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Iterative positive solutions for singular nonlinear fractional differential equation with integral boundary conditions

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Abstract

In this article, we study the existence of iterative positive solutions for a class of singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions, where the nonlinear term may be singular both for time and space variables. By using the properties of the Green function and the fixed point theorem of mixed monotone operators in cones we obtain some results on the existence and uniqueness of positive solutions. We also construct successively some sequences for approximating the unique solution. Our results include the multipoint boundary problems and integral boundary problems as special cases, and we also extend and improve many known results including singular and non-singular cases.

MSC: 34B16; 34B18

Keywords: fixed point theorem; Riemann-Stieltjes integral boundary value problem; iterative positive solution; singular fractional differential equations; cone

1 Introduction

Fractional differential equations have attracted more and more attention in recent decades, which is partly due to their numerous applications in many branches of science and engineering including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, probability, *etc.* We refer the reader to [1–9] and the references therein. On the other hand, boundary value problems with integral boundary conditions for ordinary differential equations arise often in many fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. The existence and uniqueness of positive solutions for such problems have become an important area of investigation in recent years.

In this article, we consider the existence and uniqueness of the iterative positive solutions for the following class of singular fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + p(t)f(t,x(t),x(t)) + q(t)g(t,x(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \int_0^1 k(s)x(s)\,dA(s), \end{cases} \tag{1.1}$$

where $n-1<\alpha\leq n$, $n\geq 2$, $n\in\mathbb{N}$, $p,q:(0,1)\to[0,\infty)$ are continuous, and $p(t)$, $q(t)$ are allowed to be singular at $t=0$ or $t=1$, $f:(0,1)\times(0,\infty)\times(0,\infty)\to[0,\infty)$ is continuous, and $f(t,u,v)$ may be singular at $t=0,1$ and $u=v=0$; $g:(0,1)\times(0,\infty)\to[0,\infty)$ is continuous, and $g(t,v)$ may be singular at $t=0,1$ and $v=0$; $k:(0,1)\to[0,\infty)$ is continuous with $k\in L^1(0,1)$, and $\int_0^1 k(s)x(s)\,dA(s)$ denotes the Riemann-Stieltjes integral with a signed measure, in which $A:[0,1]\to\mathbb{R}$ is a function of bounded variation.

There are various results related to the positive solutions of a nonlinear fractional differential equation with integral boundary value conditions. The number and variety of the methods for dealing with the above solutions has been constantly increasing, such as cone expansion and compression fixed point theorem [1, 10], the monotone iteration method [11–14], the properties of the Green function, and the fixed point index theory [15]. For example, by cone expansion and compression fixed point theorem Cabada and Hamdi [1] studied the existence of positive solutions of the following nonlinear fractional differential equation with integral boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t))=0, & 0<t<1,\\ x(0)=x'(0)=0, & x(1)=\lambda\int_0^1 x(s)\,ds, \end{cases}$$

where $2<\alpha\leq 3$, $0<\lambda$, $\lambda\neq\alpha$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a continuous function.

In [13], by means of the monotone iteration method, Sun and Zhao investigated the existence of positive solutions for the following fractional differential equation with integral boundary conditions:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+q(t)f(t,x(t))=0, & 0<t<1,\\ x(0)=x'(0)=0, & x(1)=\int_0^1 g(s)x(s)\,ds, \end{cases}$$

where $2<\alpha\leq 3$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative of order α, $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous, and $g,q:(0,1)\to[0,\infty)$ are also continuous with $g,q\in L^1(0,1)$.

Zhang *et al.* [15], by using the properties of the Green function and the fixed point index theory, considered the existence of a positive solution of the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+h(t)f(t,x(t))=0, & 0<t<1,\\ x(0)=x'(0)=x''(0)=0, & x(1)=\lambda\int_0^{\eta} x(s)\,ds, \end{cases}$$

where $3<\alpha\leq 4$, $0<\eta\leq 1$, $0\leq\frac{\lambda\eta^{\alpha}}{\alpha}<1$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $h:(0,1)\to[0,\infty)$ is continuous with $h\in L^1(0,1)$, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is also continuous.

