# Iteration of entire functions

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**Abstract**

These notes contain the results discussed in the lectures at the CIMPA school in Kathmandu in November 2014. They contain only some of the proofs, but some references to the literature where proofs can be found are given.

## Contents



## 1 Background from function theory

When dealing with sequences of holomorphic functions, we shall frequently use the following theorems.

**Theorem 1.1. (Weierstraß's Theorem)** *Let $D$ be a domain and let $(f_k)$ be a sequence of functions holomorphic in $D$ which converges locally uniformly to a function $f\colon D \to \mathbb{C}$. Then $f$ is holomorphic and $(f_k')$ converges locally uniformly to $f'$.*

**Theorem 1.2. (Hurwitz's Theorem)** *Let $D, f_n, f$ be as in Theorem 1.1. Then:*

(*i*) *If $f_n(z) \neq 0$ for all $z \in D$ and $n \in \mathbb{N}$, then $f(z) \neq 0$ for all $z \in D$ or $f \equiv 0$.*

*(ii) If all $f_n$ are injective, then $f$ is injective or constant.*

We use the notation $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere. The chordal metric is denoted by $\chi(\cdot,\cdot)$. The spherical derivative of a meromorphic function $f$ is defined by

$$f^{\#}(z) := \frac{2|f'(z)|}{1+|f(z)|^2} = \lim_{\zeta\to z} \frac{\chi(f(\zeta), f(z))}{|\zeta - z|}$$

**Definition 1.3.** A family of meromorphic functions is called *normal* if every sequence in the family has a subsequence which converges locally uniformly with respect to the spherical metric. The family is called normal at a point if this point has a neighborhood where it is normal.

**Theorem 1.4. (Arzelà-Ascoli Theorem)** *A family of meromorphic functions is normal if and only if it is locally equicontinuous (with respect to the spherical metric).*

**Theorem 1.5. (Marty's Theorem)** *A family $\mathcal{F}$ of functions meromorphic in a domain $D$ is normal if and only if the family $\{f^{\#} \colon f \in \mathcal{F}\}$ is locally bounded; that is, if for every $z \in D$ there exists a neighborhood $U$ of $z$ and a constant $M$ such that $f^{\#}(z) \le M$ for all $z \in U$ and for all $f \in \mathcal{F}$.*

**Theorem 1.6. (Montel's Theorem)** *Let $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ be distinct, let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions meromorphic in $D$ such that $f(z) \ne a_j$ for all $j \in \{1,2,3\}$, all $f \in \mathcal{F}$, and all $z \in D$. Then $\mathcal{F}$ is normal.*

**Theorem 1.7. (Picard's Theorem)** *Let $a_1, a_2, a_3 \in \widehat{\mathbb{C}}$ be distinct and let $f \colon \mathbb{C} \to \widehat{\mathbb{C}}$ be meromorphic. If $f(z) \ne a_j$ for all $j \in \{1,2,3\}$ and all $z \in \mathbb{C}$, then $f$ is constant.*

Theorems 1.1–1.2 and 1.4–1.7 can be found in basic function theory books such as [1]. Picard's Theorem easily follows from Montel's Theorem. The following lemma of Zalcman [38] allows to deduce Montel's Theorem from Picard's Theorem so that the theorems in some sense can be considered as equivalent.

**Theorem 1.8. (Zalcman's Lemma)** *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions meromorphic in $D$. If $\mathcal{F}$ is not normal, then there exist a sequence $(z_k)$ in $D$, a sequence $(\rho_k)$ of positive real numbers, a sequence $(f_k)$ in $\mathcal{F}$, a point $z_0 \in D$ and a nonconstant meromorphic function $f \colon \mathbb{C} \to \widehat{\mathbb{C}}$ such that $z_k \to z_0$, $\rho_k \to 0$ and $f_k(z_k + \rho_k z) \to f(z)$ locally uniformly in $\mathbb{C}$.*

*Proof.* Suppose that $\mathcal{F}$ is not normal. By Marty's criterion, there exists a sequence $(\zeta_k)$ in $D$ tending to a point $\zeta_0 \in D$ and a sequence $(f_k)$ in $\mathcal{F}$ such that $f_k^{\#}(\zeta_k) \to \infty$. Without loss of generality, we may assume that $\zeta_0 = 0$ and that $\{z : |z| \le 1\} \subset D$. Choose $z_k$ satisfying $|z_k| \le 1$ such that

$$M_k := f_k^{\#}(z_k)(1-|z_k|) = \max_{|z|\le 1} f_k^{\#}(z)(1-|z|).$$

Then $M_k \ge f_k^{\#}(\zeta_k)(1-|\zeta_k|)$ and hence $M_k \to \infty$. Define $\rho_k = 1/f_k^{\#}(z_k)$. Then $\rho_k \le 1/M_k$ so that $\rho_k \to 0$. Since $|z_k + \rho_k z| < 1$ for $|z| < (1-|z_k|)/\rho_k = M_k$ the function $g_k(z) = f_k(z_k + \rho_k z)$ is defined for $|z| < M_k$ and satisfies

$$g_k^{\#}(z) = \rho_k f_k^{\#}(z_k + \rho_k z) \le \frac{1-|z_k|}{1-|z_k+\rho_k z|} \le \frac{1-|z_k|}{1-|z_k|-\rho_k|z|} = \frac{1}{1-\frac{|z|}{M_k}}$$

there. By Marty's criterion, the sequence $(g_k)$ is normal in $\mathbb{C}$ and thus has a subsequence which converges locally uniformly in $\mathbb{C}$. Without loss of generality, we may assume that $g_k \to f$ for some $f : \mathbb{C} \to \widehat{\mathbb{C}}$ and $z_k \to z_0$ for some $z_0 \in D$. Since $g_k^{\#}(0) = 1$ for all $k$, we have $f^{\#}(0) = 1$, so that $f$ is non-constant. Clearly, we also have $f^{\#}(z) \leq 1$ for all $z \in \mathbb{C}$. $\square$

*Remark.* If $\mathcal{F}$ is not normal at a point $\xi$, then we may achieve that the sequence $(z_k)$ in Zalcman's Lemma satisfies $z_k \to \xi$.

Actually, Zalcman's Lemma can also be used to give a proof of the the theorems of Picard and Montel; see [39].

*Proof of Picard's and Montel's Theorem.* Let $f$ be as in Picard's Theorem, but not constant. Without loss of generality we may assume that $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$ and that $f'(0) \neq 0$.

Since $f(z) \neq 0$ for all $z \in \widehat{\mathbb{C}}$ there exists an entire function $h$ such that $f(z) = e^{h(z)}$ and thus for every $m \in \mathbb{N}$ an entire function $g$ with $g(z)^m = f(z)$ for all $z \in \widehat{\mathbb{C}}$, namely $g(z) = e^{h(z)/m}$. In particular there exists for $n \in \mathbb{N}$ an entire function $f_n$ with

$$f_n(z)^{2^n} = f(3^n z).$$

We show first that $\mathcal{F} := \{f_n \colon n \in \mathbb{N}\}$ is not normal in 0. Since $f_n(0)^{2^n} = f(0) \neq 0$ we see that $|f_n(0)| = \sqrt[2^n]{|f(0)|} \to 1$. Moreover,

$$3^n f'(3^n z) = 2^n f_n(z)^{2^n - 1} f_n'(z) = 2^n f_n(z)^{2^n} \frac{f_n'(z)}{f_n(z)} = 2^n f(3^n z) \frac{f_n'(z)}{f_n(z)}$$

so that

$$|f_n'(0)| = \left(\frac{3}{2}\right)^n \left|\frac{f'(0)}{f(0)}\right| |f_n(0)|$$

Since $|f_n(0)| \to 1$ we obtain $f_n^{\sharp}(0) \to \infty$. By Marty's Theorem, $\mathcal{F}$ is not normal.

We now apply Zalcman's Lemma to $\mathcal{F}$. Thus there exist $z_k \in \widehat{\mathbb{C}}$, $n_k \in \mathbb{N}$, $\varrho_k > 0$ and $g$ entire and non-constant with

$$g_k(z) := f_{n_k}(z_k + \varrho_k z) \to g(z)$$

locally uniformly for $z \in \widehat{\mathbb{C}}$.

Since $f(z) \neq 1$ for all $z \in \widehat{\mathbb{C}}$ we have $f_{n_k}(z)^{2^{n_k}} \neq 1$ for all $z \in \widehat{\mathbb{C}}$. Hence $g_k(z) \neq e^{2\pi i j/2^{n_k}}$ for all $z \in \widehat{\mathbb{C}}$ and all $j \in \{0, 1, 2, \ldots, 2^{n_k} - 1\}$. Hurwitz's Theorem implies that

$$g(z) \neq e^{2\pi i j/2^n} \text{ for } z \in \widehat{\mathbb{C}},\ n \in \mathbb{N} \text{ and } j \in \{0, 1, 2, \ldots, 2^n - 1\}.$$

Since $g$ is a non-constant holomorphic function, it is also open and thus $|g(z)| \neq 1$ for all $z \in \widehat{\mathbb{C}}$. Hence either $|g(z)| < 1$ for all $z \in \widehat{\mathbb{C}}$ or $|g(z)| > 1$ for all $z \in \widehat{\mathbb{C}}$. In the first case $g$ is bounded and in the second case $1/g$ is bounded. In both cases we obtain from Liouville's Theorem that $g$ is constant, which is a contradiction. $\square$

Besides the notation introduced above, we will write $D(a, r) = \{z \in \mathbb{C} \colon |z - a| < r\}$ for the disk of radius $r$ around a point $a$ and $\mathbb{D} = D(0, 1)$ for the unit disk.

