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ITERATION OF FUNCTIONS WHICH ARE MEROMORPHIC OUTSIDE A SMALL SET

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Abstract. In this paper, we investigate the dynamics of a broader class of functions which are meromorphic outside a compact totally disconnected set. We shall establish the connections between the Fatou components and the singularities of the inverse function and, accordingly, give sufficient conditions for the non-existence of wandering domains or Baker domains, and for the Julia set to be the Riemann sphere. Through the discussion of permutability of such functions, we shall construct several transcendental meromorphic functions which have Baker domains and wandering domains with special properties; for example, wandering and Baker domains with a critical value on the boundary and a wandering domain with the boundary being a Jordan curve (some such examples for entire functions were exhibited in other papers) and those of non-finite type which have no wandering domains.

1. Introduction. Iteration of meromorphic functions of $\boldsymbol{C} \mapsto \hat{\boldsymbol{C}}$ has developed over the last decade and continues to attract much interest. The dynamics of meromorphic functions have been revealed to have many similar results to those of rational or entire functions, such as a meromorphic function of finite type, that is, its inverse has singularities over a finite set of points, has no wandering domains (see [5]) and its Julia set is uniformly perfect (see [33] and [34]). The result that a rational function with degree at least two has no wandering domains was proved by Sullivan [29] and was extended in [18] and [15] to transcendental entire functions of finite type. However, the dynamics of transcendental meromorphic functions also possess special properties which a rational or entire function does not have; for example a transcendental meromorphic function may have the Julia set in a straight line, while the Julia set of a transcendental entire function cannot contain an isolated Jordan arc. It is obvious that the family of rational functions or entire functions is a closed system under iteration or functional composition; however, the family of meromorphic functions is not, since iteration of a transcendental meromorphic function is not in general meromorphic and may have infinitely many essential singularities. From this, it is natural to consider a broader class of functions which is closed under functional composition. We introduce the class $\mathbf{M}$ of function f, which is meromorphic outside some compact totally disconnected set $E = E(f)$, and the cluster set of f at any $a \in E(f)$ with respect to $E^c = \hat{\boldsymbol{C}} \setminus E$, that is, the set

$$C(f, E^c, a) = \{w \in \hat{\boldsymbol{C}};\ w = \lim_{n\to\infty} f(z_n) \text{ for some sequence } z_n \in E^c \text{ with } z_n \to a\}$$

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is equal to $\hat{C}$. Note that a point in E may not be isolated in E, so the final condition which a member in **M** satisfies shows that $f(z)$ at every point in E keeps the property which an isolated essential singular point possesses in the sense of the Weierstrass theorem. The class **M** was first investigated in [4] and the basic concepts, such as the Fatou set and the Julia set, and the basic properties of dynamics of functions in **M** were established there. It was proved that **M** is closed under functional composition and, for any $f, g \in \mathbf{M}$, $E(f \circ g) = E(g) \cup g^{-1}(E(f))$. The composition of a finite number of meromorphic functions is a member in **M** and has only at most countably many essential singularities. Following the discussion of [4], we investigate the dynamics of functions in **M** in this paper.

Let $f \in \mathbf{M}$ and $f^n, n \in \boldsymbol{N}$, denote the nth iterate of f. Then $f^n(z)$ is defined in $E^c(f^{n-1})$. Define the Fatou set of f by

$$F(f) = \{z \in \hat{C};\, \{f^n\} \text{ is defined and normal in some neighborhood of } z\}$$

and the Julia set of f by $J(f) = \hat{C} \setminus F(f)$.

Set $J_\infty(f) = \bigcup_{n=0}^{\infty} E(f^n)$. If $J_\infty(f)$ has at least three points, then $J(f) = \overline{J_\infty(f)}$, so $F(f)$ is the large open set in which all f^n, $n \in \boldsymbol{N}$, are meromorphic; if $J_\infty(f)$ consists of two points, then f is a holomorphic function of $\boldsymbol{C}^* = \boldsymbol{C} \setminus \{0\}$ onto itself up to a Möbius transformation; if $J_\infty(f)$ consists of one point, then f is a transcendental entire function; if $J_\infty(f)$ is empty, then f is a rational function.

It is easy to see that, for $f \in \mathbf{M}$, $F(f)$ is open and completely invariant under f, i.e., $z \in F(f)$ if and only if $f(z) \in F(f)$. Let U be a connected component of $F(f)$, called a stable domain of f, then $f^n(U)$ is contained in a component of $F(f)$, denoted by U_n. If, for some $n \geq 1$, $U_n = U$, that is, $f^n(U) \subseteq U$, then U is called periodic; if, for some pair of $n \neq m$, $U_n = U_m$, but U is not periodic, then U is called preperiodic; if for $n \neq m$, $U_n \neq U_m$, then U is called a wandering domain of f.

For a periodic component of the Fatou set we have the classification theorem. Let Ω be a periodic component of $F(f)$ of period p. Then only five possible cases occur (see [4]):

(1) Ω is a (super) attracting domain of a (super) attracting periodic point a of f of period p such that $f^{np}|_\Omega \to a$ as $n \to \infty$ and $a \in \Omega$;

(2) Ω is a parabolic domain of a rational neutral periodic point b of f of period p such that $f^{np}|_U \to b$ as $n \to \infty$ and $b \in \partial\Omega$;

(3) Ω is a Siegel disk of period p such that there exists an analytic homeomorphism $\phi : \Omega \to \Delta$, where $\Delta = \{z; |z| < 1\}$, satisfying $\phi(f^p(\phi^{-1}(z)) = e^{2\pi\alpha i} z$ for some irrational number α and $\phi^{-1}(0) \in \Omega$ is an irrational neutral periodic point of f of period p;

(4) Ω is a Herman ring of period p such that there exists an analytic homeomorphism $\phi : \Omega \to A$, where $A = \{z : 1 < |z| < r\}$, satisfying $\phi(f^p(\phi^{-1}(z)) = e^{2\pi\alpha i} z$ for some irrational number α;

(5) Ω is a Baker domain of period p such that $f^{np}|_U \to c \in J(f)$ as $n \to \infty$ but f^p is not meromorphic at c. If $p = 1$, then $c \in E(f)$.

In this paper, we mainly discuss two aspects: one is the connections between the Fatou components and the singularities of the inverse function; the other is the dynamical connection between meromorphic functions f and g satisfying the equation $h \circ f = g \circ h$. As an application of our results, we shall construct several types of Baker domains and wandering domains with special properties of transcendental meromorphic functions; a few such examples for transcendental entire functions have been exhibited in other papers [15, 9, 16, 19] and we give a transcendental meromorphic function whose Julia set coincides with the Riemann sphere.

By $\mathrm{Sing}(f^{-1})$ we denote the set of singularities of the inverse function f^{-1}, that is, the set of critical and asymptotic values and limit points of these values.

We use the following notation about singularities of the inverse function. For $f \in \mathbf{M}$, set

$$S_p(f) = \{a \in \boldsymbol{C};\, a \text{ is a finite singularity of } f^{-p}\},$$

and

$$P(f) = \bigcup_{p=1}^{\infty} S_p(f).$$

We establish the following, which are the main results of Section 2 of this paper.

THEOREM 2.1. *If $f \in \mathbf{M}$ and if U is a wandering domain of f, then every limit function of the convergent subsequence of $\{f^n|_U\}$ lies in the derived set of $P(f)$.*

Theorem 2.1 was proved in [11] for entire f and in [31] and [33] for meromorphic f. The same argument as in [33] deduces the following result. For completeness, the proof of the result will be given.

THEOREM 2.2. *Let $f \in \mathbf{M}$ and let U be a component of $F(f)$. If $f^{np}|_U \to q$ $(n \to \infty)$, then either q lies in the derived set of $S_p(f)$ or q is a periodic point of f of period $k \leq p$ and $f^p(q) = q$.*

Theorem 2.1 and Theorem 2.2 were announced at the conference 'New Direction in Dynamical Systems 2002' which was held in August 2002, at Kyoto University, Japan (see [36]). As an application of Theorems 2.1 and 2.2, we shall establish some sufficient criteria for the non-existence of wandering domains or Baker domains and for the Julia set of a transcendental meromorphic function to be the Riemann sphere.

