Iteration of Morphological Transformations

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Morphological transformations play an important role in the morphological analysis of images. Certain features of images can be revealed by putting them through a cleverly chosen sequence of morphological transformations. This paper deals with some aspects of morphological transformations, in particular with the question under what conditions iteration of such mappings yields openings or closings.

Note. This paper is dedicated to Prof. H.A. Lauwerier on occasion of his 65th birthday.

1. Introduction

Mathematical morphology is a particular branch in image processing which is concerned with the development and study of probabilistic models for images, the investigation of image transformations and functionals, and the design of specific algorithms for image analysis. It derives its tools from algebra, topology, integral geometry, and stochastic geometry. Mathematical morphology was founded by Matheron [5] and Serra [9]. After a long period of relative obscurity mathematical morphology is becoming more and more popular, not in the least because of its interesting and challenging mathematical aspects: for a number of recent results we refer to Maragos [4], Serra [10] and Heijmans and Ronse [1,2,8].

In this paper our attention will be focussed on binary (= black and white) images, although an extension to grey-level functions is straightforward. Moreover we will be concerned only with morphological image transformations. These are transformations which map the space of all subsets of $\mathbb{R}^d$(continuous case) or $\mathbb{Z}^d$ (discrete case) into itself and which are invariant under translations. Or in mathematical language: a morphological transformation is a transformation ψ which maps the space $\mathcal{P}(E)$ of all subsets of E (where $E=\mathbb{R}^d$ or $\mathbb{Z}^d$) into itself and satisfies

$$\psi(X_h)=(\psi(X))_h, \text{ for } X\subseteq E \text{ and } h\in E. \tag{1.1}$$

Here $X_h=\{x+h|x\in X\}$ is the translate of X along h. We point out that there does not exist an unambiguous definition of a morphological transformation in the literature: for our purpose the above definition will do. Morphological transformations are used to detect certain features of an object, or, as Serra puts it in the introduction to [7]:

'In order to compare bodies, to recognize them, and to uncover their genesis, or to follow their evolution in time – in brief, to reduce them to their essentials – mathematical morphology classifies them into groups of more or less similar entities by putting them through sequences of set transformations.'

A common feature of morphological transformations (we will see some typical examples in the forthcoming sections) which, from a mathematical point of view, makes them very interesting, is their nonlinear and irreversible nature. Furthermore, they are often *locally defined*: to determine if h lies in $\psi(X)$ one does not have to know the entire image X but only $X \cap M_h$, the part of X within some bounded mask M positioned at h. To be precise

$$h \in \psi(X) \text{ if and only if } h \in \psi(X \cap M_h). \tag{1.2}$$

Of course, M depends on ψ and by iteration of ψ the size of the mask may increase, and eventually it may become unbounded.

The remainder of this section will be used to introduce some further terminology. Let ψ: $\mathcal{P}(E) \to \mathcal{P}(E)$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$, be an arbitrary mapping. Then ψ is called

- *increasing* if $X \subseteq Y$ implies $\psi(X) \subseteq \psi(Y)$
- *extensive* if $X \subseteq \psi(X)$, for every $X \subseteq E$
- *anti-extensive* if $\psi(X) \subseteq X$, for every $X \subseteq E$
- *idempotent* if $\psi^2(X) = \psi(X)$, for every $X \subseteq E$.

Here $\psi^2 = \psi \circ \psi$. We denote by *id* the identity mapping given by $id\,(X) = X$. If ψ_1, ψ_2 are two mappings then we write $\psi_1 \leqslant \psi_2$ if $\psi_1(X) \subseteq \psi_2(X)$ for every $X \subseteq E$. Thus a mapping is extensive if $\psi \geqslant id$. If ψ_i is a mapping for any i in the index set I, then $\cap_{i \in I} \psi_i$ is the mapping given by

$$(\cap_{i \in I}\, \psi_i)(X) = \cap_{i \in I}\, \psi_i(X),\ X \subseteq E.$$

$\cup_{i \in I}\, \psi_i$ is defined similarly. The *dual* (or *complementary*) mapping of ψ is defined as

$$\psi^*(X) = (\psi(X^c))^c,\ X \subseteq E.$$

Here X^c stands for the complement of X.

Finally we devote some words to the discrete space $E = \mathbb{Z}^2$ subdivided by the square grid. In this case an image X can be represented as a (possibly infinite) collection of pixels. Every pixel has 4 horizontal and vertical neighbours, the so-called 4-neighbours. In addition it has 4 diagonal neighbours which together with the 4-neighbours are called the 8-neighbours. An object $X \subseteq \mathbb{Z}^2$ is called 4-connected if for every pair h, $k \in X$ there exists a sequence $h = h_0, h_1, h_2, \ldots, h_m = k$ in X such that h_{i-1} and h_i $(i = 1, \ldots, m)$ are 4-neighbours. Similarly, 8-connectedness is defined. One can still think of other neighbourhood relations. Serra and co-workers [9] have chosen to work on the hexagonal grid where every pixel has six neighbours. The main advantage of the hexagonal grid over the square grid is that it has more rotational symmetry. This is reflected in the resulting algorithms which are simpler than in the square case.

2. Dilation, erosion, and Matheron's theorem

In classical signal analysis an important role is played by linear operations such as convolution and Fourier transformation. The characteristic feature of mathematical morphology is that the object space is not a vector space but has a lattice structure [1,2,10]. It is therefore not too surprising that the two basic morphological transformations, dilation and erosion, are in a very specific manner compatible with the ordering structure of this space: dilation since it commutes with unions, and erosion since it commutes with intersections. The mathematical definitions are as follows. Let $A \subseteq E$. The *dilation* of the image X by A is defined as

$$X \oplus A = \cup_{h \in A} X_h. \tag{2.1}$$

The *erosion* of X by A is given by

$$X \ominus A = \cap_{h \in A} X_{-h}. \tag{2.2}$$

The operations $\oplus$ and $\ominus$ have first occurred in the context of *integral geometry* [5] where they were called Minkowski addition and Minkowski subtraction, respectively. We wish to point out that our definition of erosion is slightly different from that in [1,9,10]. In the above definitions A is called the *structuring element* and is usually assumed to be compact (if $E = \mathbb{R}^d$) or finite (if $E = \mathbb{Z}^d$). In the discrete case $E = \mathbb{Z}^2$ the most obvious choices for A are the set consisting of one central pixel and its four horizontal and vertical neighbours (hereafter called CROSS), and the set which contains in addition its four diagonal neighbours (hereafter called SQUARE). If $0 \in A$ then a natural though unnecessary restriction is

$$X \ominus A \subseteq X \subseteq X \oplus A.$$

The action of dilation and erosion is depicted in Figure 1. Throughout this paper we use the following convention in illustrations. Points belonging both to the initial image X and its transform $\psi(X)$ are denoted by ■, points belonging to $\psi(X)$ but not to X are denoted by ◘ and points in X which do not belong to $\psi(X)$ are denoted by □.

Let

$$\check{A} = \{-h \mid h \in A\}$$

be the reflected structuring element. The following algebraic relations hold.

