The pair $M \subset M_1$ having finite index one can construct its extension $M_1 \subset M_2$ and in fact the whole procedure may be iterated to get an increasing sequence of finite factors $N \subset M \subset M_1 \subset M_2 \subset \cdots$, and orthogonal projections $e_i \in M_{i+1}$, $i \geq 0$ ($N = M_{-1}$, $M = M_0$) in which M_{i+1} is the extension of M_i by M_{i-1} or in other words M_{i+1} and e_i are obtained by the basic construction from the pair $M_{i-1} \subset M_i$. Thus if τ denotes the unique normalized trace on $\bigcup_i M_i$ and $E_{M_{i-1}}$ the τ-preserving conditional expectation of M_i onto M_{i-1}, $i \geq 0$, then:

(a) $[e_i, y] = 0$ for $y \in M_{i-1}$;

(b) $e_i x e_i = E_{M_{i-1}}(x) e_i$, $x \in M_i$;

(c) $[M_{i+1} : M_i] = [M : N]$ and $E_{M_i}(e_i) = [M : N]^{-1} 1$.

In particular it follows that the sequence of projections e_i satisfies $[e_i, e_j] = 0$, $|i - j| \geq 2$, $e_i e_{i\pm 1} e_i = [M : N]^{-1} e_i$ and $\tau(e_i w) = [M : N]^{-1} \tau(w)$ for every word in $1, e_0, e_1, \ldots, e_{i-1}$.

2. n-step extensions. In this section we prove the main result of the paper: we show that if $N \subset M \subset M_1 \subset \cdots$ is the sequence of finite factors obtained by iterating the basic construction as in §1, then, for each $n > 0$, M_{2n+1} is the extension of M_n by N. In fact we give an explicit formula for a projection $f_n \in M_{2n+1}$ which implements the conditional expectation of M_n onto N and generates with M_n the factor M_{2n+1}: f_n will be a scalar multiple of the word of maximal length in $e_0, e_1, \ldots, e_{2n}$ where $e_i \in M_{i+1}$ are as in §1.

We define for each $n, k \geq 0$ the element

$$g_n^k = (e_{n+k} e_{n+k-1} \cdots e_k)(e_{n+k+1} e_{n+k} \cdots e_{k+1}) \cdots (e_{2n+k} e_{2n+k-1} \cdots e_{n+k})$$

(there are $n+1$ products of parentheses and in each parentheses the product of $n+1$ consecutive projections e_i in decreasing order). We put $f_n^k = [M : N]^{n(n+1)/2} g_n^k \in M_{2n+k+1}$ and $f_n = f_n^0 \in M_{2n+1}$.

To prove that the above defined f_n implements the basic construction in the extension of M_n by N, we only have to show that f_n is an orthogonal projection, that $f_n \in N' \cap M_{2n+1}$ and that $E_{M_n}(f_n) = [M_n : N]^{-1} = [M_{2n+1} : M_n]^{-1}$. (See Proposition 1.2.) Note that since $[M_{i+1} : M_i] = [M : N]$, by the multiplicative property of the index we do have $[M_n : N] = [M : N]^{n+1} = [M_{2n+1} : M_n]$. To prove the other properties, let us first recall some facts about the algebra generated by $\{e_i\}_{i \geq 0}$ (cf. [1]).

A finite product of e_i's is called a word. It is called a reduced word if it is of minimal length for the grammatical rules $e_i e_{i\pm 1} e_i \leftrightarrow e_i$, $e_i^2 \leftrightarrow e_i$ and $e_i e_j \leftrightarrow e_j e_i$ for $|i - j| \geq 2$. Note that any word is a scalar multiple of a reduced word. Jones pointed out (in [1, 4.1.4]) that reduced words can be uniquely written in the ordered form

$$(*) \qquad w = (e_{j_1} e_{j_1 - 1} \cdots e_{k_1})(e_{j_2} e_{j_2 - 1} \cdots e_{k_2}) \cdots (e_{j_p} e_{j_p - 1} \cdots e_{k_p})$$

where $j_i \geq k_i$, $j_{i+1} > j_i$, $k_{i+1} > k_i$.