# 2 Fatou and Julia sets and their basic properties

We will mainly be interested in the iteration of entire functions, but since the basic definitions and results are analogous to those for rational functions, we will concentrate on entire functions only later.

In the following, all entire and rational functions are assumed to be neither constant nor rational of degree 1.

**Definition 2.1.** Let $f$ be entire (or rational). Then

$$F(f) = \{z\colon \{f^n\} \text{ is normal at } z\}$$

is called the *Fatou set* and the complement of $F(f)$ with respect to the plane (or the sphere) is called the *Julia set* and denoted by $J(f)$.

Some basic properties of these sets are the following:

- $F(f)$ is open and $J(f)$ is closed.
- $z \in F(f) \Leftrightarrow f(z) \in F(f)$ and $z \in J(f) \Leftrightarrow f(z) \in J(f)$ . (This property is called the *complete invariance* of $F(f)$ and $J(f)$.)
- $F(f^n) = F(f)$ and $J(f^n) = J(f)$ for all $n \in \mathbb{N}$.
- If $f = T \circ g \circ T^{-1}$ for a homeomorphism $T$, then $F(f) = T(F(g))$ and $J(f) = T(J(g))$. (The functions $f$ and $g$ are then called *topologically conjugate*.)

The *forward orbit* $O^+(z_0)$ of a point $z_0$ is defined by

$$O^+(z_0) = \{f^n(z_0)\colon n \geq 0\}$$

and the *backward orbit* $O^-(z_0)$ is defined by

$$O^-(z_0) = \bigcup_{n\geq 1} f^{-n}(z_0) = \bigcup_{n\geq 1} \{z\colon f^n(z) = z_0\}.$$

The *orbit* $O(z_0)$ of $z_0$ is defined by $O(z_0) = O^+(z_0) \cup O^-(z_0)$. For a set $A$ we put $O^{(\pm)}(A) = \bigcup_{z\in A} O^{(\pm)}(z)$

**Definition 2.2.** Let $f$ be entire (or rational). Then

$$E(f) = \big\{z\colon O^-(z) \text{ is finite}\big\}$$

is called the *exceptional set*.

*Example.* For $d \in \mathbb{Z}\backslash\{0, \pm 1\}$ and $f(z) = z^d$ we have $E(f) = \{0, \infty\}$. For $m \in \mathbb{Z}$, $m \geq 0$, and $g(z) = z^m e^z$ we have $E(g) = \{0\}$.

Note that $F(f), J(f), E(f)$ are considered as subsets of the Riemann sphere $\widehat{\mathbb{C}}$ if $f$ is rational and as subsets of the plane $\mathbb{C}$ if $f$ is entire.

**Theorem 2.3.** *If $f$ is rational, then $|E(f)| \leq 2$. If $f$ is entire, then $|E(f)| \leq 1$.*

*Remark.* If $f$ is rational, then $E(f) \subset F(f)$.

**Theorem 2.4.** *If $U$ is open, $U \cap J(f) \neq \emptyset$, then $O^+(U) \supset \widehat{\mathbb{C}} \backslash E(f)$ for rational $f$ and $O^+(U) \supset \mathbb{C} \backslash E(f)$ for entire $f$.*

*Proof.* By Montel's Theorem $\widehat{\mathbb{C}} \backslash O^+(U)$ contains at most three points. Thus if $z \notin E(f)$, then $O^+(U)$ contains a point of $O^-(z)$. But this implies that $z \in O^+(U)$. □

**Theorem 2.5.** *If $U$ is open, $U \cap J(f) \neq \emptyset$, then $O^+(U \cap J(f)) \supset J(f) \backslash E(f)$.*

*Proof.* By the complete invariance of $J(f)$ we have $O^+(U \cap J(f)) = O^+(U) \cap J(f)$. The conclusion now follows from Theorem 2.4. □

**Theorem 2.6.** *Let $A$ be a closed backward-invariant set, that is, $f^{-1}(A) \subset A$. Suppose that $A$ has at least three elements if $f$ is rational and at least two elements if $f$ is entire. Then $J(f) \subset A$.*

*Proof.* For simplicity we only consider the case that $f$ is entire. Let $B = \mathbb{C} \backslash A$. The hypothesis $f^{-1}(A) \subset A$ yields $f(B) \subset B$. This implies that $f^n(B) \subset B$ for all $n \in \mathbb{N}$. So the iterates of $f$ omit the (at least two) points in $B$ as well as $\infty$. Thus $B \subset F(f)$ by Montel's Theorem. Taking complements we obtain $J(f) \subset A$. □

**Theorem 2.7.** *If $z_0 \in J(f) \backslash E(f)$, then $J(f) = \overline{O^-(z_0)}$.*

*Proof.* Since $z \notin E(f)$, the backward orbit $O^-(z_0)$ is infinite. Thus we can apply Theorem 2.6 with $A = \overline{O^-(z_0)}$ to obtain $J(f) \subset \overline{O^-(z_0)}$. Since $J(f)$ is completely invariant we have $J(f) \supset \overline{O^-(z_0)}$ and since $J(f)$ is closed this yields $J(f) \subset \overline{O^-(z_0)}$. □

*Remark.* Theorem 2.7 yields an obvious algorithm to make computer pictures of Julia sets, at least for functions for which the inverse function can be easily computed, e.g. polynomials of low degree.

The results of this section can be found (with proofs) in standard books on complex dynamics; see, e.g., [9, 30, 36].

# 3 Periodic points

**Definition 3.1.** A point $z_0$ is called a *periodic point* of $f$ if $f^n(z_0) = z_0$ for some $n \geq 1$. The smallest $n$ with this property is called the *period* of $z_0$. Let $z_0$ be a periodic point of period $n$. Then $\lambda = (f^n)'(z_0)$ is called the *multiplier* of $z_0$. (If $z_0 = \infty$, which can happen only for rational $f$ of course, this has to be modified. In this case, the multiplier is defined to be $(g^n)'(0)$ where $g(z) = 1/f(1/z)$.)

A periodic point with multiplier $\lambda$ is called *attracting*, *indifferent*, or *repelling* depending on whether $|\lambda| < 1$, $|\lambda| = 1$ or $|\lambda| > 1$.

If $z_0$ is indifferent, then $\lambda = e^{2\pi i \alpha}$ where $0 \leq \alpha < 1$, and $z_0$ is called *rationally indifferent* if $\alpha$ is rational and *irrationally indifferent* otherwise.

A point $z_0$ is called *preperiodic* if $f^n(z_0)$ is periodic for some $n \geq 1$. Finally, a periodic point of period 1 is called a *fixed point*.

**Theorem 3.2.** *A rational function of degree at least 2 has a fixed point which is repelling or has multiplier 1.*

*Proof.* We may assume that $f(\infty) \neq \infty$, since this can be achieved by conjugation.

We write $f$ in the form $f = P/Q$ with coprime polynomials $P$ and $Q$. Then

$$\deg(P) \leq \deg(Q) = \deg(f) =: d.$$

Let $z_1, \ldots, z_m \in \widehat{\mathbb{C}}$ be the fixed points of $f$. Suppose that

$$|f'(z_j)| \leq 1 \quad \text{and} \quad f'(z_j) \neq 1$$

for all $j \in \{1, 2, \ldots, m\}$. Then the function defined by

$$z \mapsto f(z) - z = \frac{P(z) - zQ(z)}{Q(z)}$$

has no multiple zeros. For $g(z) = P(z) - zQ(z)$ we find that

$$m = \deg(g) = d + 1 \geq 3.$$

For

$$R > \max_{j=1,\ldots,m} |z_j|$$

we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z - f(z)} &= \sum_{j=1}^{m} \operatorname{res}\left(\frac{1}{z - f(z)}, z_j\right) \\ &= \sum_{j=1}^{m} \lim_{z \to z_j} \frac{z - z_j}{z - f(z)} \\ &= \sum_{j=1}^{m} \frac{1}{1 - f'(z_j)}. \end{aligned}$$

On the other hand, since $f(\infty) \neq \infty$, the Laurent series of $1/(f(z) - z)$ around $\infty$ is of the form

$$\frac{1}{z - f(z)} = \frac{1}{z} + O\left(\frac{1}{z^2}\right).$$

It follows that

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{dz}{z - f(z)} = 1.$$

Thus

$$\sum_{j=1}^{m} \frac{1}{1 - f'(z_j)} = 1.$$

For $w \in \widehat{\mathbb{C}}$ with $|w| \leq 1$ and $w \neq 1$ we have

$$\operatorname{Re}\left(\frac{1}{1 - w}\right) \geq \frac{1}{2}.$$

Thus

$$1 = \operatorname{Re}\left(\sum_{j=1}^{m} \frac{1}{1 - f'(z_j)}\right) = \sum_{j=1}^{m} \operatorname{Re}\left(\frac{1}{1 - f'(z_j)}\right) \geq \sum_{j=1}^{m} \frac{1}{2} = \frac{m}{2} = \frac{d+1}{2}.$$

This is a contradiction. $\square$

*Remark.* It follows that $J(f) \neq \emptyset$ if $f$ is rational. There are several ways to prove that $J(f) \neq \emptyset$ for entire $f$. Most proofs are based on the existence of fixed points as given by the following results.

