JACOB'S LADDERS AND VECTOR OPERATOR PRODUCING NEW GENERATIONS OF L_2 -ORTHOGONAL SYSTEMS CONNECTED WITH THE RIEMANN'S $\zeta(\frac{1}{2} + it)$ FUNCTION

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ABSTRACT. In this paper we introduce a generating vector-operator acting on the class of functions $L_2([a, a + 2l])$. This operator produces (for arbitrarily fixed [a, a + 2l]) infinite number of new generation L_2 -systems. Every element of the mentioned systems depends on Riemann's zeta-function and on Jacob's ladder.

1. INTRODUCTION

1.1. In this paper we introduce vector operator \hat{G} defined on the class of all L_2 -orthogonal systems

$${f_n(t)}_{n=0}^{\infty}, t \in [a, a+2l], \forall a \in \mathbb{R}, \forall l \in \mathbb{R}^+$$

that for a fixed class $L_2([a, a + 2l])$ associates following new classes:

(1.1)
$$\{f_n(t)\}_{n=0}^{\infty} \xrightarrow{G} \{f_n^{p_1}(t)\}_{n=0}^{\infty}, \ p_1 = 1, \dots, k \}$$

(1.2)
$$\{f_n^{p_1}(t)\}_{n=0}^{\infty} \xrightarrow{\hat{G}} \{f_n^{p_1,p_2}(t)\}_{n=0}^{\infty}, \ p_1,p_2=1,\ldots,k_n\}_{n=0}^{\infty}$$

and so on up to

(1.3)

$$\{f_n^{p_1,\dots,p_{s-1}}(t)\}_{n=0}^{\infty} \xrightarrow{G} \{f_n^{p_1,\dots,p_{s-1},p_s}(t)\}_{n=0}^{\infty}, \ t \in [a,a+2l], \ p_1,\dots,p_s=1,\dots,k_s\}$$

for every fixed $k, s \in \mathbb{N}$.

Sets (1.1) - (1.3) give consequently the first generation, the second generation and so on up s^{th} generation of new L_2 -orthogonal systems. The counts of members of generations form the geometric sequence

$$k, k^2, \ldots, k^s$$
.

Remark 1. If, for example,

$$k = 10^2, \ s = 5 \times 10^4$$

i. e. for the 50 000th generation of new L_2 -orthogonal systems, we obtain

$$10^{10^{\circ}}$$
 multiple

from one and only fixed L_2 -orthogonal system.

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1.2. Let us remind the definition of the Legendre's polynomials

(1.4)
$$\{P_n(t)\}_{n=0}^{\infty}, \ t \in [-1, 1]$$

by means of the generating function that is by the formula

(1.5)
$$\frac{1}{\sqrt{1-2ut-t^2}} = \sum_{n=0}^{\infty} P_n(t)u^n, \ u \in (-1,1),$$

where, of course, the collection (1.4) represents the simple L_2 -orthogonal system.

Now, let us have a look on how the operator \hat{G} acts on the system (1.4). For given natural number k the operator \hat{G} produces as many as k^3 new species of L_2 -orthogonal systems of the third generation as follows:

(1.6)
$$P_n^{p_1, p_2, p_3}(t) = P_n(u_{p_1}(u_{p_2}(u_{p_3}(t)))) \times \prod_{r=0}^{p_1-1} |\tilde{Z}(v_{p_1}^r(u_{p_2}(u_{p_3}(t))))| \times \prod_{r=0}^{p_3-1} |\tilde{Z}(v_{p_3}^r(t))|,$$
$$p_1, p_2, p_3 = 1, \dots, k, \ t \in [-1, 1], \ a = -1, \ l = 1,$$

where

(1.7)

$$u_{p_i}(t) = \varphi_1^{p_i} \left(\frac{\overline{T+2} - T}{2}(t+1) + T \right) - T - 1, \ i = 1, 2, 3,$$
$$v_{p_1}^r(t) = \varphi_1^r \left(\frac{\overline{T+2} - T}{2}(t+1) + T \right), \ r = 0, 1, \dots, p_i - 1$$

$$t \in [-1,1] \Rightarrow u_{p_i}(t) \in [-1,1] \land v_{p_i}^r(t) \in [\overset{p_i-r}{T}, \overset{p_i-r}{T+2}].$$