From this description of reduced words it follows that if a reduced word w is written with the letters $e_r, e_{r+1}, \dots, e_s$ $(s \geq r)$ then e_{r+i} and e_{s-i} appear at most $i+1$ times in w.

To prove the theorem we first show that g_n^0 are selfadjoint elements. This will be an easy consequence of the next two lemmas.

2.1 LEMMA. *g_n^0 is the unique reduced word of maximal length in $e_0, e_1, \dots, e_{2n}$.*

PROOF. Since by definition g_n^0 is of the form $(*)$ it is a reduced word. As noted before if w is an arbitrary reduced word in $e_0, e_1, \dots, e_{2n}$ then e_0, e_{2n} appear at most once in w, e_1, e_{2n-1} at most twice and more generally e_k, e_{2n-k} at most $k+1$ times. Thus the length of w is at most equal to $1+2+\cdots+n+(n+1)+n+\cdots+2+1$ and by inspecting the conditions $j_i \geq k_i$, $j_{i+1} > j_i$, $k_{i+1} > k_i$ of $(*)$ it follows that the only reduced word w with this length is obtained when $j_i = n+i$, $k_i = i$, i.e. $w = g_n^0$. Q.E.D.

2.2 LEMMA. *If w is a reduced word in $e_0, e_1, \dots, e_{2n}$ then the reduced form of w^* has the same length as w.*

PROOF. Indeed, w^* has length at most equal to that of w and since $(w^*)^* = w$, the statement follows. Q.E.D.

To prove that g_n^0 are scalar multiples of projections we have to compute $(g_n^0)^2$. To do this we use an induction argument based on the formula

2.3 LEMMA. $g_n^0 = (e_n e_{n+1} \cdots e_{2n}) g_{n-1}^0 (e_{2n-1} \cdots e_n)$.

PROOF. The equality follows by pushing e_{2n} to the left as much as possible in the formula giving g_n^0. Q.E.D.

2.4 REMARK. Two other equalities that can be obtained in a similar fashion and seem to be of interest are

$$g_n^0 = g_{n-1}^1 (e_{2n} \cdots e_{n+1})(e_0 \cdots e_n) = (e_n e_{n-1} \cdots e_0) g_{n-1}^2 (e_1 e_2 \cdots e_n).$$

To show that g_n^0 projects on a scalar in M_n we prove

2.5 LEMMA. $E_{M_{2n}}(g_n^0) = [M:N]^{-(n+1)} g_{n-1}^1$. *More generally*

$$E_{M_{2n+k}}(g_n^k) = [M:N]^{-(n+1)} g_{n-1}^{k+1}.$$

PROOF. It is enough to prove that $E_{M_{2n}}(g_n^0) = \lambda^{n+1} g_{n-1}^1$, where $\lambda = [M:N]^{-1}$, because the rest of the statement follows by starting the sequence of factors from $M_{k-1} \subset M_k$, instead of $N = M_{-1} \subset M_0 = M$.

We first show that for $j \geq p \geq k+1$ we have

$$(**) \qquad (e_j e_{j-1} \cdots e_k)(e_p e_{p-1} \cdots e_{k+1}) = \lambda (e_{p-2} \cdots e_k)(e_j \cdots e_{k+1}).$$

Indeed we have

$$\begin{aligned}(e_j e_{j-1} \cdots e_p e_{p-1} \cdots e_k) e_p &= \lambda (e_j e_{j-1} \cdots e_p)(e_{p-2} e_{p-3} \cdots e_k) \\ &= \lambda (e_{p-2} \cdots e_k)(e_j e_{j-1} \cdots e_p),\end{aligned}$$