**Theorem 3.3. (Fatou)** *If $f$ is entire transcendental, then $f \circ f$ has a fixed point.*

**Theorem 3.4. (Rosenbloom)** *If $f$ is entire transcendental and $n \geq 2$, then $f^n$ has infinitely many fixed points.*

We postpone the proof that the Julia set of a transcendental entire function is non-empty.

**Theorem 3.5.** *Attracting periodic points are in $F(f)$ while repelling and rationally indifferent periodic points are in $J(f)$.*

*Proof.* It suffices to consider fixed points. Let $z_0$ be a fixed point of multiplier $\lambda$; that is, $f'(z_0) = \lambda$.

Suppose first that $z_0$ is attracting; that is, $|\lambda| < 1$. Then there exists $r > 0$ with

$$\frac{|f(z) - z_0|}{|z - z_0|} = \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < 1 \text{ for } z \in D(z_0, r).$$

Hence $f(D(z_0, r)) \subset D(z_0, r)$ and thus $D(z_0, r) \subset F(f)$ by Montel's Theorem.

Suppose next that $z_0$ is repelling, that is, $|\lambda| > 1$. The chain rule yields

$$(f^n)'(z_0) = f'(f^{n-1}(z_0)) \cdot f'(f^{n-2}(z_0)) \cdot \ldots \cdot f'(z_0) = f'(z_0)^n = \lambda^n \to \infty.$$

One way to deduce from this that $z_0 \in J(f)$ is using Marty's Theorem, since we then have

$$(f^n)^{\#}(z_0) = \frac{2|\lambda|^n}{1 + |z_0|^2} \to \infty.$$

We omit the proof for rationally indifferent points. □

For an attracting periodic point $z_0$ of period $p$,

$$A(z_0) = \{z \colon f^{pn}(z) \to z_0\}$$

is called the *basin of attraction* of $z_0$.

**Theorem 3.6.** *If $z_0$ is an attracting periodic point, then $A(z_0) \subset F(f)$ and $\partial A(z_0) = J(f)$.*

*Proof.* It is easy to see that $A(z_0) \subset F(f)$ and $\partial A(z_0) \subset J(f)$. The conclusion now follows from Theorem 2.6 since $\partial A(z_0)$ is closed and completely invariant. □

*Remark.* Theorem 3.6 shows that if $f$ has at least three attracting periodic points, then the Julia set must have a quite complicated structure.

**Theorem 3.7.** *$J(f)$ is the closure of the set of repelling periodic points.*

For the proof of this result we need some preparations. We restrict here to entire functions for simplicity, but the modifications needed to handle rational functions are minor.

**Definition 3.8.** Let $f$ be entire and non-constant and let $a \in \mathbb{C}$. Then $a$ is called *totally ramified* (for $f$), if $f$ has no simple $a$-point; that is, if $f(z) = a$, then $f'(z) = 0$. By $V(f)$ we denote the set of totally ramified values.

**Lemma 3.9.** *Let $f$ be entire and non-constant. Then $V(f) \cap f(\mathbb{C})$ is a discrete subset of $f(\mathbb{C})$. In particular, $V(f)$ is countable and has at most one limit point in $\mathbb{C}$.*

*Proof.* If $a = f(z_0)$, there exists a neighborhood $U$ of $z_0$ with $f(z) \in \widehat{\mathbb{C}}$ and $f'(z) \neq 0$ for $z \in U \backslash \{z_0\}$. Thus $f(U)$ is a neighborhood of $a$ which contains at most one point of $V(f)$. Hence $V(f) \cap f(\mathbb{C})$ is a discrete subset of $f(\mathbb{C})$. The second claim follows from the Theorem of Picard. $\square$

*Remark.* Actually $V(f)$ contains at most two points. One may also consider this for meromorphic functions. In this case $V(f)$ has at most four points. These results were proved by Nevanlinna in 1924; see [24, 26, 31].

*Proof of Theorem 3.7.* Let $A(f)$ be the set of repelling periodic points of $f$. We have to show that $\overline{A(f)} = J(f)$. By Theorem 3.5 we have $A(f) \subset J(f)$ and thus $\overline{A(f)} \subset J(f)$, since $J(f)$ is closed. It remains to show that $J(f) \subset \overline{A(f)}$. In order to do this, let $U \subset \mathbb{C}$ be open with $U \cap J(f) \neq \emptyset$. We have to show that $U \cap A(f) \neq \emptyset$. Let

$$C := (f')^{-}(0) = \{z \in \mathbb{C} \colon f'(z) = 0\}$$

and

$$P := O^{+}(C) \cup E(f).$$

Then $P$ is countable. Since $J(f)$ is perfect, there exists $a \in (U \cap J(f)) \backslash P$.

By Zalcman's Lemma there exists a sequence $(z_k)$ in $U$ with $z_k \to a$, a sequence $(\varrho_k)$ with $\varrho_k > 0$ and $\varrho_k \to 0$ and a sequence $(n_k)$ in $\mathbb{N}$ such that

$$f^{n_k}(z_k + \varrho_k z) \to \phi(z)$$

for a non-constant entire function $\phi$:

Since $a \notin E(f)$, we have $J(f) = \overline{O^{-}(a)}$. Thus every of the (uncountably many) points of $U \cap J(f)$ is a limit point of $O^{-}(a)$. Since $U \cap V(\phi)$ is discrete by Lemma 3.9, there exists

$$b \in (U \cap O^{-}(a)) \backslash V(\phi).$$

Since $b \in O^{-}(a)$ there exists $p \in \mathbb{N}$ with $f^p(b) = a$ and since $b \notin V(\phi)$ there exists $c \in \mathbb{C}$ with $\phi(c) = b$ and $\phi'(c) \neq 0$. Since $a \notin O^{+}(C)$ we have $(f^p)'(b) \neq 0$.

With $\psi := f^p \circ \phi$ and $m_k = p + n_k$ we have

$$f^{m_k}(z_k + \varrho_k z) = f^p(f^{n_k}(z_k + \varrho_k z)) \to f^p(\phi(z)) = \psi(z)$$

and thus

$$f^{m_k}(z_k + \varrho_k z) - (z_k + \varrho_k z) \to \psi(z) - a$$

locally uniformly in $\mathbb{C}$. Now

$$\psi(c) = f^p(\phi(c)) = f^p(b) = a$$

and

$$\psi'(c) = (f^p)'(b)\phi'(c) \neq 0.$$

for sufficiently large $k$ there exist, by Hurwitz's Theorem, $c_k \in \mathbb{C}$ with $c_k \to c$ and

$$f^{m_k}(z_k + \varrho_k c_k) - (z_k + \varrho_k c_k) = \psi(c) - a = 0.$$

Hence

$$a_k := z_k + \varrho_k c_k$$

is a periodic point of $f$ and we have $a_k \to a$, in particular $a_k \in U$ for large $k$.

Moreover,

$$\varrho_k (f^{m_k})'(z_k + \varrho_k z) \to \psi'(z)$$

locally uniformly in $\mathbb{C}$. It follows that

$$\varrho_k (f^{m_k})'(a_k) \to \psi'(c) \neq 0$$

and hence $(f^{m_k})'(a_k) \to \infty$. Thus $a_k$ is a repelling periodic point for large $k$. $\square$

For rational functions, Theorem 3.7 was proved by both Fatou and Julia. Actually, Julia started his investigations by considering the set of repelling periodic points and investigating properties of this set. Fatou started by studying the set of non-normality. This is the approach that is taken in all textbooks on the subject, and also in these lectures. Given the different approaches, it is clear that the proofs of Theorem 3.7 given by Fatou and Julia of were also quite different. However, neither one applies to transcendental entire functions.

The first proof of Theorem 3.7 for entire functions is due to Baker [3], the one presented here (based on Zalcman's lemma) is due to Berteloot and Duval [16].

The following results are corollaries of Theorem 3.7.

**Theorem 3.10.** *If $U$ is an open subset intersecting $J(f)$, then no subsequence of $(f^n)$ is normal in $U$.*

**Theorem 3.11.** *If $U$ is an open subset intersecting $J(f)$ and $K$ is a compact subset of $\mathbb{C}\backslash E(f)$, then $f^n(U) \supset K$ for all large $n$.*

Theorem 3.11 sharpens Theorem 2.4. If $f$ is rational, then we may take $K = J(f)$ and obtain that $f^n(U \cap J(f)) = J(f)$ for all large $n$. This can be considered as an explanation of the self-similarity observed in Julia sets.