Based on a method originally due to Zhai and Hao [16], Jleli and Samet [17] presented the existence and uniqueness criteria for positive solutions to the following nonlinear arbitrary order fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t),x(t))+g(t,x(t))=0, & t\in(0,1),\\ x^{(i)}(0)=0, & i=0,1,2,\ldots,n-2,\\ [D_{0^+}^{\beta}x(t)]_{t=1}=0, & 2\leq\beta\leq n-2, \end{cases}$$

where $n-1<\alpha\leq n$, $n>3$, $n\in\mathbb{N}$, and $f:[0,1]\times[0,\infty)\times[0,\infty)\to[0,\infty)$ and $g:[0,1]\times[0,\infty)\to[0,\infty)$ are given continuous functions.

Recently, by cone expansion fixed point theorem, Li *et al.* [10], obtained the positive solutions of the following class of singular fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+p(t)f(t,x(t))+q(t)g(t,x(t))=0, & 0<t<1,\\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0, \\ x(1)=\int_0^1 k(s)x(s)\,dA(s), \end{cases}$$

where $n-1<\alpha\leq n$, $n\geq 2$, $n\in\mathbb{N}$, $p,q:(0,1)\to[0,\infty)$ are continuous with $p,q\in L^1(0,1)$ and are allowed to be singular at $t=0$ or $t=1$, $f,g:[0,1]\times(0,\infty)\to[0,\infty)$ are continuous, and $f(t,u)$, $g(t,u)$ may be singular at $u=0$; $k:(0,1)\to[0,\infty)$ is continuous with $k\in L^1(0,1)$, and $\int_0^1 k(s)x(s)\,dA(s)$ denotes the Riemann-Stieltjes integral with a signed measure, in which $A:[0,1]\to\mathbb{R}$ is a function of bounded variation. In that paper, they needed the following two conditions to prove the operator to be completely continuous:

(a) $p,q:(0,1)\to[0,\infty)$ are continuous, $p(t)\not\equiv 0$, $q(t)\not\equiv 0$, $t\in[0,1]$, and

$$\int_0^1 \phi(s)p(s)\,ds<\infty,\qquad \int_0^1\phi(s)q(s)\,ds<\infty;$$

(b) $f,g:[0,1]\times(0,\infty)\to[0,\infty)$ are continuous, and for any $0<r<R<\infty$,

$$\lim_{m\to\infty}\sup_{u\in\overline{K_R}\setminus K_r}\int_{H(m)}\big(p(s)f\big(s,u(s)\big)+q(s)g\big(s,u(s)\big)\big)\,ds=0,$$

where

$$H(m)=\left[0,\frac{1}{m}\right]\cup\left[\frac{m-1}{m},1\right],\qquad \phi(s)=\phi_0(s)+\frac{g_A(s)}{1-M},$$

$$\phi_0(s)=\frac{\tau(s)^{\alpha-2}s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)},\qquad \tau(s)=\frac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}},$$

$$M=\int_0^1 t^{\alpha-1}k(t)\,dA(t)<1,\qquad g_A(s)=\int_0^1 G_0(t,s)k(t)\,dA(t),$$

$$G_0(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(t(1-s))^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1,\\ (t(1-s))^{\alpha-1}, & 0\leq t\leq s\leq 1.\end{cases}$$

Then, by using cone expansion fixed point theorem they obtain the existence of positive solutions.

However, up to now, the singular fractional differential equations with Riemann-Stieltjes integral conditions have seldom been considered by using fixed point theorem. In particular, we consider that $f(t,u,v)$ has singularity at $t=0$ or 1 and $v=0$, $g(t,v)$ has singularity at $t=0$ or 1 and $v=0$. In this article, we apply the fixed point theorem of mixed monotone operators to get the existence and uniqueness of the iterative solutions for singular fractional differential equations (1.1) without using the above condition (b).

Obviously, what we discuss is different from those in [1, 10, 13, 15, 17–20]. Comparing with the results in [10], we are based on a new method dealing with problem (1.1). Moreover, $f(t,u,v)$ not only has three variables, but also is singular both for time and space variables. Comparing with the results in [13, 15, 17], we do not need $f(t,u,v)$ to be continuous at $t = 0$ or 1 and at $u = v = 0$. The main new features presented in this article are as follows. Firstly, the boundary value problem has a more general form in which $p(t)$, $q(t)$ are allowed to be singular at $t = 0,1$ and f may be singular for time and space variables, that is, $f(t,u,v)$ and $g(t,v)$ may be singular at $t = 0$ or 1 and $v = 0$. Secondly, by using the fixed point theorem of mixed monotone operators, we obtain a unique positive solution of the boundary value problem (1.1), and we also construct successively some sequences for approximating the unique positive solution. Thirdly, let $\int_0^1 x(s)\,dA(s)$ denote the Riemann-Stieltjes integral, where A is a function of bounded variation, and dA may be a signed measure. As applications, the multipoint problems and integral problems are particular cases. In this paper, we also extend and improve many known results including singular and nonsingular cases.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are to be used to prove our main results. In Section 3, we discuss the existence and uniqueness positive solution of the BVP (1.1) and also construct successively some sequences for approximating the unique positive solution. In Section 4, we give an example to demonstrate the application of our theoretical results.