which easily implies (**). Applying recursively (**) we get

$$\begin{aligned}
E_{M_{2n}}(g_n^0) &= (e_ne_{n-1}\cdots e_0)\cdots(e_{2n-1}\cdots e_{n-1})E_{M_{2n}}(e_{2n})(e_{2n-1}\cdots e_n)\\
&= \lambda(e_n\cdots e_0)\cdots(e_{2n-1}\cdots e_{n-1})(e_{2n-1}\cdots e_n)\\
&= \lambda^2(e_n\cdots e_0)\cdots(e_{2n-2}e_{2n-3}\cdots e_{n-2})(e_{2n-3}\cdots e_{n-1})(e_{2n-1}\cdots e_n)\\
&= \lambda^3(e_n\cdots e_0)\cdots(e_{2n-5}\cdots e_{n-2})(e_{2n-2}\cdots e_{n-1})(e_{2n-1}\cdots e_n)\\
&= \cdots = \lambda^n(e_n\cdots e_0)e_1(e_{n+1}\cdots e_2)\cdots(e_{2n-2}\cdots e_{n-1})(e_{2n-1}\cdots e_n)\\
&= \lambda^{n+1}(e_n\cdots e_1)(e_{n+1}\cdots e_2)\cdots(e_{2n-1}\cdots e_n) = \lambda^{n+1}g_{n-1}^1. \quad \text{Q.E.D.}
\end{aligned}$$

We can now prove the theorem.

2.6 THEOREM. *Let $N \subset M$ be a pair of finite factors with $[M : N] < \infty$. Let $N \subset M \subset M_1 \subset \cdots$ be the sequence of finite factors obtained by iterating the basic construction and $e_i \in M_{i+1}$ the projection implementing the conditional expectation of M_i onto M_{i-1} at each step of the basic construction as in* §1, *for $i \geq 0$ $(M_{-1} = N,\ M_0 = M)$. Let*

$$f_n = [M : N]^{n(n+1)/2}(e_ne_{n-1}\cdots e_0)(e_{n+1}e_n\cdots e_1)\cdots(e_{2n}e_{2n-1}\cdots e_n) \in M_{2n+1}.$$

Then M_{2n+1} is the extension of M_n by N and $f_n \in M_{2n+1}$ is the projection that implements the conditional expectation of M_n onto N, i.e. $f_n \in N' \cap M_{2n+1}$, $f_nxf_n = E_N(x)f_n$, $x \in M_n$, $E_{M_n}(f_n) = [M_n : N]^{-1}$ and $M_{2n+1} = (M_n \cup \{f_n\})''$.

PROOF. We will prove the theorem by induction over $n \geq 0$. If $n = 0$ then $f_0 = e_0$ and we have nothing to prove. Assume the statement is true up to $n-1$. Let $\lambda = [M : N]^{-1}$ and $c_n = \lambda^{-n(n+1)/2}$. Since $f_n = c_ng_n^0$ and g_n^0 is a word in $e_0, e_1, \ldots, e_{2n}$, which all commute with N, it follows that $f_n \in N' \cap M_{2n+1}$. Note also that since $e_{2n} \in M_{2n-1}' \cap M_{2n+1}$, e_{2n} commutes with $g_{n-1}^0 \in M_{2n-1}$. To see that g_n^0 is selfadjoint we use Lemma 2.2 to obtain that g_n^{0*} has the same length as g_n^0 and thus by Lemma 2.1 $g_n^0 = (g_n^0)^*$. Further, Lemma 2.3 implies that

$$\begin{aligned}
(g_n^0)^2 &= g_n^{0*}g_n^0\\
&= (e_ne_{n+1}\cdots e_{2n-1})g_{n-1}^0(e_{2n}e_{2n-1}\cdots e_{n+1}e_ne_{n+1}\\
&\qquad\qquad\cdots e_{2n-1}e_{2n})g_{n-1}^0(e_{2n-1}\cdots e_n)\\
&= \lambda^n(e_ne_{n+1}\cdots e_{2n-1})g_{n-1}^0e_{2n}g_{n-1}^0(e_{2n-1}\cdots e_n)\\
&= \lambda^n(e_ne_{n+1}\cdots e_{2n})(g_{n-1}^0)^2(e_{2n-1}\cdots e_n)\\
&= \lambda^nc_{n-1}^{-1}(e_ne_{n+1}\cdots e_{2n})g_{n-1}^0(e_{2n-1}\cdots e_n)\\
&= \lambda^nc_{n-1}^{-1}g_n^0 = c_n^{-1}g_n^0.
\end{aligned}$$