THEOREM 2.4. *Let $f(z) = \mu + z + e^z + \lambda/(e^z - 1)$. Then there exist μ and λ such that $J(f) = \hat{\boldsymbol{C}}$.*

Note that $f(z)$ in Theorem 2.4 is not of bounded type, that is, $\mathrm{Sing}(f^{-1})$ is unbounded. In Section 3, we discuss two permutable meromorphic functions in M. Two meromorphic functions $f(z)$ and $g(z)$ in M are said to be permutable if

$$f \circ g(z) = g \circ f(z) \quad \text{in} \quad \hat{\boldsymbol{C}} \setminus (E(f) \cup E(g) \cup g^{-1}(E(f)) \cup f^{-1}(E(g))).$$

Julia [20] and Fatou [17] proved that two permutable rational functions with degree at least two have the same Julia sets. However, it is still open as to whether or not two transcendental entire functions have the same Julia sets, although some important progress has been made towards a solution of this problem; see [12, 22, 25] for further detail. We establish the following result in Section 3.

THEOREM 3.2. *Let $f(z)$ and $g(z)$ both be transcendental meromorphic functions in* **M** *and permutable. If $J(f) = \overline{J_\infty(f)}$, then $J(f) = J(g)$.*

Here we give an example of two transcendental meromorphic functions which are permutable. Given a periodic meromorphic function $H(z)$ with period τ, it is easy to see that $z + H(z)$ and $m\tau + z + H(z)$ are permutable. If, in addition, $H(z)$ is odd, that is, $H(-z) = -H(z)$, then $m\tau - z - H(z)$ is also permutable with $z + H(z)$.

For an application of the construction of special Baker domains, we shall consider a generalization of the subject and establish the following.

THEOREM 3.1. *Let $f(z)$ and $g(z)$ be meromorphic functions in* **M** *such that, for a meromorphic function $h(z)$ in* ***C***, *we have $h(f(z)) = g(h(z))$. If $J(f) = \overline{J_\infty(f)}$, and either $\infty \in E(f)$ or $f(\infty) \neq \infty$, then $h(J(f)) = J(g)$ and $h(F(f)) = F(g)$.*

From the special version of Theorem 3.1 for $h(z) = e^z$ and by using a logarithmic change of variables, we can construct several types of Baker domains; for example, we can construct a transcendental meromorphic function in ***C*** which has a Baker domain U such that $\mathrm{dist}(U, P(f)) > 0$ and $f(z)$ is univalent in U.

Herman [19] proved that, for $1 + 2\pi\lambda = e^{2\pi i\alpha}$ and a suitable real number α,

$$f(z) = z + \lambda\sin(2\pi z) + 1$$

has a wandering domain U in which all iterates $f^n(z)$ are univalent. We can also construct a non-entire meromorphic function with such properties; for example, for a suitable real number α,

$$f(z) = z + (e^{2\pi i\alpha} - 1)\tan z + 2\pi i\ ,$$

has a wandering domain which has such properties. Eremenko and Ljubich [16] constructed, by the theory of complex approximation, a transcendental entire function $f(z)$ which has a wandering domain in which all iterates $f^n(z)$ are univalent.

For Herman's example and ours and in the construction of [16], we cannot know whether there exists any relationship between the boundaries of the wandering domains and $\mathrm{Sing}(f^{-1})$ and/or $P(f)$. Question 8 raised in [8] asks whether there is some relation between ∂U_n and $\mathrm{Sing}(f^{-1})$ if U is a wandering domain such that $U_n \cap \mathrm{Sing}(f^{-1}) = \emptyset$. However, here we can prove the following by a result of Rippon [26].

THEOREM 3.4. *For almost all λ on $\{|\lambda| = 1\}$, the function*

$$f_\lambda(z) = z + e^z + 1 - \lambda + 2\pi i$$

has a wandering domain U in which all iterates $f_\lambda^n(z)$ are univalent and such that every ∂U_n contains a critical value and $\partial U_n \subset \overline{P(f_\lambda)}$. We also have $\overline{P(f_\lambda)} \subseteq J(f_\lambda)$.

We naturally raise the following question.

QUESTION. Is there a transcendental meromorphic function $f(z)$ which has a wandering domain U such that $\overline{U_n} \cap \mathrm{Sing}(f^{-1}) = \emptyset$?

However, it is easy to construct an example which has a wandering domain U such that $\mathrm{dist}(\partial U_n, P(f)) > 0$, for instance, $z + e^z + 2\pi i$ is such an example. We shall prove that a non-entire meromorphic function $z - 4[e^z/(e^z - 1)] + 2(1+\lambda) + 2\pi i$ has the same properties as in Theorem 3.4.

Finally, we mention that we can construct a transcendental meromorphic function which has a wandering domain U such that all ∂U_n are Jordan curves.

2. Fatou components and singularities of the inverse function. In this section, we first prove the following theorem and, as an application of this result, we give a sufficient condition for the non-existence of wandering domains.

THEOREM 2.1. *If $f \in \mathbf{M}$ and if U is a wandering domain of f, then every limit function of the convergent subsequence of $\{f^n|_U\}$ lies in the derived set of $P(f)$.*

We shall prove Theorem 2.1 by using the hyperbolic metric and, to this end, recall some basic knowledge about the hyperbolic metric.

Let Ω be a hyperbolic domain in the complex plane $\boldsymbol{C}$, that is, $\boldsymbol{C} \setminus \Omega$ contains at least two points. There exists the hyperbolic metric $\lambda_\Omega(z)|dz|$ on Ω with Gaussian curvature -4. Throughout, we use the notation $B(a, \delta) = \{z; |z-a| < \delta\}$, $B^*(a, \delta) = B(a, \delta) \setminus \{a\}$ and $D_R^* = \{z; |z| > R\}$. Then it is well-known that

$$\lambda_{B^*(0,\delta)}(z) = \frac{1}{|z| \log(\delta/|z|)} \quad \text{and} \quad \lambda_{D_R^*}(z) = \frac{1}{|z| \log(|z|/R)} . \tag{1}$$

For any hyperbolic simply connected domain Ω, by the Koebe 1/4 theorem we can easily prove that

$$\lambda_\Omega(z)\delta_\Omega(z) \geq \frac{1}{4} , \quad z \in \Omega , \tag{2}$$

where $\delta_\Omega(z)$ is the Euclidean distance of z from the boundary of Ω.

The following version of the Schwarz-Pick lemma (see [1]) will play a key role in the proof of our theorems.

LEMMA 2.1. *Let U and Ω both be hyperbolic domains and let h be an analytic function in U such that $h(U) \subseteq \Omega$. Then*

$$\lambda_\Omega(h(z))|h'(z)| \leq \lambda_U(z) , \quad z \in U , \tag{3}$$

with equality if and only if h is an unbranched covering map of Ω from U.

Now we can proceed to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Suppose conversely that there exists a limit function a of some subsequence of $\{f^n|_U\}$ which is not in the derived set of $P(f)$. Then we can take a positive number δ such that $B^*(a, 2\delta) \cap P(f) = \emptyset$, where $B^*(a, r) = \{z; 0 < |z-a| < r\}$.

Now we want to prove that U is simply connected. Suppose the contrary, then we can draw a Jordan curve γ in U which is not null-homotopic in U.

Since γ is a compact subset of U, we always have sufficient large n such that $f^n(\gamma) \subset B^*(a, \delta)$. Take a point $b \in \gamma$. Let g_n be an analytic branch of f^{-n} in some neighborhood of $f^n(b)$ such that $g_n(f^n(b)) = b$. Then there exist three possibilities:

(i) a is an analytic point of g_n, that is, g_n can be analytically continued to $B(a, 2\delta)$;

(ii) a is an algebraic branch point of g_n;

(iii) a is a transcendental branch point of g_n.

In Case (i), $g_n(B(a, 2\delta))$ is simply connected and $\mathrm{int}(\gamma) \subset g_n(B(a, 2\delta))$, and then $f^n(z)$ is meromorphic in $\overline{\mathrm{int}(\gamma)}$.

In Case (ii), g_n can be analytically continued throughout $B^*(a, 2\delta)$. We can produce a Jordan curve Γ such that $\gamma \subset \mathrm{int}(\Gamma)$ in the way that g_n is continued along $\partial B(a, \delta)$ finitely many times. Thus $f^n(\Gamma)$ covers the circle $\partial B(a, \delta)$,

$$f^n : \mathrm{int}(\Gamma) \setminus \{\alpha\} \to B^*(a, \delta) \tag{4}$$

is proper and $f^n(\alpha) = a$. Application of Picard's theorem to (4) yields that f^n is meromorphic in $\mathrm{int}(\Gamma)$, so f^n is meromorphic in $\overline{\mathrm{int}(\gamma)}$.