$$(X \oplus A)^c = X^c \ominus \check{A} \tag{2.3}$$

$$(X \oplus A) \oplus B = X \oplus (A \oplus B) \tag{2.4a}$$

$$(X \ominus A) \ominus B = X \ominus (A \oplus B) \tag{2.4b}$$

$$X_h \oplus A = (X \oplus A)_h \tag{2.5a}$$

$$X_h \ominus A = (X \ominus A)_h \tag{2.5b}$$

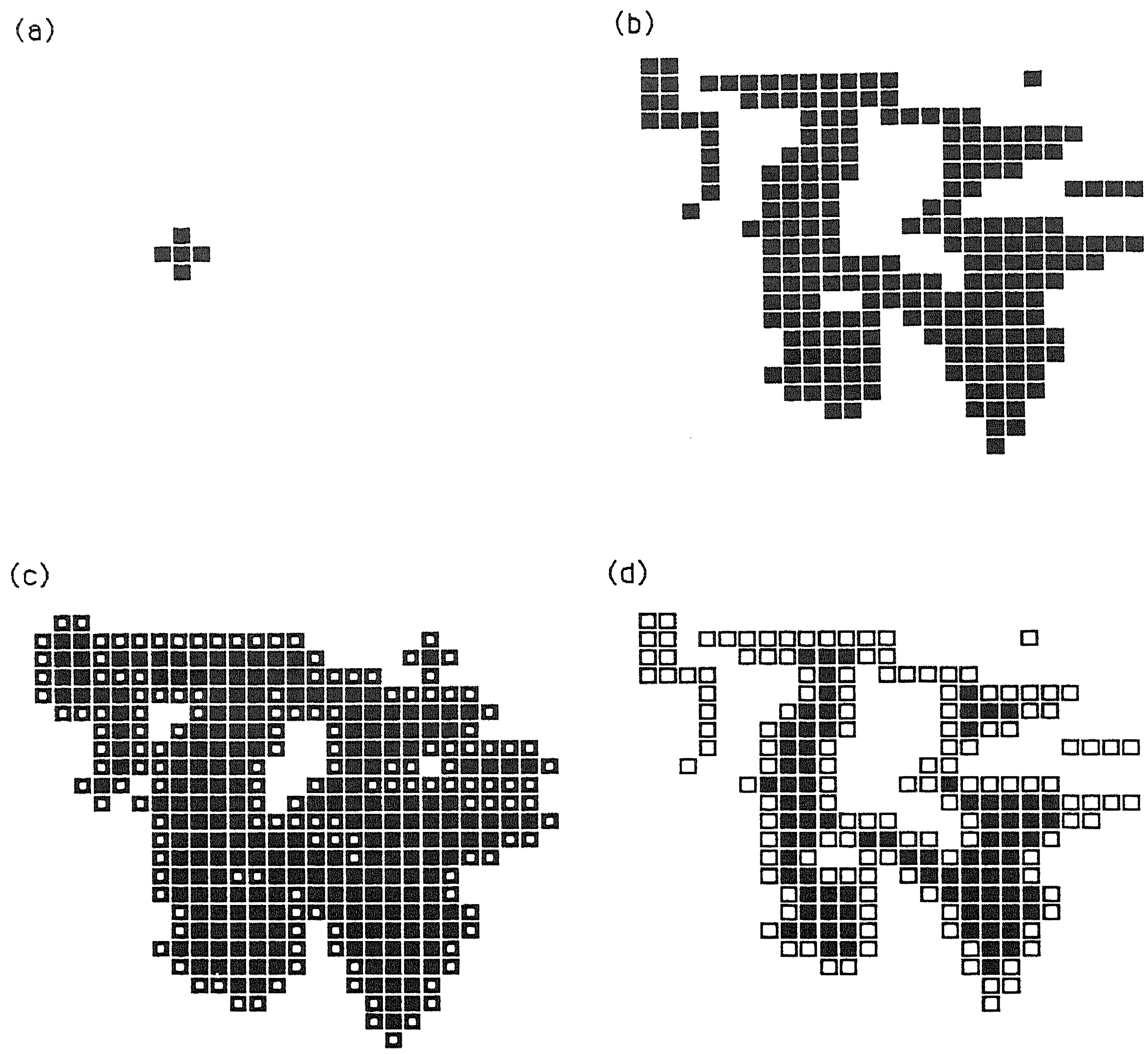


FIGURE 1. Dilation and erosion of X by A. (a) Structuring element A. (b) Original image X. (c) Dilated set $X \oplus A$. (d) Eroded set $X \ominus A$.

$$(\cup_{i \in I} X_i) \oplus A = \cup_{i \in I}(X_i \oplus A) \tag{2.6a}$$

$$(\cap_{i \in I} X_i) \ominus A = \cap_{i \in I}(X_i \ominus A). \tag{2.6b}$$

Here $X_i \subseteq E$ for every i in the (finite or infinite) index set I. The duality relation (2.3) states that dilation of an object yields the same result as the erosion of its background. The main implication of this relation is that properties of dilations and erosions always occur in pairs: to every property of dilations there corresponds a dual property of erosions and vice versa. Yet there exists a second duality relation between dilations and erosion which is only based on the (partial) ordering relation $\subseteq$ and which for that reason can also be used in more general situations:

$$X \oplus A \subseteq Y \text{ if and only if } X \subseteq Y \ominus A. \tag{2.7}$$

In a slightly different setting, this relation is also known as *Galois connection* or *adjunction*: see [1]. From (2.7) it follows immediately that

$$(X \ominus A) \oplus A \subseteq X \subseteq (X \oplus A) \ominus A, \tag{2.8}$$

where, in general, the inclusions may be strict. Both dilation and erosion belong to the class of increasing and translation-invariant transformations. It is easily seen that by taking intersections and unions of dilations and erosions, respectively, we stay within this class. But the converse also holds: every increasing translation-invariant mapping on $\mathcal{P}(E)$ can be decomposed as an intersection of dilations or, equivalently, as a union of erosions. This is essentially the content of Matheron's theorem which we state below. Before doing so we introduce the notion of a kernel. Let $\psi: \mathcal{P}(E) \to \mathcal{P}(E)$ be an arbitrary mapping. Then its kernel $\mathcal{V}[\psi]$ is defined as

$$\mathcal{V}[\psi] = \{A \subseteq E \mid 0 \in \psi(A)\}. \tag{2.9}$$

By $\mathcal{V}^*$ we denote the kernel of the dual mapping i.e. $\mathcal{V}^* = \mathcal{V}[\psi^*]$.

THEOREM 2.1 (MATHERON'S THEOREM). *Let $\psi: \mathcal{P}(E) \to \mathcal{P}(E)$ be an increasing translation-invariant mapping with kernel $\mathcal{V}$. Then*

$$\psi(X) = \cup_{A \in \mathcal{V}} (X \ominus A) = \cap_{A \in \mathcal{V}} (X \oplus \check{A}).$$

We give an illustration of this theorem by applying it to the so-called *median filter*. Let $A \subseteq \mathbb{Z}^d$ be a finite structuring element containing an odd number of points, say $2p-1$. For an image X we define $\mu_A(X)$ by

$$h \in \mu_A(X) \text{ if and only if } \#(X \cap A_h) \geqslant p.$$

Here $\#(Y)$ denotes the number of points of Y. In Figure 2 we have depicted the action of the median filter μ_A in the 2-dimensional case with CROSS as structuring element. This particular example is called the 4-median filter. Before we decompose μ_A as a union of erosions we state a trivial but, from a practical point of view, very important result.

PROPOSITION 2.2. *Let ψ be an increasing translation-invariant transformation with kernel $\mathcal{V}$. If $\mathcal{V}_0 \subseteq \mathcal{V}$ is such that for every $A \in \mathcal{V}$ there exists an $A_0 \in \mathcal{V}_0$ such that $A_0 \subseteq A$, then*

$$\psi(X) = \cup_{A_0 \in \mathcal{V}_0} (X \ominus A_0).$$

In [4] Maragos has shown that under some extra assumption on ψ one can always choose $\mathcal{V}_0$ to be minimal. If the underlying space is discrete and if the transformation ψ is locally defined (c.f. Section 1) then $\mathcal{V}_0$ can be chosen finite. This is the content of the next theorem.

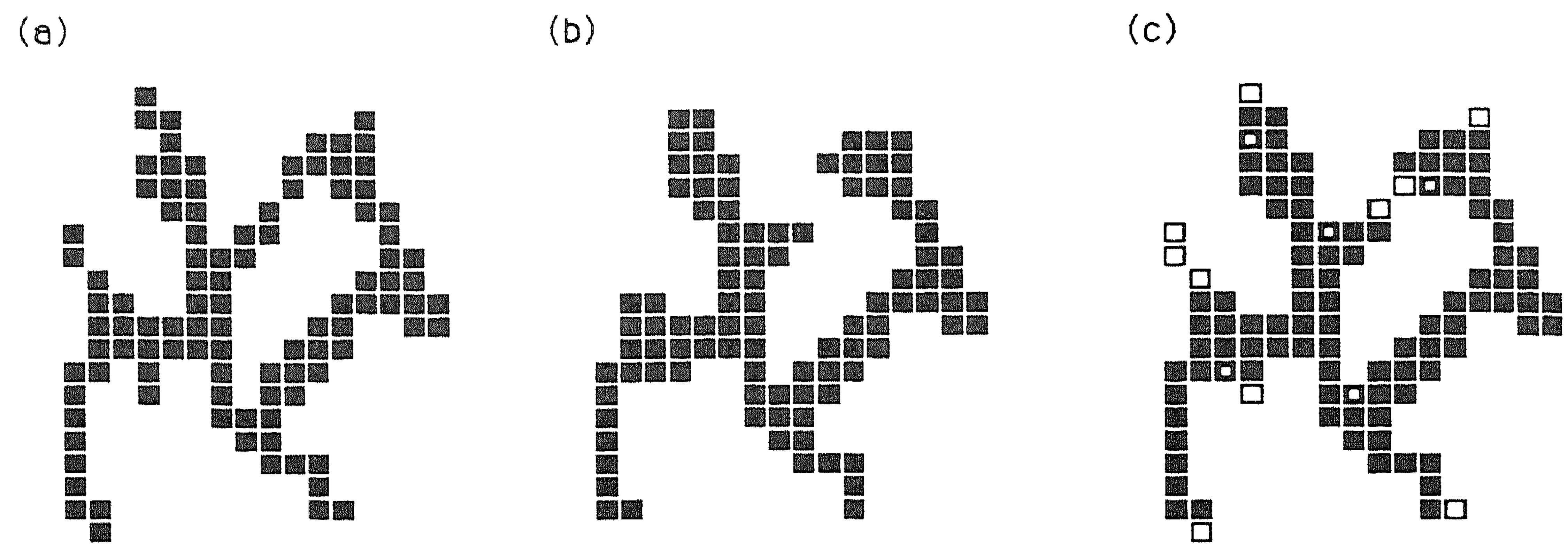


FIGURE 2. The 4-median filter. (a) Original image X. (b) 4-median of X. (c) Comparison of the original and the transformed image.