# 4 Classification of Fatou components

It follows from the complete invariance of $F(f)$ and $J(f)$ that if $U_0$ is a connected component of $F(f)$ and $n \in \mathbb{N}$, then $f^n(U_0) \subset U_n$ for some connected component $U_n$ of $F(f)$. (Actually we have $f^n(U_0) = U_n$ if $f$ is rational and $|U_n \backslash f^n(U_0)| \leq 1$ if $f$ is entire.)

**Definition 4.1.** Let $U_0$ be a connected component of $F(f)$ and let $U_n$ be as above.

- If $U_m \neq U_n$ for $m \neq n$, then $U_0$ is called *wandering*.

- If there exist $m \neq n$ such that $U_m = U_n$, then $U_0$ is called *preperiodic*.
- If there exist $n \geq 1$ such that $U_n = U_0$, then $U_0$ is called *periodic*. The minimal $n$ with this property is called the *period* of $U_0$.

A famous theorem of Sullivan [37] says that rational functions have no wandering domains. As we shall see later, entire functions functions may have wandering domains. The behavior in components of the Fatou set which are not wandering is well understood.

**Theorem 4.2.** *Let $f$ be entire and let $U$ be a periodic component of $F(f)$ of period $p$. Then $U$ is of one of the following types:*

- *There exists $\xi \in U$ satisfying $f^p(\xi) = \xi$ and $|(f^p)'(\xi)| < 1$. Then $f^{pn}|_U \to \xi$ as $n \to \infty$. In this case, $U$ is called an* immediate attracting basin.
- *There exists $\xi \in \partial U$ satisfying $f^p(\xi) = \xi$ and $(f^p)'(\xi) = 1$ such that $f^{pn}|_U \to \xi$ as $n \to \infty$. In this case, $U$ is called an* (immediate) parabolic basin.
- *There exists $\xi \in U$ satisfying $f^p(\xi) = \xi$ and $(f^p)'(\xi) = e^{2\pi i\alpha}$ where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and there exists a biholomorphic function $\tau\colon \mathbb{D} \to U$ such that $\tau^{-1}(f(\tau(z))) = e^{2\pi i\alpha}z$ for all $z \in \mathbb{D}$. In this case, $U$ is called a* Siegel disk.
- *$f^{pn}|_U \to \infty$ as $n \to \infty$. In this case, $U$ is called a* Baker domain.

*Remark.* If $f$ is rational, then there is the additional possibility of a *Herman ring*. On the other hand, Baker domains are not a separate case for rational functions, since if $f^{pn}|_U \to \infty$ as $n \to \infty$ for a rational function $f$, then $\infty$ is a fixed point of $f^p$ which is attracting or has multiplier 1 so that $U$ is an (immediate) attracting or parabolic basin in this case.

**Theorem 4.3.** *If $f$ is entire and $U$ is a multiply connected component of $F(f)$, then $f^n|_U \to \infty$ as $n \to \infty$.*

The proof of Theorem 4.2 can be found in textbooks on complex dynamics [9, 30, 36]. There it is given for rational functions, but the modifications for entire functions are minor. Actually the case of entire functions is simpler because of Theorem 4.3 whose proof is easy.

*Proof of Theorem 4.3.* Suppose that $f^n|_U \not\to \infty$ as $n \to \infty$. Then there exists a sequence $(n_k)$ tending to infinity such that $f^{n_k}|_U \to \phi$ for some holomorphic function $\phi\colon U \to \mathbb{C}$. Let $\gamma$ be (the trace of) a closed curve in $U$ or, more generally, a compact subset of $U$. Then there exists $C > 0$ such that $|f^{n_k}(z)| \leq C$ for all $z \in \gamma$ and all $k \in \mathbb{N}$. By the maximum principle, this implies that $|f^{n_k}(z)| \leq C$ for all $z$ in the interior $\text{int}(\gamma)$ of $\gamma$. This implies that $(f^{n_k})$ is normal in $\text{int}(\gamma)$. We deduce from Theorem 3.10 that $\text{int}(\gamma) \subset F(f)$. Since $\gamma$ was an arbitrary closed curve, we conclude that $U$ is simply connected. $\square$

**Lemma 4.4.** *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions which are holomorphic and injective in $D$. Suppose that $f(z) \neq 0$ for all $f \in \mathcal{F}$ and all $z \in D$. Then $\mathcal{F}$ is normal.*

One way to prove this result is to use the Koebe distortion theorem. Alternatively, we may use Zalcman's lemma.

*Proof.* Suppose that $\mathcal{F}$ is not normal and let $z_k$, $\rho_k$, $f_k$ and $f\colon \mathbb{C} \to \mathbb{C}$ be as in Zalcman's lemma. By Hurwitz's theorem, $f$ is injective and $f(z) \neq 0$ for all $z \in \mathbb{C}$. This is a contradiction, since an injective entire function $f$ has the form $f(z) = az + b$ where $a, b \in \mathbb{C}$ and $a \neq 0$, and we have $f(-b/a) = 0$ for such $f$. □

*Proof of Theorem 4.2.* Without loss of generality we may assume that $U$ is invariant; that is, $p = 1$. We consider the set $L$ of all function $\phi\colon U \to \widehat{\mathbb{C}}$ for which there exists a sequence $(n_k)$ in $\mathbb{N}$ with $n_k \to \infty$ such that $f^{n_k} \to \phi$ locally uniformly in $U$. For $\phi \in L$ we then have $\phi(U) \subset \overline{U}$.

*Case* 1. The set $L$ contains a non-constant function $\phi$. Since $\phi$ is an open mapping, we have $\phi(U) \subset U$. Let $n_k$ be as in the definition of $L$ and put $m_k := n_{k+1} - n_k$. Without loss of generality we may assume that $m_k \to \infty$, since otherwise we may pass to a subsequence of $(n_k)$. By normality there exists a subsequence $(m_{k_j})$ of $(m_k)$ with

$$f^{m_{k_j}} \to \psi \text{ for some } \psi\colon U \to \widehat{\widehat{\mathbb{C}}}.$$

Since

$$f^{n_{k_j+1}} = f^{m_{k_j}+n_{k_j}} = f^{m_{k_j}} \circ f^{n_{k_j}} (= f^{n_{k_j}} \circ f^{m_{k_j}})$$

we have

$$\phi = \psi \circ \phi.$$

This implies that $\psi = \mathrm{id}_U$, that is, $\psi(z) = z$ for all $z \in U$.

Since $f^{m_{k_j}} \to \mathrm{id}_U$ we conclude that $f$ is injective. Hurwitz's theorem yields that $f$ is also surjective.

Since $U$ is simply connected by Theorem 4.3, there exists, by the Riemann Mapping Theorem, a biholomorphic function $\tau\colon \mathbb{D} \to U$. Let $g\colon \mathbb{D} \to \mathbb{D}$, $g = \tau^{-1} \circ f \circ \tau$. Then $g$ is biholomorphic and thus a Möbius transformation of the form

$$g(z) = c\frac{z-a}{1-\overline{a}z}$$

where $a \in \mathbb{D}$ and $c \in \mathbb{C}$ with $|c| = 1$.

First we show that $f$ has a fixed point in $U$. Suppose that this is not the case. Then $g$ has no fixed point in $\mathbb{D}$. Since $g(1/\overline{z}) = 1/\overline{g(z)}$, that is, $g = T^{-1} \circ f \circ T$ with $T(z) = 1/\overline{z}$, the Möbius transformation $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has no fixed point in $\widehat{\mathbb{C}} \backslash \overline{\mathbb{D}}$. Thus it has a fixed point $z_1 \in \partial D$. Let now $M$ be Möbius transformation with $M(z_1) = \infty$, $M(\partial\mathbb{D}) = \mathbb{R} \cup \{\infty\}$ and $M(\mathbb{D}) = H := \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$. With $z_2, z_3 \in \partial\mathbb{D}$ in suitable orientation one can take

$$M(z) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z - z_2}{z - z_1}.$$

Put $h := M \circ g \circ M^{-1}$. Then $h(\infty) = \infty$ and hence $h$ is of the form $h(z) = \alpha z + \beta$. Since $h(\mathbb{R}) = \mathbb{R}$ we have $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$ and $h(H) \subset H$ yields $\alpha > 0$.

If $\alpha > 1$, then $h^n|_H \to \infty$. If $\alpha = 1$, then $\beta \neq 0$ and we also have $h^n|_H \to \infty$. If $\alpha < 1$, then $h^n|_H \to \beta/(1-\alpha)$. In all three cases, all limit functions of $\{h^n|_H\}$, and hence those of $\{g^n|_{\mathbb{D}}\}$ and $\{f^n|_U\}$, are constant. This is a contradiction.

Hence $f$ has a fixed point $\xi \in U$. Then we may choose $\tau$ such that $\tau(0) = \xi$. This implies that $g(0) = 0$ and hence that $g(z) = cz$. Since $|c| = 1$ we have $c = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{Q}$, say $\alpha = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then $g^q = \mathrm{id}|_{\mathbb{D}}$ and hence

$f^q = \mathrm{id}|_{\mathbb{C}}$, contradicting our assumption that $f$ is not a polynomial of degree 1. Hence $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Altogether we see that $U$ is a Siegel disk.

