# JACOB'S LADDERS AND VECTOR OPERATOR PRODUCING NEW GENERATIONS OF $L_{2}$-ORTHOGONAL SYSTEMS CONNECTED WITH THE RIEMANN'S $\zeta\left(\frac{1}{2}+i t\right)$ FUNCTION 

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#### Abstract

In this paper we introduce a generating vector-operator acting on the class of functions $L_{2}([a, a+2 l])$. This operator produces (for arbitrarily fixed $[a, a+2 l])$ infinite number of new generation $L_{2}$-systems. Every element of the mentioned systems depends on Riemann's zeta-function and on Jacob's ladder.


## 1. Introduction

1.1. In this paper we introduce vector operator $\hat{G}$ defined on the class of all $L_{2^{-}}$ orthogonal systems

$$
\left\{f_{n}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], \forall a \in \mathbb{R}, \forall l \in \mathbb{R}^{+}
$$

that for a fixed class $L_{2}([a, a+2 l])$ associates following new classes:

$$
\begin{gather*}
\left\{f_{n}(t)\right\}_{n=0}^{\infty} \xrightarrow{\hat{G}}\left\{f_{n}^{p_{1}}(t)\right\}_{n=0}^{\infty}, p_{1}=1, \ldots, k,  \tag{1.1}\\
\left\{f_{n}^{p_{1}}(t)\right\}_{n=0}^{\infty} \xrightarrow{\hat{G}}\left\{f_{n}^{p_{1}, p_{2}}(t)\right\}_{n=0}^{\infty}, p_{1}, p_{2}=1, \ldots, k, \tag{1.2}
\end{gather*}
$$

and so on up to

$$
\begin{equation*}
\left\{f_{n}^{p_{1}, \ldots, p_{s-1}}(t)\right\}_{n=0}^{\infty} \xrightarrow{\hat{G}}\left\{f_{n}^{p_{1}, \ldots, p_{s-1}, p_{s}}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], p_{1}, \ldots, p_{s}=1, \ldots, k, \tag{1.3}
\end{equation*}
$$

for every fixed $k, s \in \mathbb{N}$.
Sets (1.1) - (1.3) give consequently the first generation, the second generation and so on up $s^{\text {th }}$ generation of new $L_{2}$-orthogonal systems. The counts of members of generations form the geometric sequence

$$
k, k^{2}, \ldots, k^{s}
$$

Remark 1. If, for example,

$$
k=10^{2}, s=5 \times 10^{4}
$$

i. e. for the $50000^{\text {th }}$ generation of new $L_{2}$-orthogonal systems, we obtain

$$
10^{10^{5}} \text { multiple }
$$

from one and only fixed $L_{2}$-orthogonal system.

[^0]1.2. Let us remind the definition of the Legendre's polynomials
\[

$$
\begin{equation*}
\left\{P_{n}(t)\right\}_{n=0}^{\infty}, t \in[-1,1] \tag{1.4}
\end{equation*}
$$

\]

by means of the generating function that is by the formula

$$
\begin{equation*}
\frac{1}{\sqrt{1-2 u t-t^{2}}}=\sum_{n=0}^{\infty} P_{n}(t) u^{n}, u \in(-1,1) \tag{1.5}
\end{equation*}
$$

where, of course, the collection (1.4) represents the simple $L_{2}$-orthogonal system.
Now, let us have a look on how the operator $\hat{G}$ acts on the system (1.4). For given natural number $k$ the operator $\hat{G}$ produces as many as $k^{3}$ new species of $L_{2}$-orthogonal systems of the third generation as follows:

$$
\begin{align*}
& P_{n}^{p_{1}, p_{2}, p_{3}}(t)=P_{n}\left(u_{p_{1}}\left(u_{p_{2}}\left(u_{p_{3}}(t)\right)\right)\right) \times \prod_{r=0}^{p_{1}-1}\left|\tilde{Z}\left(v_{p_{1}}^{r}\left(u_{p_{2}}\left(u_{p_{3}}(t)\right)\right)\right)\right| \times \\
& \prod_{r=0}^{p_{2}-1}\left|\tilde{Z}\left(v_{p_{2}}^{r}\left(u_{p_{3}}(t)\right)\right)\right| \times \prod_{r=0}^{p_{3}-1}\left|\tilde{Z}\left(v_{p_{3}}^{r}(t)\right)\right|,  \tag{1.6}\\
& p_{1}, p_{2}, p_{3}=1, \ldots, k, t \in[-1,1], a=-1, l=1,
\end{align*}
$$