THEOREM 2.3. *Let $E = \mathbb{Z}^d$. If ψ is an increasing translation-invariant transformation which is locally defined, then there exists a finite subset $\mathcal{V}_0$ of $\mathcal{V}[\psi]$ such that*

$$\psi(X) = \cup_{A_0 \in \mathcal{V}_0} (X \ominus A_0).$$

PROOF. Let $\mathcal{V}$ be the kernel of ψ, i.e. $A \in \mathcal{V}$ if and only if $0 \in \psi(A)$. Since ψ is locally defined there exists a bounded (hence finite) mask $M \subseteq \mathbb{Z}^d$ such that

$$0 \in \psi(A) \text{ if and only if } 0 \in \psi(A \cap M).$$

Define

$$\mathcal{V}_0 = \{A \cap M | A \in \mathcal{V}\}.$$

Then $\mathcal{V}_0 \subseteq \mathcal{V}$, and for every $A \in \mathcal{V}$ there exists an $A_0 \in \mathcal{V}_0$ such that $A_0 \subseteq A$, namely $A_0 = A \cap M$. Since $\mathcal{V}_0 \subseteq \mathcal{P}(M)$ and M is finite, also $\mathcal{V}_0$ is finite. □

In the case of μ_A we have

$$\mathcal{V}[\mu_A] = \{B \subseteq \mathbb{Z}^d | \#(A \cap B) \geq p\},$$

and we may choose

$$\mathcal{V}_0 = \{B \subseteq \mathbb{Z}^d | B \subseteq A \text{ and } \#(B) = p\}.$$

Thus, by a well-known combinatorial result,

$$\# \mathcal{V}_0 = \begin{bmatrix} 2p-1 \\ p \end{bmatrix}.$$

In the example of Figure 2, we have $p = 3$ and hence $\#(\mathcal{V}_0) = 10$. The number $\#(\mathcal{V}_0)$ increases exponentially with p, and eventually there will be more economic ways to perform μ_A.

3. Iteration of transformations which are not increasing

Although a substantial part of mathematical morphology is concerned with increasing transformations, many important algorithms use transformations which are not increasing. We mention $X \to X^c$ as an obvious but also important member of this class. Adding this prototype example to the increasing transformations of the previous section and taking unions, intersections and compositions, we obtain a large family of morphological transformations which are not necessarily increasing. A member of this family which deserves extra attention is the so-called *hit-or-miss transformation*. Let A,B be two non-intersecting structuring elements. We define

$$X \circledast (A,B) = \{h \in E \mid A_h \subseteq X \text{ and } B_h \subseteq X^c\}. \tag{3.1}$$

One can easily give a description of this mapping in terms of erosions and dilations by using the following (geometric) characterization of erosion:

$$X \ominus A = \{h \in E \mid A_h \subseteq X\}. \tag{3.2}$$

Combining (3.1), (3.2) and (2.3) we get

$$\begin{aligned} X \circledast (A,B) &= (X \ominus A) \cap (X^c \ominus B) = (X \ominus A) \cap (X \oplus \check{B})^c \\ &= (X \ominus A) \setminus (X \oplus \check{B}). \end{aligned}$$

Putting $T = (A,B)$ we may also write $X \circledast T$ instead of $X \circledast (A,B)$. The hit-or-miss transformation is well suited to locate within an object points with certain (local) geometric properties, e.g. isolated points, border points, or corner points. In Figure 3 below, a hit-or-miss transformation is used to locate all lower left corner points of an object: note that the structure of (A,B) reflects the structure one is looking for within the object.

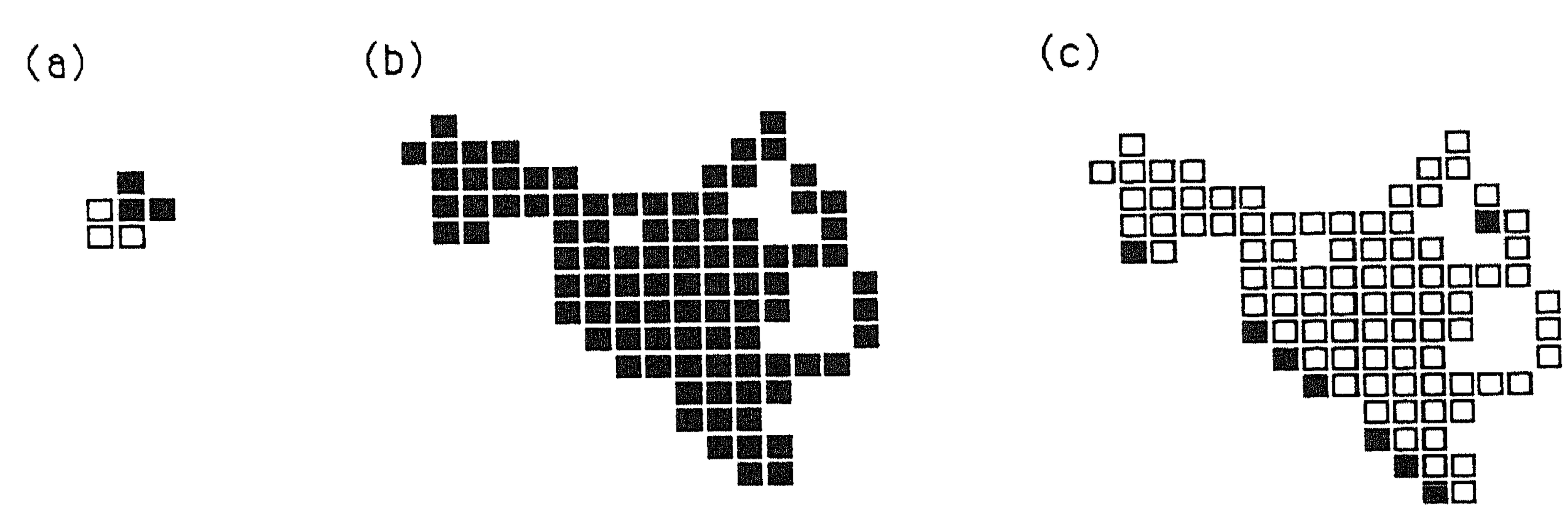


FIGURE 3. Hit-or-miss transformation. (a) Structuring element (A,B); the points of A are denoted by ■, the points of B by □. (b) Original image. (c) The hit-or-miss transformation $X \circledast (A,B)$ consists of all lower left corner points of X.

As we already mentioned, the hit-or-miss transformation plays a prominent role in quite a number of morphological image processing algorithms. Below, we shall describe in some detail two such algorithms: the (finite) convex hull and the skeleton. In both examples (and many others not mentioned here) the basic idea is to add or delete points from an object depending on the state of their neighbouring pixels. This is also the key feature of the so-called *cellular automata*: the state of a cell (pixel) at time $t+1$ is determined by its own state and those of its neigbours at time t: see [4]. Probably the best known example is Conway's game of life.

We introduce two other transformations which are not increasing, namely the *thickening* and the *thinning*. The thickening and the thinning of an image X with the structuring element $T=(A,B)$ are respectively defined as

$$X \odot T = X \cup (X \circledast T) \tag{3.3}$$

$$X \bigcirc T = X \setminus (X \circledast T). \tag{3.4}$$

Since $X \circledast T \subseteq X$ if $0 \in A$ and $X \circledast T = \varnothing$ if $A \cap B \neq \varnothing$, the thickening operator yields a non-trivial result only if $A \cap B = \varnothing$ and $0 \notin A$. Similarly, the thinning is a non-trivial operation if $A \cap B = \varnothing$ and $0 \notin B$. It is not difficult to check that thickening and thinning are dual operations:

$$X^c \odot (A,B) = (X \bigcirc (B,A))^c. \tag{3.5}$$

The examples which we give below involve a sequential application of the thickening and thinning operation. So let us first make some general remarks about iteration of morphological transformations. Let $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ be a translation-invariant (not necessarily increasing) transformation. Suppose furthermore that ψ is anti-extensive, i.e. $\psi(X) \subseteq X$ for all $X \subseteq E$. Then, for every non-negative integer k, $\psi^{k+1} \leqslant \psi^k$ and we define ψ^∞ by

$$\psi^\infty(X) = \bigcap_{k \geqslant 1} \psi^k(X). \tag{3.6}$$

Then ψ^∞ is translation-invariant and anti-extensive. Furthermore, ψ^∞ is increasing if ψ is. It is rather easy to show that ψ^∞ is idempotent if and only if $\psi \circ \psi^\infty = \psi^\infty$. In Section 4 we present an example of a morphological transformation ψ which is translation-invariant, increasing, and anti-extensive, but for which ψ^∞ is *not* idempotent.