In Case (iii), g_n can also be analytically continued throughout $B^*(a, 2\delta)$. There exists a component W of $f^{-n}(B(a, \delta))$ such that $\gamma \subset W$ and

$$f^n : W \to B^*(a, \delta) \tag{5}$$

is a covering. It is easy to see that ∂W is a Jordan curve and ∂W tends to $E(f^n)$ along both directions from a fixed point in ∂W. Since $E(f^n)$ is totally disconnected, along both directions ∂W tends to two points e and e' in $E(f^n)$. If $e \neq e'$, then there exists an $\alpha \in E(f^n) \setminus \{e, e'\}$ in W and, by noting that $C(f^n, E(f^n)^c, \alpha) = \hat{\boldsymbol{C}}$, this hence derives a contradiction to (5). Thus we have proved that $e = e'$, and W is simply connected. It follows that $\mathrm{int}(\gamma) \subset W$ and, by applying Picard's theorem to (5), f^n is also meromorphic in $\overline{\mathrm{int}(\gamma)}$.

In one word, we have proved that f^n is meromorphic in $\overline{\mathrm{int}(\gamma)}$. Since n can be assumed to be arbitrary sufficiently large, for each positive integer n, f^n is meromorphic in $\mathrm{int}(\gamma)$. This implies that $\mathrm{int}(\gamma) \subset F(f)$, and then γ is contractible in U. This is a contradiction, from which it follows that U is simply connected.

Set $f^{n_k}|_U \to a$ $(k \to \infty)$. By the argument as above, U_k, the component of $F(f)$ containing $f^{n_k}(U)$, is also simply connected. Assume there exists a point $b \in U$ such that $f^{n_k}(b) \in B^*(a, \delta)$. Applying Lemma 2.1 to $f^{n_k} : U \to U_k$ implies that

$$\frac{|(f^{n_k})'(b)|}{4|f^{n_k}(b) - a|} \leq \lambda_{U_k}(f^{n_k}(b))|(f^{n_k})'(b)| \leq \lambda_U(b)\,. \tag{6}$$

On the other hand, let W^* and W be respectively the components of, $f^{-n_k}(B^*(a, \delta))$ and $f^{-n_k}(B(a, \delta))$ containing b. Then $W^* \subset W$ and $f^{n_k} : W^* \to B^*(a, \delta)$ is an unbranched covering. It follows from Lemma 2.1 that

$$\begin{aligned}\lambda_W(b) < \lambda_{W^*}(b) &= \lambda_{B^*(a,\delta)}(f^{n_k}(b))|(f^{n_k})'(b)| \\ &= \frac{|(f^{n_k})'(b)|}{|f^{n_k}(b)-a|\log(\delta/|f^{n_k}(b)-a|)}.\end{aligned} \tag{7}$$

Combining (6) with (7) implies that

$$\log(\delta/|f^{n_k}(b)-a|) < \frac{4\lambda_U(b)}{\lambda_W(b)}.$$

This is impossible, since $f^{n_k}(b) \to a$ $(k \to \infty)$. Hence Theorem 2.1 follows. □

By the method from [32], we can extend Theorem 2 in [32] for a meromorphic function to a function in **M**.

THEOREM 2.2. *Let $f \in \mathbf{M}$ and let U be a component of $F(f)$. If $f^{np}|_U \to q$ $(n \to \infty)$, then either q lies in the derived set of $S_p(f)$ or q is a periodic point of f of period $k \le p$ and $f^p(q) = q$.*

In order to prove Theorem 2.2, we need the following lemmas, which are of independent significance.

LEMMA 2.2. *Let $f \in \mathbf{M}$. If $S_p(f) \cap B^*(0,\delta) = \emptyset$, then each component of $f^{-p}(B(0,\delta))$ is simply connected in $\boldsymbol{C}$.*

Lemma 2.2 can be proved by the method from Zheng [32]. The following is an immediate product of the combination of Lemmas 2.2 and 2.1.

LEMMA 2.3. *Let $f \in \mathbf{M}$. If $S_p(f) \cap B^*(0,\delta) = \emptyset$, f^p is analytic at b such that $f^p(b) \in B^*(0,\delta)$ and 0 is not in the component of $f^{-p}(B(0,\delta))$ containing b, then we have*

$$|(f^p)'(b)| > \frac{|f^p(b)|\log(\delta/|f^p(b)|)}{4|b|}. \tag{8}$$

If $S_p(f) \subset B(0,R)$ and $|f^p(c)| < R$, then, for any analytic point z of f^p, we have

$$|(f^p)'(z)| > \frac{|f^p(z)|\log(|f^p(z)|/R)}{4|z-c|}. \tag{9}$$

Equation (9) was also established in [27] for a meromorphic function in $\boldsymbol{C}$. Now we are in a position to prove Theorem 2.2.

PROOF OF THEOREM 2.2. We prove Theorem 2.2 for the case when q is a finite number; the same argument can show Theorem 2.2 for $q = \infty$.

Suppose that $f^p(q) \neq q$ and q is not in the derived set of $S_p(f)$. Then, from the classification theorem of periodic components of a Fatou set, it is easy to see that $q \notin F(f)$. Assume that $q = 0$ without any loss of generality. Since $f^{np}|_U \to 0$ as $n \to \infty$, we can take a $b \in f^{n_0 p}(U)$ $(n_0 > 0)$ and two positive numbers δ and r such that $S_p(f) \cap B^*(0,\delta) = \emptyset$, $f^{np}(B(b,r)) \subset B^*(0,\delta)$, $n = 0, 1, 2, \ldots$ and $0 \notin f^{-p}(B(0,\delta))$. Thus, for $n > 0$, applying

Lemma 2.3 to f^p at $f^{(s-1)p}(b) \in B^*(0,\delta)$ $(s = 1, 2, \dots, n)$ gives

$$\begin{aligned} |(f^{np})'(b)| &= \prod_{s=1}^{n} |(f^p)'(f^{(s-1)p}(b))| \geq \prod_{s=1}^{n} \frac{|f^{sp}(b)|\log(\delta/|f^{sp}(b)|)}{4|f^{(s-1)p}(b)|} \\ &= \frac{|f^{np}(b)|\log(\delta/|f^{np}(b)|)}{4|b|} \prod_{s=1}^{n-1} \frac{1}{4} \log(\delta/|f^{sp}(b)|)\,. \end{aligned} \tag{10}$$

On the other hand, since $f^{np} : B(b,r) \to B^*(0,\delta)$, from Lemma 2.1 it follows that

$$\frac{|(f^{np})'(b)|}{|f^{np}(b)|\log(\delta/|f^{np}(b)|)} \leq \frac{1}{r}\,. \tag{11}$$

Equation (11) contradicts (10), for $\prod_{s=1}^{n-1}(1/4)\log(\delta/|f^{sp}(b)|) \to \infty$ as $n \to \infty$. Thus Theorem 2.2 follows. □

As we did in [32], from Theorems 2.1 and 2.2 we can obtain several consequences, whose proofs are omitted.

COROLLARY 2.1. *For $e \in J_\infty(f)$, if $e \notin (S_p(f))'$, then there exist no components of $F(f)$ in which $f^{np}(z) \to e$ as $n \to \infty$.*

This is a generalization of Theorem F in [4].

