Transactions on Modelling and Simulation vol 7, © 1994 WIT Press, www.witpress.com, ISSN 1743-355X

Iterative solutions of boundary integral equations

S. Amini & N.D. Maines
Department of Mathematics and Computer Science, University of Salford, Salford, M5 4WT, UK

ABSTRACT

Collocation discretisation of boundary integral equations leads to fully populated complex valued non-hermitian boundary element equations. In this paper we study the efficient solution of these linear systems by various iterative methods based on the splitting of the discrete operators. In particular the boundary element solution of the Burton and Miller formulation for the exterior Helmholtz equation is considered where the *hypersingular operator*, the derivative of the double layer Helmholtz potential, is present. The choice of the coupling parameter in the formulation and the splitting of the operator are shown to play an important role in the convergence of the iterative methods.

1 INTRODUCTION

General boundary integral equations of interest can be written in operator form as $\mathcal{A}\phi = f$, where $\mathcal{A} : \mathcal{H}^r(\Gamma) \to \mathcal{H}^{r-\alpha}(\Gamma)$ is a pseudodifferential operator of order α. The advantage of the pseudodifferential operator framework is that much of the operator properties and convergence analyses of boundary element methods can be treated in a unified fashion, Schatz *et al* [10]. Discretisation of these equations using the piecewise polynomial approximation spaces leads to fully populated linear systems $\mathcal{A}_n\phi_n = f_n$ where n is the number of degrees of freedom. For large n the cost of their solution by direct methods is prohibitive.

Well-known iterative techniques such as multi-grid and conjugate gradient type methods, developed for the solution of the large, sparsely populated finite element and finite difference systems, have been applied to boundary element equations. For second kind Fredholm equations, corresponding to $\alpha = 0$, these methods have been shown to be efficient (see Amini and Chen [3, 4], and references therein). For the case of the first kind equation, corresponding to $\alpha = -1$, or the hypersingular equation ($\alpha = +1$), these meth-

ods, without appropriate modifications, either diverge or converge slowly, Amini *et al* [5].

We study various iterative methods based on the splitting of $\mathcal{A}_n$ as $\mathcal{D}_n - \mathcal{C}_n$, where $\mathcal{D}_n$ corresponds to the discretisation of the most dominant part of $\mathcal{A}$, with the restriction that solutions of systems $\mathcal{D}_n v_n = b_n$ can be carried out in $\mathcal{O}(n^p)$ operations with $p \leq 2$. Although our results are applicable to general boundary element equations we concentrate on the solution of the exterior Helmholtz equation. In Section 2 we introduce the boundary integral equation for the Helmholtz equation together with its numerical approximation. In Section 3, our iterative schemes are introduced and finally we present the results of numerical experiments in Section 4.

2 EXTERIOR HELMHOLTZ PROBLEM

The direct boundary integral solution of the Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0, \quad p \in D_+$$

subject to appropriate boundary and radiation conditions, is given by

$$\phi(p) = \int_\Gamma \phi(q) \frac{\partial G_k}{\partial n_q}(p,q) d\Gamma_q - \int_\Gamma G_k(p,q) \frac{\partial \phi}{\partial n}(q) d\Gamma_q, \quad p \in D_+.$$

D_+ is the unbounded region exterior to the closed boundary Γ, G_k is the fundamental solution of the Helmholtz equation and $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the normal to Γ pointing towards D_+.

A uniquely solvable direct boundary integral equation, due to Burton and Miller [6], which relates the Cauchy data ϕ and $\frac{\partial \phi}{\partial n}$ is

$$\left\{ -\frac{1}{2} I + M_k + i\eta N_k \right\} \phi(p) = \left\{ L_k + i\eta \left(\frac{1}{2} I + M_k^T \right) \right\} \frac{\partial \phi}{\partial n}(p), \quad p \in \Gamma. \qquad (1)$$

In the above, L_k and M_k are the *single* and *double*-layer Helmholtz potentials respectively and M_k^T and N_k are their respective normal derivatives whilst η is a real constant coupling parameter. We may write the above equation in shorthand as

$$A_{k,\eta} \phi = B_{k,\eta} \frac{\partial \phi}{\partial n} \qquad (2)$$

The case $\eta = 0$ gives the classical Surface Helmholtz Equation (SHE) which is singular for a countable set, I_Γ, of real values of k. For the case where Γ is a unit circle it is easy to show that $I_\Gamma = \{k | J_n(k) = 0, \text{ for some } n = 0, 1, \ldots\}$, where J_n are the Bessel functions, see Amini [2] and Kress [9].

The operators $L_k, M_k, M_k^T : \mathcal{H}^r(\Gamma) \to \mathcal{H}^{r+1}(\Gamma)$ are pseudodifferential operators of order -1 (once smoothing), whilst N_k has order $+1$, essentially a once differentiating operator, Colton and Kress [8].

