Iterative Methods for Toeplitz Systems

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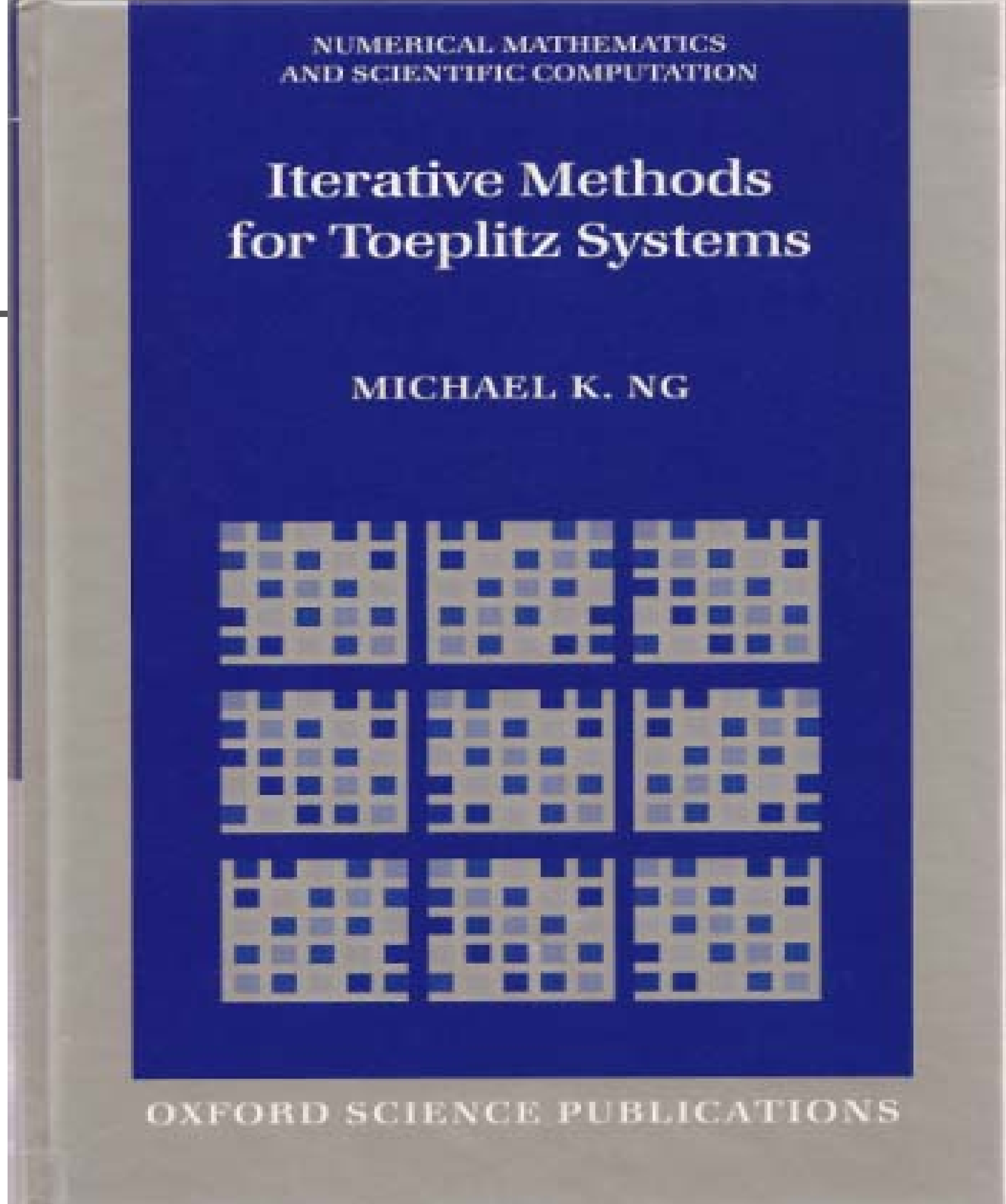
Outline

- Structured matrices have been around for a long time and are encountered in various fields of application.
- Toeplitz matrices, circulant matrices, Hankel matrices, semiseparable matrices, Kronecker product matrices, 2-by-2 block matrices …

Outline

- Toeplitz Matrices
- Overview
- Theory
- Applications
- Research Problems


NUMERICAL MATHEMATICS
AND SCIENTIFIC COMPUTATION
Iterative Methods
for Toeplitz Systems
MICHAEL K. NG
OXFORD SCIENCE PUBLICATIONS

Toeplitz Matrices

- A matrix is said to be a Toeplitz matrix if it is constant along its diagonals

$$\begin{bmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & & t_{2-n} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{bmatrix}$$

Background

- The name Toeplitz originates from the work of Otto Toeplitz (1911) on blinear forms related to Laurent series
- Time series: Yule-Walker Equations (1927)
- Levinson’s work (1947) in formulating the Wiener filtering problem

An example of Toeplitz system

- Linear prediction is a particularly important topic in digital signal processing
- The determination of the optimal linear filter for prediction requires the solution of a set of linear equations having a Toeplitz structure
- Stationary time series

$$\hat{x}(t) = -\sum_{k=1}^{n} h_n(k)x(t-k)$$

where $-h_n(k)$, $k = 1, \ldots, n$, represent the weights in the linear combination. These weights are called the prediction coefficients of the one-step forward predictor of order n.