*Case* 2. All functions in $L$ are constant. If $L = \{\infty\}$, then $U$ is a Baker domain. (Here and in the following we identify the constant $c \in \widehat{\mathbb{C}}$ with the constant function $c\colon U \to \widehat{\mathbb{C}}$, $z \mapsto c$.) We thus assume that $L \cap \mathbb{C} \neq \emptyset$. Clearly, $L \cap \mathbb{C} \subset \overline{U}$.

Let $z_0 \in U$ and $\gamma$ a compact, connected subset of $U$ containing $z_0$ and $f(z_0)$, e.g. the trace of a curve connecting $z_0$ and $f(z_0)$ in $U$. Then

$$L = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{k\geq n} f^k(\gamma)}.$$

This implies that $L$ is a non-empty, connected subset of $\overline{U}$.

Let $a \in L \cap \mathbb{C}$. Then there exists a sequence $(n_k)$ in $\mathbb{N}$ with $n_k \to \infty$ and $f^{n_k}|_U \to a$, in particular $f^{n_k}(z_0) \to a$ and $f^{n_k}(f(z_0)) = f(f^{n_k}(z_0)) \to a$. Since $f$ is continuous in $a$ we obtain $f(a) = a$. Thus $L \cap \mathbb{C}$ consists of fixed points of $f$ and hence is a discrete subset of $\mathbb{C}$. Since $L$ is also connected, we have $L = \{\xi\}$ for a fixed point $\xi$ of $f$. Hence $f^n|_U \to \xi \in \overline{U}$.

Suppose first that $\xi \in U$. Since $(f^n)'|_U \to 0$ we have $|(f^n)'(\xi)| < 1$ for large $n$. By the chain rule, we have $(f^n)'(\xi) = \prod_{k=0}^{n-1} f'((f^k)(\xi)) = f'(\xi)^n$. Thus $|f'(\xi)^n| = |(f^n)'(\xi)| < 1$ which implies that $|f'(\xi)| < 1$. Hence $U$ is an attracting basin.

Suppose now that $\xi \in \partial U$. We shall show that $U$ is a parabolic domain. In order to so, it remains to show that $\xi$ has multiplier 1. We may assume without loss of generality that $\xi = 0$. Let $\lambda = f'(0)$ be the multiplier. So we have to show that $\lambda = 1$.

It is easy to see that $|\lambda| \geq 1$, since otherwise 0 would be an attracting fixed point and thus in $F(f)$, contrary to $0 = \xi \in \partial U \subset J(f)$. On the other hand, it is not difficult to see that $f^n|_U \to 0 \in \partial U$ implies that $|\lambda| \leq 1$. Thus $|\lambda| = 1$.

Thus there exists $\delta > 0$ such that $f|_{D(0,\delta)}$ is injective. Choose $v \in U$ and a domain $V$ with $\overline{V} \subset U$ and $\{v, f(v)\} \subset V$. Then there exist $n_0 \in \mathbb{N}$ with $f^n(V) \subset D(0,\delta)$ for $n \geq n_0$. Put

$$W = \bigcup_{n=n_0}^{\infty} f^n(V).$$

Then $W$ is a domain satisfying $f(W) \subset W$ and $f^n|_W \to 0$. Moreover, we have $W \subset D(0,\delta)$ so that $f|_W$ is injective. We fix $w \in W$ and consider the functions $\phi_n\colon W \to \mathbb{C}$, $\phi_n(z) = f^n(z)/f^n(w)$. Then the $\phi_n$ are injective and we have $\phi_n(z) \neq 0$ for all $z \in W$ and $n \in \mathbb{N}$. By Lemma 4.4 the $\phi_n$ form a normal family. Thus $(\phi_n)$ has a convergent subsequence, say $\phi_{n_k} \to \phi$. Since $\phi_n(w) = 1$ for all $n$ we have $\phi(w) = 1$. By Hurwitz's Theorem $\phi$ is constant or injective.

For $z \in W$ we have

$$\lambda = f'(0) = \lim_{\zeta\to 0} \frac{f(\zeta)}{\zeta} = \lim_{n\to\infty} \frac{f(f^n(z))}{f^n(z)} = \lim_{n\to\infty} \frac{f^{n+1}(z)}{f^n(z)}$$

and hence

$$\phi(f(z)) = \lim_{k\to\infty} \frac{f^{n_k}(f(z))}{f^{n_k}(w)} = \lim_{k\to\infty} \frac{f^{n_k+1}(z)}{f^{n_k}(w)} = \lim_{k\to\infty} \frac{f^{n_k+1}(z)}{f^{n_k}(z)} \frac{f^{n_k}(z)}{f^{n_k}(w)} = \lambda\phi(z).$$

If $\phi$ is not constant, then $\phi\colon W \to \phi(W)$ is bijective. For $m \in \mathbb{N}$ and $z \in W$ we have $\phi(f^m(z)) = \lambda^m \phi(z)$, in particular $\phi(f^m(w)) = \lambda^m \phi(w) = \lambda^m$ and thus $\lambda^m \in \phi(W)$. Since $|\lambda| = 1$ there exists a sequence $(m_k)$ with $m_k \to \infty$ and $\lambda^{m_k} \to 1$. We deduce that

$$f^{m_k}(w) = \phi^{-1}(\lambda^{m_k}) \to \phi^{-1}(1) = w.$$

On the other hand, we have $f^{m_k}(w) \to 0$ since $w \in U$. This is a contradiction. Hence $\phi$ is constant. Since $\phi(w) = 1$ we thus have $\phi(z) \equiv 1$. The equation $\phi(f(z)) = \lambda\phi(z)$ now yields immediately that $\lambda = 1$. □

# 5 Connectivity of Fatou components

**Theorem 5.1.** *If $f$ is entire and $U$ is a multiply connected component of $F(f)$, then $U$ is wandering.*

In order to prove this theorem, we begin with the following result.

**Theorem 5.2.** *Let $f$ be entire and let $U$ be a multiply connected component of $F(f)$. Let $\gamma$ be a Jordan curve in $U$ which is not null-homotopic. Then $\gamma_k = f^k(\gamma)$ is not null-homotopic in the Fatou component $U_k$ which contains $f^k(U)$ and it satisfies $n(\gamma_k, 0) \neq 0$ for large $k$ and $\operatorname{dist}(\gamma_k, 0) \to \infty$ as $k \to \infty$.*

*In particular, $U_k$ is also multiply-connected for all $k \in \mathbb{N}$.*

*Proof.* By assumption, there is a closed curve $\gamma$ in $U$ such that $n(\gamma, a) \neq 0$ for some point $a \in \mathbb{C}\backslash U$. We may assume that $a \in \partial U \subset J(f)$. We may also assume without loss of generality that $n(\gamma, a) \geq 1$. By Theorem 3.7 the point $a$ is limit of repelling periodic points. Thus there exists a repelling periodic point $b$ such that $n(\gamma, b) \geq 1$. Now, by the argument principle,

$$n(\gamma_k, f^k(b)) = \frac{1}{2\pi i}\int_{f^k\circ\gamma} \frac{dw}{w - f^k(b)} = \frac{1}{2\pi i}\int_{\gamma} \frac{(f^k)'(z)}{f^k(z) - f^k(b)} dz$$

equals the number of $f^k(b)$-points of $f^k$ in $\operatorname{int}(\gamma)$, and thus this number is at least 1. Hence $n(\gamma_k, f^k(b)) \geq 1$ for all $k \in \mathbb{N}$.

Let $p$ be the period of $b$ and let $R > \max\{|b|, |f(b)|, \ldots, |f^{p-1}(b)|\}$. By Theorem 4.3, we have $\operatorname{dist}(f^k(\gamma), 0) \to \infty$ as $k \to \infty$ and thus $\operatorname{dist}(f^k(\gamma), 0) > R$ for large $k$. Since also $|f^k(b)| < R$ and $n(\gamma_k, f^k(b)) \geq 1$ for all $k \in \mathbb{N}$ we conclude that $n(\gamma_k, 0) \neq 0$ for large $k$. □

The proof of Theorem 5.1 also requires the hyperbolic metric, which we briefly introduce. Define

$$\rho_{\mathbb{D}}\colon \mathbb{D} \to (0, \infty), \quad \rho_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2}.$$

Let $U$ be a domain in $\mathbb{C}$ such that $\mathbb{C}\backslash U$ contains at least two points. A domain with this property is called hyperbolic.

The uniformization theorem says that for a hyperbolic domain $U$ there exists a covering map $h\colon \mathbb{D} \to U$. If $U$ is simply connected, then $h$ is just the map from the Riemann Mapping Theorem.