The choice of the coupling parameter affects the conditioning of equation (2). Based on exact analysis for the case of a circle (or a sphere in 3-D), it

has been shown that the choice $\eta = \frac{1}{k}$ almost minimises the conditioning of the equation, see Amini [1, 2] and Kress [9].

We attempt the numerical solution of equation (2) for the Neumann problem by the collocation method based on piecewise constant approximations.

3 ITERATIVE METHODS

We are concerned with the iterative solution of equation

$$\mathcal{A}_n \phi_n = f_n, \tag{3}$$

the boundary element discretisation of equation (2). Various iterative techniques have been analysed and implemented successfully for second kind equations where there is clustering of the eigenvalues of the integral operator (see for example, [3, 4]). The problem here is the presence of the operator N_k as the eigenvalues of its discrete approximation grow with n. This means that without modification these methods will not work efficiently.
We split the matrix $\mathcal{A}_n$ into $\mathcal{D}_n - \mathcal{C}_n$ and write equation (3) as

$$\mathcal{D}_n \phi_n = \mathcal{C}_n \phi_n + f_n.$$

We attempt two iterative methods for the solution of the above, namely

- direct iteration as
$$\mathcal{D}_n \phi_n^{(\ell+1)} = \mathcal{C}_n \phi_n^{(\ell)} + f_n, \tag{4}$$
- two-grid method, using equation (4) in the *smoothing* and *interpolation* step [7].

For the $\mathcal{D}_n$ matrix we propose to take three possible choices:

(a) the diagonal elements of $\mathcal{A}_n$,

(b) the tridiagonal band, or

(c) the tridiagonal band together with the $(1, n)$ and $(n, 1)$ corner elements, which will be referred to as periodic tridiagonal splitting.

Note (c) reflects taking an element on either side of any collocation point and (a) is recognised to be the well-known Jacobi method. The periodic tridiagonal system has a simple LU factorisation of the form

$$\begin{pmatrix} \times & & & & \\ a_{21} & \times & & & \\ & a_{32} & \times & & \\ & & \ddots & \times & \\ \times & \times & \times & \times & \times \end{pmatrix} \begin{pmatrix} 1 & \times & & & \times \\ & 1 & \times & & \times \\ & & \ddots & \times & \times \\ & & & \ddots & \times \\ & & & & \ddots \end{pmatrix}$$

which can be stored in 4 vectors.

3.1 2-GRID SCHEMES

Suppose we discretise the boundary integral equation (2) on the two grids G_n and G_m with n and m elements respectively ($n < m$). The idea is to obtain the solution of $\mathcal{A}_m \phi_m = f_m$ on the fine grid G_m using information from the coarse grid G_n.
We can briefly describe our 2-grid cycle in the form:

Initialize:
$r_m = f_m$, $\phi_m = 0$

Begin Iteration:
Restrict r_m to coarse grid; $r_m \rightarrow r_n$
Solve on G_n; $\mathcal{A}_n v_n = r_n$
Interpolate v_n to fine grid; $v_n \rightarrow v_m$
Smooth v_m; $v_m \rightarrow \bar{v}_m$
Correct ϕ_m; $\phi_m := \phi_m + \bar{v}_m$
If $\|\bar{v}_m\| < TOL$ exit with solution
Calculate residual on G_m; $r_m = f_m - \mathcal{A}_m \phi_m$
End Iteration.

If we choose nodes of G_n to coincide with those on G_m then the restriction step can be performed by simple injection. The solution on the coarse grid is carried out by a direct method eg. Gaussian elimination with partial pivoting, the LU factors being stored for use at each iteration. The interpolation of values from the coarse to the fine grid is carried out using the generalised Picard iteration $\mathcal{D}_m v_m = \mathcal{C}_n v_n + r_m$. The smoothing process can be written in the form $\mathcal{D}_m v_m^{(\ell+1)} = \mathcal{C}_m v_m^{(\ell)} + r_m$ where $v_m^{(0)} = v_m$ and $v_m^{(p)} = \bar{v}_m$, p is the number of smoothing iterations.

4 NUMERICAL EXPERIMENTS

For the first test problem we take Γ as the unit circle or an ellipse of approximate length 2π with minor to major axis ratio of 1:2 (i.e. $a = 0.65$, $b = 1.30$ with $(x/a)^2 + (y/b)^2 = 1$). Both are divided into equal length elements and the collocation points are the element midpoints. We consider the field produced on the surface of Γ due to a source at $(0.5, 0.0)$ at 4 wavenumbers $k = 3, 5, 8$ and 10.