1.3. Now we give the following.

Property 1. (a) Every member of every new L_2 -orthogonal system

$$\{P_n^{p_1,p_2,p_3}(t)\}_{n=0}^{\infty}, t \in [-1,1], p_1, p_2, p_3 = 1, \dots, k$$

contains the function

$$\left|\zeta\left(\frac{1}{2}+it\right)\right|_{t=\tau}$$

for corresponding τ since (comp. [3], (9.1), (9.2))

(1.8)
$$|\tilde{Z}(t)| = \sqrt{\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t}} = \{1 + o(1)\}\frac{1}{\sqrt{\ln t}}|\zeta\left(\frac{1}{2} + it\right)|, \ t \to \infty.$$

(b) property (a) holds true due to the Theorem of this paper for every generation

$${f_n^{p_1,\dots,p_s}(t)}_{n=0}^{\infty}, t \in [a, a+2l], s \in \mathbb{N}$$

Remark 2. The main aim of this paper is expressed by the Property 1. Namely, that there is a close binding between the theory of the Riemann's zeta-function on the critical line and the theory of L_2 -orthogonal systems. Let us notice also that this paper finishes preparatory papers [5] and [6].

2. Main result

- 2.1. We use the following notions:
 - (a) Jacob's ladder $\varphi_1(t)$,
 - (b) the function

(2.1)
$$\tilde{Z}^{2}(t) = \frac{\mathrm{d}\varphi_{1}(t)}{\mathrm{d}t} = \frac{1}{\omega(t)} \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2},$$
$$\omega(t) = \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t} \right) \right\} \ln t, \ t \to \infty,$$

(c) direct iterations of the Jacob's ladder

(2.2)
$$\varphi_1^0(t) = t, \ \varphi_1^1(t) = \varphi_1(t), \ \varphi_1^2(t) = \varphi_1(\varphi_1(t)), \dots, \varphi_1^k(t) = \varphi_1(\varphi_1^{k-1}(t))$$

for every fixed $k \in \mathbb{N}$,

(d) reverse iterations (by means of $\varphi_1^{-1}(t)$)

$$\begin{bmatrix} 0 & 0 \\ [T, \overline{T+U}], [T, \overline{T+U}], \dots, [T, \overline{T+U}], \\ U = o\left(\frac{T}{\ln T}\right), \ T \to \infty$$

of the basic segment

$$[T, T+U] = [\overset{0}{T}, \overbrace{T+U}^{0}],$$

that we have introduced into the theory of the Riemann's zeta-function, see [1] - [5].

- 2.2. Next we use the following analytic properties of Jacob's ladder $\varphi_1(t)$:
 - (e) $\varphi_1(t) \in C^{\infty}([T_0, \infty])$ and it is strongly increasing function, see [1],
 - (f) $\varphi_1^p(t) \in C^{\infty}([T_0, \infty]), \ p = 1, \dots, k$ and it is again strongly increasing (this property follows easily from (e))
 - (g) as a consequence of (f) we have next: every function

$$\varphi_1^p(t), t \in [A, B], A > T_0$$

is absolutely continuous and strongly increasing on every segment [A, B] with $A > T_0$ (namely, the Lipschitz condition holds true by (f)),

(h) the composite function F[f(t)], where F is absolutely continuous and f is absolutely continuous and monotonic is again absolutely continuous function. 2.3. Finally, we introduce the following functions (comp. (1.7)) together with some of their properties:

(2.3)
$$u_{p_{i}}(t) = \varphi_{1}^{p_{i}} \left(\frac{\widehat{T+2l} - T}{2l}(t-a) + T \right) - T + a,$$
$$v_{p_{i}}^{r}(t) = \varphi_{1}^{r} \left(\frac{\widehat{T+2l} - T}{2l}(t-a) + T \right),$$
$$t \in [a, a+2l], \ i = 1, \dots, s, \ r = 0, 1, \dots, p_{i-1}, \ p_{i} = 1, \dots, k,$$

where

(2.4)
$$u_{p_i}(t) \in [a, a+2l], \ v_{p_i}^r(t) \in \begin{bmatrix} p_i - r, \overbrace{T+2l}^{p_i - r}, \overbrace{T+2l}^{p_i - r} \end{bmatrix}$$

and, with regard to the second inclusion, see [5], Property 2, the segments

$$\begin{bmatrix} p_i - r & p_i - r \\ T & T + 2l \end{bmatrix}$$

represent corresponding components of the disconnected set (see [5], (2.9))

(2.5)
$$\Delta(T,k,l) = \bigcup_{r=0}^{k} \begin{bmatrix} r & T \\ T, T+2l \end{bmatrix}$$

and the following properties of the above mentioned set hold true¹:

(2.6)
$$l = o\left(\frac{T}{\ln T}\right), \ T \to \infty \ \Rightarrow$$

(2.7)
$$\left| \begin{bmatrix} r & r \\ T, \overline{T+2l} \end{bmatrix} \right| = \overline{T+2l} - \overline{T} = o\left(\frac{T}{\ln T}\right),$$

(2.8)
$$\left| \left[\overbrace{T+2l}^{r-1}, \stackrel{r}{T} \right] \right| = \stackrel{r}{T} - \overbrace{T+2l}^{r-1} \sim (1-c) \frac{T}{\ln T},$$

(2.9)
$$\begin{bmatrix} 0 & 0 \\ T, \overline{T+2l} \end{bmatrix} \prec \begin{bmatrix} 1 & 1 \\ T, \overline{T+2l} \end{bmatrix} \prec \cdots \prec \begin{bmatrix} k & k \\ T, \overline{T+2l} \end{bmatrix},$$

where c is the Euler's constant and the property (2.9) follows from (2.8).

Remark 3. Asymptotic behavior of the disconnected set (2.5) is as follows: if $T \to \infty$, then the components of this set recede unboundedly each from other and all together are receding to infinity. Hence the set (2.5) behaves at $T \to \infty$ as one-dimensional Friedmann-Hubble expanding universe.

 $^{^{1}[5], (2.5), (2.6)}$

2.4. The following Theorem is the main result of this paper.

Theorem 1. There is such a generating vector-operator \hat{G} that for every fixed L_2 -orthogonal system

(2.10)
$$\{f_n(t)\}_{n=0}^{\infty}, t \in [a, a+2l], a \in \mathbb{R}, l \in \mathbb{R}^+$$

and for every fixed $k \in \mathbb{N}$ the operator \hat{G} associates following orthogonal systems:

(a) the first generation of the following new species of L_2 -orthogonal systems

(2.11)
$$\{f_n^{p_1}(t)\}_{n=0}^{\infty}, \ t \in [a, a+2l], \ p_1 = 1, \dots, k$$

i. e.

(2.12)
$$\hat{G}[\{f_n(t)\}] = (\{f_n^1(t)\}, \{f_n^2(t)\}, \dots, \{f_n^k(t)\}); \{f_n(t)\} = \{f_n(t)\}_{n=0}^{\infty},$$

where²

(2.13)
$$f_n^{p_1}(t) = f_n(u_{p_1}(t)) \prod_{r=0}^{p_1-1} \left| \tilde{Z}(v_{p_1}^r(t)) \right|$$

and every of the functions $u_{p_1}(t)$ defines an automorphism on [a, a + 2l],

(b) the second generation of the following L_2 -orthogonal systems

(2.14)
$$\{f_n^{p_1,p_2}(t)\}_{n=0}^{\infty}, t \in [a, a+2l], p_1, p_2 = 1, \dots, k,$$

i. e.