2 Preliminaries and lemmas

In this section, we present some definitions, lemmas, and basic results that will be used in the article. For convenience of readers, we refer to [6, 8, 21, 22] for details.

Let $(E,\|\cdot\|)$ be a Banach space. We denote the zero element of E by θ. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (1) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (2) $x \in P$, $-x \in P \Rightarrow x = \theta$. In this paper, suppose that $(E,\|\cdot\|)$ is a Banach space partially ordered by a cone $P \subset E$, that is, $x \leq y$ if and only if $y - x \in P$.

For $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > 0$, we denote by P_h the set $P_h = \{x \in P \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is a component of P. A cone P is called normal if there exists a constant $N > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; the smallest such N is called the normality constant of P.

Definition 2.1 ([6]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 ([6]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $x : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3 ([23]) Suppose that $(E, \|\cdot\|)$ is a Banach space, P is a cone in E, and $D \subset P$. An operator $A : D \to P$ is said to be α-concave if there exists $\alpha \in (0,1)$ such that

$$A(tx) \geq t^{\alpha} Ax, \quad \forall t \in (0,1), x \in D.$$

Definition 2.4 ([23]) Suppose that $(E, \|\cdot\|)$ is a Banach space, P is a cone in E, and $D \subset P$. An operator $B : D \to P$ is said to be subhomogeneous if

$$B(tx) \geq tBx, \quad \forall t \in (0,1), x \in D.$$

Definition 2.5 ([24]) Suppose that $(E, \|\cdot\|)$ is a Banach space, P is a cone in E, and $D \subset P$. An operator $A : D \times D \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, that is, for all $x_i, y_i \in D$ $(i = 1,2)$, $x_1 \leq x_2$, $y_1 \geq y_2$ imply $A(x_1, y_1) \leq A(x_2, y_2)$.

Lemma 2.1 ([6]) *Let $\alpha > 0$. If $x \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} x(t) = 0$$

has

$$x(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R},\ i = 1,2,\ldots,N,$$

as the unique solution, where $N = [\alpha] + 1$.

From the definition of the Riemann-Liouville derivative we obtain the following result.

Lemma 2.2 ([6]) *Assume that $x \in C(0,1) \cap L^1(0,1)$ is a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$ $(i = 1,2,\ldots,N)$, where $N = [\alpha] + 1$.

In the following, we present the Green function of the fractional differential equation boundary value problem.

Lemma 2.3 ([10]) *Let $M \neq 1$ and $y \in C(0,1) \cap L^1(0,1)$, $n-1 < \alpha \leq n$. Then the problem*

$$\begin{cases} D_{0^+}^{\alpha} x(t) + y(t) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ x(1) = \int_0^1 k(s)x(s)\,dA(s) \end{cases} \tag{2.1}$$

is equivalent to

$$x(t) = \int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s) = G_0(t,s) + \frac{t^{\alpha-1}}{1-M} g_A(s),$$

in which

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ (t(1-s))^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

$$M = \int_0^1 t^{\alpha-1} k(t)\, dA(t), \qquad g_A(s) = \int_0^1 G_0(t,s) k(t)\, dA(t).$$

Lemma 2.4 *Let* $0 \le M < 1$ *and* $g_A(s) \ge 0$ *for* $s \in [0,1]$. *Then the Green function* $G(t,s)$ *defined by* (2.2) *satisfies the following*:

(1) $G : [0,1] \times [0,1] \to [0,\infty)$ *is continuous*;

(2) *For any* $t,s \in [0,1]$, *we have* $\frac{t^{\alpha-1}}{1-M} g_A(s) \le G(t,s) \le \frac{t^{\alpha-1}}{(1-M)\Gamma(\alpha)}(1-s)^{\alpha-1}$.