For extensive mappings ψ we define ψ_∞ by

$$\psi_\infty(X) = \bigcup_{k \geqslant 1} \psi^k(X). \tag{3.7}$$

3.1. Example 1: Convex hull

Convexity of an object is a global property. Local information does not suffice to decide about the convexity of an object. So how can morphological transformations, which essentially only require local knowledge of an image (at least in the practical cases where the structuring elements are bounded), be used to construct the convex hull of an object? We consider the case $E = \mathbb{Z}^2$ and define the convex hull $CH(X)$ as the intersection of all discrete halfplanes which contain X: by a discrete halfplane we mean a set of points $(x,y) \in \mathbb{Z}^2$ satisfying the requirement $ax + by \leqslant c$, where $a,b,c \in \mathbb{Z}$. A set $X \subseteq \mathbb{Z}^2$ is called convex if $X = CH(X)$. For a thorough exposition on discrete convexity we refer to [7].

Obviously the mapping $X \rightarrow CH(X)$ is a translation-invariant and increasing mapping, and application of Matheron's theorem shows that $CH(X)$ can be obtained as a union of erosions. So far, so good. Unfortunately, the kernel $\mathcal{V}$ of CH contains infinitely many structuring elements and cannot be reduced to a finite set $\mathcal{V}_0$ as in Proposition 2.2: in fact, this would contradict the global character of convexity. Therefore the representation of CH as a union of erosions is not of any practical value.

If, in the definition of the discrete convex hull we only admit those halfplanes for which $|a|, |b| \leqslant 1$ (that are halfplanes bounded by lines whose angle with the positive x-axis is an integer multiple of 45°) then the situation changes drastically. We call the resulting notion 45-convexity and denote the 45-convex hull of X by $CH_{45}(X)$. There exists an algorithm which is essentially an iteration of thickenings and which yields the 45-convex hull if the initial object is 4-connected. Let $T_1, ..., T_8$ be as depicted in Figure 4a and define

$$\psi(X) = (...((X \odot T_1) \odot T_2) \odot \cdots \odot T_8).$$

Then ψ is an extensive, translation-invariant mapping and $\psi_\infty(X) = CH_{45}(X)$ if X is 4-connected. We shall not prove this result, but rather illustrate it by means of an example: see Figure 4b.

3.2. Example 2: Skeleton

Our second example handles the computation of the skeleton of a discrete object. Our use of words ('the' skeleton) might suggest that this notion is well-defined, but unfortunately this is far beyond the truth, both in the continuous and the discrete case. We refer the reader to chapter XI of [9] and chapters 11-13 of [10] for an interesting discussion on both the theoretical and practical aspects of the skeleton. A possible definition of the skeleton in the continuous case $E = \mathbb{R}^d$ goes as follows: The skeleton of an object X is the set of all points x such that the maximal ball B centered at x and contained in X, intersects the boundary of X in two or more points. One may think of the skeleton as a set which is a union of arcs and which has the same homotopy as the set X.

Algorithms for the computations of 'the' skeleton (at least, something which looks like it) in the discrete case are sometimes called *homotopic thinnings*. As an example we mention the sequential thinning by the structuring elements $T_1, T_2, ..., T_8$ of Figure 5a. Formally, this algorithm, which is originally due to Levialdi [3], transforms X into $\psi^\infty(X)$: here $\psi(X) = (...((X \bigcirc T_1) \bigcirc T_2) \bigcirc \cdots \bigcirc T_8)$. Again, a figure can explain more than thousand words.

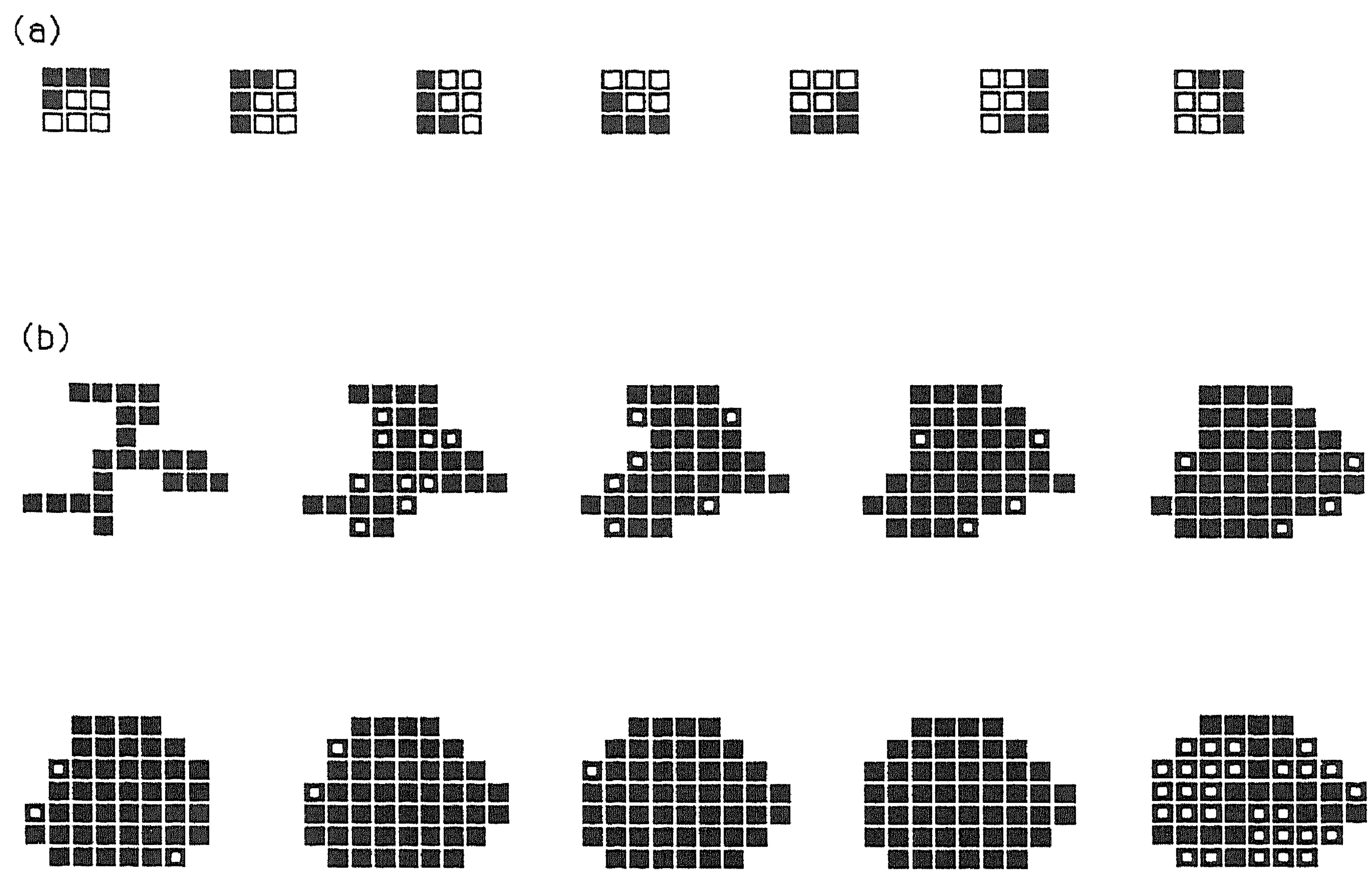


FIGURE 4: (a) Structuring elements for the computation of the 45-convex hull; T_2, T_3, T_4, etc. are the 45°-, 90°-, 135°-, etc. rotations of T_1. (b) The 45-convex hull can be computed by iteration of $\psi(X)=(...((X \odot T_1) \odot T_2) \odot ... \odot T_8)$, with $T_1, T_2, ..., T_8$ as depicted in (a). The first object is the initial image X, the second object is $\psi(X)$, etc. The one but last object shows the final result $\psi_\infty(X)=CH_{45}(X)$, which is reached after 7 iterations. The last object compares the original image to its 45-convex hull.

by iteration of $\psi(X)=(...((X \odot T_1) \odot T_2) \odot \cdots \odot T_8)$, with $T_1, T_2, ..., T_8$ as depicted in (a). The first object is the initial image X, the second object is $\psi(X)$, etc. The one but last object shows the final result $\psi_\infty(X)=CH_{45}(X)$, which is reached after 7 iterations. The last object compares the original image to its 45-convex hull.