(2.15)
$$\hat{G}[\{f_n^{p_1}(t)\}] = (\{f_n^{p_1,1}(t)\}, \{f_n^{p_1,2}(t)\}, \dots, \{f_n^{p_1,k}(t)\}),$$

where

(2.16)
$$f_n^{p_1,p_2}(t) = f_n(u_{p_1}(u_{p_2}(t))) \prod_{r=0}^{p_1-1} \left| \tilde{Z}(v_{p_1}^r(u_{p_2}(t))) \right| \prod_{r=0}^{p_2-1} \left| \tilde{Z}(v_{p_2}^r(t)) \right|$$

and each of the functions $u_{p_1}(u_{p_2}(t))$ defines an automorphism on [a, a+2l], (c) and so on up to the s^{th} generation of the L_2 -orthogonal systems

(2.17)
$$\{f_n^{p_1, p_2, \dots, p_s}(t)\}_{n=0}^{\infty}, \ t \in [a, a+2l], \ p_1, \dots, p_s = 1, \dots, k,$$

i. e.
(2.18)
$$\hat{G}[\{f_n^{p_1,\dots,p_{s-1}}(t)\}] = (\{f_n^{p_1,\dots,p_{s-1},1}(t)\}, \{f_n^{p_1,\dots,p_{s-1},2}(t)\},\dots, \{f_n^{p_1,\dots,p_{s-1},k}(t)\}),$$

 2 Comp. (2.3).

where

$$f_n^{p_1, p_2, \dots, p_s}(t) = f_n(u_{p_1}(u_{p_2}(\dots(u_{p_s}(t))\dots))) \times \prod_{r=0}^{p_1-1} \left| \tilde{Z}(v_{p_1}^r(u_{p_2}(u_{p_3}(\dots(u_{p_s}(t))\dots)))) \right| \times \prod_{r=0}^{p_2-1} \left| \tilde{Z}(v_{p_2}^r(u_{p_3}(u_{p_3}(\dots(u_{p_s}(t))\dots)))) \right| \times$$

(2.19)

$$\begin{split} & \vdots \\ & \prod_{r=0}^{p_{s-1}-1} \left| \tilde{Z}(v_{p_s-1}^r(u_{p_s}(t))) \right| \times \\ & \prod_{r=0}^{p_s-1} \left| \tilde{Z}(v_{p_s}(t)) \right| \end{split}$$

and each of the functions

$$u_{p_1}(u_{p_2}(\ldots(u_{p_s}(t))\ldots))$$

defines an automorphism on [a, a + 2l],

(d) for every fixed $k, s \in \mathbb{N}$ the L_2 -orthonormal system

(2.20)
$$\left(\prod_{i=0}^{s} \sqrt{\frac{2l}{\widehat{T+2l}-\stackrel{i}{T}}}\right) f_{n}^{p_{1},\dots,p_{s}}(t), \ t \in [a,a+2l], \ p_{1},\dots,p_{s}=1,\dots,k$$

is corresponding with (2.19),

- (e) finally, all these formulas are true for all sufficiently big T > 0, that is we have the continuum set of possibilities how to construct new classes of L_2 -orthogonal systems (2.17).
- 2.5. Let us denote the set of all L_2 -orthogonal systems not containing the functions

(2.21)
$$\varphi_1(t), \ \tilde{Z}^2(t) = \frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t}$$

as

$$L_2^0([a, a+2l]).$$

Now it is true by Theorem 1 that for every fixed

$$\{f_n(t)\}_{n=0}^{\infty} = \{f_n(t)\} \in L_2^0([a, a+2l]),$$

$$\{f_n(t)\} \xrightarrow{\hat{G}} \{f_n^{p_1}(t)\} \xrightarrow{\hat{G}} \dots \xrightarrow{\hat{G}} \{f_n^{p_1,\dots,p_s}(t)\}.$$

We instantly get the following.

Corollary 1.