Proof

(1) By Lemma 2.4 in [10] we have that $G : [0,1] \times [0,1] \to [0,\infty)$ is continuous.

(2) From (2.2) we have

$$0 \le G_0(t,s) \le \frac{1}{\Gamma(\alpha)} \bigl(t(1-s)\bigr)^{\alpha-1}, \quad \forall t,s \in [0,1]. \tag{2.3}$$

So for all $t,s \in [0,1]$, we have

$$\begin{aligned} \frac{t^{\alpha-1}}{1-M} g_A(s) \le G(t,s) &= G_0(t,s) + \frac{t^{\alpha-1}}{1-M} g_A(s) \\ &\le t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{1}{1-M} g_A(s) \right). \end{aligned} \tag{2.4}$$

By (2.2) it is obvious that

$$\begin{aligned} g_A(s) &= \int_0^1 G_0(t,s) k(t)\, dA(t) \\ &\le \int_0^1 \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-1} k(t)\, dA(t) \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1} \int_0^1 t^{\alpha-1} k(t)\, dA(t) \\ &= \frac{M}{\Gamma(\alpha)}(1-s)^{\alpha-1}. \end{aligned}$$

Thus, for all $t,s \in [0,1]$, we have

$$\frac{t^{\alpha-1}}{1-M} g_A(s) \le G(t,s) = G_0(t,s) + \frac{t^{\alpha-1}}{1-M} g_A(s) \le \frac{t^{\alpha-1}}{(1-M)\Gamma(\alpha)}(1-s)^{\alpha-1}. \tag{2.5}$$

□

Taking $A : P_h \times P_h \to P_h$, $B : P_h \to P_h$ in Theorem 3.1 in [23] and Corollary 3.7 in [25], and also taking $A : P_h \times P_h \to P_h$, $B : P_h \to P_h$, $C = \theta$ $\varphi(t) = t^\gamma$ in Corollary 3.4 in [26], it is easy to get the following lemma.

Lemma 2.5 *Let $(E, \|\cdot\|)$ be a Banach space, and P be a normal cone in E. Suppose that there exist $h \in P$ and $h > \theta$ such that $A : P_h \times P_h \to P_h$ is a mixed monotone operator and $B : P_h \to P_h$ is a decreasing operator satisfying the following conditions*:

(1) *There exists $\gamma \in (0,1)$ such that $A(tx, t^{-1}y) \geq t^{\gamma} A(x,y)$, $t \in (0,1)$, $x, y \in P_h$.*
(2) *$B(t^{-1}y) \geq tBy$, $t \in (0,1)$, $y \in P_h$.*
(3) *There exists a constant $\delta_0 > 0$ such that $A(x,y) \geq \delta_0 By$, $\forall x, y \in P_h$.*

Then the operator equation $A(x,x) + Bx = x$ has a unique solution $x^* \in P_h$, and for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}) + By_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + Bx_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

3 Main result

Theorem 3.1 *Assume that the following conditions hold.*

(H_1) *$p, q : (0,1) \to [0, \infty)$ are continuous, and $p(t)$, $q(t)$ are allowed to be singular at $t = 0$ or $t = 1$.*
(H_2) *$f : (0,1) \times (0,\infty) \times (0,\infty) \to [0,\infty)$, $g : (0,1) \times (0,\infty) \to [0,\infty)$ are continuous, and $f(t,u,v)$, $g(t,v)$ may be singular at $t = 0$ and $v = 0$.*
(H_3) *For fixed $t \in (0,1)$, and $v \in (0,\infty)$, $f(t,u,v)$ is increasing in $u \in (0,\infty)$; for fixed $t \in (0,1)$ and $u \in (0,\infty)$, $f(t,u,v)$ is decreasing in $v \in (0,\infty)$; and for fixed $t \in (0,1)$, $g(t,v)$ is decreasing in $v \in (0,\infty)$.*
(H_4) *There exists a constant $\gamma \in (0,1)$ such that for all $\lambda, t \in (0,1)$ and $u, v \in (0,\infty)$,*

$$f\bigl(t, \lambda u, \lambda^{-1} v\bigr) \geq \lambda^{\gamma} f(t,u,v), \qquad g\bigl(t, \lambda^{-1} v\bigr) \geq \lambda g(t,v). \tag{3.1}$$