where

$$
\begin{align*}
& u_{p_{i}}(t)=\varphi_{1}^{p_{i}}\left(\frac{\stackrel{p_{i}}{T+2}-\stackrel{p_{i}}{T}}{2}(t+1)+\stackrel{p_{i}}{T}\right)-T-1, i=1,2,3, \\
& v_{p_{1}}^{r}(t)=\varphi_{1}^{r}\left(\frac{\stackrel{p_{i}}{T+2}-\stackrel{p_{i}}{T}}{2}(t+1)+\stackrel{p_{i}}{T}\right), r=0,1, \ldots, p_{i}-1,  \tag{1.7}\\
& t \in[-1,1] \Rightarrow u_{p_{i}}(t) \in[-1,1] \wedge v_{p_{i}}^{r}(t) \in\left[\stackrel{p_{i}-r}{T}, \widehat{T+2}\right]
\end{align*}
$$

1.3. Now we give the following.

Property 1. (a) Every member of every new $L_{2}$-orthogonal system

$$
\left\{P_{n}^{p_{1}, p_{2}, p_{3}}(t)\right\}_{n=0}^{\infty}, t \in[-1,1], p_{1}, p_{2}, p_{3}=1, \ldots, k
$$

contains the function

$$
\left|\zeta\left(\frac{1}{2}+i t\right)\right|_{t=\tau}
$$

for corresponding $\tau$ since (comp. [3], (9.1), (9.2))

$$
\begin{equation*}
|\tilde{Z}(t)|=\sqrt{\frac{\mathrm{d} \varphi_{1}(t)}{\mathrm{d} t}}=\{1+o(1)\} \frac{1}{\sqrt{\ln t}}\left|\zeta\left(\frac{1}{2}+i t\right)\right|, t \rightarrow \infty \tag{1.8}
\end{equation*}
$$

(b) property (a) holds true due to the Theorem of this paper for every generation

$$
\left\{f_{n}^{p_{1}, \ldots, p_{s}}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], s \in \mathbb{N}
$$

Remark 2. The main aim of this paper is expressed by the Property 1. Namely, that there is a close binding between the theory of the Riemann's zeta-function on the critical line and the theory of $L_{2}$-orthogonal systems. Let us notice also that this paper finishes preparatory papers [5] and 6].

## 2. Main Result

2.1. We use the following notions:
(a) Jacob's ladder $\varphi_{1}(t)$,
(b) the function

$$
\begin{align*}
& \tilde{Z}^{2}(t)=\frac{\mathrm{d} \varphi_{1}(t)}{\mathrm{d} t}=\frac{1}{\omega(t)}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2}  \tag{2.1}\\
& \omega(t)=\left\{1+\mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right\} \ln t, t \rightarrow \infty
\end{align*}
$$

(c) direct iterations of the Jacob's ladder

$$
\begin{equation*}
\varphi_{1}^{0}(t)=t, \varphi_{1}^{1}(t)=\varphi_{1}(t), \varphi_{1}^{2}(t)=\varphi_{1}\left(\varphi_{1}(t)\right), \ldots, \varphi_{1}^{k}(t)=\varphi_{1}\left(\varphi_{1}^{k-1}(t)\right) \tag{2.2}
\end{equation*}
$$

for every fixed $k \in \mathbb{N}$,
(d) reverse iterations (by means of $\varphi_{1}^{-1}(t)$ )

$$
\begin{aligned}
& {\left[\frac{0}{T}, \widehat{T+U}\right],[\stackrel{1}{T}, \widehat{T+U}], \ldots,\left[\stackrel{k}{T}, \frac{k}{T+U}\right]} \\
& U=o\left(\frac{T}{\ln T}\right), T \rightarrow \infty
\end{aligned}
$$

of the basic segment

$$
[T, T+U]=\left[\stackrel{0}{T}, \widehat{0}, \frac{0}{T+U}\right]
$$

that we have introduced into the theory of the Riemann's zeta-function, see [1] -5.
2.2. Next we use the following analytic properties of Jacob's ladder $\varphi_{1}(t)$ :
(e) $\varphi_{1}(t) \in C^{\infty}\left(\left[T_{0}, \infty\right]\right)$ and it is strongly increasing function, see [1],
(f) $\varphi_{1}^{p}(t) \in C^{\infty}\left(\left[T_{0}, \infty\right]\right), p=1, \ldots, k$ and it is again strongly increasing (this property follows easily from (e))
(g) as a consequence of (f) we have next: every function