(2.22)
$$L_2^0([a,a+2l]) \xrightarrow{\hat{G}} L_2^1([a,a+2l]) \xrightarrow{\hat{G}} \dots \xrightarrow{\hat{G}} L_2^s([a,a+2l]),$$

where

$$L_{2}^{i}([a, a+2l])$$

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stands for the i^{th} generation of the image of $L_2^0([a, a+2l])$. Consequently, the union

(2.23)
$$\bigcup_{i=1}^{s} L_{2}^{i}([a, a+2l])$$

represents a kind of ζ -extension of the s^{th} order of $L_2^0([a, a + 2l])$ for every fixed segment [a, a + 2l] and every fixed $k, s \in \mathbb{N}$. Finally, the union over all the segments

(2.24)
$$\bigcup_{a \in \mathbb{R}, l > 0} \left\{ \bigcup_{i=1}^{s} L_2^i([a, a+2l]) \right\}$$

represents the complete ζ -extension of the s^{th} order of the set

(2.25)
$$\bigcup_{a \in \mathbb{R}, l > 0} L_2^0([a, a + 2l])$$

2.6.

Remark 4. We may select another function

$$\psi(t) \neq \varphi_1(t), \ \psi(t) \nsim \varphi_1(t), \ t \to \infty$$

instead of the Jacob's ladder and use the pair

$$\psi(t), \ \frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$$

to extend $L_2^0([a, a + 2l])$. This way however will not attain ζ -extensions (2.13), (2.16), (2.19), (2.23) and (2.24) and therefore will be irrelevant.

3. Jacob's ladders

3.1. Let us remind that the Jacob's ladder

$$\varphi_1(t) = \frac{1}{2}\varphi(t)$$

was introduced in [1], see also [3], where the function $\varphi(t)$ is an arbitrary continuous solution of the nonlinear integral equation³

(3.1)
$$\int_{0}^{\mu[x(T)]} Z^{2}(t) e^{-\frac{2}{x(T)}t} dt = \int_{0}^{T} Z^{2}(t) dt,$$

where

(3.2)
$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right),$$
$$\vartheta(t) = -\frac{t}{2}\ln\pi + \operatorname{Im}\left\{\ln\Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)\right\}$$

and the class of functions $\{\mu\}$ is specified as

$$\mu \in C^{\infty}([y_0, +\infty))$$

being monotonically increasing, unbounded from above and obeying the inequality

(3.3) $\mu(y) \ge 7y \ln y.$

Every admissible function $\mu(y)$ generates a solution

 $y = \varphi(T; \mu) = \varphi(T).$

³Also introduced in [1].

Remark 5. The function $\varphi_1(T)$ is called Jacob's ladder as an analogue of the Jacob's dream in Chumash, Bereishis, 28:12.

3.2. Let us remind that the Hardy-Littlewood integral (1918)

(3.4)
$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 \mathrm{d}t$$

can be expressed as follows:

(3.5)
$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 dt = T \ln T + (2c - 1 - \ln 2\pi)T + R(T),$$

with, for example, Ingham's error term

(3.6)
$$R(T) = \mathcal{O}(T^{1/2} \ln T) = \mathcal{O}(T^{1/2+\delta}), \ \delta > 0, \ T \to \infty$$

for arbitrary small δ .

Next, it is true by Good's Ω -theorem (1977), that

(3.7)
$$R(T) = \Omega(T^{1/4}), \ T \to \infty.$$

Remark 6. Let

(3.8)
$$R_a(T) = \mathcal{O}(T^{1/4+a}), \ a \in \left[\delta, \frac{1}{4} + \delta\right], \ T \to \infty.$$

Then, by (3.7), one obtain for every valid estimate of the type (3.8) that

(3.9)
$$\limsup_{T \to \infty} |R_a(T)| = +\infty.$$

In other words, every expression of the type (3.5) and (3.9) possesses an unbounded error at infinity.

3.3. Under the circumstances (3.5) and (3.9) we have shown in our paper [1] that the Hardy-Littlewood integral (3.4) has an infinite set of almost exact representations expressed by the following formula.

Formula1.

(3.10)
$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^2 \mathrm{d}t = \varphi_1(T) \ln\{\varphi_1(T)\} + (c - \ln 2\pi)\varphi_1(T) + c_0 + \mathcal{O}\left(\frac{\ln T}{T}\right), \ T \to \infty$$

(c is the Euler's constant and c_0 is the constant from the Titchmarsh-Kober-Atkinson formula) with the error term vanishing at infinity:

(3.11)
$$\tilde{R}(T) = \mathcal{O}\left(\frac{\ln T}{T}\right) \xrightarrow{T \to \infty} 0.$$

Remark 7. The comparison of (3.9) and (3.11) completely characterizes the level of exactness of our representation (3.10) of the Hardy-Littlewood integral (3.4).