The difference between the value $x(t)$ and the predicted value $\hat{x}(t)$ is called the forward prediction error, and is denoted by $e(t)$. Thus

$$e(t) = x(t) - \hat{x}(t) = x(t) + \sum_{k=1}^{n} h_n(k)x(t-k).$$

The mean square value of the forward prediction error is

$$\mathsf{E}(|e(t)|^2) = \mathsf{E}\left(\left|x(t) + \sum_{k=1}^{n} h_n(k)x(t-k)\right|^2\right),$$

where E is the expectation operator. Since this is a quadratic function of the predictor coefficients, the minimization of $\mathsf{E}(|e(t)|^2)$ yields the set of linear equations

$$\gamma_{xx}(j) = -\sum_{k=1}^{n} h_n(k)\gamma_{xx}(j-k), \quad j = 1, 2, \ldots, n.$$

Here

$$\gamma_{xx}(k) = \mathsf{E}(x(t)\overline{x(t-k)})$$

Direct Methods

- Schur algorithm (1917) – a test for determining the positive definiteness of a Toeplitz matrix
- Levinson (1947)
- Durbin (1960)
- Trench (1964)
- $O(n^2)$ algorithms
- Small → Large systems (Recursive)

Direct Methods

- The Gohberg-Semencul Formula

$$T_{m+1}^{-1} = \frac{1}{\delta_m}\left\{L_1^t L_1 - L_2^t L_2\right\}, \quad m = 0, 1, \ldots, n-1,$$

L_1 and L_2 are lower triangular Toeplitz matrices given by

$$L_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ r_{m-1} & 1 & 0 & & 0 \\ \vdots & r_{m-1} & 1 & \ddots & \vdots \\ r_2 & & \ddots & \ddots & 0 \\ r_1 & r_2 & \cdots & r_{m-1} & 1 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ r_1 & 0 & 0 & & 0 \\ \vdots & r_1 & 0 & \ddots & \vdots \\ r_{m-2} & & \ddots & \ddots & 0 \\ r_{m-1} & r_{m-2} & \cdots & r_1 & 0 \end{bmatrix},$$

Superfast Direct Toeplitz Solvers

- Brent et al. (1980)
- Bitmead and Anderson (1980)
- Morf (1980)
- de Hong (1986)
- Ammar Gragg (1988)
- $O(n\log^2 n)$ algorithms
- Recursive from $n \rightarrow n/2 \rightarrow n/4 \rightarrow n/8 \rightarrow \ldots$

Look-ahead algorithms

- Singular or ill-conditioned principal submatrices
- Avoid breakdowns or near-breakdowns by skipping such submatrices
- Gueguen (1981), Delsarte et al (1985), Chan and Hansen (1992), Sweet (1993)
- Worst case: $O(n^3)$ algorithms

Stability

- The stability properties of symmetric positive definite Toepltiz matrices: Sweet (1984), Bunch (1985), Cybenko (1987), Bojanczyk et al (1995)
- Weakly stable (residual is small for well-conditioned matrices)
- Look-ahead methods are stable

Stability

- Toeplitz matrices → Cauchy matrices
- Partial pivoting → stable ? Gohberg et al (1995)
- Displacement representation → error growth
- Gu (1995), Chandrasekaran and Sayed (1996), Park and Elden (1996): QR-type algorithm on displacement representation → stable

Iterative Methods

- Rino (1970) and Ekstrom (1974): a decomposition of Toeplitz matrix into a circulant matrices and iterative methods
- Strang (1986), Olkin (1986): the use of preconditioned conjugate gradient method with circulant matrices as preconditioners for Toeplitz systems

Circulant Preconditioners

- Circulant matrices: Toeplitz matrices where each column is a circular shift of its preceding column

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & & c_2 \\ c_2 & c_1 & c_0 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ c_{n-2} & & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 & c_0 \end{bmatrix}$$

Circulant Preconditioners

$$C_n = F_n^* \Lambda_n F_n$$

where the Fourier matrix F_n is the matrix with entries given by

$$[F_n]_{j,k} = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n}, \quad 0 \le j, k \le n-1.$$

- Design of circulant matrices, Strang's preconditioners (1986), R. Chan's preconditioners (1988), Ku and Kuo's preconditioners (1992) …

Circulant Preconditioners

- T. Chan preconditioners (1988) → optimal preconditioners

  $\min \| C - T \|_F$

- Tyrtyshnikov preconditioners (1992) → superoptimal preconditioners

  $\min \| I - C^{-1}T \|_F$

Results

$$t_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{ik\theta}d\theta, \quad k = 0, \pm1, \pm2, \ldots.$$

Theorem 4.10. (Performance of T. Chan's preconditioners) *(Chan and Yeung, 1992b) Let f be a 2π-periodic continuous positive function. Then the spectra of $c(T_n)^{-1}T_n$ are clustered around 1 for large n.*

n	No	S_n	$c(T_n)$	R_n	$K_n^{(2)}$	P_n	Y_n	M_n
16	8	8	8	6	6	8	8	14
32	20	8	7	5	5	10	16	14
64	37	6	7	5	5	7	18	11
128	56	5	6	5	5	7	13	9
256	67	5	6	5	5	6	10	8
512	70	5	6	5	5	6	8	8