(H_5) *$\int_0^1 (1-s)^{\alpha-1} p(s) s^{\gamma(1-\alpha)} f(s,1,1)\,ds < \infty$ and $\int_0^1 (1-s)^{\alpha-1} q(s) s^{1-\alpha} g(s,1)\,ds < \infty$.*
(H_6) *There exists a constant $\delta > 0$ such that, for all $t \in (0,1)$ and $u, v \in (0,\infty)$, $f(t,u,v) \geq \delta g(t,v)$.*

Then the singular fractional differential equation (1.1) *has a unique positive solution x^*, which satisfies $at^{\alpha-1} \leq x^*(t) \leq bt^{\alpha-1}$, $t \in [0,1]$, for two constants $a, b > 0$. Moreover, for any initial values $x_0, y_0 \in P_h$, $h = t^{\alpha-1}$, the sequences $\{x_n\}$, $\{y_n\}$ of successive approximations defined by*

$$\begin{aligned} x_n(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, x_{n-1}(s), y_{n-1}(s)\bigr)\,ds + \int_0^1 G(t,s)q(s)g\bigl(s, y_{n-1}(s)\bigr)\,ds, \\ y_n(t) &= \int_0^1 G(t,s)p(s)f\bigl(s, y_{n-1}(s), x_{n-1}(s)\bigr)\,ds + \int_0^1 G(t,s)q(s)g\bigl(s, x_{n-1}(s)\bigr)\,ds, \\ &\quad n = 1, 2, \ldots, \end{aligned}$$

both convergence uniformly to x^* on $[0,1]$ as $n \to \infty$.

Proof Let $E = C[0,1]$ and $\|u\| = \sup_{0 \leq t \leq 1} |u(t)|$. Obviously, $(E, \|\cdot\|)$ is a Banach space. Let $P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$, and $h(t) = t^{\alpha-1}$. Define

$$P_h = \left\{ x \in C[0,1] \,\middle|\, \exists D \geq 1 : \frac{1}{D} t^{\alpha-1} \leq x(t) \leq Dt^{\alpha-1}, t \in [0,1] \right\}.$$

Then P is a cone of E, and P_h is a component of P. From Lemma 2.3 we have that $x(t)$ is the solution of the singular fractional differential equations (1.1) if and only if it satisfies the following integral equation:

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)\big[p(s)f\big(s,x(s),x(s)\big) + q(s)g\big(s,x(s)\big)\big]\,ds \\ &= \int_0^1 G(t,s)p(s)f\big(s,x(s),x(s)\big)\,ds + \int_0^1 G(t,s)q(s)g\big(s,x(s)\big)\,ds, \end{aligned}$$

where $G(t,s)$ is given in Lemma 2.3. Define two operators $A : P_h \times P_h \to P$ and $B : P_h \to P$ by

$$A(x,y)(t) = \int_0^1 G(t,s)p(s)f\big(s,x(s),y(s)\big)\,ds, \qquad By(t) = \int_0^1 G(t,s)q(s)g\big(s,y(s)\big)\,ds,$$

respectively. Then it is easy to prove that x is the solution of the singular fractional differential equations (1.1) if it satisfies $x = A(x,x) + Bx$.

(1) Firstly, we show that A, B are well defined. From (H_4) we have that, for all $\lambda \in (0,1)$, $t \in (0,1)$, and $u, v \in (0,\infty)$,

$$f(t,u,v) = f\big(t,\lambda\lambda^{-1}u,\lambda^{-1}\lambda v\big) \geq \lambda^{\gamma} f\big(t,\lambda^{-1}u,\lambda v\big), \tag{3.2}$$

$$g(t,v) = g\big(t,\lambda^{-1}\lambda v\big) \geq \lambda g(t,\lambda v). \tag{3.3}$$

So by (3.2), for all $\lambda \in (0,1)$, $t \in (0,1)$, and $u, v \in (0,\infty)$, we have

$$f\big(t,\lambda^{-1}u,\lambda v\big) \leq \lambda^{-\gamma} f(t,u,v), \quad g(t,\lambda v) \leq \lambda^{-1} g(t,v). \tag{3.4}$$

Taking $u = v = 1$ in (3.1), (3.3), and (3.4), we have

$$f\big(t,\lambda,\lambda^{-1}\big) \geq \lambda^{\gamma} f(t,1,1), \qquad f\big(t,\lambda^{-1},\lambda\big) \leq \lambda^{-\gamma} f(t,1,1), \quad \forall \lambda \in (0,1), t \in (0,1), \tag{3.5}$$

$$g\big(t,\lambda^{-1}\big) \geq \lambda g(t,1), \qquad g(t,\lambda) \leq \lambda^{-1} g(t,1), \quad \forall \lambda \in (0,1), t \in (0,1). \tag{3.6}$$