In Table 1 the results are shown for both boundaries (the circle being represented by ratio 1, the ellipse by ratio 2) for the Burton and Miller equation (1) with coupling parameter $\eta = 1/k$ for the direct iterative schemes (4) and number of elements n. The values shown are the number of iterations and the resulting error achieved. The measure of error quoted in all

tables is $\log_2$ RME where the Relative Mean Error is defined as:

$$\text{RME} = \sum_{i=1}^{n} \left| \phi(p_i) - \tilde{\phi}_n(p_i) \right| \; / \; \sum_{i=1}^{n} |\phi(p_i)| \, .$$

The p_i are the collocation points, ϕ is the exact solution and $\tilde{\phi}_n$ denotes the iterative method approximation to the collocation solution ϕ_n. The same schemes applied to either the Surface Helmholtz Equation or the B&M equation with $\eta = 1$ generally diverge.

$k(n)$	ratio	tridiagonal	periodic tridiagonal
3(36)	1	11(-10.39)	11(-10.44)
	2	10(-8.65)	10(-8.68)
3(72)	1	50(-12.34)	53(-12.36)
	2	49(-10.65)	49(-10.66)
5(60)	1	12(-11.51)	11(-11.71)
	2	12(-9.87)	11(-9.89)
8(96)	1	14(-11.96)	14(-11.97)
	2	14(-10.18)	12(-10.17)
10(120)	1	14(-11.99)	12(-12.10)
	2	14(-10.34)	13(-10.31)

Table 1: Standard Iterations $\eta = 1/k$

The results for the direct iteration with Jacobi splitting are not presented here as the iteration matrix $\mathcal{D}_n^{-1}\mathcal{C}_n$ had spectral radius very close to 1. When convergence was achieved the number of iterations was > 100. A disappointing feature of the direct iteration scheme is that as n is increased in order to improve accuracy (for fixed k) the number of iterations rises dramatically.

$k(m)$	ratio	SHE	$\eta = 1$	$\eta = 1/k$
3(36)	1	2(-9.64)	8(-8.92)	5(-10.35)
	2	2(-7.63)	15(-8.16)	6(-8.47)
3(72)	1	2(-11.59)	7(-11.11)	5(-11.55)
	2	2(-9.66)	11(-9.10)	6(-10.06)
5(60)	1	4(-7.87)	14(-8.45)	7(-11.41)
	2	4(-7.31)	20(-7.82)	6(-9.47)
8(96)	1	2(-9.00)		6(-11.98)
	2	2(-7.38)		6(-10.15)
10(120)	1	4(-7.23)		5(-11.97)
	2	3(-7.84)		6(-10.22)

Table 2: 2-Grid Iterations with 2 Jacobi Smoothing Steps

$k(m)$	ratio	SHE	$\eta = 1$	$\eta = 1/k$
3(36)	1	3(-9.62)	3(-8.80)	2(-10.26)
	2	4(-7.59)	2(-8.09)	2(-8.57)
3(72)	1	2(-11.59)	3(-11.44)	3(-12.31)
	2	2(-8.82)	3(-9.10)	2(-9.59)
5(60)	1	4(-7.87)	3(-8.55)	2(-11.47)
	2	4(-7.31)	3(-7.85)	2(-9.98)
8(96)	1	2(-9.00)	2(-8.41)	2(-12.06)
	2	3(-7.68)	2(-7.57)	2(-10.17)
10(120)	1	4(-7.23)	3(-9.45)	2(-12.06)
	2	3(-8.01)	2(-7.55)	2(-10.25)

Table 3: 2-Grid Iterations with 2 Tridiagonal Smoothing Steps

$k(m)$	ratio	SHE	$\eta = 1$	$\eta = 1/k$
3(36)	1	2(-9.66)	2(-9.02)	2(-10.41)
	2	3(-7.61)	3(-8.14)	2(-8.62)
3(72)	1	2(-11.59)	3(-11.46)	3(-12.32)
	2	3(-9.63)	3(-10.31)	3(-10.58)
5(60)	1	3(-7.85)	2(-8.68)	2(-11.50)
	2	3(-7.37)	2(-8.11)	2(-9.86)
8(96)	1	2(-9.01)	2(-8.42)	2(-12.07)
	2	3(-7.69)	3(-7.80)	3(-10.18)
10(120)	1	4(-7.23)	3(-9.46)	2(-12.08)
	2	3(-8.01)	3(-7.63)	2(-10.30)

Table 4: 2-Grid Iterations with 2 PT Smoothing Steps

Tables 2 to 4 give the results for the 2-Grid Scheme using the Jacobi, tridiagonal or periodic tridiagonal smoothing for the SHE and the B&M equation with the coupling parameter $\eta = 1$ or $1/k$. The number of elements is m for the fine grid and $n = m/3$ on the coarse grid. Blank spaces in Table 2 mean that the particular problem diverged.