Table 4.1 *Number of iterations for different preconditioned systems.*

Results

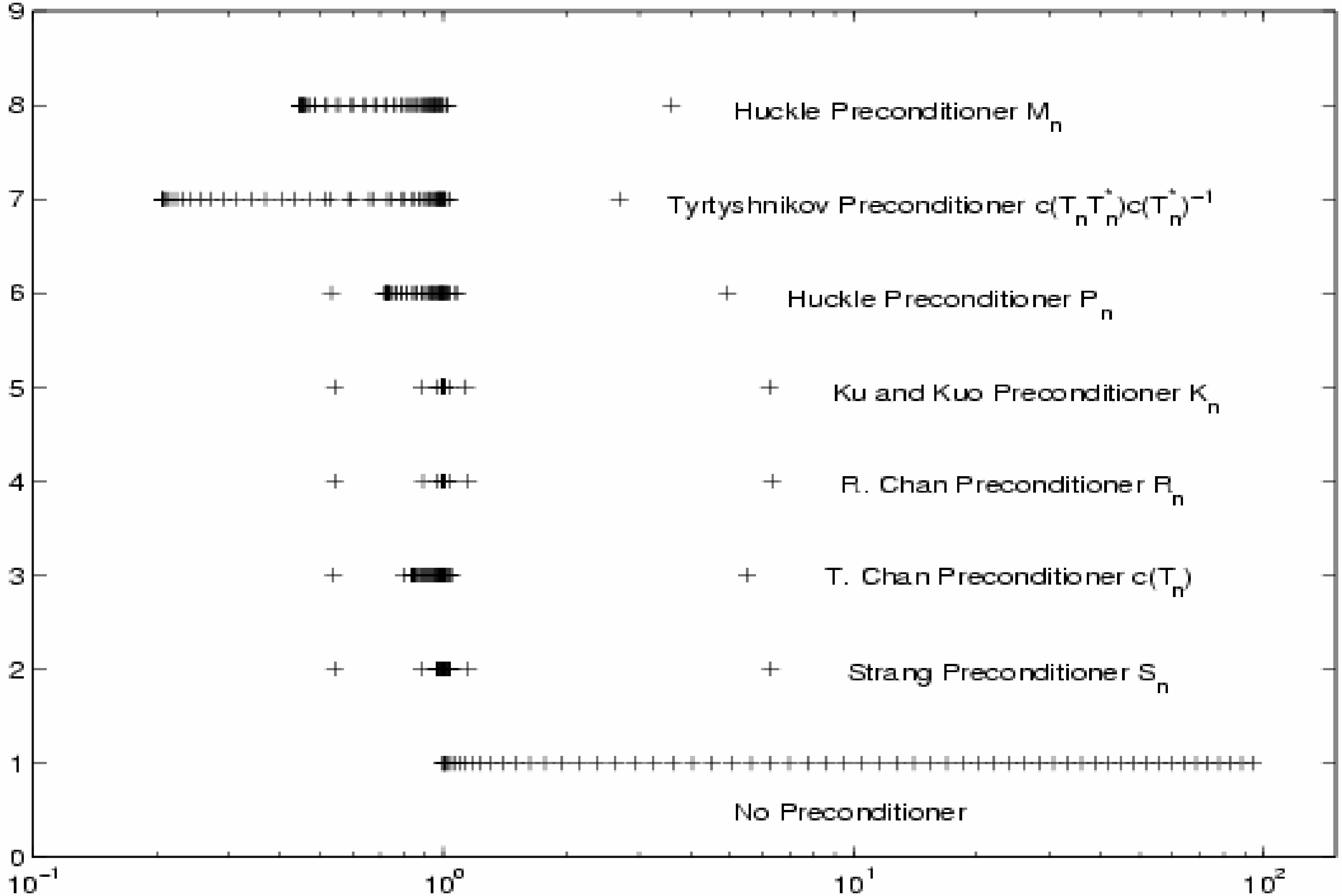


FIG. 4.1. Spectra of preconditioned matrices for $n = 64$.

Circulant Preconditioners

- The eigenvalues of Strang’s preconditioner is the values of the convolution product of the Dirichlet kernel
- The eigenvalues of T. Chan’s preconditioner is the values of the convolution product of the Fejer kernel
- Convergence results/different conditions

Transform-based Preconditioners

- Circulant matrices are precisely those matrices that can diagonalized by the discrete Fourier transform
- Sine transform
- Cosine transform
- Hartley transform
- An effective basis (e.g., shift matrices)

Ill-conditioned systems

- Zeros of f

Theorem 3.8. (Spectra and Zeros) *(Kac et al., 1953; Kesten, 1962; Parter, 1962) Suppose that $f(\theta) - f_{\min}$ has a unique zero of order 2ν at $\theta = \theta_0$. Then $\lambda_{\min}(T_n)$ has the asymptotic expansion*

$$\lambda_{\min}(T_n) = f_{\min} + c\frac{f^{(2\nu)}(\theta_0)}{(2\nu)!}n^{-2\nu} + O(n^{-2\nu-1}), \quad n = 1, 2, \ldots$$

where c is a constant dependent on f and ν.