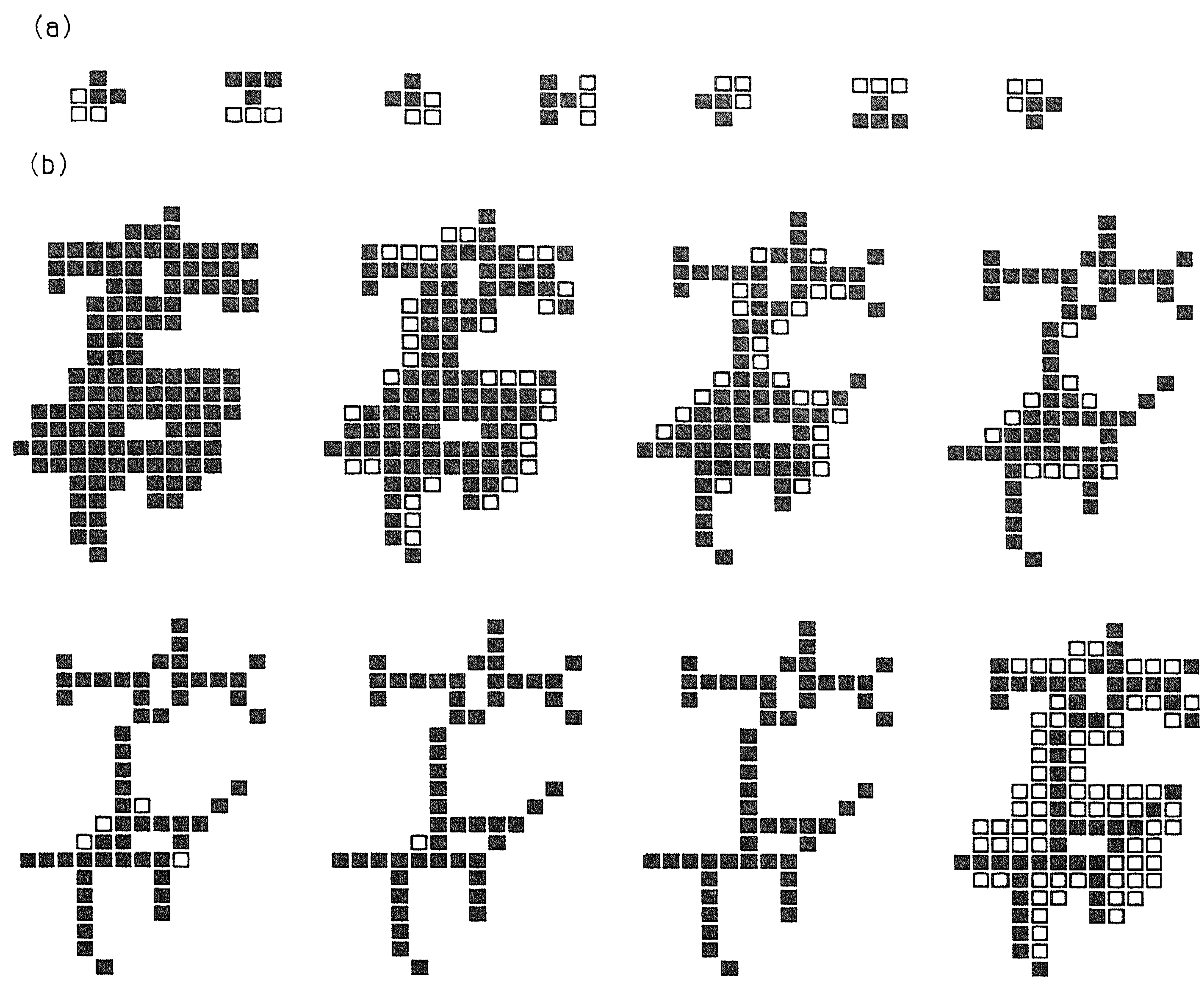


FIGURE 5. (a) Structuring elements for the computation of the homotopic thinning; T_3, T_5, T_7 are the 90°-, 180°-, 270°-, rotations of T_1. The same applies to T_2, T_4, T_6, T_8. (b) Homotopic thinning obtained by iteration of the mapping $\psi(X) = (...((X \bigcirc T_1) \bigcirc T_2) \bigcirc \cdots \bigcirc T_8)$, with $T_1, T_2, ..., T_8$ as depicted in (a) (Levialdi's algorithm). The first object is the initial image, the second $\psi(X)$, etc. The one but last object is $\psi\downarrow(X)$, the homotopic thinning of X, and is reached after 5 iterations. The last object compares the homotopic thinning $\psi^\infty(X)$ to the initial image X.

4. Opening, closing and iteration of increasing transformations

In Section 2 we saw that dilation and erosion are irreversible operations and that

$$(X \ominus A) \oplus A \subseteq X \subseteq (X \oplus A) \ominus A$$

where in general the inclusions are strict. The operations

$$X^A = (X \oplus A) \ominus A \qquad (4.1)$$

$$X_A = (X \ominus A) \oplus A \tag{4.2}$$

are called the *closing* resp. *opening* of X by A. Like dilation and erosion, closing and opening are translation-invariant increasing transformations, and they are related by the duality relation

$$(X^c)_A = (X^{\check{A}})^c. \tag{4.3}$$

Furthermore, one can easily show that

$$X_A = \cup \{A_h | h \in E, A_h \subseteq X\}, \tag{4.4}$$

or in words, the opening of X by A consists of all translates of A which are contained in X. Depending on the structure of A, one can think of the opening as an operation which deletes small isolated particles and removes thin recesses from an object. On the other hand, the action of the closing is best understood by interpreting it as the opening of the complement (background) with the reflected structuring element: it fills small holes and narrow coves in an object. In Figure 6 the closing and opening of the object of Figure 1 is performed: again we have chosen CROSS as structuring element.

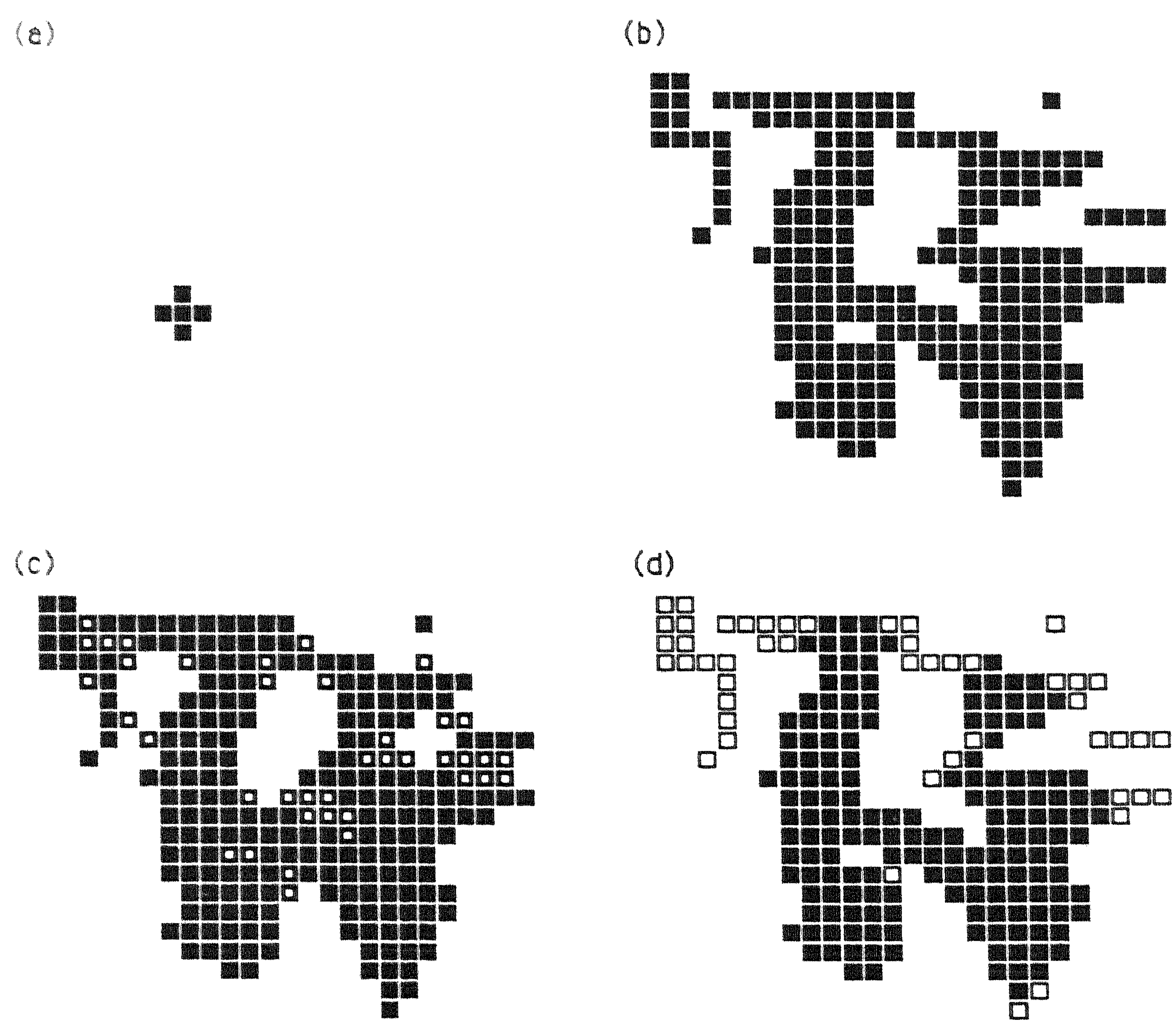


FIGURE 6: Closing and opening of X by A. (a) Structuring element A. (b) Orginal image X. (c) Closing X^A. (d) Opening X_A.