Thus $f_n = c_ng_n^0$ is a selfadjoint projection in $N' \cap M_{2n+1}$. Next we apply recursively Lemma 2.5 to get

$$\begin{aligned}
E_{M_n}(f_n) &= c_nE_{M_n}(g_n^0) = c_nE_{M_n}E_{M_{2n}}(g_n^0) = c_n\lambda^{n+1}E_{M_n}(g_{n-1}^1)\\
&= c_n\lambda^{n+1}E_{M_n}E_{M_{2n-1}}(g_{n-1}^1) = c_n\lambda^{(n+1)+n}E_{M_n}(g_{n-2}^2)\\
&= \cdots = c_n\lambda^{(n+1)+n+\cdots+2}E_{M_n}(g_0^n) = c_n\lambda^{(n+1)+n+\cdots+2}E_{M_n}(e_n)\\
&= c_n\lambda^{(n+2)(n+1)/2}1_{M_n} = \lambda^{n+1}1_{M_n}
\end{aligned}$$

(we used $g_0^n = e_n$).

Moreover by [1],

$$[M_{2n+1} : M_n] = \prod_{n\le i\le 2n} [M_{i+1} : M_i] = [M : N]^{n+1}$$
$$= \prod_{0\le i\le n} [M_{i+1} : M_i] = [M_n : N].$$

By Proposition 1.2 the rest of the properties of f_n follow automatically. Q.E.D.

2.7 REMARK. We could include the proof of $g_n^0 = g_n^{0*}$ in the induction argument. Indeed by Lemma 2.3 and using $g_{n-1}^0 = (g_{n-1}^0)^*$ and $[e_{2n}, g_{n-1}^0] = 0$ we get

$$(g_n^0)^* = e_n e_{n+1} \cdots e_{2n} (g_{n-1}^0)^* e_{2n-1} e_{2n-2} \cdots e_n$$
$$= e_n e_{n+1} \cdots e_{2n} g_{n-1}^0 e_{2n-1} \cdots e_n = g_n^0.$$

We preferred however the deductive argument of Lemmas 2.1 and 2.2 as it points out some properties of f_n.

3. Some applications. In this section we derive some consequences on the inclusion $N \subset M_n$. We consider the case when the relative entropy $H(M|N)$ considered in [3] satisfies $H(M|N) = \ln[M : N]$. An important case when this equality occurs is when $N' \cap M = \mathbf{C}$ (cf. [3]). First we compute the relative entropy from n to n steps.

3.1 THEOREM. *If $H(M|N) = \ln[M : N]$ then*

$$H(M_{n+k}|M_{k-1}) = \ln[M_{n+k} : M_{k-1}], \quad \textit{for every } n, k \ge 0.$$

In particular $H(M_n|N) = \ln[M_n : N]$ and $H(M_k|M_{k-1}) = \ln[M_k : M_{k-1}]$, for every $k \ge 0$.

PROOF. Since $H(M,N) = \ln[M : N]$, $E_{N'\cap M}(e_0) = \lambda 1$ and the anti-isomorphism $N' \cap M \ni x' \mapsto \theta_0(x') = J_M x' J_M \in M' \cap M_1$ is trace preserving (cf. 4.5 in [3]). To show that $E_{M'\cap M_1}(e_1) = \lambda 1$ it suffices to prove that $M' \cap M_1 \ni y' \mapsto \theta_1(y') = J_{M_1} y' J_{M_1} \in M' \cap M_1$ is also trace preserving (cf. [3]). But $\theta_1\theta_0 = \sigma'$, where σ' is the restriction to $N' \cap M$ of the isomorphism σ defined in [3, 1.3], $\sigma'(x') = \lambda^{-1} \sum_i m_i e_0 e_1 x' e_0 m_i^*$, with $\{m_i\}$ an orthonormal basis of M over N. Indeed if $x' = \sum_i m_i n_i \in N' \cap M$, with $n_i \in N$, then $\theta_0(x') \in M_1$ implies $\theta_0(x') = \sum_{i,j} m_i E_N(m_i^* m_j x'^*) e_N m_j^*$ and thus in $L^2(M_1, \tau)$ we have