Ill-conditioned Systems

$$s(\theta) = \prod_{j=1}^{k} [2 - 2\cos(\theta - \theta_j)]^{\nu_j}$$

Theorem 6.3. (Band-Toeplitz Preconditioner II) *(Chan, 1993) Let f be a nonnegative periodic function defined in $[-\pi, \pi]$ with zeros attained at $\{\theta_j\}_{j=1}^{k}$ and their orders are $\{\nu_j\}_{j=1}^{k}$. Define $s(\theta)$ as in (6.6). Then there exist constants $c_1, c_2 > 0$, such that*

$$c_1 \leq \frac{f(\theta)}{s(\theta)} \leq c_2, \quad \forall \theta \in [-\pi, \pi].$$

In particular, $\kappa(T_n[s]^{-1}T_n[f]) \leq c_2/c_1$ for all $n > 0$.

Ill-conditioned Systems

- The generalized Jackson kernel forms an approximate convolution identity → match the zeros automatically

	θ^2						θ^4					
n	32	64	128	256	512	1024	32	64	128	256	512	1024
I	36	79	170	362	753	1544	63	209	790	2149	†	†
S	–	–	–	–	–	–	–	–	–	–	–	–
C	12	16	19	23	29	39	26	42	71	161	167	247
$K_{N,4}$	8	9	10	9	9	9	15	17	20	24	26	26
$K_{N,6}$	10	10	10	10	9	9	15	16	18	18	17	18
$K_{N,8}$	9	10	10	10	10	10	16	17	19	19	19	20

Table 6.1 *Numbers of iterations required for difference preconditioners.*

Multigrid Methods

- Use projection/restriction operators to generate a sequence of sub-systems
- The zeros can be matched (zeros of f)

Theorem 6.25. (Level-Independent Convergence) *Let $f(\theta)$ be such that*

$$c_2(1 \pm \cos(l\theta)) \geq f(\theta) \geq c_1(1 \pm \cos(l\theta)),$$

for some integer l and positive constants c_1 and c_2. Then for any $1 \leq m \leq q$,

$$\|G^m T^m\|_1 \leq \sqrt{1 - \frac{c_1}{2c_2}}.$$

Recursive Preconditioners

- Use the principal submatrices as preconditioners
- Match the zeros automatically
- Solve the subsystems recursively
- Idea of direct methods
- Use the Gohberg-Semencul formula to represent the inverses of submatrices

Block-Toeplitz-Toeplitz-block Systems

$$T_{mn} = \begin{bmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \ddots & \ddots & \vdots \\ T_{m,1} & T_{m,2} & \cdots & T_{m,m} \end{bmatrix}$$

$$= \begin{bmatrix} T_{(0)} & T_{(1)} & \cdots & T_{(m-1)} \\ T_{(1)} & T_{(0)} & \cdots & T_{(m-2)} \\ \vdots & \ddots & \ddots & \vdots \\ T_{(m-1)} & T_{(m-2)} & \cdots & T_{(0)} \end{bmatrix}.$$

Results

Block-circulant-circulant-block preconditioners

can be defined based on the block structure

Theorem 7.14. (Clustered Spectra) *Let T_{mn} be given by (7.38) with an absolutely summable generating sequence. Then for all $\varepsilon > 0$, there exists an $N > 0$ such that for all $n > N$ and all $m > 0$, at most $O(m)$ eigenvalues of $c_F^{(1)}(T_{mn}) - T_{mn}$ have absolute values exceeding ϵ. Therefore if T_{mn} are positive definite for all m and n and $\lambda_{\min}(T_{mn}) \geq \delta > 0$, then for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n > N$ and all $m > 0$, at most $O(m)$ eigenvalues of $[c_F^{(1)}(T_{mn})]^{-1} T_{mn} - I$ have absolute value larger than ϵ.*

Toeplitz Least Squares Problems

- Min || T x – b ||$^2_2$

$$c(k) = e^{-0.1*k^2}, \quad k = 1, 2, \ldots, m$$
$$r(k) = e^{-0.1*k^2}, \quad k = 1, 2, \ldots, n.$$

		Example (ii)		
n	m	no prec	disp prec	part prec
64	128	24	15	12
64	256	46	15	11
64	512	79	13	10
64	1024	132	11	9
64	2048	177	10	9

Applications

- PDEs/ODEs
- Queueing Systems
- Time Series Analysis
- Signal and Image Processing
- Integral Equations

Applications to PDEs

- An elliptic problem on the unit-square with Dirichlet boundary conditions
- Circulant preconditioners are not optimal → condition number O(n)
- Sine transform based preconditioners are optimal → condition number O(1)
- Boundary conditions are matched

Example

$$\frac{\partial}{\partial x}\left[\left(1+\varepsilon e^{x+y}\right)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(1+\frac{\varepsilon}{2}\sin(2\pi(x+y))\right)\frac{\partial u}{\partial y}\right]=g(x,y)$$

	$\varepsilon = 0.01$					$\varepsilon = 1.0$				
$1/h$	No	C_B	P	MILU	MINV	No	C_B	P	MILU	MINV
4	12	9	3	7	4	15	10	5	6	3
8	25	12	3	9	5	29	13	7	9	4
16	47	15	3	13	7	54	18	9	14	6
32	90	20	3	20	11	107	25	11	20	10
64	186	25	3	28	16	209	35	12	28	15
128	363	33	3	41	24	419	50	13	41	22

Table 8.1 *Number of iterations for the unit square.*

	$\varepsilon = 0.01$				$\varepsilon = 1.0$			
$1/h$	No	P	MILU	MINV	No	P	MILU	MINV
8	22	3	9	4	24	7	9	4
16	40	3	12	6	45	9	13	6
32	80	4	17	9	86	10	18	8
64	155	4	25	14	169	12	26	12
128	311	4	36	21	338	14	37	19

Table 8.2 *Number of iterations for the L-shaped domain.*

Sinc-Galerkin Methods for BVPs

- Toeplitz-plus-diagonal systems
- Toeplitz ←→ known f / banded prec.