It follows from the geometric interpretation that a repeated application of the closing (or opening) has no further effect: both are idempotent transformations. Or in mathematical terms:

$$(X^A)^A = X^A, \ (X_A)_A = X_A.$$

DEFINITION. A mapping ψ: $\mathcal{P}(E) \to \mathcal{P}(E)$ is called a closing (opening) if ψ is translation-invariant, increasing, idempotent, and extensive (anti- extensive).

Generally we shall denote a closing by ϕ and an opening by α. In the exposition below we shall merely be concerned with openings, but it should be clear that by duality all that is going to be said about openings has its counterpart for closings: if α is an opening then α^* is a closing and vice versa.

The opening given by (4.2) is a very special one: we call it a *structural opening* to indicate that it uses one structuring element A. Below we shall prove that the class of structural openings forms a basis for the overall class of openings: every opening can be obtained as a union of structural openings. To do this we need some auxiliary results.

PROPOSITION 4.1. *Let α_i be an opening for every i in the index set I. Then $\alpha = \bigcup_{i \in I} \alpha_i$ is an opening as well.*

PROOF. Translation-invariance, increasingness, and anti-extensivity of α are trivial. We only prove the idempotence. Since $\alpha \leqslant id$ we immediately get that $\alpha^2 \leqslant \alpha$. On the other hand $\alpha^2 \geqslant \alpha_i \alpha_i = \alpha_i^2 = \alpha_i$, hence $\alpha^2 \geqslant \bigcup_{i \in I} \alpha_i = \alpha$. □

The action of a particular opening is best understood by studying its *domain of invariance*. Let ψ: $\mathcal{P}(E) \to \mathcal{P}(E)$ be an arbitrary mapping. The domain of invariance (= set of fixed points) of ψ is defined as

$$\mathrm{Inv}(\psi) = \{X \subseteq E \mid \psi(X) = X\}.$$

If α is an opening, then $\mathrm{Inv}(\alpha)$ is closed under translations and arbitrary unions (in particular, $\varnothing \in \mathrm{Inv}(\alpha)$). We say that $\mathrm{Inv}(\alpha)$ is *invariant under dilations*, since $A \in \mathrm{Inv}(\alpha)$ and $B \subseteq E$ arbitrary implies that $A \oplus B \in \mathrm{Inv}(\alpha)$. Our next result shows among others, that every opening is uniquely characterized by its domain of invariance. The proof is straightforward and we omit it.

PROPOSITION 4.2. *Let α_1, α_2 be openings. Then $\alpha_1 \geqslant \alpha_2$ if and only if $\mathrm{Inv}(\alpha_2) \subseteq \mathrm{Inv}(\alpha_1)$, and in that case $\alpha_1 \alpha_2 = \alpha_2 \alpha_1 = \alpha_2$. In particular, $\alpha_1 = \alpha_2$ if and only if $\mathrm{Inv}\alpha_1) = \mathrm{Inv}(\alpha_2)$.*

If $\mathbf{A} \subseteq \mathcal{P}(E)$ is invariant under dilations, then

$$\alpha_{\mathbf{A}}(X) = \bigcup \{A \in \mathbf{A} \mid A \subseteq X\} \tag{4.5}$$

defines an opening with $\mathrm{Inv}(\alpha_{\mathbf{A}}) = \mathbf{A}$. In particular, $\alpha_{\mathbf{A}}(X) \in \mathbf{A}$ for every $X \subseteq E$. From (4.5) it follows immediately that $X_A \subseteq \alpha_{\mathbf{A}}(X)$, for $A \in \mathbf{A}, X \subseteq E$, where for

$A = \alpha_{\mathbf{A}}(X)$ equality holds. This shows that

$$\alpha_{\mathbf{A}}(X) = \cup_{A \in \mathbf{A}} X_A. \tag{4.6}$$

THEOREM 4.3. *Every opening α can be written as the union of structural openings in the following way*

$$\alpha(X) = \cup_{A \in \mathrm{Inv}(\alpha)} X_A. \tag{4.7}$$

PROOF. Defining $\mathbf{A} = \mathrm{Inv}(\alpha)$ we get that $\mathbf{A} = \mathrm{Inv}(\alpha) = \mathrm{Inv}(\alpha_{\mathbf{A}})$, hence $\alpha = \alpha_{\mathbf{A}}$ by Proposition 4.2. Now the result follows immediately from (4.6). □

Since in most applications, $\mathrm{Inv}(\alpha)$ is very large, the practical use of (4.7) is rather limited. Fortunately, it is often possible to reduce the number of structuring elements in (4.7) considerably.

PROPOSITION 4.4. *Let α be an opening and let $\mathbf{A}_0 \subseteq \mathcal{P}(E)$ be such that* $\mathrm{Inv}(\alpha) = \{A \oplus B \mid A \in \mathbf{A}_0, B \subseteq E\}$, *i.e.,* $\mathrm{Inv}(\alpha)$ *is the smallest dilation-invariant family in $\mathcal{P}(E)$ which contains $\mathbf{A}_0$. Then*

$$\alpha(X) = \cup_{A \in \mathbf{A}_0} X_A. \tag{4.8}$$

The proof is left to the reader. Note that, because of Proposition 4.1, (4.8) defines an opening for every collection $\mathbf{A}_0$ of structuring elements.

EXAMPLE. The mapping $X \rightarrow \mathrm{int}(X)$, where $\mathrm{int}(X)$ denotes the interior of X, defines an opening on $\mathcal{P}(\mathbb{R}^d)$. For $\mathbf{A}_0$ we can choose the family of open balls B_r with radius $r > 0$. Then

$$\mathrm{int}(X) = \cup_{r>0} X_{B_r}, \quad X \subseteq \mathbb{R}^d.$$

So far, the results stated in this section are due to Matheron [5]. We now show how to construct openings from arbitrary increasing translation-invariant mappings. We refer to Section 5.7 of [10] for some related results. To our knowledge the results given below are new.

Let ψ be an increasing translation-invariant mapping which is anti-extensive. Then $\mathrm{Inv}(\psi)$ is a dilation-invariant subset of $\mathcal{P}(E)$. If α is the opening 'generated' by $\mathrm{Inv}(\psi)$, i.e. α is the opening with $\mathrm{Inv}(\alpha) = \mathrm{Inv}(\psi)$, then $\alpha \leqslant \psi$. Note that $\alpha = \psi$ if and only if ψ is an opening. Under some extra assumption on ψ, α can be obtained by iteration of ψ. Or, to put it differently: iteration of ψ yields an idempotent mapping. We refer to Theorem 4.5 below for a precise statement. We recall that

$$\psi^{\infty} = \bigcap_{n \geqslant 1} \psi^n.$$

In order for ψ^{∞} to be idempotent it is necessary and sufficient that

$\psi \circ \psi^\infty = \psi^\infty$, i.e., $\psi(\cap_{n \geqslant 1} \psi^n(X)) = \cap_{n \geqslant 1} \psi^n(X)$. The following counterexample illustrates that this identity does not hold in general.

COUNTEREXAMPLE. Let $E = \mathbb{Z}$ and let the structuring element A be given by $A = \{..., -7, -5, -3, -1, 2\}$. Define $\psi(X) = (X \oplus A) \cap X$. Then ψ is an increasing, translation-invariant, anti-extensive mapping on $\mathcal{P}(\mathbb{Z})$. However, ψ^∞ is not idempotent as we show now. Let $X = \{0, 1, 3, 5, 7, ...\}$. Then (see Figure 7) $\psi(X) = \{0, 3, 5, 7, ...\}$, $\psi^2(X) = \{0, 5, 7, ...\}$, etc. Hence $\psi^\infty(X) = \{0\}$. But $\psi(\psi^\infty(X)) = \psi(\{0\}) = \varnothing$.