All the methods work well for the SHE, as to be expected for a second kind Fredholm problem. For equation (1) the choice of coupling parameter is very important if Jacobi smoothing is considered. It has less effect on the number of iterations for the other cases but greatly increases the accuracy due to the improved conditioning of equation (1). Increasing the collocation points does not result in a marked increase in the number of iterations as happens for the direct iterative scheme.

The second test problem considers the field produced on the surface of a square with vertices $(0,0),(0,1),(1,0)$ and $(1,1)$, by a source at the centre. We take a uniform mesh with collocation at the element midpoints. In tables 5 and 6 the number of iterations are given for the 2-Grid methods

applied to the SHE and B&M at 3 wavenumbers with m elements on the fine grid and $m/3$ on the coarse. The direct iterative schemes either do not converge or converge too slowly.

k	m	SHE	m	B&M ($\eta = 1$)	B&M ($\eta = 1/k$)
1	12	3(-6.91)	24	3(-5.57)	3(-5.57)
3	36	3(-8.04)	72	3(-6.60)	3(-7.04)
3	72	3(-9.82)	144	3(-7.56)	3(-8.04)
5	60	3(-7.91)	120	3(-6.43)	3(-7.67)
5	120	2(-9.80)	240	3(-7.35)	3(-8.52)

Table 5: 2-Grid Iterations with 2 Jacobi Smoothing Steps

k	m	SHE	m	B&M ($\eta = 1$)	B&M ($\eta = 1/k$)
1	12	3(-7.08)	24	3(-5.56)	3(-5.56)
3	36	3(-8.03)	72	3(-6.52)	3(-6.96)
3	72	3(-9.78)	144	3(-7.45)	3(-7.89)
5	60	3(-7.94)	120	3(-6.18)	3(-7.38)
5	180	3(-9.76)	240	3(-7.16)	3(-8.31)

Table 6: 2-Grid Iterations with 1 PT Smoothing Step

In all cases we find that the use of one periodic tridiagonal smoothing step gives very similar results to that of 2 Jacobi iterations. Taking $\eta = 1/k$ again improves the accuracy though to a lesser extent to that of the smooth boundary.

5 CONCLUSIONS

Several iterative methods based on the splitting of the discrete operator have been implemented for the solution of boundary element equations. The direct iterative schemes (4) are efficient only for the case of smooth boundaries and whenever a modest level of accuracy is required. However, the application of these splitting schemes as smoothers within a 2-Grid cycle results in methods which converge in a small number of iterations in all cases considered. For the SHE we see that the Jacobi smoother performs well, whilst for the Burton and Miller equation (involving the N_k operator) the choice of periodic tridiagonal with $\eta = 1/k$ gives consistently good results. We are at present studying the design of appropriate preconditioners to be used with conjugate gradient methods for the efficient solution of systems involving the hypersingular operator.

REFERENCES

[1] Amini, S. 'On the Choice of the Coupling Parameter in Boundary Integral Formulations of the Exterior Acoustic Problem' *Appl. Anal.*, vol. 35, pp.75-92, 1990

[2] Amini, S. 'Boundary Integral Solution of the Exterior Helmholtz Problem' *Comp. Mech.*, no. 13, pp.2-11, 1993

[3] Amini, S. and Chen Ke 'Iterative Methods for Boundary Element Methods - Applications to the Exterior Acoustic Problem' in *Boundary Elements X*, no. 4, pp.317-331 (Brebbia, C. A. ed.) CMP, Springer-Verlag, 1988

[4] Amini, S. and Chen Ke 'Conjugate Gradient Method for the Second Kind Integral Equations - Applications to the Exterior Acoustic Problem' *Engineering Analysis with Boundary Elements*, no. 6, pp.72-77, 1989

[5] Amini, S., Chen Ke and Harris, P. J. 'Iterative Solution of Boundary Element Equations for the Exterior Acoustic Problem' *ASME. Vib. Acous.*, no. 112, pp.257-262, 1990

[6] Burton, A. J. and Miller, G. F. 'The Application of Integral Equation Methods for the Numerical Solution of Boundary Value Problems' *Proc. Roy. Soc. London*, A232, pp.201-210, 1971

[7] Chen Ke and Amini, S. 'Numerical Analysis of Boundary Integral Solution of the Helmholtz Equation in Domains with Non-Smooth Boundaries' *IMA. J. N.A.*, no. 13, pp.43-66, 1993

[8] Colton, D.L. and Kress, R. *Integral Equation Methods in Scattering Theory*, Wiley, New York, 1983

[9] Kress, R., 'Minimising the Condition Number of Boundary Integral Operators in Acoustic and Electromagnetic Scattering' *Q. J. Mech. Appl. Math.*, vol. 38, pp.232-341, 1985

[10] Schatz, A.H., Thomée, V. and Wendland W.L. *Mathematical Theory of Finite and Boundary Element Methods*, DMV Seminar, Band 15, Birkhäuser Verlag, 1990