M	No	E_s	E_u	B	E_s	E_u
4	79	4.1×10^{-3}	8.9×10^{-2}	9	4.1×10^{-3}	8.9×10^{-2}
8	522	3.7×10^{-4}	1.8×10^{-2}	9	3.7×10^{-4}	1.8×10^{-2}
16	>1000	***	***	9	1.1×10^{-5}	1.4×10^{-3}
32	>1000	***	***	9	7.6×10^{-8}	2.2×10^{-5}
64	>1000	***	***	9	1.1×10^{-9}	7.4×10^{-8}

Table 8.3 *Number of iterations required for convergence and the errors E_s and E_u between the numerical approximation and the true solution.*

M	No	E_s	E_u	B	E_s	E_u
4	108	6.6×10^{-4}	8.5×10^{-3}	17	6.6×10^{-4}	8.5×10^{-3}
8	961	1.1×10^{-4}	1.3×10^{-4}	21	1.1×10^{-4}	1.3×10^{-4}
16	>1000	***	***	25	5.8×10^{-6}	7.2×10^{-5}
32	>1000	***	***	30	6.0×10^{-8}	8.9×10^{-7}
64	>1000	***	***	31	5.3×10^{-9}	6.9×10^{-8}

Table 8.4 *Number of iterations required for convergence and the errors E_s and E_u between the numerical approximation and the true solution.*

PDEs

- Hyperbolic and parabolic equations

$$\frac{\partial u}{\partial t} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = g(x,y)$$

- Block-circulant preconditioners by Holmgren and Otto (1992), Jin and Chan (1992), Hemmingsson (1996)

Applications to ODEs

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = J\mathbf{y}(t) + \mathbf{g}(t),\ t \in (t_0, T]; \\ \mathbf{y}(t_0) = \mathbf{z}, \end{cases}$$

$$M_s = A_s \otimes I_m - hB_s \otimes J_m$$

Boundary value methods

Results

m	s	$\kappa(A_s)$	$\kappa(\hat{A}_s)$	$\kappa(B_s)$	$\kappa(\hat{B}_s)$	$\kappa(M_s)$	$\kappa(\hat{M}_s)$
10	8	10.6	10.7	3.2	9.3	44.6	49.0
10	16	21.0	21.0	3.7	9.5	44.5	48.0
10	32	41.3	41.0	3.9	9.5	44.1	48.0
20	8	10.6	10.7	3.2	9.3	164.2	181.0
20	16	21.0	21.0	3.7	9.5	163.8	179.0
20	32	41.3	41.0	3.9	9.5	163.6	178.0

Table 8.5 *Heat equation. Condition numbers for different sizes of the underlying Toeplitz matrices and their small rank perturbations counterparts.The formula (8.52) with $k = 4$ is used here.*

		CGN			BiCGStab		GMRES	
m	s	I	P	S	I	P	I	P
10	8	128	24	26	51	7	63	9
10	16	171	23	23	63	7	79	8
10	32	186	20	20	70	8	89	8
20	8	439	28	29	130	7	130	8
20	16	613	31	37	162	6	173	8
20	32	679	29	31	184	6	207	8

Table 8.6 *Number of iterations for the heat equation problem. The formula (8.52) with $k = 4$ is used here.*

Applications to Integral Equations

- Displacement kernel k(s,t)=k(s-t)
- Circulant integral operator
- Discretization schemes (modified prec.)

Accuracy	Rectangular B	Rectangular S	Rectangular I	Trapezoidal B	Trapezoidal S	Trapezoidal I	Simpson's B	Simpson's S	Simpson's I
10^{0}	65.1	61.2	492.5	1.49	1.69	5.50	0.50	5.15	5.76
10^{-1}	**	**	**	5.62	6.09	51.28	0.84	9.27	9.89
10^{-2}	**	**	**	16.94	18.35	157.51	1.18	13.44	14.48
10^{-3}	**	**	**	89.48	98.62	930.19	2.68	27.04	29.21
10^{-4}	**	**	**	318.02	332.33	**	5.61	57.49	61.88
10^{-5}	**	**	**	**	**	**	11.64	109.98	116.66
10^{-6}	**	**	**	**	**	**	28.18	270.56	285.09

Table 12.2 *Number of megaflops for different quadrature rules and preconditioners.*