One can show that for a covering map $h$ as above the map $\rho_U\colon U \to (0,\infty)$ defined by

$$\rho_U(h(z))|h'(z)| = \rho_{\mathbb{D}}(z)$$

is well-defined; that is, if $w \in U$ and if $h_1, h_2\colon \mathbb{D} \to U$ are covering maps and if $z_1, z_2 \in \mathbb{D}$ satisfy $h_1(z_1) = h_2(z_2) = w$, then $\rho_{\mathbb{D}}(z_1)/h_1'(z_1) = \rho_{\mathbb{D}}(z_2)/h_2'(z_2)$, and this common value is defined to be $\rho_U(w)$. The map $\rho_U$ is called the *density of the hyperbolic metric*

For a (rectifiable) curve $\gamma$ in $U$ we call

$$\ell_U(\gamma) = \int_\gamma \rho_U(z)|dz|$$

the *hyperbolic length* of a curve $\gamma$ in $U$. Finally, for $a, b \in U$ the *hyperbolic distance* of $a$ and $b$ is defined by

$$\lambda_U(a,b) = \inf_\gamma \ell_U(\gamma),$$

where the infinimum is taken over all curves connecting $a$ with $b$.

It is not difficult to see that $\lambda_U$ is indeed a metric on $U$. It is called the *hyperbolic metric* or *Poincaré metric.*

Schwarz's Lemma says that if $F\colon \mathbb{D} \to \mathbb{D}$ is holomorphic with $F(0) = 0$, then $|F'(0)| \leq 1$, with equality only in the case where $F$ has the form $F(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$. Now if $U$ and $V$ are hyperbolic domains and $f\colon U \to V$ is holomorphic, then there exists coverings $\phi\colon \mathbb{D} \to U$ and $\psi\colon \mathbb{D} \to V$, which one may choose such that $\phi(0) = z$ and $\psi(0) = f(z)$ for a given point $z \in U$. It is a basic result about covering maps that $f$ may now be *lifted* to a holomorphic map $F\colon \mathbb{D} \to \mathbb{D}$; that is, there exists a holomorphic map $F\colon \mathbb{D} \to \mathbb{D}$ satisfying $F(0) = 0$ such that $f \circ \phi = \psi \circ F$; see Figure 1. Applying Schwarz's Lemma to $F$ then yields the following result.

**Lemma 5.3.** *If $U$ and $V$ are hyperbolic domains and $f\colon U \to V$ is holomorphic, then*

(*i*) $\rho_V(f(z))|f'(z)| \leq \rho_U(z)$ *for* $z \in U$,

(*ii*) $\ell_V(f(\gamma)) \leq \ell_U(\gamma)$ *for a curve* $\gamma$ *in* $U$,

(*iii*) $\lambda_V(f(a), f(b)) \leq \lambda_U(a,b)$ *for* $a, b \in U$.

*Equality can occur (for $a \neq b$) only if $f$ is a covering.*

The choice $f(z) = z$ leads to the following result.

**Lemma 5.4.** *If $U$ and $V$ are hyperbolic domains, $U \subset V$, then*

(*i*) $\rho_V(z) \leq \rho_U(z)$ *for* $z \in U$,

(*ii*) $\ell_V(\gamma) \leq \ell_U(\gamma)$ *for a curve* $\gamma$ *in* $U$,

(*iii*) $\lambda_V(a,b) \leq \lambda_U(a,b)$ *for* $a, b \in U$.

**Lemma 5.5.** *Let $U$ be a hyperbolic domain.*

(*i*) *If $U$ is simply connected, then*

$$\frac{1}{2\operatorname{dist}(z,\partial U)} \leq \rho_U(z) \leq 2\operatorname{dist}(z,\partial U).$$

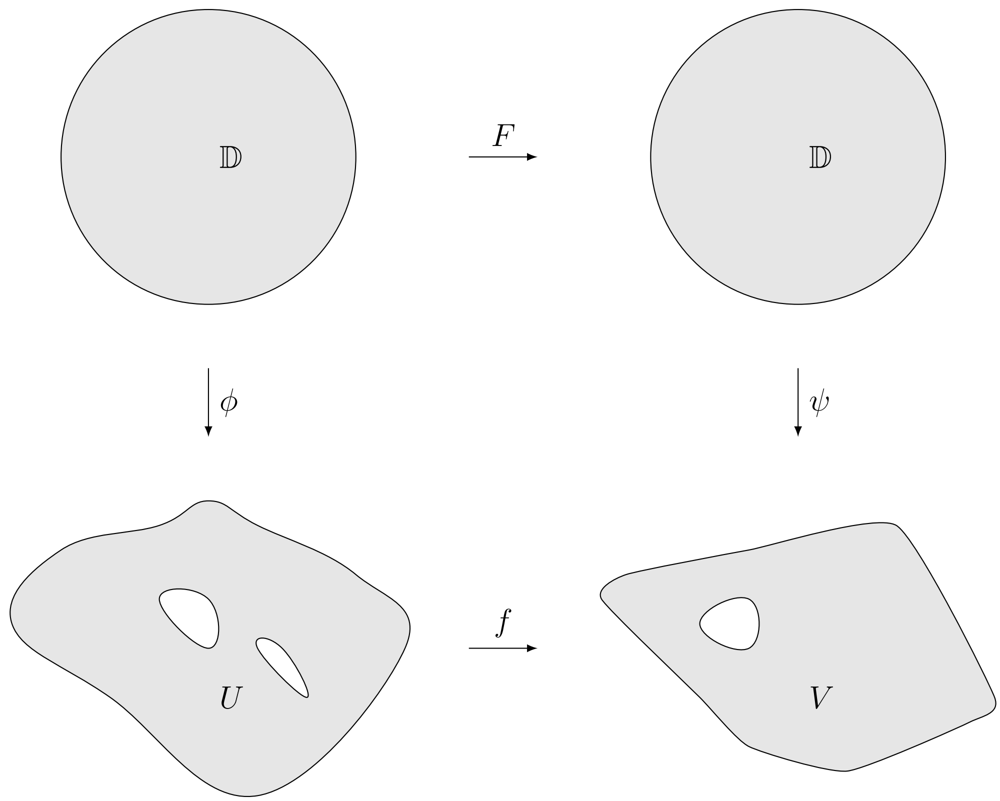


Figure 1: The lift $F\colon \mathbb{D} \to \mathbb{D}$ of a map $f\colon U \to V$

(*ii*) *There exist* $a, b, c > 0$ *such that if* $|z| > c$, *then*

$$\frac{a}{|z| \log |z|} \leq \rho_{\mathbb{C}\setminus\{0,1\}}(z) \leq \frac{b}{|z| \log |z|}.$$

*Sketch of proof.* By Lemma 5.4 we have $\rho_U(z) \leq \rho_{D(z,\mathrm{dist}(z,\partial U))}(z)$. Since $\zeta \to z + r\zeta$ is a covering (in fact a biholomorphic map) from $\mathbb{D}$ to $D(z,r)$ one easily finds that $\rho_{D(z,r)}(z) = 2r$. This proves the right inequality of (*i*).

The left inequality of (*i*) uses the Koebe one quarter theorem which says that if $h\colon \mathbb{D} \to \mathbb{C}$ is holomorphic and injective, then $h(\mathbb{D}) \supset D(h(0), |h'(0)|/4)$; see Theorem 8.4 below. Now Lemma 5.4 is applied to this inclusion.

The upper bound in (*ii*) can be obtained by using Lemma 5.4 with $U = \mathbb{C}\setminus\overline{\mathbb{D}}$, noting that $z \mapsto \exp((z-1)/(z+1))$ is a covering from $\mathbb{D}$ to $\mathbb{C}\setminus\overline{\mathbb{D}}$. We omit the proof of the lower bound in (*ii*). $\square$

In the following, let

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

be the winding number of a closed curve $\gamma$ around a point $a$.

**Theorem 5.6.** *Let* $f, U, \gamma$ *and* $\gamma_k$ *be as in Theorem 5.2. Then there exists* $\alpha > 0$ *and a sequene* $(r_k)$ *tending to* $\infty$ *such that* $\gamma_k \subset \{z \in \mathbb{C}\colon r_k \leq |z| \leq r_k^\alpha\}$.

*Proof.* Without loss of generality we may assume that $0, 1 \in J(f)$. Let

$$r_k = \min\{|z| \colon z \in \gamma_k\} \quad \text{and} \quad R_k = \max\{|z| \colon z \in \gamma_k\}$$

and let $U_k$ be the Fatou component which contains $f^k(U)$ and hence in particular $\gamma_k$. Choose $a_k, b_k \in \gamma_k$ with $|a_k| = r_k$ and $|b_k| = R_k$. By Lemma 5.3 we have

$$\lambda_{U_k}(a_k, b_k) \le \ell_{U_k}(\gamma_k) \le \ell_U(\gamma)$$

while by Lemmas 5.4 and 5.5 we have

$$\lambda_{U_k}(a_k, b_k) \ge \lambda_{\mathbb{C}\setminus\{0,1\}}(a_k, b_k) \ge a \int_{|a_k|}^{|b_k|} \frac{dt}{t \log t} = a \log\left(\frac{\log R_k}{\log r_k}\right).$$

With $\alpha = \exp(\ell_U(\gamma)/a)$ we obtain

$$\frac{\log R_k}{\log r_k} \le \alpha$$

from which the conclusion follows. $\square$

*Proof of Theorem 5.1.* Suppose that $U$ is invariant; that is, $f(U) \subset U$. Let $\gamma$ and $\gamma_k$ be as in Theorems 5.2 and 5.6. Replacing $\gamma$ by a longer curve if necessary we may achieve that $\gamma \cap \gamma_1 \ne \emptyset$. This implies that $\gamma_k \cap \gamma_{k+1} \ne \emptyset$ for $k \in \mathbb{N}$. With

$$r_k = \min\{|z| \colon z \in \gamma_k\} \quad \text{and} \quad R_k = \max\{|z| \colon z \in \gamma_k\}$$

as before we obtain $r_{k+1} \le R_k$. Since $R_k \le r_k^\alpha$ by Theorem 5.6 we obtain $\square$

*Example.* Let $a_1 = 1$, $a_2 = 2$ and

$$a_{n+1} = \frac{a_n^n}{\prod_{k=1}^{n-1} a_k}.$$

Then the Fatou set of

$$f(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right)$$

has a multiply connected component. (This component is a wandering domain by Theorem 5.1.)

