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ITERATIVE APPROXIMATION OF FIXED POINTS OF LIPSCHITZIAN STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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ABSTRACT. Suppose $X = L_p$ (or l_p), $p \geq 2$, and K is a nonempty closed convex bounded subset of X. Suppose $T: K \to K$ is a Lipschitzian strictly pseudo-contractive mapping of K into itself. Let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying:

(i) $0 < C_n < 1$ for all $n \geq 1$,
(ii) $\sum_{n=1}^{\infty} C_n = \infty$, and
(iii) $\sum_{n=1}^{\infty} C_n^2 < \infty$.

Then the iteration process, $x_0 \in K$,

$$x_{n+1} = (1 - C_n)x_n + C_n T x_n$$

for $n \geq 1$, converges strongly to a fixed point of T in K.

1. Introduction. Let X be a Banach space, $K \subseteq X$. A mapping $T: K \to K$ is said to be a *strict pseudo-contraction* if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1}$$

holds for all x, y in K and $r > 0$. If, in the above definition, $t = 1$, then T is said to be a *pseudo-contractive* mapping. Pseudo-contractive mappings have been studied by various authors (see e.g., [**1**, **5**, **6**, **10**, **11**, **12**]). Interest in such mappings stems mainly from the fact that a mapping T is pseudo-contractive if and only if $(I - T)$ is accretive [**5**, Proposition 1], where, for a mapping U with domain $D(U)$ and range $R(U)$ in an arbitrary Banach space X, U is said to be accretive [**5**] if the inequality

$$\|x - y\| \leq \|x - y + s(Ux - Uy)\| \tag{2}$$

holds for every x and y in $D(U)$ and for *all* $s > 0$. If (2) holds only for *some* $s > 0$, U is said to be *monotone* [**12**]. The mapping theory for accretive mappings is thus closely related to the fixed point theory of pseudo-contractive mappings.

The accretive operators were introduced independently by T. Kato [**12**] and F. E. Browder [**5**] in 1967. An early fundamental result in the theory of accretive operators, due to Browder [**5**], states that the initial value problem

$$du/dt + Tu = 0, \qquad u(0) = \omega$$

is solvable if T is locally Lipschitzian and accretive on X, a result which was subsequently generalized by R. H. Martin [**16**] to the continuous accretive operators.

In [**2**], J. Bogin considered the connection between strict pseudo-contractions and strictly accretive operators (defined below). He proved that U is a strict pseudo-contraction if and only if $(I - U)$ is a strict accretive operator. He further proved a

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fixed point theorem in Banach spaces for Lipschitz strict psuedo-contractions, and as a consequence obtained a mapping theorem of Browder for Lipschitzian strictly accretive operators.

Suppose now $X = L_p$ (or l_p), $p \geq 2$, and $K \subseteq X$. Suppose further that $T\colon K \to K$ is a Lipschitz strict pseudo-contraction with a nonempty fixed point set in K. Our objective in this paper is to prove that an iterative process of the type introduced by W. R. Mann [**15**] converges strongly to a fixed point of T.

REMARK. For Lipschitz pseudo-contractions with a nonempty fixed point set, it is an open question, even in Hilbert spaces, whether or not an iteration process of the Mann-type will converge strongly to a fixed point of T (see [**11**, p. 504].

2. Preliminaries. For a Banach space X we shall denote by J the duality mapping from X to 2^{X^*} given by

$$Jx = (f^* \in X^*\colon \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle),$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If X^* is strictly convex, then J is single-valued, and if X^* is uniformly convex, then J is uniformly continuous on bounded sets (see [**18, 19**]). Thus, by a single-valued normalized duality mapping, we shall mean a mapping $j\colon X \to X^*$ such that for each u in X, $j(u)$ is an element of X^* which satisfies the following two conditions:

$$\langle j(u), u \rangle = \|j(u)\| \cdot \|u\|, \qquad \|j(u)\| = \|u\|.$$

The accretiveness (or monotonicity) for U defined in (2) can also be expressed in terms of the duality map J as follows (see [**12**]). For each $x, y \in D(U)$, there is some $j \in J(x - y)$ such that

$$\mathrm{Re}\langle Ux - Uy, j \rangle \geq 0 \tag{3}$$

and (as was observed in [**12**]) if X is a Hilbert space, (3) is equivalent to the monotonicity of U in the sense of Minty [**17**].

Now let $K \subseteq X$. A mapping $A\colon K \to X$ is said to be *strictly accretive* if for each x, y in K there exists $\omega \in J(x - y)$ such that

$$(Ax - Ay, \omega) \geq k\|x - y\|^2 \tag{4}$$

for some constant $k > 0$. Without loss of generality we shall assume $k \in (0, 1)$.