Boundary Integral Equations

$$g(x) = -\frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\sigma(y)\,dS_y + \eta, \qquad x \in \partial\Omega.$$

$$(\mathcal{B}u)(\phi) = \int_0^{2\pi} b(\theta-\phi)u(\theta)\,d\theta, \qquad 0 \le \phi \le 2\pi$$

with 2π-periodic kernel function $b(\phi)$. The *optimal circulant integral operator* for $\mathcal{A}$ is the unique circulant integral operator $\mathcal{C}$ that minimizes the Hilbert–Schmidt norm $|||\mathcal{B}-\mathcal{A}|||$ over all circulant integral operators $\mathcal{B}$, where

$$|||\mathcal{B}-\mathcal{A}|||^2 \equiv \int_0^{2\pi}\int_0^{2\pi} |a(\theta,\phi) - b(\theta-\phi)|^2\,d\theta\,d\phi,$$

Theorem 12.12. (Spectra of the Preconditioned Operators) *There exist positive constants $\gamma_2 \ge \gamma_1 > 0$ such that the spectrum of $\mathcal{C}^{-1}\mathcal{A}$ lies in $[\gamma_1, \gamma_2]$.*

Applications to Queueing Networks

- Kolmogorov balance equations for networks
- Stationary probability distribution
- Sparse/Tensor structure
- Nonsymmetric, Toeplitz-like matrix
- Fast convergence when circulant preconditioners are used

Applications to Signal Processing

- Linear prediction filter
- Circulant preconditioners can be applied
- Probabilistic convergence result

Theorem 10.1. (Clustered Spectra of Preconditioned Matrices) *(Ng and Chan, 1994) Let the discrete-time process satisfy the above assumptions. Then for any given* $\epsilon > 0$ *and* $0 < \eta < 1$, *there exist positive integers* ρ_1 *and* ρ_2 *such that for* $n > \rho_1$, *the probability that at most* ρ_2 *eigenvalues of the matrix* $I - c(\bar{A})^{-1}(A^*A)$ *have absolute value greater than* ϵ, *is greater than* $1-\eta$, *provided that* $m = O(n^{3+\nu})$ *with* $\nu > 0$.

Applications to Image Processing

- Deconvolution problem
- Point spread functions ←→ Toeplitz matrices subject to boundary conditions

$$
\begin{bmatrix}
h_m & \cdots & h_0 & \cdots & h_{-m} & & & & \\
 & h_m & & h_0 & & h_{-m} & & 0 & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & 0 & & & h_m & & h_0 & & h_{-m} & \\
 & & & & & h_m & \cdots & h_0 & \cdots & h_{-m}
\end{bmatrix}
\begin{bmatrix}
f_{-m+1} \\ f_{-m+2} \\ \vdots \\ f_0 \\ f_1 \\ \vdots \\ f_n \\ f_{n+1} \\ \vdots \\ f_{n+m-1} \\ f_{n+m}
\end{bmatrix}
=
\begin{bmatrix}
g_1 \\ \vdots \\ g_n
\end{bmatrix}
$$

Deconvolution Problems

- Regularization
- Very ill-conditioned Toeplitz matrices
- Direct inversion → noises amplification
- Many possible solutions
- Regularization restricts the set of admissible solutions
- Tikhonov regularization: L_2 or H_1 norm

Deconvolution Problems

- Periodic boundary condition
- Zero boundary condition
- Reflective boundary condition

$$\min_{\mathbf{f}(\mu)} \left\{ \mu \| D\mathbf{f}(\mu) \|_2^2 + \| \mathbf{g} - A\mathbf{f}(\mu) \|_2^2 \right\}$$

$$(\mu D^t D + A^t A)\mathbf{f}(\mu) = A^t \mathbf{g}.$$

Example

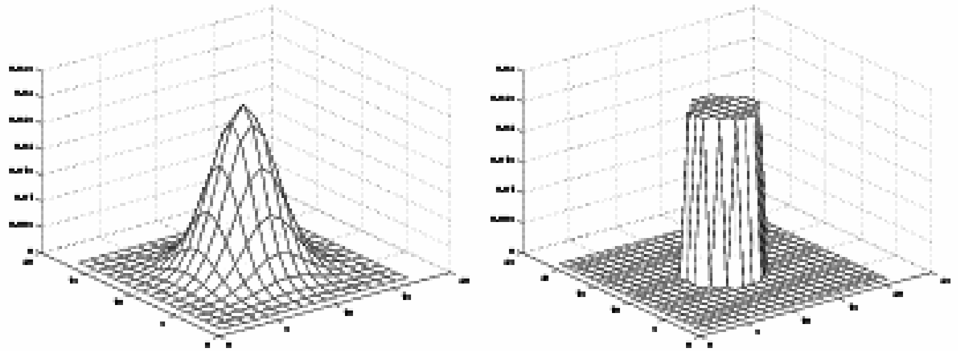


FIG. 11.2. Gaussian (atmospheric turbulence) blur (left) and out-of-focus blur (right).

FIG. 11.3. Noisy and blurred image by Gaussian (left) and out-of-focus blur (right).

Example

rel. error $= 1.24 \times 10^{-1}$ rel. error $= 1.15 \times 10^{-1}$ rel. error $= 6.59 \times 10^{-2}$

FIG. 11.4. Restoring Gaussian blur with zero boundary (left), periodic boundary (middle) and Neumann boundary (right) conditions.

Example

rel. error $= 1.20 \times 10^{-1}$ rel. error $= 1.09 \times 10^{-1}$ rel. error $= 4.00 \times 10^{-2}$

FIG. 11.5. Restoring out-of-focus blur with zero boundary (left), periodic boundary (middle) and Neumann boundary (right) conditions.