COROLLARY 2.2. *Let $f \in \mathbf{M}$. If $(\mathrm{Sing}(f^{-1}))' \cap E(f) = \emptyset$, $J(f) \cap (P(f))'$ is finite and $(P(f))' \cap J_\infty(f) \setminus E(f) = \emptyset$, then f has no wandering domains.*

If $\infty \in E(f)$, then the condition '$(\mathrm{Sing}(f^{-1}))' \cap E(f) = \emptyset$' implies that $\mathrm{Sing}(f^{-1})$ is bounded. In what follows, we discuss the connection between singularities and Baker domains. Given a cycle of the Baker domains, $\{B_0, B_1, \dots, B_{p-1}\}$, with period p of $f(z)$ in $\mathbf{M}$, we have

$$f^{np}|_{B_j} \to a_j, \quad n \to +\infty\,, \quad 0 \leq j \leq p-1\,.$$

We draw a curve γ_0 in B_0 to connect a point z_0 and $f^p(z_0)$, and set

$$\Gamma_0 = \bigcup_{n=0}^{\infty} f^{np}(\gamma_0)\,, \quad \Gamma_{j+1} = f(\Gamma_j)\,, \quad 0 \leq j \leq p-1\,.$$

It is easy to see that $\Gamma_p \subset \Gamma_0$ and

$$f(z) \to a_{j+1}\,, \quad z \in \Gamma_j \to a_j\,, \quad 0 \leq j \leq p-1\,, \tag{12}$$

where $a_p = a_0$. Therefore, we call $\{a_0, a_1, \dots, a_{p-1}\}$ a cycle of periodic points for $\{B_0, B_1, \dots, B_{p-1}\}$ with period p. According to the definition of the Baker domains, at least one of $\{a_0, a_1, \dots, a_{p-1}\}$ is in $E(f)$. If $a_j \in E(f)$, then from (12) a_{j+1} is an asymptotic value of $f(z)$, and so the inverse f^{-1} has a singularity over a_{j+1}, that is, $a_{j+1} \in \mathrm{Sing}(f^{-1})$.

So from Theorem 2.2, we have

$$\{a_0, a_1, \dots, a_{p-1}\} \subset (S_p(f))' \cap J(f)\,.$$

COROLLARY 2.3. *Let $f \in \mathbf{M}$. Then f has no Baker domains of period $k \leq p$ if one of the following statements holds*:

(1) *$f(z)$ has no asymptotic values which lie in $J_\infty(f, p) = \bigcup_{n=0}^{p-1} f^{-n}(E(f))$;*

(2) *$(S_p(f))' \cap E(f) = \emptyset$.*

As an application of Corollaries 2.2 and 2.3, we give a sufficient condition to determine the Julia set of a transcendental meromorphic function equal to the Riemann sphere.

THEOREM 2.3. *Let $f(z) \in \mathbf{M}$ with $(\mathrm{Sing}(f^{-1}))' \cap E(f) = \emptyset$. Assume that $J(f) \cap (P(f))'$ is finite, $(P(f))' \cap J_\infty(f) \setminus E(f) = \emptyset$ and, for every $b \in \mathrm{Sing}(f^{-1})$, b is pre-periodic, $b \in J_\infty(f)$ or $f^n(b) \to E(f)$ as $n \to \infty$. Then $J(f) = \hat{\boldsymbol{C}}$.*

When $f(z)$ is a transcendental meromorphic function, then the condition '$(\mathrm{Sing}(f^{-1}))' \cap E(f) = \emptyset$' is equivalent to the fact that $\mathrm{Sing}(f^{-1})$ is bounded, that is, $f(z)$ is of bounded type. From Theorem 2.3, it is easy to deduce that $J(n\pi i \tan z) = \hat{\boldsymbol{C}}$ and $J(e^z) = \hat{\boldsymbol{C}}$, which was conjectured by Fatou [17] and proved by Misiurewicz [24]. However, $n\pi i \tan z$ and e^z are both of finite type. From Theorem 2.3, we construct a transcendental meromorphic function which is not of bounded type and whose Julia set is the Riemann sphere.

THEOREM 2.4. *Let $f(z) = \mu + z + e^z + \lambda/(e^z - 1)$. Then there exist μ and λ such that $J(f) = \hat{\boldsymbol{C}}$.*

PROOF. We consider the function

$$g(z) = z \exp\left(\mu + z + \frac{\lambda}{z-1}\right),$$

where μ and λ are chosen to be two non-negative real numbers. 0 is a unique asymptotic value of $g(z)$ and the critical points of $g(z)$ are solutions of the equation

$$z^3 - z^2 - (\lambda + 1)z + 1 = 0\,. \tag{13}$$

By calculation, we can deduce that Equation (13) has only three real roots which lie respectively in the intervals $(-\infty, -1)$, $(0, 1)$ and $(1, +\infty)$, denoted in turn by x_1, x_2 and x_3. Since, for $\lambda = 0$, $x_1 = -1$ and $x_2 = x_3 = 1$, we have $x_1 \to -1^-$, $x_2 \to 1^-$ and $x_3 \to 1^+$ as $\lambda \to 0^+$.

Obviously, the roots of $\mu + z + \lambda/(z - 1) = 0$ are fixed points of $g(z)$, and so

$$z_0 = \frac{-(\mu-1) - \sqrt{(\mu-1)^2 + 4(\mu-\lambda)}}{2} = -\frac{2(\mu-\lambda)}{\sqrt{(\mu-1)^2 + 4(\mu-\lambda)} - (\mu-1)}$$

is a fixed point of $g(z)$ and $z_0 \to -\mu$ as $\lambda \to 0^+$. Now we want to choose μ and λ such that $g^2(x_1) = z_0$, that is, the following equation holds:

$$g^2(x_1) = -\frac{2(\mu-\lambda)}{\sqrt{(\mu-1)^2 + 4(\mu-\lambda)} - (\mu-1)}\,. \tag{14}$$

We rewrite (14) to give

$$\left(\sqrt{(\mu-1)^2 + 4(\mu-\lambda)} - (\mu-1)\right) g^2(x_1) + 2(\mu-\lambda) = 0\,. \tag{15}$$

We denote the function on the left-hand side of the above equation by $F_\lambda(\mu)$. Since

$$g(x_1) = x_1 \exp\left(\mu + x_1 + \frac{\lambda}{x_1 - 1}\right) \to -e^{\mu-1}$$

as $\lambda \to 0^+$, we have

$$F_\lambda(\mu) \to -2\exp(2\mu - 1 - e^{\mu-1}) + 2\mu, \quad \lambda \to 0^+ .$$

Therefore, for sufficiently small $\lambda > 0$, $F_\lambda(\mu) \to +\infty$ as $\mu \to +\infty$. On the other hand,

$$p(\mu) = -2\exp(2\mu - 1 - e^{\mu-1}) + 2\mu \to -2\exp(-1 - e^{-1})$$

as $\mu \to 0^+$. There exists a $\mu_0 > 0$ such that $p(\mu_0) = -\exp(-1 - e^{-1}) < 0$. Thus, given a sufficiently small $\lambda > 0$, we have a $\mu > \mu_0$, which is a zero of $F_\lambda(\mu)$. Thus, we have found λ and $\mu > \mu_0 > 0$ such that $g^2(x_1) = z_0$. Since x_1 is independent of μ, we have $x_1 \neq z_0$, and then x_1 is a preperiodic point of $g(z)$.

By noting that

$$x_j + \frac{\lambda}{x_j - 1} = 2x_j - \frac{1}{x_j}, \quad j = 1, 2, 3 \tag{16}$$

and $x_j \to 1$ ($j = 2, 3$) as $\lambda \to 0^+$, we deduce that

$$f(x_j) = x_j \exp\left(\mu + x_j + \frac{\lambda}{x_j - 1}\right) > x_j e^{\mu+(1/2)} > 1, \quad j = 2, 3$$

so that

$$f^n(x_j) > x_j e^{n\mu} \to +\infty, \quad j = 2, 3 .$$

By applying Theorem 2.4, we obtain $J(g) = \hat{\boldsymbol{C}}$.

Since $\exp f(z) = g(e^z)$, it follows from Theorem 3.1 in the next section that $\exp J(f) = J(g) = \hat{\boldsymbol{C}}$, and then $J(f)$ must have interior points. This implies $J(f) = \hat{\boldsymbol{C}}$. □

To the best of our knowledge, this seems to be the first example of a transcendental meromorphic function which is not of bounded type and whose Julia set coincides with the Riemann sphere.

3. Semiconjugation of functions in M. In order to construct some special wandering domains in this section, and Baker domains with special properties in the next section and for application in the proof of Theorem 2.4, we are motivated to investigate the connection between the dynamics of two functions $f(z)$ and $g(z)$ in **M** satisfying the functional equation

$$h(f(z)) = g(h(z)), \tag{17}$$

where h is meromorphic in $\boldsymbol{C}$. It is the discussion of this general family **M** that leads us to the possibility of constructing some meromorphic functions in $\boldsymbol{C}$ with special dynamical properties. We first want to establish a relation between $J(f)$ and $J(g)$.

THEOREM 3.1. *Let f, g and h satisfy* (17). *If $J(f) = \overline{J_\infty(f)}$ and either $\infty \in E(f)$ or $f(\infty) \neq \infty$, then $h(J(f)) = J(g)$ and $h(F(f)) = F(g)$.*

PROOF. It is clear that we only need to prove the equality $h(J(f)) = J(g)$.