In the sequel we shall assume that L_p, $p \geq 2$, has at least two disjoint sets of positive finite measure, X will denote L_p or l_p $(p \geq 2)$, and j will always denote the single-valued normalized duality maping of X into X^*. We shall need the following results.

LEMMA 1. *For the Banach space X, the following inequality holds for all x, y in X:*

$$\|x + y\|^2 \leq (p - 1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y) \rangle. \tag{5}$$

PROOF. For $X = L_p$ or l_p, $p \geq 2$, the following inequality holds (see, e.g., [**7**]). For all $x, y \in X$,

$$(p - 1)\|x + y\|^2 \geq \|x\|^2 + \|y\|^2 + 2\langle y, j(x) \rangle.$$

Now, replace x by y and y by $x - y$ to get

$$\|x - y\|^2 \leq (p - 1)\|x\|^2 + \|y\|^2 + 2\langle -x, j(y) \rangle.$$

Now replace x by $-x$ to obtain (5).

LEMMA 2 (DUNN [**9**, p. 41]). *Let β_n be recursively generated by*

$$\beta_{n+1} = (1-\delta_n)\beta_n + \sigma_n^2$$

with $n \geq 1$, $\beta_1 \geq 0$, $\{\delta_n\} \subseteq [0,1]$, and

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty, \tag{7a}$$

$$\sum_{n=1}^{\infty} \delta_n = \infty. \tag{7b}$$

Then $\beta_n \geq 0$, for $n \geq 1$, and $\beta_n \to 0$ as $n \to \infty$.

LEMMA 3 (BOGIN [**2**]). *Let X be a Banach space, K a subset of X, and $U\colon K \to X$. Then, if U is a strict pseudo-contraction, $T = I - U$ is strictly monotone, with $k = (t-1)/t$.*

PROOF. U is a strict contraction implies that for all $x, y \in K$, and $r > 0$, $t > 0$, we have

$$\begin{aligned}\|x-y\| &\geq \|(1+r)(x-y) - rt(Ux - Uy)\| \\ &= \|(1+r)(x-y) - r(tUx - tUy)\|.\end{aligned}$$

Thus, the mapping (tU) is pseudo-contractive, so by [**5**], the mapping T_t defined by $T_t = I - (tU)$ is monotone. So, for each x, y in K there exists $j \in J(x-y)$ such that

$$\langle T_t x - T_t y, j(x-y)\rangle \geq 0.$$

Observe that $T_t = I - (tU) = I - t(I-T) = tT - (t-1)I$, so that the above inequality yields

$$t\langle Tx - Ty, j(x-y)\rangle - (t-1)\langle x-y, j(x-y)\rangle \geq 0$$

which simplifies to

$$\langle Tx - Ty, j(x-y)\rangle \geq \frac{(t-1)}{t}\langle x-y, j(x-y)\rangle = k\|x-y\|^2,$$

where $k = (t-1)/t$, establishing the lemma.

3. Main result.

THEOREM. *Suppose K is a nonempty closed bounded convex subset of X and $T\colon K \to K$ is a Lipschitz strictly psuedo-contractive mapping of K into itself. Let $\{C_n\}$ be a real sequence satisfying:*

(i) $0 < C_n < 1$ *for all* $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} C_n = \infty$,

(iii) $\sum_{n=1}^{\infty} C_n^2 < \infty$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by $x_1 \in K$,

$$x_{n+1} = (1-C_n)x_n + C_n T x_n, \tag{8}$$

converges strongly to a fixed point of T.

PROOF. The existence of a fixed point follows from Deimling [**8**].

Let p be a fixed point of T. Since T is strictly pseudo-contractive, then $(I-T)$ is strictly accretive. Thus, there exists some $k \in (0,1)$ such that for each x, y in K

$$\operatorname{Re}\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq k\|x-y\|^2.$$

In particular,

$$\operatorname{Re}\langle (I-T)x - (I-T)p, j(x-p)\rangle \geq k\|x-p\|^2. \tag{9}$$