Image Restoration Problems

- Other deblurring matrices: spatial variant matrices
- Other measures in the fitting term: L1 norm (non-Gaussian noises)
- Other regularization methods: TV norm, edge-preserving methods (convex, nonconvex), Lipschitz regularization methods
- Other constraints: nonnegativity

Data-fitting term

- Data-fitting term is L1 norm
- $|| A f - g ||_1$ + regularization
- Non-Guassian noises
- Nonlinear problems
- Nonsmooth
- nonnegativity

Spatial-variant Matrices

- Example: Superresolution imaging
- Several low-resolution images
- Downsampling, missing pixels, motions, zooming, etc
- Transformed based preconditioners are not effective

TV-norm

$$\min_f F(f) = \min_u \frac{1}{2}\|\mathcal{H}f - g\|^2_{L^2(\Omega)} + \alpha \int_\Omega |\nabla f| \; dx \; dy$$

$$G(f) \equiv \mathcal{H}^*(\mathcal{H}f - g) - \alpha \nabla \cdot \left(\frac{\nabla f}{|\nabla f|} \right) = 0, \quad (x, y) \in \Omega,$$

$$\frac{\partial f}{\partial n} = 0, \quad (x, y) \in \partial\Omega$$

$$\kappa_\beta(f) = \frac{1}{\sqrt{|\nabla f|^2 + \beta}}, \quad \mathcal{L}_f v = -\nabla \cdot (\kappa_\beta(f) \nabla v)$$

$$\mathcal{A}_f v \equiv (\mathcal{H}^*\mathcal{H} + \alpha \mathcal{L}_f) v,$$

(a)

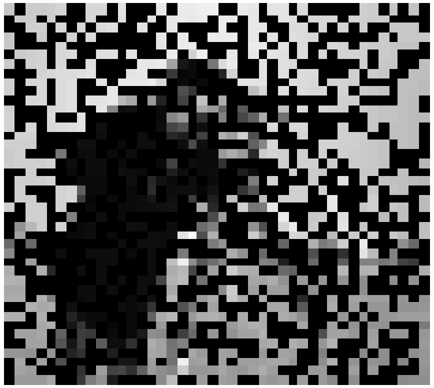
(b)

(c)

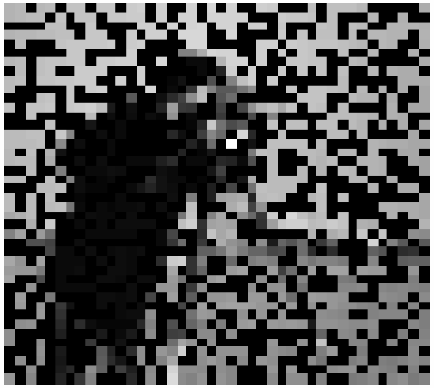
(d)

Figure 4: Multi-frame with a single blur. 4(a)-4(c) Observed image frames with blurring $\sigma = 0.8$. 4(d) Restored image with $\lambda = 0.0004$ reconstructed in 42.92 seconds, relative error $= 0.0874$ and PSNR $= 24.06$ dB.

(a)



(b)



(c)

(d)

Figure 7: Random missing pixels without blur. 7(a)-7(c) Observed image frames. 7(d) Restored image with $\lambda = 0.0052$ reconstructed in 31.22 seconds, relative error = 0.110 and PSNR = 22.91 dB.

Results

Method	Our method (s)	Artificial time marching scheme (s)
Multi-frame, no blur	21.67	696.64
Multi-frame, single blur	42.92	950.55
Multi-frame, multi-blur	95.98	1870.00
Single-frame, multi-blur	110.47	1826.40

Table 3: Comparison with the artificial time marching scheme.

Other Regularization Methods

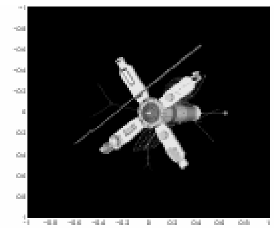


Figure 1: Original satellite image.

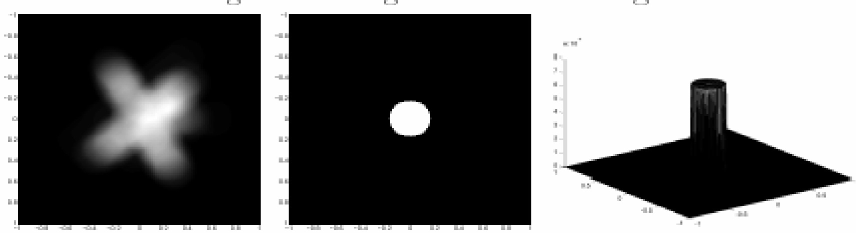


Figure 2: Degraded image(left), image relative error=0.6622, out of focus PSF(middle) and 3D graph of PSF(right).