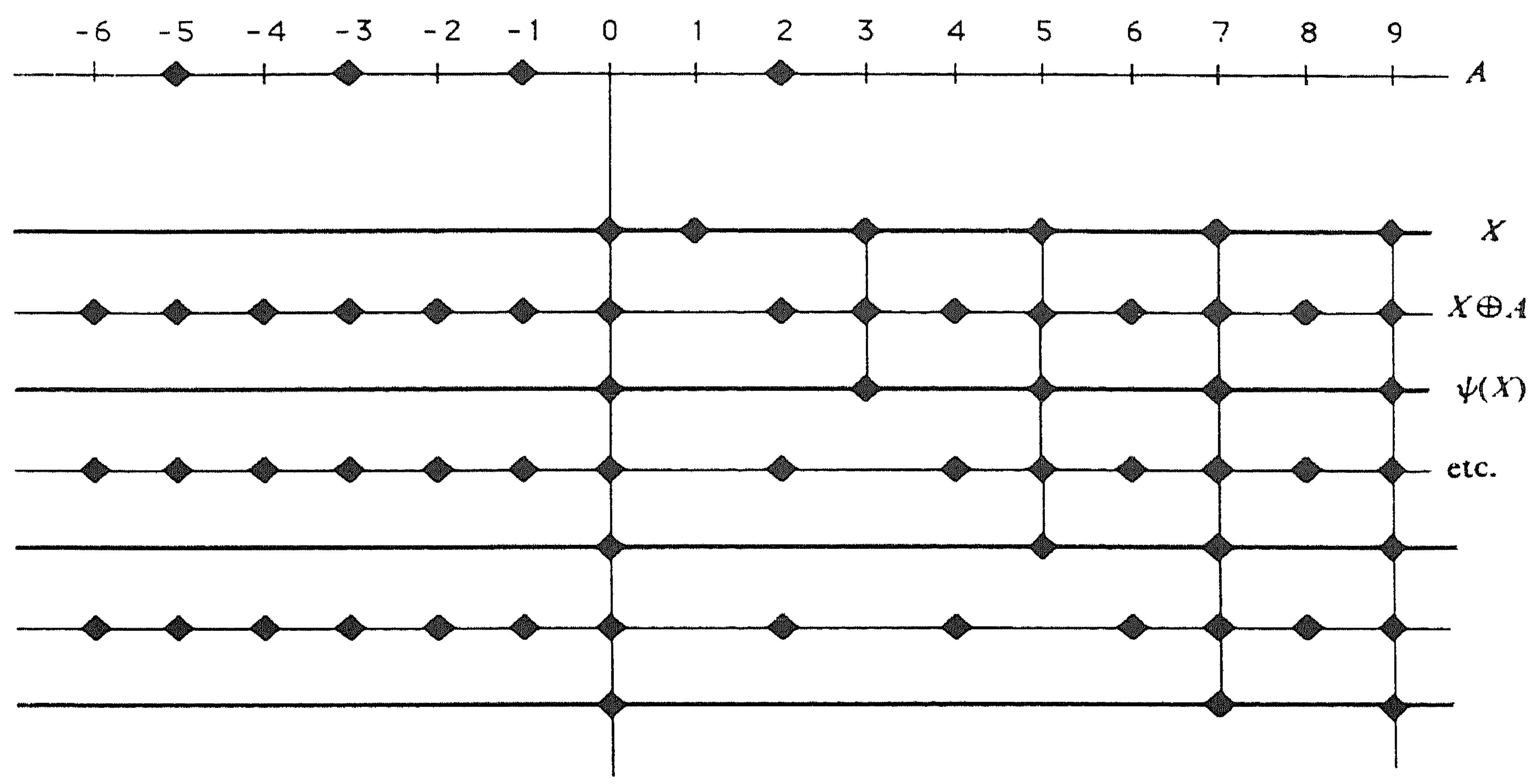


FIGURE 7. $\psi(X) = (X \oplus A) \cap X$, where $A = \{..., -5, -3, -1, 2\}$. If $X = \{0, 1, 3, 5, ...\}$ then $\psi^n(X) = \{0, 2n+1, 2n+3, ...,\}$. Hence $\psi^\infty(X) = \{0\}$. But $\psi \circ \psi^\infty(X) = \varnothing \neq \psi^\infty(X)$.

In order for ψ^∞ to be idempotent we have to impose an extra condition on ψ. Let $X_n \subseteq E$, $n \in \mathbb{N}$, and $X \subseteq E$. By $X_n \downarrow X$ we mean that

$$... \subseteq X_{n+1} \subseteq X_n \subseteq X_{n-1} \subseteq ... \subseteq X_1,$$

and that

$$\cap_{n \geqslant 1} X_n = X,$$

i.e., X_n is a non-increasing sequence in $\mathcal{P}(E)$ which 'converges' to X. The mapping $\psi: \mathcal{P}(E) \to \mathcal{P}(E)$ is called *$\downarrow$-continuous* if $X_n \downarrow X$ implies that $\psi(X_n) \downarrow \psi(X)$ for every non-increasing sequence $\{X_n\}$.

THEOREM 4.5. *Let ψ be an increasing, translation-invariant, anti-extensive mapping which is $\downarrow$-continuous. Then ψ^∞ is an opening (in particular, ψ^∞ is idempotent) and* $\mathrm{Inv}(\psi^\infty) = \mathrm{Inv}(\psi)$.

PROOF. For every $X \subseteq E$ we have by definition that $\psi^n(X) \downarrow \psi^\infty(X)$, so by the $\downarrow$-continuity of ψ we get that $\psi^{n+1}(X) \downarrow \psi(\psi^\infty(X))$. Since the limits must be the same we find that $\psi(X) = \psi(\psi^\infty(X))$. The relation $\text{Inv}(\psi) = \text{Inv}(\psi^\infty)$ follows immediately. □

Since erosions commute with intersections (see (2.6b)) they automatically belong to the class of $\downarrow$-continuous mappings. But this class is actually much larger.

PROPOSITION 4.6. *Compositions, arbitrary intersections, and finite unions of $\downarrow$-continuous mappings are $\downarrow$-continuous. In particular, if A is finite, then the dilation $X \to X \oplus A$ is $\downarrow$-continuous.*

PROOF. We only prove that the union of a finite number of $\downarrow$-continuous mappings is $\downarrow$-continuous. The other statements are almost trivial.

Let ψ_i be $\downarrow$-continuous for $i = 1,...,p$, and define $\psi = \cup_{i=1}^p \psi_i$. We show that ψ is $\downarrow$-continuous. Let $X_n \downarrow X$. Since $X \subseteq X_n$ we have $\psi_i(X) \subseteq \psi_i(X_n)$ $(i = 1,...,p)$ and so $\psi(X) \subseteq \psi(X_n)$. This proves that $\psi(X) \subseteq \cap_{n \geq 1} \psi(X_n)$.

To prove the other inclusion, assume that $y \in \cap_{n \geq 1} \psi(X_n)$. Thus $y \in \psi(X_n) = \cup_{i=1}^p \psi_i(X_n)$, for every $n \geq 1$. So there must be some index i $(1 \leq i \leq p)$ and an infinite subsequence X_{n_k} $(k \geq 1)$ such that $y \in \psi_i(X_{n_k})$. But, since $\psi_i(X_{n_k})$ is decreasing, $y \in \psi_i(X_n)$ for all n. Now it follows from the $\downarrow$-continuity of ψ_i that $y \in \cap_{n \geq 1} \psi_i(X_n) = \psi_i(X) \subseteq \psi(X)$, and the result is proved. □

Combining these results one ends up with a large class of $\downarrow$-continuous mappings, in particular if the underlying space E is discrete. The following result is an immediate consequence of Theorem 2.3.