For any $x, y \in P_h$, we can choose a constant $D_1 \geq 1$ such that $\frac{1}{D_1}t^{\alpha-1} \leq x, y \leq D_1 t^{\alpha-1}$, $t \in [0,1]$. On the one hand, from (H_3), (H_4), (3.4), and (3.5) we have

$$\begin{aligned} f\big(t,x(t),y(t)\big) &\leq f\big(t,D_1t^{\alpha-1},D_1^{-1}t^{\alpha-1}\big) \leq f\big(t,D_1t^{1-\alpha},D_1^{-1}t^{\alpha-1}\big) \\ &\leq t^{\gamma(1-\alpha)} f\big(t,D_1,D_1^{-1}\big) \leq t^{\gamma(1-\alpha)} D_1^{\gamma} f(t,1,1), \quad t \in (0,1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} f\big(t,x(t),y(t)\big) &\geq f\big(t,D_1^{-1}t^{\alpha-1},D_1t^{\alpha-1}\big) \geq f\big(t,D_1^{-1}t^{\alpha-1},D_1t^{1-\alpha}\big) \\ &\geq t^{\gamma(\alpha-1)} f\big(t,D_1^{-1},D_1\big) \geq t^{\gamma(\alpha-1)} D_1^{-\gamma} f(t,1,1), \quad t \in (0,1). \end{aligned} \tag{3.8}$$

On the other hand, from (H_3), (H_4), (3.4), and (3.6), we get

$$\begin{aligned} g\big(t,y(t)\big) &\leq g\big(t,D_1^{-1}t^{\alpha-1}\big) \leq t^{1-\alpha} g\big(t,D_1^{-1}\big) \\ &\leq t^{1-\alpha} D_1 g(t,1) \leq t^{1-\alpha} D_1^2 g(t,1), \quad t \in (0,1), \end{aligned} \tag{3.9}$$

$$\begin{aligned} g\bigl(t,y(t)\bigr) &\ge g\bigl(t,D_1t^{\alpha-1}\bigr) \ge g\bigl(t,D_1t^{1-\alpha}\bigr) \ge t^{\alpha-1}g(t,D_1) \\ &\ge t^{\alpha-1}D_1^{-1}g(t,1) \ge t^{\alpha-1}D_1^{-2}g(t,1), \quad t\in(0,1). \end{aligned} \tag{3.10}$$

By (H_4), (3.7), and (3.9) we get that

$$\begin{aligned} &\int_0^1 G(t,s)p(s)f\bigl(s,x(s),y(s)\bigr)\,ds \\ &\quad\le \int_0^1 G(t,s)p(s)s^{\gamma(1-\alpha)}D_1^{\gamma}f(s,1,1)\,ds \\ &\quad\le \frac{t^{\alpha-1}}{(1-M)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}p(s)s^{\gamma(1-\alpha)}D_1^{\gamma}f(s,1,1)\,ds \\ &\quad< \infty, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\int_0^1 G(t,s)q(s)g\bigl(s,y(s)\bigr)\,ds \\ &\quad\le \int_0^1 G(t,s)q(s)s^{1-\alpha}D_1^{2}g(s,1)\,ds \\ &\quad\le \frac{t^{\alpha-1}}{(1-M)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}q(s)s^{1-\alpha}D_1^{2}g(s,1)\,ds \\ &\quad< \infty. \end{aligned} \tag{3.12}$$

From Lemma 2.4 we have that $G:[0,1]\times[0,1]\to[0,\infty)$ is continuous. So $A:P_h\times P_h\to P$ and $B:P_h\to P$ are well defined.