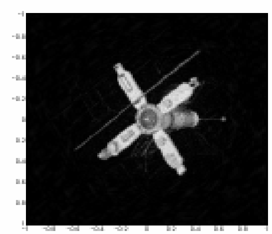


Figure 3: Recovered image using Lipschitz regularization method, $\alpha = 0.417$, $C_t = 0.99$, image relative error=0.2064, ISNR=10.1248

Blind Deconvolution Example

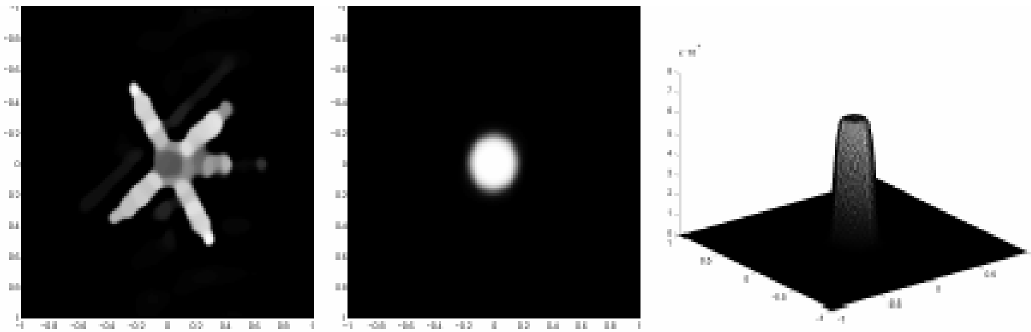


Figure 8: Recovered PSF(left), recovered image(middle) and 3D graph of recovered PSF, when number of AIM iteration=2, $\alpha_1 = 10^{-5}$ and $\alpha_2 = 5.10 \times 10^{-3}$,image relative error=0.3601, ISNR=5.2903, PSF relative error=0.2841, TV norm is used in both image and PSF.

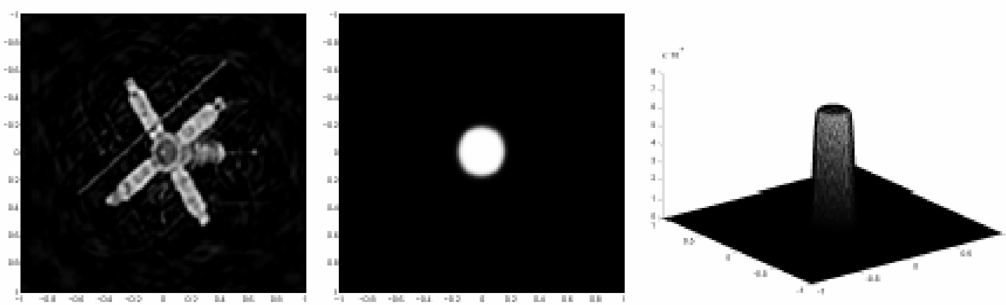


Figure 4: Recovered image(left), recovered PSF (middle) and 3D graph of recovered PSF, when number of AIM iteration=2, $\alpha_1 = 1.2 \times 10^{-6}$ and $\alpha_2 = 6.40 \times 10^{-3}$, image relative error=0.3545, ISNR=5.4265, PSF relative error=0.2055, Lipschitz regularization is used in image and TV norm is used in PSF.

Current Research Directions

- Toeplitz-plus-diagonal systems
- Weighted Toeplitz least squares problems
- Destroy the structure
- Direct methods ?
- Iterative methods: matrix-vector multiplications

Research Directions

- HSS preconditioners
- Approximate the inverse of circulant-plus-diagonal matrix

Formulation

- Constraint Preconditioning:

$$\begin{bmatrix} D^{-2} & K \\ K^T & -\mu I \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

- HSS Preconditioning:

$$M = \begin{bmatrix} W & K \\ -K^T & \mu I \end{bmatrix} = \begin{bmatrix} W & O \\ O & \mu I \end{bmatrix} + \begin{bmatrix} O & K \\ -K^T & O \end{bmatrix} = H + S$$

Comparison

Comparison of preconditioners, 1D problem with well-conditioned K, $\mu = 0.001$.

n	CG	GMRES	HSS($\alpha = \mu$)	HSS($\alpha = 0.05$)	HSS($\alpha = \sqrt{\mu}$)	CP
64	159	48	13	7	6	3
128	424	66	13	7	7	3
256	> 1000	90	18	7	7	3
512	> 1000	132	57	16	17	3
1024	> 1000	168	72	14	16	3

Comparison of preconditioners, 1D problem with ill-conditioned K, $\mu = 0.001$.

n	CG	GMRES	HSS($\alpha = \mu$)	HSS($\alpha = 6 \cdot 10^{-5}$)	HSS($\alpha = \alpha_{best}$)	CP
64	761	117	55	43	43	37
128	> 1000	224	106	74	74	67
256	> 1000	410	159	95	84	125
512	> 1000	770	236	127	117	271
1024	> 1000	> 1000	250	129	117	553

Nonlinear Image Restoration

Comparison of preconditioners for the nonlinear image restoration problem.

Method	Bridge		Cameraman	
	30dB	40dB	30dB	40dB
CG	289	273	334	768
Circulant-PCG	183	167	189	195
GMRES	> 1000	> 1000	> 1000	> 1000
HSS ($\alpha = 0.01$)	82	76	98	69
HSS ($\alpha = 0.05$)	48	46	51	45
HSS ($\alpha = 0.10$)	39	37	38	49
HSS ($\alpha = 0.50$)	95	87	105	270
CP	312	295	378	> 1000

Thank you very much !