3.4. Now, let us remind other formulae demonstrating the power of Jacob's ladder $\varphi_1(t)$.

First we have obtained the following⁴

Formula2.

(3.12)
$$\int_{T}^{T+U} \left| \zeta \left(\frac{1}{2} + i\varphi_{1}(t) \right) \right|^{4} \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2} \mathrm{d}t \sim \frac{1}{2\pi^{2}} U \ln^{5} T,$$
$$U = T^{7/8 + 2\delta}, \ T \to \infty.$$

Remark 8. The formula (3.12) is the first asymptotic formula of the sixth order in $|\zeta|$ on the critical line in the theory of the Riemann's zeta-function.

Next, let

$$S(t) = \frac{1}{\pi} \arg\left\{\zeta\left(\frac{1}{2} + it\right)\right\}, \ S_1(T) = \int_0^T S(t) dt,$$

where the function arg is defined in the usual way. We have obtained the following two formulae concerning S(t) function:⁵

Formula3.

(3.13)
$$\int_{T}^{T+U} \left[\arg \left\{ \zeta \left(\frac{1}{2} + i\varphi_{1}(t) \right) \right\} \right]^{2k} \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2} \mathrm{d}t \sim \frac{1}{2^{k}} U \ln T (\ln \ln T)^{k}, \ U \in \left[T^{1/3+\delta}, \frac{T}{\ln T} \right], \ T \to \infty,$$

for every fixed $k \in \mathbb{N}$.

Formula4.

(3.14)
$$\int_{T}^{T+U} \{S_1[\varphi_1(t)]\}^{2k} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \mathrm{d}t \sim a_k U \ln T, \ T \to \infty.$$

Remark 9. New kind of the classical A. Selberg's formulae (1946) are expressed by means of our results (3.13) and (3.14).

Remark 10. Let us notice explicitly, that formulae (3.12) - (3.14) are ζ -correlation formulae on the critical line. For example, (3.13) at k = 1 describes interaction between values of the functions

$$\left[\arg\left\{\zeta\left(\frac{1}{2}+i\varphi_1(t)\right)\right\}\right]^k, \ \left|\zeta\left(\frac{1}{2}+it\right)\right|^2.$$

4. Proof of Theorem 1

4.1. In [5], (7.1) (7.2), we have shown the following results concerning direct and reverse iterations:

Lemma 1. If

(4.1)
$$U = o\left(\frac{T}{\ln T}\right), \ T \to \infty,$$

then for every function

$$g(t) \in L([T, T+U])$$

 $^{^{4}}$ See [3], (8.3).

⁵See [2], (5.4), (5.5).

the following holds true:

(4.2)
$$\int_{T}^{T+U} g(t) dt = \int_{T}^{p} f(\tau) g[\varphi_{1}^{p}(\tau)] \prod_{r=0}^{p-1} \tilde{Z}^{2}[\varphi_{1}^{r}(\tau)] d\tau, \ p = 1, \dots, k$$

for every fixed $k \in \mathbb{N}$.

Remark 11. Let us notice also the subsection 2.2, (g).

4.2. Proof of Theorem 1.

4.2.1. Let

(4.3)
$$\{f_n(t)\}_{n=0}^{\infty} \subset L_2([a, a+2l])$$

is arbitrary fixed system of orthogonal functions. Then l is also fixed positive number and condition (4.1) is fulfilled for U = 2l for all sufficiently big and positive T. Now we have $(m \neq n)$