(1) We prove that $h(J(f)) \subseteq J(g)$. Take an arbitrary point $z_0 \in h(J(f))$. There exists a $w_0 \in J(f)$ such that $h(w_0) = z_0$. Since $J(f) = \overline{J_\infty(f)}$, we have a sequence $\{w_k\}$ in $J_\infty(f)$ such that $w_k \to w_0$ $(k \to +\infty)$, and $f^{n_k}(w_k) \in E(f)$. From (17), we have

$$h(f^{n_k+1}(z)) = g^{n_k+1}(h(z)) . \tag{18}$$

Since $w_k \in E(h(f^{n_k+1}(z))) = E(g^{n_k+1}(h(z)))$ and h is meromorphic at w_k, $h(w_k) \in E(g^{n_k+1}) \subset J(g)$, and hence $h(w_0) \in (J(g))' = J(g)$. Thus we prove $h(J(f)) \subset J(g)$.

(2) Now we prove that $J(g) \subseteq h(J(f))$. From $h(J(f)) \subseteq J(g)$ and $J(f) = \overline{J_\infty(f)}$ and by noting that $h(z)$ is non-constant, it follows that $J(g) = \overline{J_\infty(g)}$. Therefore, we need only to prove that $J_\infty(g) \subset h(J(f))$. Take an arbitrary point $a \in J_\infty(g)$ which is not the Picard exceptional value of $h(z)$, and assume a is at a point b under $h(z)$, that is, $h(b) = a$. There exists a positive integer n such that $g^n(a) \in E(g)$, and hence b is not a meromorphic point of $g^{n+1}(h(z)) = h(f^{n+1}(z))$. This implies that $f^{n+1}(b) = \infty$ or $b \in E(f^{n+1})$.

Assume that $f^{n+1}(b) = \infty$. If $\infty \notin E(f)$ and ∞ is not a fixed point of $f(z)$, then $c = f^{n+2}(b) \neq \infty$ is a meromorphic point of $h(z)$ and hence of $h(f^{n+2}(z)) = g^{n+2}(h(z))$. This is a contradiction; if $\infty \in E(f)$, we have $b \in J_\infty(f) \subset J(f)$.

Therefore, we always have $b \in J_\infty(f) \subset J(f)$ and $a = h(b) \in h(J(f))$. This completes the proof of Theorem 3.1. □

From Theorem 3.1 and the result in Bergweiler [10] for entire functions $f(z)$ and $g(z)$, we have the following

COROLLARY 3.1. *Let $f(z)$ and $g(z)$ both be in* **M** *with either $\infty \in E(f)$ or $f(\infty) \neq \infty$. If* $\exp f(z) = g(e^z)$, *then we have*

$$\exp J(f) = J(g) \quad \text{and} \quad \exp F(f) = F(g) . \tag{19}$$

As an immediate application of Theorem 3.1, we have the following.

THEOREM 3.2. *Let $f(z)$ and $g(z)$ both be transcendental meromorphic functions in* **C** *and permutable. If $J(f) = \overline{J_\infty(f)}$, then $J(f) = J(g)$.*

PROOF. From Theorem 3.1 it follows that $g^{-1}(J(f)) = J(f)$, that is, $J(f)$ is a completely invariant closed set of g, and therefore $J(g) \subseteq J(f)$. From $J(f) = \overline{J_\infty(f)}$ and $f \circ g = g \circ f$, it is easy to see that $J(g) = \overline{J_\infty(g)}$, and so we also have $J(f) \subseteq J(g)$. Thus $J(g) = J(f)$. □

Since $E(f) = E(g) = \{\infty\}$, another immediate approach to prove Theorem 3.2 is available by noting the fact that

$$\bigcup_{j=1}^{n} f^{-j}(\infty) = \bigcup_{j=1}^{n} g^{-j}(\infty) .$$

Next we discuss a dynamical connection between Fatou components of f and g which satisfy (17).

THEOREM 3.3. *Let $f(z)$, $g(z)$ and $h(z)$ be in* **M** *such that* (17) *holds. If $h(z)$ maps any component of $F(f)$ onto a hyperbolic domain, then the following hold.*

(i) *If f has no wandering domains, then g has no wandering domains.*

(ii) *If U is a periodic component of $F(f)$ for $f(z)$, then $h(U)$ is contained in a periodic component V of $F(g)$ for $g(z)$, and they are of the same type, unless U is a Baker domain or Herman ring. If U is a Herman ring, then V must be a Siegel disk or Herman ring.*

PROOF. Suppose that g has a wandering domain V. Let U be a component of $h^{-1}(V)$. Since $h \circ f^n(U) = g^n(V)$, for $m \neq n$ we have $h \circ f^n(U) \cap h \circ f^m(U) = \emptyset$, so that $f^n(U) \cap f^m(U) = \emptyset$ and, from the Montel theorem, $U \subseteq F(f)$. Let $\tilde{U}$ be a component of $F(f)$ containing U.

Since $f(z)$ has no wandering domains, $\tilde{U}$ is periodic for $f(z)$. We assume without any loss of generality that $\tilde{U}$ is invariant, that is, $f(\tilde{U}) \subseteq \tilde{U}$. This implies that

$$g \circ h(\tilde{U}) = h \circ f(\tilde{U}) \subseteq h(\tilde{U}),$$

and so $h(\tilde{U})$ is invariant under $g(z)$. Since $V = h(U) \subseteq h(\tilde{U})$, $V = h(\tilde{U})$ and so V is invariant under g. We derive a contradiction.

From the above discussion, if U is a periodic component of $F(f)$ for $f(z)$, then $h(U)$ is contained in a periodic component of $F(g)$ for $g(z)$. If a is a periodic point of $f(z)$ with period p, then $h(a)$ is also a periodic point of $g(z)$ and $((f^p)'(a))^m = (g^p)'(h(a))$, where m is the multiplicity of $h(z)$ over a. Therefore, it is not difficult to prove (ii). □

The statement (i) in Theorem 3.3 was proved in [12] for $f(z)$ and $g(z)$ being entire. From Theorem 3.3 and Theorem E in [4] (that if it is of finite type, then the composition of two transcendental meromorphic functions has no wandering domains), we immediately deduce that if $f(z)$ and $g(z)$ are both meromorphic functions in $\boldsymbol{C}$ and $f(z)$ is of finite type such that $f(g(z))$ is of finite type, then $g(f(z))$ has no wandering domains, for $g(z)$ must map any component of $F(f(g))$ onto a hyperbolic domain. The important significance of this result is that $g(f(z))$ may not be of finite type, not even of bounded type. Thus, from this point of view, we easily obtain some families of meromorphic functions in $\boldsymbol{C}$ which are not of finite type without any wandering domains. Here are two examples to describe this situation. Let $f(z) = \exp(R(e^{P(z)}) + kP(z))$, where $R(z)$ is a rational function with at least one non-zero pole, $P(z)$ is a polynomial and k is an integer. Since $f(z) = w^k e^{R(w)} \circ e^{P(z)}$, $f(z)$ is of finite type and, therefore, from $f(z) = e^w \circ (R(e^{P(z)}) + kP(z))$ and by using Theorem 3.3, the non-entire meromorphic function $R(\exp P(e^z)) + kP(e^z)$ has no wandering domains. By a simple calculation, $R(\exp P(e^z)) + kP(e^z)$ is not of bounded type. We can easily check that $e^w \circ (\tan z - iz)$ is of finite type, therefore $\tan e^z - ie^z$ has no wandering domains. This method was first used in Baker and Singh [6] to construct a class of transcendental entire functions which have no wandering domains and has been developed in many other papers.