From (8),

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-C_n)(x_n-p) + C_n(Tx_n - Tp)\|^2 \\
&= (1-C_n)^2\|(x_n-p) + C_n(1-C_n)^{-1}(Tx_n-Tp)\|^2 \\
&\leq (1-C_n)^2[\|x_n-p\|^2 + C_n^2(1-C_n)^{-2}(p-1)\|Tx_n-Tp\|^2 \\
&\qquad + 2C_n(1-C_n)^{-1}\langle (Tx_n-Tp), j(x_n-p)\rangle]
\end{aligned}$$

so that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-C_n)^2\|x_n-p\|^2 + C_n^2(p-1)L^2\|x_n-p\|^2 \\
&\quad - 2C_n(1-C_n)\langle Tp - Tx_n, j(x_n-p)\rangle \\
&= (1-C_n)^2\|x_n-p\|^2 + (p-1)C_n^2L^2\|x_n-p\|^2 \\
&\quad - 2C_n(1-C_n)\langle Tp - Tx_n, j(x_n-p)\rangle \\
&\quad - 2C_n(1-C_n)\langle x_n - p, j(x_n-p)\rangle \\
&\quad + 2C_n(1-C_n)\langle x_n - p, j(x_n-p)\rangle \\
&= (1-C_n)^2\|x_n-p\|^2 + (p-1)L^2C_n^2\|x_n-p\|^2 \\
&\quad + 2C_n(1-C_n)\|x_n-p\|^2 \\
&\quad - 2C_n(1-C_n)\langle x_n - Tx_n - p + Tp, j(x_n-p)\rangle \\
&= [(1-C_n)^2 + 2C_n(1-C_n)]\|x_n-p\|^2 + (p-1)L^2C_n^2\|x_n-p\|^2 \\
&\quad - 2C_n(1-C_n)\langle (I-T)x_n - (I-T)p, j(x_n-p)\rangle \\
&\leq [(1-C_n)^2 + 2(1-k)C_n(1-C_n)]\|x_n-p\|^2 \\
&\quad + (p-1)L^2C_n^2\|x_n-p\|^2 \\
&\leq [(1-C_n)^2 + 2(1-k)C_n(1-C_n)]\|x_n-p\|^2 + d^2C_n^2,
\end{aligned}$$

where

$$d = (p-1)^{1/2}L\sup_{n\geq 1}\|x_n-p\|$$

and clearly, by adding $(1-k)^2C_n^2\|x_n-p\|^2$ to the right side of the above inequality, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq [(1-C_n)^2 + 2^{(1-k)}C_n(1-C_n) + (1-k)^2C_n^2]\|x_n-p\|^2 + d^2C_n^2 \\
&= [1-(1-k)C_n]^2\|x_n-p\|^2 + d^2C_n^2.
\end{aligned}$$

Set $\rho_n = \|x_n-p\|^2$, $1-\gamma_n = [1-(1-k)C_n]^2 \geq 0$ to obtain

$$\rho_{n+1} \leq (1-\gamma_n)\rho_n + C_n^2d^2. \tag{10}$$

The inequality (10) and a simple induction now yield

$$0 \leq \rho_n \leq B^2\alpha_n \quad \text{for all } n \geq 1, \tag{11}$$

where $\alpha_n \geq 0$ is recursively generated by

$$\alpha_{n+1} = (1-\gamma_n)\alpha_n + C_n^2, \qquad \alpha_1 = 1, \tag{12}$$

and $B^2 = \max\{\rho_1, d^2\}$.

Observe that $1 - \gamma_n = [1-(1-k)C_n]^2$ so that

$$\gamma_n = (1-k)C_n[2-(1-k)C_n]$$

and

$$\sum_{n=1}^{\infty} \gamma_n = 2(1-k)\sum_{n=1}^{\infty} C_n - (1-k)^2 \sum_{n=1}^{\infty} C_n^2 = \infty. \tag{13}$$

Furthermore, $\sum_{n=1}^{\infty} C_n^2 < \infty$ implies $\lim_{n\to 0} C_n = 0$.

Consequently, there is a sufficiently large N such that $n \geq N$ implies $\gamma_n \in [0,1]$. For $j \geq 1$, put $\beta_j = \alpha_{N+j}$, $\delta_j = \gamma_{N+j}$, and $\sigma_j = C_{N+j}$. Observe that (iii) implies

$$\sum_{j=1}^{\infty} \sigma_j^2 = \sum_{j=1}^{\infty} C_{N+j}^2 < \infty. \tag{14}$$

So, from $\beta_1 = \alpha_{N+1} \geq 0$, (13), and (14), it follows from Lemma 2 that $\alpha_n \to 0$ as $n \to \infty$, so that (11) implies $\rho_n \to 0$ as $n \to \infty$, i.e., $\|x_n - p\| \to 0$ as $n \to \infty$, so that $\{x_n\}_{n=1}^{\infty}$ converges strongly to p.

REMARK 2. It is a consequence of the above proof that, under the hypotheses of the theorem, the fixed point of T must be *unique*. The element $p \in F(T)$, where $F(T)$ denotes the set of fixed points of T, was arbitrarily chosen. Suppose now there is a $p^* \in F(T)$ with $p^* \neq p$. Repeating the argument of the theorem relative to p^*, one sees that (8) converges to both p^* and p, showing that $F(T) = \{p\}$.

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