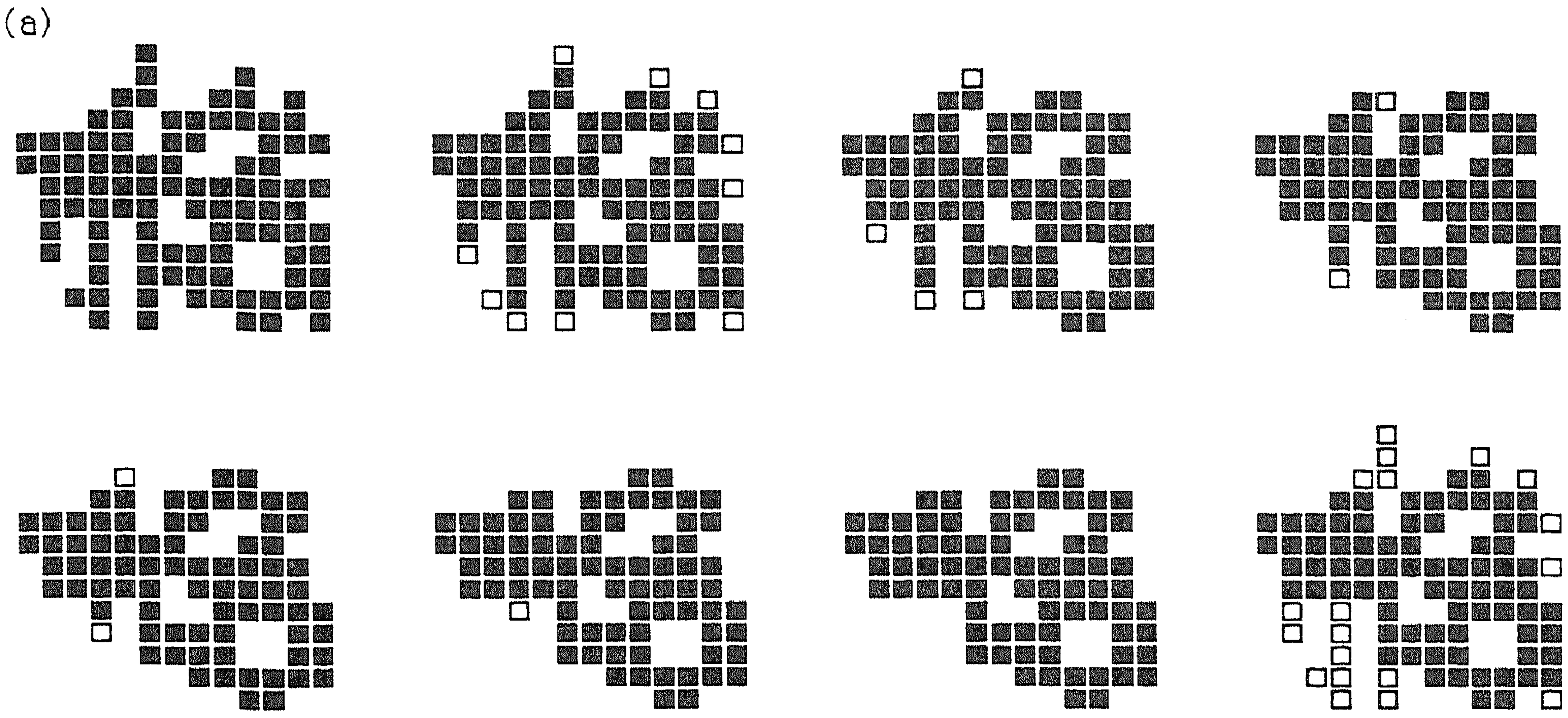
THEOREM 4.7. *Every increasing translation-invariant transformation on $\mathcal{P}(\mathbb{Z}^d)$ which is locally defined, is $\downarrow$-continuous.*

We conclude with two examples.

4.1. Example 1: The median opening

The median filter μ_A of Section 2, with $\#(A)$ finite and odd is a $\downarrow$-continuous mapping since it can be written as a finite union of erosions. So it follows from Theorem 4.5 that $\cap_{n \geq 1} (\mu_A \cap id)^n$ is an opening, the median opening. A good understanding of this operation can be achieved by finding its domain of invariance. In the example below (see Figure 8), where $E = \mathbb{Z}^2$ and A is the CROSS, the domain of invariance consists of all objects in which every pixel has at least two 4-neighbours.

(a)



(b)

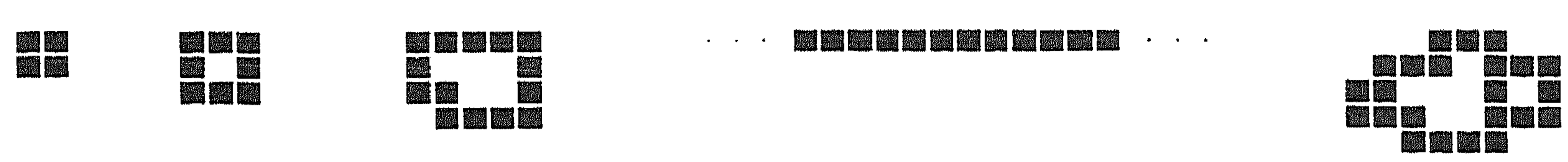


FIGURE 8. The median opening $(\mu_A \cap id)^{\infty}$ where $A = \text{CROSS}$. (a) The first object is the original image X, the second object is $\mu_A(X) \cap X$, etc. The one but last object is the median opening of X by A. The last object compares X with its median opening. (b) Some examples of objects which lie in the invariance domain of the median opening.

4.2. Example 2: Closing by iteration

By duality, all the results mentioned above for openings carry over to closings: if ψ is an increasing, translation-invariant, extensive mapping which is $\uparrow$-continuous, then $\psi_{\infty} = \cap_{n \geqslant 1} \psi^n$ is a closing with $\text{Inv}(\psi_{\infty}) = \text{Inv}(\psi)$. As an example we consider the mapping $\psi(X) = (X \oplus A) \ominus B$ where $B \subseteq A$ and B is finite. Then ψ is extensive since

$$\psi(X) \supset (X \oplus A) \ominus A \; = \; X^A \supset X. \qquad (*)$$

Moreover, ψ is $\uparrow$-continuous and we may therefore conclude that ψ_{∞} is a closing with invariant elements $\text{Inv}(\psi)$. It follows from $(*)$ that $X \in \text{Inv}(\psi_{\infty})$ implies $X^A = X$. In Figure 9 below we have chosen SQUARE and CROSS for A and B respectively. Then X is invariant under ψ_{∞} if and only if X consists of disjoint rectangles which lie 'far enough apart'. If desired the reader may give a precise description of $\text{Inv}(\psi_{\infty})$.

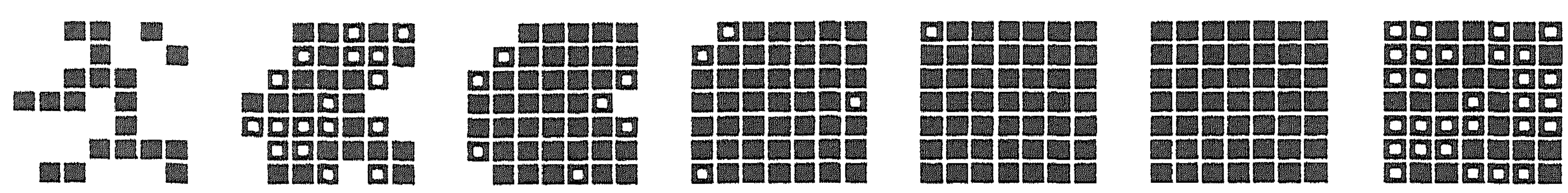


FIGURE 9. Iteration of the mapping $\psi(X)=(X \oplus A) \ominus B$ with $A=$SQUARE and $B=$CROSS. The final result $\psi_\infty(X)$ consists of a family of rectangles which surround the original image. The first object is X, the second $\psi(X)$, etc. The one but last object is $\psi_\infty(X)$ reached after 4 iterations. The last object compares X and $\psi_\infty(X)$.

REFERENCES

1. H.J.A.M. HEIJMANS (1987). Mathematical morphology: an algebraic approach. *CWI Newsletter 14,* 7-27.
2. H.J.A.M. HEIJMANS, C. RONSE (1988). *The Algebraic Basis of Mathematical Morphology. Part I: Dilations and Erosions,* CWI Report AM-R8807.
3. S. LEVIALDI (1971). Parallel pattern processing. *IEEE Trans. Syst. Man. Cybern SMC-1,* 292-296.
4. P. MARAGOS (1988). A representation theory for morphological image and signal processing. To appear in *IEEE Trans. Patt. Anal. Mach. Intell.*
5. G. MATHERON (1975). *Random Sets and Integral Geometry,* J. Wiley & Sons, New York.
6. K. PRESTON JR., M.J.B. DUFF (1984). *Modern Cellular Automata: Theory and Applications,* Plenum Press, New York and London.
7. C. RONSE (1985). Definitions of convexity and convex hulls in digital images. *Bull. Soc. Math. Belg. Ser. B-37,* 71-85.
8. C. RONSE, H.J.A.M. HEIJMANS (1989). The algebraic basis of mathematical morphology. Part II: Openings and closings. In preparation.
9. J. SERRA (1982). *Image Analysis and Mathematical Morphology,* Academic Press, London.
10. J. SERRA (ed.) (1988). *Image Analysis and Mathematical Morphology. Volume 2: Theoretical Advances,* Academic Press, London.