**Theorem 5.7.** *Let $f$ be entire and let $U$ be a Baker domain of $f$. Then there exists a curve $\gamma$ tending to $\infty$ in $U$ and $c > 0$ such that $|f(z)| \le c|z|$ for $z \in \gamma$.*

The theorems in this section are due to Baker [4, 8]. Baker [2] has also shown that multiply connected components of $F(f)$ actually exist for entire $f$ by examples similar to the one above. These were the first examples [5] of wandering domains. The lemmas on the hyperbolic metric can be found in many textbooks on complex analysis and, e.g., [10].

## 6 Examples of Baker and wandering domains

*Example* 6.1. $f_1(z) = z + e^z - 1$ has a Baker domain containing $\{z\colon \operatorname{Re} z < 0\}$.

*Example* 6.2. $f_2(z) = z - e^z + 1$ has the attracting fixed points $2\pi ik$, $k \in \mathbb{Z}$.

*Example* 6.3. $f_3(z) = z - e^z + 1 + 2\pi i$ has a simply connected wandering domain $U_0$ such that $f^k(U_0)$ contains $2\pi ik$.

The functions $f_j$ satisfy $\exp f_j(z) = g_j(\exp z)$ with $g_1(z) = z\exp(z-1)$ and $g_2(z) = g_3(z) = z\exp(1-z)$.

The Fatou-Julia theory can also be developed for holomorpic self-maps of $\mathbb{C}\backslash\{0\}$, see, e.g., [21, 22, 27, 28, 34]. In particular, the Julia set can be defined for such maps.

**Theorem 6.4.** *Let $f$ be an entire (non-linear) function such that* $\exp f_j(z) = g_j(\exp z))$ *for some holomorphic function $g\colon \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$. Then $J(f) = \exp^{-1}(J(g))$.*

*Example* 6.5. $f_4(z) = 2z - e^z + 2 - \log 2$ has a Baker domain $U$ such that $f|_U$ is univalent and $\partial U$ is a Jordan curve in $\widehat{\mathbb{C}}$.

**Theorem 6.6.** *There exists an entire function with an infinitely connected wandering domain.*

**Theorem 6.7.** *There exists an entire function with a doubly connected wandering domain. More generally, for any $m \in \mathbb{N}$ there exists an entire function with a wandering domain of connectivity $m$.*

Example 6.1 is, up to a change of variables, due to Fatou [23]. Example 6.3 is due to Herman, see [37] and [6]. Theorem 6.4 is from [12] and Example 6.5 is from [13]. Theorem 6.6 is due to Baker [7] and Theorem 6.7 is a result of Kisaka and Shishikura [29].

## 7 The singularities of the inverse function

**Definition 7.1.** Let $f$ be entire and $a \in \mathbb{C}$. Then $a$ is called a *critical value* of $f$ if there exists $z \in \mathbb{C}$ with $f(z) = a$ and $f'(z) = 0$ and $a$ is called an *asymptotic value* if there exists a curve $\gamma\colon [0,\infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to a$ as $t \to \infty$.

The set of critical and asymptotic values is also called the set of singularities of the inverse of $f$ and denoted by $\operatorname{sing}(f^{-1})$ for the following reason.

**Proposition 7.2.** *Let $\varphi$ be a branch of the inverse of $f$ defined in some domain $U$ and let $\gamma$ be a curve in $\mathbb{C}\backslash\operatorname{sing}(f^{-1})$ starting in $U$. Then $\varphi$ can be continued analytically along $\gamma$.*

The following result is know a the monodromy theorem.

**Theorem 7.3. (Monodromy Theorem)** *Let $U$ be a simply connected domain and let $V$ be a subdomain of $U$. Let $g\colon V \to \mathbb{C}$ be holomorphic and suppose that $g$ can be continued analytically along any path in $U$. Then there exists a holomorphic function $G\colon V \to \mathbb{C}$ such that $G|_V = g$.*

The monodromy theorem yields the following result.

**Proposition 7.4.** *Let $f$ be entire and let $U \subset \mathbb{C} \setminus \operatorname{sing}(f^{-1})$ be a simply connected domain and let $z_0 \in \mathbb{C}$ with $f(z_0) \in U$. Then there exists a branch $\varphi\colon U \to \mathbb{C}$ of the inverse of $f$ satisfying $\varphi(f(z_0)) = z_0$.*

**Theorem 7.5.** *Let $f$ be a meromorphic function and let $C = \{U_0, U_1, \dots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$; that is, $f(U_j) \subset U_{j+1}$, with $U_p = U_0$.*

- *If $C$ is a cycle of attracting or parabolic basins, then $\bigcup_{j=0}^{p-1} U_j \cap \operatorname{sing}(f^{-1}) \neq \emptyset$.*
- *If $C$ is a cycle of Siegel discs, then $\partial U_j \subset \overline{O^+(\operatorname{sing}(f^{-1}))}$ for $j \in \{0, 1, \dots, p-1\}$.*

*Remark.* For the Baker domain $U$ of the function $f$ from Example 6.5 we have

$$\operatorname{dist}\left(U, \overline{O^+(\operatorname{sing}(f^{-1}))}\right) > 0.$$

So there is no analog of Theorem 7.5 for Baker domains.

Theorem 7.5 is classical. The analogous result for rational functions can be found in standard textbooks on complex dynamics [9, 30, 36].

# 8 The Eremenko-Lyubich class

**Definition 8.1.** The set $B$ of all entire functions $f$ for which $\operatorname{sing}(f^{-1})$ is bounded is called the *Eremenko-Lyubich class.*

**Theorem 8.2.** *Let $f \in B$ and let $U$ be a component of $F(f)$. Then $f^n|_U \not\to \infty$.*

The main tool in the proof is the *logarithmic change of variable* described in the following theorem, cf. Figure 2.

**Theorem 8.3.** *Let $f \in B$ be transcendental and $R > |f(0)|$ such that $\operatorname{sing}(f^{-1}) \subset D(0, R)$. Put $H = \{z\colon \operatorname{Re} z > \log R\}$, let $W$ be a component of $\{z\colon |f(z)| > R\}$ and let $V$ be a component of $\exp^{-1}(W)$. Then $V$ and $W$ are simply connected and there exist a univalent map $F\colon V \to H$ such that $\exp F(z) = f(\exp z)$ for $z \in V$.*

*Proof.* We put $\Delta = \{z\colon |z| > R\}$. Let $u_0 \in H$ and let $w_0 \in W$ such that $f(w_0) = \exp u_0 \in \Delta$. Since $w_0$ is not a critical value there exists a branch $\psi$ of $f^{-1} \circ \exp$ defined in some neighborhood $U$ of $u_0$; that is, we have $f(\psi(u)) = e^u$ for all $u \in U$. Since $\Delta \cap \operatorname{sing}(f^{-1}) = \emptyset$ we can continue $\psi$ analytically along any path in $H$. The Monodromy Theorem now implies that $\psi$ has a holomorphic continuation to $H$, which we again denote by $\psi$.

Suppose that $\psi$ is not injective, say $\psi(c_1) = \psi(c_2)$ where $c_1, c_2 \in H$ with $c_1 \neq c_2$. Then $\exp(c_1) = f(\psi(c_1)) = f(\psi(c_2)) = \exp c_2$ so that $c_1 = c_2 + 2\pi i m$ for some $m \in \mathbb{Z}$. For the branches $\log_1 f$ and $\log_2 f$ of the logarithm of $f$ which are defined in some neighborhood of $d := \psi(c_1) = \psi(c_2)$ and which satisfy $\log_j f(d) = c_j$ we thus have $\log f_1 = \log f_2 + 2\pi i m$. Their inverse functions $\psi_1$ and $\psi_2$ thus satisfy $\psi_1(z) = \psi_2(z + 2\pi i m)$, first in a neighborhood of $c_1$, but by the identity theorem also in $H$.