The above method applies to the construction of transcendental meromorphic functions which have unbounded Siegel disk with a singular value on its boundary. Rippon [26] discussed the Siegel disk with singular value on its boundary and as an application of his result showed that $f_\lambda(z) = e^{\lambda z} - 1$ has a Siegel disk containing Siegel point 0 with unique asymptotic value -1 on its boundary for almost all λ on $\{|\lambda| = 1\}$, so that the Siegel disk is unbounded. We take into account the function $g_\lambda(z) = \lambda(e^z - 1)$. Note that $-\lambda$ is a unique asymptotic value of $g_\lambda(z)$. Let $f(z) = e^z - 1$ and $g(z) = \lambda z$. Then $f_\lambda(z) = f \circ g$ and $g_\lambda(z) = g \circ f$. We denote the Siegel disk of $f_\lambda(z)$ containing 0 by U. From $f \circ g(U) = U$, it follows that $g_\lambda(g(U)) = g(U)$, and so $g(U) \subseteq F(g_\lambda)$. Let V be the component of $F(g_\lambda(z))$ containing $g(U)$. Then $U = f \circ g(U) \subseteq f(V)$. On the other hand, since $g_\lambda(V) = V$, we have $f_\lambda(f(V)) = f(V)$, which implies $f(V) \subseteq F(f_\lambda)$. Since $0 \in f(V)$, we have $f(V) \subseteq U$ and so $f(V) = U$, $V = g(U)$. Thus $g(U)$ is the Siegel disk of $g_\lambda(z)$ and $-\lambda \in \partial g(U)$ and $g(U)$ is unbounded. The result in [26] does not seem to be available for deducing the existence of the Siegel disk of $g_\lambda(z)$ with $-\lambda$ on its boundary.

Let $T(z)$ be a Möbius transformation. The result in [26] also applies to the function $f_\lambda(T(z))$, and so $f_\lambda(T(z))$ has a Siegel disk with unique asymptotic value on its boundary for almost all λ on $\{|\lambda| = 1\}$. The same argument as in the above implies that $T(f_\lambda(z))$ and $\lambda T(e^z - 1)$ have an unbounded Siegel disk with unique asymptotic value on its boundary for almost all λ on $\{|\lambda| = 1\}$.

By using the quasi-conformal surgery of Shishikura [28] (cf. [35]), we can construct an unbounded Herman ring from an unbounded Siegel disk, that is, we have a transcendental meromorphic function which has an unbounded Herman ring. However, we do not know whether such a Herman ring has a singular value on its boundary.

Next, by the result of Rippon [26], we consider wandering domains as mentioned in the introduction and prove the following.

THEOREM 3.4. *For almost all λ on $\{|\lambda| = 1\}$, the function*

$$f_\lambda(z) = z + e^z + 1 - \lambda + 2\pi i$$

has a wandering domain U in which all iterates $f^n(z)$ are univalent and such that every ∂U_n contains a critical value and $\partial U_n \subseteq \overline{P(f_\lambda)}$. We also have $\overline{P(f_\lambda)} \subseteq J(f_\lambda)$.

PROOF. Let $g_\lambda(z) = ze^{z+1-\lambda}$ and $F_\lambda(z) = z + e^z + 1 - \lambda$. $g_\lambda(z)$ has a fixed point $\lambda - 1$ with multiplier λ and $\mathrm{Sing}(g_\lambda^{-1}) = \{0, -e^{-\lambda}\}$. By the theorem of Rippon [26], $g_\lambda(z)$ has a Siegel disc U_λ containing $\lambda - 1$ with $-e^{-\lambda}$ on ∂U_λ for almost all λ on $\{|\lambda| = 1\}$. Taking a point z_0 such that $e^{z_0} + 1 - \lambda = 0$, we then have $F_\lambda(z_n) = z_n$ and $F'_\lambda(z_n) = \lambda$, where $z_n = z_0 + 2n\pi i$, $n \in \boldsymbol{Z}$. Therefore, for almost all λ on $\{|\lambda| = 1\}$, $F_\lambda(z)$ has a Siegel disc $V_\lambda^{(n)}$ containing z_n and $\partial V_\lambda^{(n)} \subseteq \overline{P(F_\lambda(z))}$. Since $\exp F_\lambda(z) = g_\lambda(e^z)$, we have $\exp F(F_\lambda) = F(g_\lambda)$ and so $\exp V_\lambda^{(n)} = U_\lambda$. Noting that $\mathrm{Sing}(F_\lambda^{-1}) = \{(2n+1)\pi i - \lambda; n \in \boldsymbol{Z}\}$ and $\exp \mathrm{Sing}(F_\lambda^{-1}) = -e^{-\lambda}$, it follows that the critical value $(2n+1)\pi i - \lambda$ stands on $\partial V_\lambda^{(n)}$. Since $f_\lambda(z) = F_\lambda(z) + 2\pi i$ and $f_\lambda(F_\lambda) = F_\lambda(f_\lambda)$, we have $J(f_\lambda) = J(F_\lambda)$ so that $f_\lambda(V_\lambda^{(n)}) = V_\lambda^{(n+1)}$.

Since $f_\lambda^n(z) = F_\lambda^n(z) + 2n\pi i$ and

$$F_\lambda^n(\mathrm{Sing}(F_\lambda^{-1})) = F_\lambda^n(\mathrm{Sing}(F_\lambda^{-1})) + 2n\pi i$$
$$= f_\lambda^n(\mathrm{Sing}(F_\lambda^{-1})) = f_\lambda^n(\mathrm{Sing}(f_\lambda^{-1}))\,,$$

we have $\partial V_\lambda^{(n)} \subseteq \overline{P(F_\lambda(z))} = \overline{P(f_\lambda(z))}$. Thus Theorem 3.4 follows. □

In order to construct a non-entire meromorphic function with the properties of Theorem 3.4, we consider the function

$$h_\lambda(z) = z\exp\left(-4\frac{z}{z-1} + 2(1+\lambda)\right)\,.$$

Then $\mathrm{Sing}(h_\lambda^{-1}) = \{0, -e^{2\lambda}\}$, where 0 is an asymptotic value and $h_\lambda(z)$ has a fixed point $(\lambda+1)/(\lambda-1)$ with multiplier λ^2. Since 0 is a fixed point of $h_\lambda(z)$, the immediate basin of attraction of $h_\lambda(z)$ for $(\lambda+1)/(\lambda-1)$, when $0 < |\lambda| < 1$, contains only one critical value $-e^{2\lambda}$ and therefore is simply connected (see [36, Theorem 3]). Thus we can use the method as in [26] for almost all λ on $\{|\lambda| = 1\}$ to prove that $h_\lambda(z)$ has a Siegel disc U_λ with critical value $-e^{2\lambda}$ on its boundary. By the same argument as in the proof of Theorem 3.4, we can deduce that, for almost all λ on $\{|\lambda| = 1\}$,

$$f_\lambda(z) = z - 4\frac{e^z}{e^z-1} + 2(1+\lambda) + 2\pi i$$

has a wandering domain V such that every ∂V_n contains a critical value and f_λ is univalent in V_n.

In the following, we construct a meromorphic function which has a Baker domain with a critical value on its boundary.

THEOREM 3.5. *For almost all λ on $\{|\lambda| = 1\}$, the function*

$$f_\alpha(z) = 2\pi i\alpha + z - e^{z+1+2\pi i\alpha}\,, \quad \lambda = e^{2\pi i\alpha}\,,$$

has a Baker domain U in which $f_\alpha(z)$ is univalent with critical value on ∂U. We also have $\partial U = \overline{P(f_\lambda)}$.

PROOF. Let $g_\alpha(z) = \lambda z\exp(-\lambda e z)$, $\lambda = e^{2\pi i\alpha}$. A simple calculation implies that $g_\alpha(0) = 0$ and $g'_\alpha(0) = \lambda$, and $\mathrm{Sing}(g_\alpha^{-1}) = \{0, e^{-2}\}$, where 0 is an asymptotic value, e^{-2} is a critical value and $(\lambda e)^{-1}$ is a critical point of $g_\alpha(z)$ such that $g_\alpha((\lambda e)^{-1}) = e^{-2}$. When $0 < |\lambda| < 1$, 0 is an attracting fixed point and hence we have an immediate basin of attraction U_λ of 0 for $g_\alpha(z)$. Then U_λ must contain a singular value which is not (pre)periodic and hence U_λ contains e^{-2}. Thus we can use the theorem of Rippon [26] to deduce that, for almost all λ on $\{|\lambda| = 1\}$, $g_\alpha(z)$ has a Siegel disc V at 0 with e^{-2} on its boundary. Since $\exp f_\alpha(z) = g_\alpha(e^z)$, there exists a unique component U of $F(f_\alpha)$ such that $\exp U = V$ and $\exp\partial U = \partial V$. It follows that $f_\alpha(z)$ is univalent in U and $\mathrm{Sing}(f_\alpha^{-1}) \subset \partial U$, therefore $\overline{P(f_\alpha)} \subseteq \partial U$. On the other hand, from $\partial V \subseteq \overline{P(g_\alpha)}$ we have $\partial U \subseteq \overline{P(f_\alpha)}$ and so $\partial U = \overline{P(f_\alpha)}$.