$$\theta_1(\theta_0(x'))(m_p n e_0 m_r^*) = \sum_{i,j} m_p n e_0 m_r^* m_j E_N(x' m_j^* m_i) e_0 m_i^*$$
$$= \sum_i m_p n e_0 E_N(x' m_r^* m_i) m_i^* = m_p n e_0 x' m_r^*$$
$$= m_p e_0 x' n m_r^* = \sigma'(x')(m_p n e_0 m_r^*),$$

for all $n \in N$. Thus, since σ', θ_0 are trace preserving, θ_1 is also trace preserving. Induction now shows that $E_{M_k'\cap M_{k+1}}(e_k) = \lambda 1$, $k \ge -1$, and thus $H(M_{k+1}, M_k) = \ln[M_{k+1} : M_k]$.

To prove that $H(M_{n+k}|M_{k-1}) = \ln[M_{n+k} : M_{k-1}]$ it now suffices to prove that $H(M_n|N) = \ln[M_n : M]$ or, by [3], $E_{M_n'\cap M_{2n+1}}(f_n) = \lambda^{n+1} 1_{M_{2n+1}}$. Since

$M'_n \cap M_{2n+1} \subset M'_{n-1} \cap M_{2n+1} \subset \cdots \subset M' \cap M_{2n+1}$ we have $E_{M'_n \cap M_{2n+1}} = E_{M'_n \cap M_{2n+1}} E_{M'_{n-1} \cap M_{2n+1}} \cdots E_{M' \cap M_{2n+1}}$. Since e_0 appears only once in g_n^0 and $E_{M' \cap M_{2n+1}}(e_0) = \lambda 1$ and $e_i \in M'_{i-1}$, it follows that

$$E_{M' \cap M_{2n+1}}(g_n^0) = (e_n \cdots e_1 E_{M' \cap M_{2n+1}}(e_0))(e_{n+1} \cdots e_1) \cdots (e_{2n} e_{2n-1} \cdots e_n).$$

Using now the same computations as in the proof of 2.6 it follows that

$$E_{M' \cap M_{2n+1}}(g_n^0) = \lambda^{n+1} g_{n-1}^1.$$

By induction it follows that

$$\begin{aligned} E_{M'_n \cap M_{2n+1}}(g_n^0) &= \lambda^{n+1} E_{M'_n \cap M_{2n+1}}(g_{n-1}^1) = \lambda^{n+1} E_{M'_n \cap M_{2n+1}} E_{M'_1 \cap M_{2n+1}}(g_{n-1}^1) \\ &= \lambda^{n+1} \lambda^n E_{M'_n \cap M_{2n+1}}(g_{n-2}^2) = \cdots = \lambda^{(n+1)+n+\cdots 1} I \end{aligned}$$

and thus $E_{M'_n \cap M_{2n+1}}(f_n) = \lambda^{n+1} I$. Q.E.D.

3.2 COROLLARY. *Let $N \subset M$ be as in Theorem* 3.1. *Let J_n be the canonical conjugation on $L^2(M_n, \tau)$. Suppose M_{2n+1} is represented on $L^2(M_n, \tau)$ so that to coincide with the basic construction of $N \subset M_n$. Then we have*

(i) *For every projection $f \in N' \cap M_n$, $[(M_n)_f : N_f] = [M_n : N]\tau(f)^2$.*

(ii) *The anti-isomorphism $N' \cap M_n \ni x \mapsto J_n x J_n \in M'_n \cap M_{2n+1}$ is trace preserving.*

(iii) *For every $k \geq 0$ there exists a trace preserving isomorphism $N' \cap M \ni x \mapsto x' \in M'_{k-1} \cap M_k$ so that for every minimal projection $f \in N' \cap M$, $[M_f : N_f] = [(M_k)_f : (M_{k-1})_{f'}]$.*

PROOF. By 4.5 in [**3**] the condition $H(M_n|N) = \ln[M_n : N]$ is equivalent to the above conditions (i) and (ii). Then (iii) follows by (i), (ii) and by the fact that given any trace preserving anti-isomorphism between two finite-dimensional algebras there exists a trace preserving isomorphism between them which acts on the centers in the same way the anti-isomorphism does. Q.E.D.

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