Let $R' > R$ and $H' = \{z\colon \operatorname{Re} z > \log R'\}$. Then $\psi(\partial H'))$ is bounded. Since $\psi$ is holomorphic and thus open, we have $\partial\psi(H') \subset \psi(\partial H')$. Thus $\partial\psi(H') \subset D(0, r')$ for

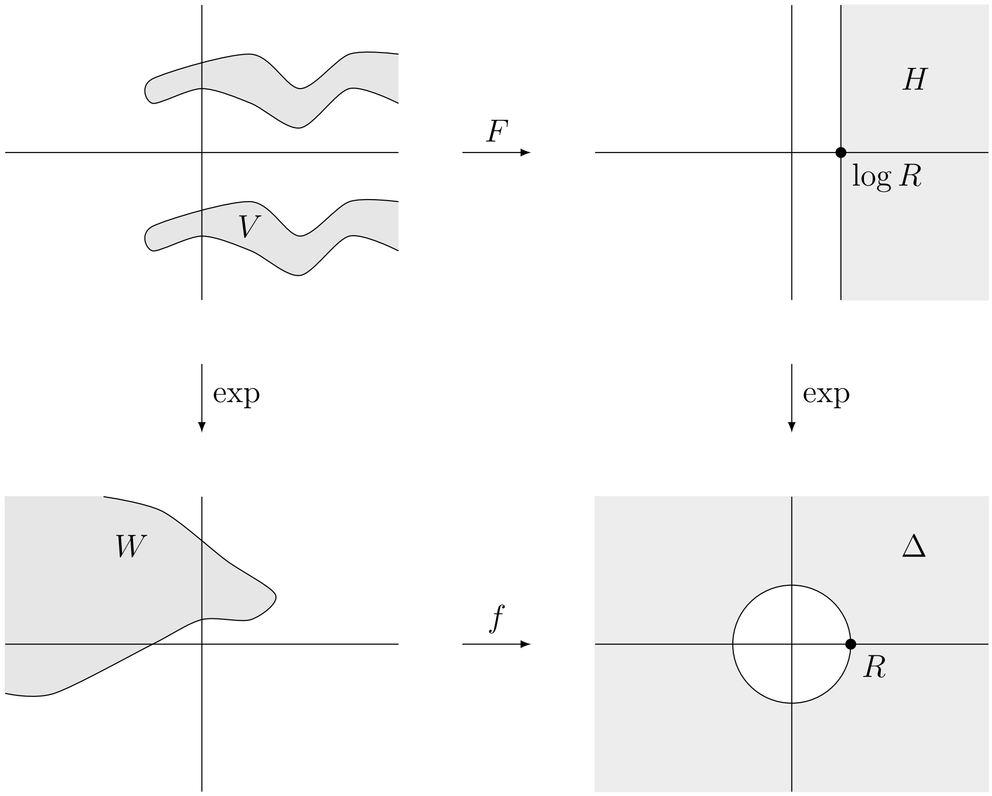


Figure 2: The logarithmic change of variable.

some $r' > 0$. Since $\psi(H')$ is unbounded, this implies that $\psi(H') \supset \{z \colon |z| > r'\}$. Thus $|f(z)| > R'$ for $|z| > r'$. This implies that $f$ is a polynomial, contradicting our hypotheses. Thus $\psi$ is univalent. This implies that $W$ is simply connected.

Since $|f(0)| < R$ we have $0 \notin W$. This implies that there exists a branch of the logarithm which maps $W$ onto $V$. Then $\log \circ \psi$ maps $H$ univalently onto $V$. Denoting by $F$ the inverse function of this map the conclusion follows. $\square$

*Remark.* We have define the map $F$ in one component $V$ of $\exp(f^{-1}(\Delta))$ and shown that it is univalent there. Thus we can define a map $F \colon \exp(f^{-1}(\Delta)) \to H$ such that $\exp F(z) = f(\exp z)$ and $F$ is univalent in any component $V$ of $\exp(f^{-1}(\Delta))$

The proof of Theorem 8.2 requires the following result known as the Koebe's One Quarter Theorem.

**Theorem 8.4. (Koebe One Quarter Theorem)** *Let $f \colon D(a, r) \to \mathbb{C}$ be holomorphic and injective. Then $f(D(a, r)) \supset D(f(a), \frac{1}{4}|f'(a)|r)$.*

Usually only the special case $a = 0$, $r = 1$ and $f'(a) = 1$ is stated, but the above version follows easily from this. The constant $\frac{1}{4}$ is best possible here. However, for our applications it would be enough to know that the conclusion holds with $\frac{1}{4}$ replaced by some positive constant $K$. This weaker result can easily be deduced from Lemma 4.4.

**Theorem 8.5.** *Let $f$, $R$ and $F \colon \exp(f^{-1}(\Delta)) \to H$ be as in Theorem 8.3 and the remark following it. Then*

$$|F'(v)| \geq \frac{1}{4\pi}(\operatorname{Re} F(v) - \log R)$$

*for* $v \in \exp(f^{-1}(\Delta))$.

*Proof.* Let $V$ be a component of $\exp(f^{-1}(\Delta))$, $v \in V$ and $u = F(v) \in H$. Let $G$ be the branch of $F^{-1}$ that maps $H$ to $V$. Then $G$ is univalent in $D(u, \operatorname{Re} u - \log R)$. Since $V$ contains no disc of radius greater than $\pi$, Koebe's Theorem implies that

$$\pi \geq \frac{1}{4}|G'(u)|(\operatorname{Re} u - \log R) = \frac{1}{4|F'(v)|}(\operatorname{Re} F(v) - \log R). \qquad \square$$

*Proof of Theorem 8.2.* Suppose that $z_0 \in U$ and $f^n(z_0) \to \infty$. Then there exists $\varepsilon > 0$ such that $f^n(z) \to \infty$ uniformly for $z \in D(z_0, \varepsilon)$. Let $R$, $H$, $F$ and $V$ be as in Theorem 8.3 and let $v_0 \in V$ with $\exp v_0 = z_0$. Then there exists $\delta > 0$ such that $\operatorname{Re} F^n(v) \to \infty$ uniformly for $v \in D(v_0, \delta)$. We may assume that $F^n(D(v_0, \delta)) \subset H$ for all $n \in \mathbb{N}$. Thus $F^n(D(v_0, \delta)) \subset V_n$ for some component $V_n$ of $\exp(f^{-1}(\Delta))$. This implies that $F^n(D(v_0, \delta))$ contains no disk of radius greater than $\pi$. Since $F^n$ is univalent in $D(v_0, \delta)$ we deduce from Koebe's Theorem that $|(F^n)'(v_0)| \leq 4\pi/\delta$. On the other hand, by Theorem 8.5,

$$|(F^n)'(v_0)| = \prod_{j=1}^{n} |F'(F^j(v_0))| \geq \prod_{j=1}^{n} \frac{\operatorname{Re} F^k(v_0) - \log R}{4\pi} \to \infty$$

as $n \to \infty$ since $\operatorname{Re} F^k(v_0) \to \infty$ as $k \to \infty$. This is a contradiction. $\square$

The results in this section are due to Eremenko and Lyubich [20]. The class $B$ and the logarithmic change of variable were introduced by them to the subject in the same paper. Eremenko and Lyubich [20], as well as Goldberg and Keen [25], also proved that if $\operatorname{sing}(f^{-1})$ is finite, then $f$ has no wandering domains. This is an analog of Sullivan's result [37] that rational functions have no wandering domains for entire functions. Bishop [17] has recently constructed an example of an entire function in $B$ which has wandering domains.

# 9 The escaping set

**Definition 9.1.** For entire $f$ the set

$$I(f) = \{z \colon f^n(z) \to \infty \text{ as } \infty\}$$

is called the *escaping set*.

Theorem 8.2 thus says that if $f \in B$, then $I(f) \subset J(f)$.

**Theorem 9.2.** *$I(f) \neq \emptyset$ and $J(f) = \partial I(f)$.*

**Theorem 9.3.** *$I(f)$ has at least one unbounded component.*

*Remark.* This is a partial answer to a question of Eremenko who had asked whether *all* components of $I(f)$ are unbounded. (Another result towards this conjecture is given in [32] where it is shown that all components of $I(f)$ are unbounded if $f \in B$ and if $f$ has finite order.)

Let

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|$$

be the maximum modulus and let $\rho > 0$ be such that $M(r) > r$ for $r \geq \rho$. Then $M^n(r) \to \infty$ as $n \to \infty$ for $r \geq \rho$. Define

$$A(f) = \{z\colon \text{ there exists } L \in \mathbb{N} \text{ such that } |f^n(z)| \geq M^{n-L}(\rho) \text{ for } n \geq L\}.$$

Then $A(f)$ does not depend on $\rho$. Also, one may replace $M^{n-L}(\rho)$ by $M(\rho, f^{n-L})$.

**Theorem 9.4.** *$A(f) \neq \emptyset$ and $J(f) = \partial A(f)$.*

The proof uses the following result of Bohr.

**Theorem 9.5.** *There exists $c > 0$ such that if $f$ is entire and $r$ is sufficiently large, then there exists $R \geq cM(r/2)$ such that $\partial D(0, R) \subset f(D(0, r))$.*

Theorem 9.2 is due to Eremenko [19] who made the first systematic study of the escaping set. The proof that $I(f) \neq 0$ given in the lecture is due to Domínguez [18]. Theorem 9.3 is due to Rippon and Stallard [33]. They actually prove that every component of $A(f)$ is unbounded. The set $A(f)$ was introduced in [15] and Theorem 9.4 can be found there, as well as in [33]. Theorem 9.5 is a classical result of Bohr; see, e.g., [26].

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