(2) Secondly, we prove that $A:P_h\times P_h\to P_h$ and $B:P_h\to P_h$. Let $D\ge 1$ be such that

$$\begin{aligned} D > \max\biggl\{ &\frac{1}{(1-M)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}p(s)s^{\gamma(1-\alpha)}D_1^{\gamma}f(s,1,1)\,ds, \\ &\biggl(\frac{1}{1-M}\int_0^1 g_A(s)p(s)s^{\gamma(\alpha-1)}D_1^{-\gamma}f(s,1,1)\,ds\biggr)^{-1}, \\ &\frac{1}{(1-M)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}q(s)s^{1-\alpha}D_1^{2}g(s,1)\,ds, \\ &\biggl(\frac{1}{1-M}\int_0^1 g_A(s)q(s)s^{\alpha-1}D_1^{-2}g(s,1)\,ds\biggr)^{-1}\biggr\}. \end{aligned}$$

Then from Lemma 2.4 and (3.11) and (3.12), for all $t\in[0,1]$, $x,y\in P_h$, we have

$$\begin{aligned} A(x,y)(t) &= \int_0^1 G(t,s)p(s)f\bigl(s,x(s),y(s)\bigr)\,ds \\ &\le \int_0^1 G(t,s)p(s)s^{\gamma(1-\alpha)}D_1^{\gamma}f(s,1,1)\,ds \\ &\le \frac{t^{\alpha-1}}{(1-M)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}p(s)s^{\gamma(1-\alpha)}D_1^{\gamma}f(s,1,1)\,ds \\ &\le Dt^{\alpha-1}, \end{aligned}$$

$$\begin{aligned}
A(x,y)(t) &= \int_0^1 G(t,s)p(s)f\big(s,x(s),y(s)\big)\,ds \\
&\geq \int_0^1 G(t,s)p(s)s^{\gamma(\alpha-1)}D_1^{-\gamma}f(s,1,1)\,ds \\
&\geq \frac{t^{\alpha-1}}{1-M}\int_0^1 g_A(s)p(s)s^{\gamma(\alpha-1)}D_1^{-\gamma}f(s,1,1)\,ds \\
&\geq \frac{1}{D}t^{\alpha-1},
\end{aligned}$$

$$\begin{aligned}
By(t) &= \int_0^1 G(t,s)q(s)g\big(s,y(s)\big)\,ds \\
&\leq \int_0^1 G(t,s)q(s)s^{1-\alpha}D_1^2 g(s,1)\,ds \\
&\leq \frac{t^{\alpha-1}}{(1-M)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}q(s)s^{1-\alpha}D_1^2 g(s,1)\,ds \\
&\leq Dt^{\alpha-1},
\end{aligned}$$

$$\begin{aligned}
By(t) &= \int_0^1 G(t,s)q(s)g\big(s,y(s)\big)\,ds \\
&\geq \int_0^1 G(t,s)q(s)s^{\alpha-1}D_1^{-2}g(s,1)\,ds \\
&\geq \frac{t^{\alpha-1}}{1-M}\int_0^1 g_A(s)q(s)s^{\alpha-1}D_1^{-2}g(s,1)\,ds \\
&\geq \frac{1}{D}t^{\alpha-1}.
\end{aligned}$$

So $A: P_h \times P_h \to P_h$ and $B: P_h \to P_h$.

(3) Next, by (H_3) it is easy to prove that A is a mixed monotone operator and B is an decreasing operator.

(4) From (H_4), for any $t \in [0,1]$ and $x, y \in P_h$, we have

$$\begin{aligned}
A\big(\lambda x, \lambda^{-1}y\big)(t) &= \int_0^1 G(t,s)f\big(s,\lambda x(s),\lambda^{-1}y(s)\big)\,ds \geq \lambda^\gamma \int_0^1 G(t,s)f\big(s,x(s),y(s)\big)\,ds \\
&= \lambda^\gamma A(x,y)(t),
\end{aligned}$$

$$B\big(\lambda^{-1}y\big)(t) = \int_0^1 G(t,s)g\big(s,\lambda^{-1}y(s)\big)\,ds \geq \lambda \int_0^1 G(t,s)g\big(s,y(s)\big)\,ds = \lambda By(t),$$

that is, $A(\lambda x, \lambda^{-1}y)(t) \geq \lambda^\gamma A(x,y)(t)$, $B(\lambda^{-1}y)(t) \geq \lambda By(t)$ for all $t \in [0,1]$, $x, y \in P_h$.

(5) By (H_6), for all $t \in [0,1]$, $x, y \in P_h$,

$$\begin{aligned}
A(x,y)(t) &= \int_0^1 G(t,s)f\big(s,x(s),y(s)\big)\,ds \\
&\geq \delta \int_0^1 G(t,s)g\big(s,y(s)\big)\,ds \\
&= \delta By(t).
\end{aligned}$$

Then by Lemma 2.5 the conclusions of Theorem 3.1 hold. □

Remark 3.1 The fractional differential equation with Riemann-Stieltjes integral conditions considered in Theorem 3.1 is singular, that is, $f(t,u,v)$ has singularity at $t = 0$ or $t = 1$ and $v = 0$, and $g(t,v)$ has singularity at $t = 0$ or $t = 1$ and $v = 0$, which generalizes and improves the known results for continuous functions in [16, 27–29].