It is easy to see that U does not contain any periodic points and any singular values, so U is a Baker domain for f_α. □

In the following we consider the boundary of a wandering domain of an entire function by constructing examples. The complexity of the boundary of an unbounded periodic component of an entire function was revealed in [7] and [3].

THEOREM 3.6. *There exists a polynomial $P(z)$ such that $f(z) = z + P(e^z)$ has a wandering domain with the boundary being a quasicircle.*

PROOF. We consider the disk $D = B(a, 2)$, where a is a sufficiently large positive number. By Runge's theorem in complex approximation theory, we have a polynomial $P(z)$ such that

$$|P(z) - \log\{a + (z-a)^2\} - \log z| < \frac{0.01}{2(a+4)} \quad \text{on } \bar{D}.$$

Set

$$g(z) = ze^{P(z)}.$$

Then on $\bar{D}$, we have

$$\begin{aligned} |g(z) - \{a + (z-a)^2\}| &= |ze^{P(z)} - \{a + (z-a)^2\}| \\ &= |a + (z-a)^2||z \exp[P(z) - \log\{a + (z-a)^2\}] - 1| \\ &< (a+4)2|P(z) - \log\{a + (z-a)^2\} - \log z| \\ &< 0.01, \end{aligned} \tag{20}$$

where we have used the inequality $|e^z - 1| < |z|/(1 - |z|)$, $|z| < 1$. From (20), it is easy to see that $g(B(a, 1/2)) \subset B(a, 1/2)$ and then there exists a component U of $F(g)$ containing $B(a, 1/2)$ which is an immediate attracting basin of $g(z)$ for an attracting fixed point α in $B(a, 1/2)$. A simple calculation deduces that $g(z)$ maps the circle $\{|z - a| = 3/2\}$ into $\{|z - a| > 2\}$ and hence $U \subset B(a, 3/2)$. We have a component V of $g^{-1}(B(a, 3/2))$ containing a and certainly $\bar{V} \subset B(a, 3/2)$. Since $g(z)$ is analytic on $\overline{B(a, 3/2)}$, V is simply connected. This implies that $g : V \to B(a, 3/2)$ is a polynomial-like mapping (see [13, p. 99]). Then there exist a polynomial $h(z)$ and a quasi-conformal mapping ϕ such that $g(z) = \phi^{-1} \circ h \circ \phi$ on V (see [13, Theorem 3.1]). $\phi(U)$ is an immediate attracting basin of $h(z)$ for $\phi(\alpha)$. From Rouché's theorem, $h(z)$ is of degree two and hence $\partial\phi(U)$ is a quasicircle (see [13, Theorem 2.1]). Thus we have that $\partial U = \phi^{-1}(\partial\phi(U))$ is also a quasicircle. Since $0 \notin U$, by the logarithmic change of variables $\exp f(z) = g(e^z)$, for suitable k we have a sequence $\{z_n\}$ with $z_{n+1} = z_n + 2\pi i$ such that $f_1(z_n) = z_n$, $f_1'(z_n) = g'(\alpha)$ and $\exp z_n = \alpha$, where $f_1(z) = z + P(e^z) - 2k\pi i$. U_n is the immediate attracting basin of $f_1(z)$ for z_n and hence $U_{n+1} = U_n + 2\pi i$ and $\exp U_n = U$ and $\exp \partial U_n = \partial U$. Since ∂U does not go around 0, ∂U_n is a quasicircle.

Let $f(z) = f_1(z) + 2\pi i$. Certainly, $f(U_n) = U_{n+1}$, that is, U_n is a wandering domain of $f(z)$. □

Finally, we mention that the function $z + e^z + 2\pi i$ possesses a wandering domain which has a complicated boundary. In fact, ze^z has a parabolic domain V for 0 containing the negative real axis $(-\infty, 0)$. From [3, Theorem 1.1], the set Θ is dense in ∂V, where

$$\Theta(V) = \left\{e^{i\theta};\ \lim_{r\to 1^-} \psi(re^{i\theta}) = \infty\right\}$$

and ψ is a Riemann map from the unit disk Δ onto V. Then $z + e^z + 2\pi i$ has a wandering domain U such that $\exp U = V$. Since $\operatorname{Log} z$ can be separated into analytic branches in V, we denote by $\log_k z$ the branch such that $\log_k V = U$. Thus $\log_k \circ \psi$ is a Riemann map from Δ onto U and so $\Theta(U)$ is dense in ∂U.

4. Conformal conjugacies in Baker domains. We consider the holomorphic mapping $f(z)$ of a hyperbolic domain U onto itself. A domain $D \subset U$ is called an attracting disk of U (under $f(z)$), provided that D is simply connected, $f(D) \subset D$ and it satisfies the attracting property: given an arbitrary compact set $K \subset U$, there exists $N = N(K)$ such that $f^N(K) \subset D$. Then (D, ϕ, T, Ω) is called a conformal conjugacy (of $f(z)$ in U) if

(A1) D is an attracting disk of U;

(A2) $\phi : U \to \Omega$ is holomorphic and univalent in D and Ω is the right half-plane $H = \{z \in \boldsymbol{C}; \operatorname{Re} z > 0\}$ or $\boldsymbol{C}$;

(A3) T is a Möbius transformation mapping Ω onto itself and $\phi(D)$ is an attracting disk of Ω under T;

(A4) for any $z \in U$, we have

$$\phi(f(z)) = T(\phi(z)). \tag{21}$$

The function $\phi(z)$ may not be univalent in U, but the functional equation (21) does hold in U. The dynamics of $f(z)$ in D are equivalent to those of a Möbius transformation T in $\phi(D)$. After a suitable further conjugacy with a Möbius transformation, we formulate the conformal conjugacy (D, ϕ, T, Ω) (if it exists) into the occurrence of the following three cases:

(B1) $T(z) = z + 1$, $\phi : U \to \boldsymbol{C}$;

(B2) $T(z) = z \pm i$, $\phi : U \to H$;

(B3) $T(z) = \lambda z$, $\lambda > 1$, $\phi : U \to H$.

Then the hyperbolic domain U is said to be of type I, II or III for $f(z)$, if, in turn, Cases B1, B2 or B3 take place. The holomorphic self-mapping $f : U \to U$ may not have any conformal conjugacy; for example, a meromorphic function $f(z)$ in its Herman ring and super-attracting stable domain has no conformal conjugacy, while in the attracting stable domain, parabolic domain or the Siegel disk, conformal conjugacy exists. We give a criterion for the existence of conformal conjugacy by using the results of Marden and Pommerenke [23] and Cowen [14].

THEOREM 4.1. *Let $f(z)$ be a holomorphic self-mapping of U and assume that U does not contain any super-attracting fixed points of $f(z)$. Then $f(z)$ in U possesses conformal* *conjugacy if and only if, for an arbitrary closed curve γ in U, there exists a positive integer $m = m(\gamma)$ such that $f^m(\gamma) \sim 0$ in U.*

Therefore, the existence of an attracting disk of U for $f(z)$ suffices to deduce the existence of conformal conjugacy. It was proved by König [21] that the Baker domain of a transcendental meromorphic function with only finitely many poles possesses conformal conjugacy and an example of a Baker domain which has no conformal conjugacy was constructed.

We consider the connection between the Baker domain which has conformal conjugacy and the singular value. The Baker domain of type I contains at least one singular value; the Baker domains of types II and III may not contain any singular values. We shall describe these situations through examples.

By using the logarithmic change of variables, that is, $\log g(e^z)$, we construct Baker domains of types I, II and III which have special properties. This method was used in Herman [19]. To this end, we take into account the connection between $f(z)$ and $g(z)$ such that

$$\exp f(z) = g(e^z) . \tag{22}$$

By a simple calculation, we deduce that e^a is a critical value of $g(z)$, if a is a critical value of $f(z)$; also, if b is a non-zero critical point of $g(z)$, then $z_n = \log b + 2n\pi i$, where n is an integer, is a critical point of $g(e^z)$ and hence of $f(z)$, and $f(z_n)$ is a critical value of $f(z)$.