(4.4)
$$0 = \int_{a}^{a+2l} f_m(t) f_n(t) dt = \int_{T}^{T+2l} f_m(\tau - T + a) f_n(\tau - T + a) d\tau =$$

next we obtain by Lemma 1 for any sufficiently big T

(4.5)
$$= \int_{T}^{\frac{\nu}{T+2l}} f_m[\varphi_1^p(\rho) - T + a] f_n[\varphi_1^p(\rho) - T + a] \prod_{r=0}^{p-1} \tilde{Z}^2[\varphi_1^r(\rho)] d\rho =$$

and next, by simple sunstitution

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$$\rho = \rho(t) = \frac{\widehat{T+2l} - \widehat{T}}{2l}(t-a) + \widehat{T}, \ \rho \in [\stackrel{p}{T}, \widehat{T+2l}], \ t \in [a, a+2l],$$

where $\rho(t)$ is absolutely continuous and increasing, we obtain

$$= \frac{\widehat{T+2l} - \widehat{T}}{2l} \int_{a}^{a+2l} f_{m} \left[\varphi_{1}^{p} \left(\frac{\widehat{T+2l} - \widehat{T}}{2l} (t-a) + \widehat{T} \right) - T + a \right] \times$$

$$(4.6) \qquad f_{n} \left[\varphi_{1}^{p} \left(\frac{\widehat{T+2l} - \widehat{T}}{2l} (t-a) + \widehat{T} \right) - T + a \right] \times$$

$$\prod_{r=0}^{p-1} \widetilde{Z}^{2} \left[\varphi_{1}^{r} \left(\frac{\widehat{T+2l} - \widehat{T}}{2l} (t-a) + \widehat{T} \right) \right] dt =$$

and, in the next step of the first cycle, we finish with

(4.7)
$$= \frac{\widehat{T+2l} - \widehat{T}}{2l} \int_{a}^{a+2l} f_{m}^{p}(t) f_{n}^{p}(t) dt \Rightarrow \int_{a}^{a+2l} f_{m}^{p}(t) f_{n}^{p}(t) dt = 0,$$

where

(4.8)
$$f_{n}^{p}(t) = f_{n} \left[\varphi_{1}^{p} \left(\frac{\widehat{T+2l} - \widehat{T}}{2l}(t-a) + \widehat{T} \right) - T + a \right] \times$$
$$\prod_{r=0}^{p-1} \left| \widetilde{Z} \left[\varphi_{1}^{r} \left(\frac{\widehat{T+2l} - \widehat{T}}{2l}(t-a) + \widehat{T} \right) \right] \right|, \ t \in [a, a+2l], \ p = 1, \dots, k$$

4.2.2. Now we give the following

Definition 1. The symbol \hat{G} stands for vector operator defined on the set all L_2 -orthogonal systems

$${f_n(t)}_{n=0}^{\infty}, t \in [a, a+2l], a \in \mathbb{R}, l > 0$$

defined by three integral transformatrions (4.4) – (4.6). \hat{G} maps an L_2 -orthogonal system into k-tuple of new orthogonal systems

$$\hat{G}[\{f_n(t)\}] = (\{f_n^1(t)\}, \{f_n^2(t)\}, \dots, \{f_n^k(t)\}) = \{f_n^p(t)\}_{n=0}^{\infty}, p = 1, \dots, k, \ t \in [a, a+2l]; \ \{f_n(t)\}_{n=0}^{\infty} = \{f_n(t)\}, \dots$$

for every fixed $k \in \mathbb{N}$.

4.2.3. Let us notice that the transformation⁶

(4.9)
$$u_p(t) = \varphi_1^p\left(\frac{\widehat{T+2l} - T}{2l}(t-a) + T\right) - T + a, \ t \in [a, a+2l]$$

has the following properties:

(a) By [5], subsection 6.1

$$u_p(a) = \varphi_1^p(\tilde{T}) - T + a = T - T + a = a; \quad \tilde{T} = \varphi_1^{-p}(T),$$
$$u_p(a+2l) = \varphi_1(\tilde{T+2l}) - T + a = a + 2l.$$

(b) Since the continuous function $\varphi_1^p(t)$ is increasing and

$$\rho = \frac{\widehat{T+2l} - \widehat{T}}{2l}(t-a) + \widehat{T}, \ t \in [a, a+2l]$$

it is evident that the composite function

$$u_p(t), t \in [a, a+2l]$$

is also increasing and therefore

$$t \in [a, a+2l] \implies u_p(t) \in [a, a+2l].$$

Remark 12. We have as a consequence of (a) and (b) that new automorphism on [a, a+2l] is defined by the one-to-one correspondence (4.9) for every fixed sufficiently big positive T. Of course, every function $u_{p_i}(t)$ defines an automorphism on [a, a+2l] too.