Remark 3.2 The function $g(t,v)$ we considered in Theorem 3.1 is decreasing and has singularity at $t = 0$ or $t = 1$ and $v = 0$, which generalizes and improves the results in [30].

Remark 3.3 Comparing with the main results in [31, 32], the nonlinear fractional differential equation we considered is also continuous. However, we get the iterative positive solutions for boundary value problem (1.1) by using the fixed point theorem of the mixed monotone operator, which generalizes and improves the results including singular and nonsingular cases in [17, 31–33].

4 An example

Let $\alpha = \frac{7}{6}$, $p(t) = q(t) = t^{-\frac{1}{5}}$, $f(t,u,v) = \sqrt[6]{\frac{u}{tv}} + \frac{1}{\sqrt{tv}}$, $g(t,v) = \frac{1}{\sqrt{tv(v+1)}}$.

(1) It is obvious that $p(t)$, $q(t)$ are singular at $t = 0$. The functions $f : (0,1) \times (0,\infty) \times (0,\infty) \to [0,\infty)$ and $g : (0,1) \times (0,\infty) \to [0,\infty)$ are continuous. So the conditions (H_1) and (H_2) hold.

(2) It is obvious that, for fixed $t \in (0,1)$ and $v \in (0,\infty)$, $f(t,u,v)$ is increasing in $u \in (0,\infty)$, for fixed $t \in (0,1)$ and $u \in (0,\infty)$, $f(t,u,v)$ is decreasing in $v \in (0,\infty)$, and, for fixed $t \in (0,1)$, $g(t,v)$ is decreasing in $v \in (0,\infty)$.

(3) Taking $\gamma = \frac{1}{2} \in (0,1)$, for all $t \in (0,1)$ and $u,v \in (0,\infty)$, we have

$$\begin{aligned} f\left(t,\lambda u,\lambda^{-1}v\right) &= \sqrt[6]{\frac{\lambda u}{t\lambda^{-1}v}} + \frac{1}{\sqrt{t\lambda^{-1}v}} \geq \lambda^{\frac{1}{3}}\sqrt[6]{\frac{u}{tv}} + \lambda^{\frac{1}{2}}\frac{1}{\sqrt{tv}} \geq \lambda^{\frac{1}{2}}\left(\sqrt[6]{\frac{u}{tv}} + \frac{1}{\sqrt{tv}}\right) \\ &= \lambda^{\gamma}f(t,u,v), \end{aligned}$$

and, for all $t \in (0,1)$ and $v \in (0,\infty)$,

$$g\left(t,\lambda^{-1}v\right) = \frac{1}{\sqrt{t\lambda^{-1}v(\lambda^{-1}v+1)}} \geq \lambda\frac{1}{\sqrt{tv(v+1)}} = \lambda g(t,v).$$

(4) It is easy to prove that

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-1}p(s)s^{\gamma(1-\alpha)}f(s,1,1)\,ds = \int_0^1 (1-s)^{\frac{1}{6}}s^{-\frac{1}{5}}s^{\frac{1}{2}\times(-\frac{1}{6})}\left(s^{-\frac{1}{6}} + s^{-\frac{1}{2}}\right)ds < \infty, \\ &\int_0^1 (1-s)^{\alpha-1}q(s)s^{1-\alpha}g(s,1)\,ds = \int_0^1 (1-s)^{\frac{1}{6}}s^{-\frac{1}{5}}s^{-\frac{1}{6}}(2s)^{-\frac{1}{2}}\,ds < \infty. \end{aligned}$$

(5) Take $\delta = 1 > 0$ such that, for all $t \in (0,1)$, $u,v \in (0,\infty)$,

$$f(t,u,v) = \sqrt[6]{\frac{u}{tv}} + \frac{1}{\sqrt{tv}} \geq \frac{1}{\sqrt{tv}} \geq \frac{1}{\sqrt{tv(v+1)}} = g(t,v).$$

Therefore, the assumptions of Theorem 3.1 are satisfied. Thus, the conclusions follow from Theorem 3.1.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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