It follows from Corollary 3.1 that $\exp J(f) = J(g)$ and $\exp F(f) = F(g)$. Given a component V of $F(g)$, we have a component U of $F(f)$ such that $V = \exp U$ and $\partial V = \exp \partial U$. From the discussion in the above paragraph, it is easy to see that, if U contains a critical value a, then $V = \exp U$ contains a critical value e^a of $g(z)$.

EXAMPLE 1. Set

$$g_1(z) = 3z^k \exp\left(\frac{1}{2} - \frac{z}{z-1}\right)$$

and

$$f_1(z) = kz + \log 3 + \frac{1}{2} - \frac{e^z}{e^z - 1} + 2m\pi i ,$$

where k is an integer larger than 1 and m is an integer. Then $f_1(z)$ possesses two invariant Baker domains U_1 and U_2 of type III such that

$$\operatorname{dist}(U_1, P(f_1)) > 0 , \tag{23}$$

∂U_1 is a quasi-circle, $f_1(z)$ is univalent in U_1 and U_2 contains infinitely many critical values.

PROOF. The point 0 is a super-attracting fixed point of $g_1(z)$. Assume that V_1 is the component of $F(g_1)$ containing 0. We want to prove that V_1 does not contain any non-zero critical points. From the equation

$$g_1'(z) = 3z^{k-1} \exp\left(\frac{1}{2} - \frac{z}{z-1}\right)\left[k + \frac{z}{(z-1)^2}\right]$$

we solve the two critical points of $g_1(z)$,

$$z_{1,2} = \frac{2k - 1 \pm i\sqrt{4k-1}}{2k} .$$

We can check that $|z_{1,2}| = 1$. Since $w = 1/2 - z/(z-1)$ conformally maps the unit circle $\{|z| = 1\}$ onto the imaginary axis $x = 0$, $g_1(z)$ maps the unit circle $\{|z| = 1\}$ into the circle $\{|z| = 3\}$, and it is clear that the unit circle $\{|z| = 1\}$ cannot intersect the boundary of V_1.

The Möbius transformation $M(z) = 1/2 - z/(z-1)$ keeps the real axis invariant. For any positive number $R \geq 3$, $M(z)$ maps the circle $\{|z| = R\}$ onto one circle which goes through the points $z = 1/2 - R/(R-1)$ and $z = 1/2 - R/(R+1)$ orthogonal to the real axis. By noting that $1/2 - R/(R-1) \geq -1$, we have

$$|g_1(z)| \geq 3R^k e^{-1} > R\,, \quad \text{on } |z| = R\,,$$

and hence $f_1(\{|z| \geq 3\}) \subset \{|z| > 3\}$. It is easy to see that ∞ is also a super-attracting fixed point of $g_1(z)$ and $\{|z| \geq 3\}$ is contained in an invariant attracting component V_2 of $F(g_1)$ of ∞. Thus, $g_1^n(z_{1,2}) \to \infty$ $(n \to \infty)$, and $\mathrm{dist}(V_1, P(g_1)) > 0$. We can prove that the boundary ∂V_1 is a quasi-circle.

There exists a unique invariant component U_1 of $F(f_1)$ such that $\exp U_1 = V_1$. It is obvious that U_1 is an invariant Baker domain of $f_1(z)$ of type III such that (23) holds and $\partial U_1 = \mathrm{Log}\, \partial V_1$. Since U_1 is simply connected and does not contain any singularities of the inverse f_1^{-1}, it is easy to see that $f_1(z)$ is univalent in U_1.

There exists a unique invariant component U_2 of $F(f_1)$ such that $\exp U_2 = V_2$. Certainly, U_2 is an invariant Baker domain of $f_1(z)$ of type III containing infinitely many critical values. □

An entire function was constructed in [9] to have a Baker domain such that (23) holds.

EXAMPLE 2. Set

$$g_2(z) = e^{2\pi i\alpha} z \exp\left(z + \frac{1}{z-1} + 1\right)$$

and

$$f_2(z) = 2\pi i\alpha + 1 + z + e^z + \frac{1}{e^z - 1}\,,$$

where α is an irrational number satisfying the condition of Siegel type. Then $f_2(z)$ has an invariant Baker domain U of type II in which $f_2(z)$ is univalent and $\partial U \subseteq \overline{P(f_2)}$.

PROOF. By a simple calculation, 0 is a Siegel fixed point of $g_2(z)$. Then there exists a Siegel disk V of $g_2(z)$ containing 0. Note that $g_2(z)$ is univalent in V and

$$\partial V \subseteq \overline{P(g_2)}. \tag{24}$$

There exists a unique invariant component U of $F(f_2)$ such that $\exp U = V$. It is obvious that U is an invariant Baker domain of $f_2(z)$ of type II. For any two points z_1 and z_2 in U such that $f_2(z_1) = f_2(z_2)$, from $\exp f_2(z) = g_2(e^z)$ we have $e^{z_1} = e^{z_2}$, so $z_1 = z_2 + 2m\pi i$. Therefore, using $f_2(z_1) = f_2(z_2)$ implies $z_1 = z_2$, that is, $f_2(z)$ is univalent in U. From (24), $\partial U \subseteq \overline{P(f_2)}$. □

EXAMPLE 3. Set

$$g_3(z) = e^{2\pi i\alpha} z \exp\left(z\frac{2-z}{2z-1} - \frac{2z-1}{z(2-z)}\right)$$

and

$$f_3(z) = 2\pi i\alpha + z + e^z\frac{2-e^z}{2e^z-1} - \frac{2e^z-1}{e^z(2-e^z)}\,.$$

By choosing an appropriate α, $g_3(z)$ has a Herman ring which contains the unit circumference and $f_3(z)$ has an invariant Baker domain U of type II containing the imaginary axis, whose boundary consists of two components in $\boldsymbol{C}$ and in which $f_3(z)$ is univalent and $\partial U \subseteq \overline{P(f_3)}$.

PROOF. By a simple calculation, $g_3(z)$ maps the unit circumference onto itself and $\arg g_3(e^{i\theta}) = 2\pi i\alpha + \theta + 2\sin(\arg(2-e^{i\theta}))\ (\mathrm{mod}\ 2\pi)$. From the result of Yoccoz [30], for an appropriate α, $g_3(z)$ is analytically conjugate in a neighbourhood $\tilde{V}$ of the unit circumference to a rotation $z \to e^{2\pi i\beta}z$, where β is the rotation number of $g_3(z)$ depending on α. Thus, $g_3(z)$ has a Herman ring V which contains the unit circumference and $\partial V \subseteq \overline{P(g_3)}$. From $\exp f_3(z) = g_3(e^z)$, we can deduce that $f(z)$ possesses the desired properties. □

In the Baker domain U in Examples 2 and 3, we take a point a and $f_j(a)$ $(j = 2, 3)$ and draw a simple curve γ connecting a and $f_j(a)$ in U. We have an analytic branch of f^{-1} which maps U onto itself, which we still denote by f^{-1}. Set

$$\Gamma = \bigcup_{j=-\infty}^{\infty} f^j(\gamma)\,.$$

Then Γ goes forward to infinity on two sides. In Example 2, Γ does not separate the boundary of U, while in Example 3 it does.

We do not give here an example of a Baker domain of type II which contains singular values or such that (23) holds.

EXAMPLE 4. Set

$$g_4(z) = z\exp\left(z + \frac{z-1}{z+1} + 1\right)$$

and

$$f_4(z) = 1 + z + e^z + \frac{e^z-1}{e^z+1}\,.$$

Then $f_4(z)$ has an infinite number of invariant Baker domains of type I containing at least one critical value.

PROOF. Obviously, $g_4(0) = 0$ and $g_4'(0) = 1$. There exists a parabolic domain V of $g_4(z)$ for 0 containing at least one critical value of $g_4(z)$. From (22), we can deduce that $f_4(z)$ possesses an infinite number of invariant Baker domains U_n of type I such that $\exp U_n = V$ $(n = 0, \pm 1, \pm 2, \ldots)$. □

Finally, we mention that by using the logarithmic change of variables, we can construct some transcendental meromorphic functions which have wandering domains. We can prove that $2m\pi i + f_4(z)$ and $2m\pi i + z + e^z + 1/(e^z - 1) + 1$, $m \neq 0$, have wandering domains which contain critical values. As Baker did in [2], we can prove that $2m\pi i + z + K^2(e^z)$, $m \neq 0$, has an infinity of different families of wandering domains, if $K(z)$ is a meromorphic function with infinitely many zeros.

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