 $^{^{6}}$ Comp. (2.3), (4.8).

4.2.4. By making use of the operator \hat{G} on the system

$${f_n^{p_1}(t)}_{n=0}^{\infty}, t \in [a, a+2l], p_1 = 1, \dots, k$$

(the second cycle) we obtain

(4.10)
$$\hat{G}[\{f_n^{p_1}(t)\}] = \{f_n^{p_1, p_2}(t)\}_{n=0}^{\infty}, \ p_1, p_2 = 1, \dots, k,$$

where

$$\begin{aligned} &(4.11) \\ &f_n^{p_1,p_2}(t) = \\ &f_n \left[\varphi_1^{p_1} \left(\frac{\overline{T+2l} - T}{2l} \left(\varphi_1^{p_2} \left(\frac{\overline{T+2l} - T}{2l} (t-a) + T \right) - T \right) + T \right) - T + a \right] \times \\ &\prod_{r=0}^{p_1-1} \left| \tilde{Z} \left[\varphi_1^r \left(\left(\frac{\overline{T+2l} - T}{2l} \left(\varphi_1^{p_2} \left(\frac{\overline{T+2l} - T}{2l} (t-a) + T \right) - T \right) + T \right) \right) \right) \right] \right| \times \\ &\prod_{r=0}^{p_2-1} \left| \tilde{Z} \left[\varphi_1^r \left(\frac{\overline{T+2l} - T}{2l} (t-a) + T \right) \right) \right] \right|. \end{aligned}$$

It is clear that there is need for simplification of our formulae (4.8) and (4.11). For this purpose we use functions of $(2.3)^7$ that provides us with the results

(4.12)
$$f_n^{p_1}(t) = f_n(u_{p_1}(t)) \prod_{r=0}^{p_1-1} |\tilde{Z}(v_{p_1}^r(t))|,$$
$$f_n^{p_1,p_2}(t) = f_n(u_{p_1}(u_{p_2}(t))) \prod_{r=0}^{p_1-1} |v_{p_1}^r(u_{p_2}(t))| \prod_{r=0}^{p_2-1} |\tilde{Z}(v_{p_2}^r(t))|,$$

where, for example,

$$\{u_{p_1}(t_1), \ t_1 = u_{p_2}(t_2), \ t_1, t_2 \in [a, a+2l]\} \ \Rightarrow \ u_{p_1}(u_{p_2}(t_2)), \ t_2 = t,$$

i. e. we have formulae (2.13) and (2.16).

 $^{^7\}mathrm{See}$ also Remark 11.

4.2.5. Next, in the $(s-1)^{\text{th}}$ -cycle we get

$$f_n^{p_1, p_2, \dots, p_{s-1}}(t) = f_n(u_{p_1}(u_{p_2}(\dots(u_{p_{s-1}}(t))\dots))) \times \prod_{r=0}^{p_1-1} \left| \tilde{Z}(v_{p_1}^r(u_{p_2}(u_{p_3}(\dots(u_{p_{s-1}}(t))\dots)))) \right| \times \prod_{r=0}^{p_2-1} \left| \tilde{Z}(v_{p_2}^r(u_{p_3}(u_{p_4}(\dots(u_{p_{s-1}}(t))\dots)))) \right| \times$$

(4.13)

$$\begin{split} &\prod_{r=0}^{p_{s-2}-1} \left| \tilde{Z}(v_{p_s-2}^r(u_{p_{s-1}}(t))) \right| \times \\ &\prod_{r=0}^{p_{s-1}-1} \left| \tilde{Z}(v_{p_{s-1}}(t)) \right|, \ s > 2, \end{split}$$

then, if we use the operator \hat{G} on (4.13) to obtain the sth cycle, we get the set of k^s formulas (2.19). That means the formula (2.15) holds true for every $s \in \mathbb{N}$.

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