$$
\varphi_{1}^{p}(t), t \in[A, B], A>T_{0}
$$

is absolutely continuous and strongly increasing on every segment $[A, B]$ with $A>T_{0}$ (namely, the Lipschitz condition holds true by (f)),
(h) the composite function $F[f(t)]$, where $F$ is absolutely continuous and $f$ is absolutely continuous and monotonic is again absolutely continuous function.
2.3. Finally, we introduce the following functions (comp. (1.7)) together with some of their properties:

$$
\begin{align*}
& u_{p_{i}}(t)=\varphi_{1}^{p_{i}}\left(\frac{\frac{p_{i}}{T+2 l}-\stackrel{p_{i}}{T}}{2 l}(t-a)+\stackrel{p_{i}}{T}\right)-T+a \\
& v_{p_{i}}^{r}(t)=\varphi_{1}^{r}\left(\frac{\frac{p_{i}}{T+2 l}-\stackrel{p_{i}}{T}}{2 l}(t-a)+\stackrel{p_{i}}{T}\right),  \tag{2.3}\\
& t \in[a, a+2 l], i=1, \ldots, s, r=0,1, \ldots, p_{i-1}, p_{i}=1, \ldots, k
\end{align*}
$$

where

$$
u_{p_{i}}(t) \in[a, a+2 l], v_{p_{i}}^{r}(t) \in\left[\begin{array}{c}
p_{i}-r  \tag{2.4}\\
T
\end{array}, \widehat{T+2 l}\right]
$$

and, with regard to the second inclusion, see [5], Property 2, the segments

$$
\left[\begin{array}{c}
p_{i}-r \\
T
\end{array}, \frac{p_{i}-r}{T+2 l}\right]
$$

represent corresponding components of the disconnected set (see [5], (2.9))

$$
\begin{equation*}
\Delta(T, k, l)=\bigcup_{r=0}^{k}[\stackrel{r}{T}, \widehat{T+2 l}] \tag{2.5}
\end{equation*}
$$

and the following properties of the above mentioned set hold trus

$$
\begin{gather*}
l=o\left(\frac{T}{\ln T}\right), T \rightarrow \infty \Rightarrow  \tag{2.6}\\
|[\stackrel{r}{T}, \widehat{T+2 l}]|=\frac{r}{T+2 l}-\stackrel{r}{T}=o\left(\frac{T}{\ln T}\right)  \tag{2.7}\\
\left|\left[\frac{r-1}{T+2 l}, \stackrel{r}{T}\right]\right|=\stackrel{r}{T}-\frac{r-1}{T+2 l} \sim(1-c) \frac{T}{\ln T}  \tag{2.8}\\
{\left[\stackrel{0}{T}, \frac{0}{T+2 l}\right] \prec\left[\stackrel{1}{T}, \frac{1}{T+2 l}\right] \prec \cdots \prec\left[\frac{k}{T}, \frac{k}{T+2 l}\right],} \tag{2.9}
\end{gather*}
$$

where $c$ is the Euler's constant and the property (2.9) follows from (2.8).
Remark 3. Asymptotic behavior of the disconnected set (2.5) is as follows: if $T \rightarrow$ $\infty$, then the components of this set recede unboundedly each from other and all together are receding to infinity. Hence the set (2.5) behaves at $T \rightarrow \infty$ as onedimensional Friedmann-Hubble expanding universe.

[^1]2.4. The following Theorem is the main result of this paper.

Theorem 1. There is such a generating vector-operator $\hat{G}$ that for every fixed $L_{2}$-orthogonal system

$$
\begin{equation*}
\left\{f_{n}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], a \in \mathbb{R}, l \in \mathbb{R}^{+} \tag{2.10}
\end{equation*}
$$

and for every fixed $k \in \mathbb{N}$ the operator $\hat{G}$ associates following orthogonal systems:
(a) the first generation of the following new species of $L_{2}$-orthogonal systems

$$
\begin{equation*}
\left\{f_{n}^{p_{1}}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], p_{1}=1, \ldots, k \tag{2.11}
\end{equation*}
$$

i. e.

$$
\begin{equation*}
\hat{G}\left[\left\{f_{n}(t)\right\}\right]=\left(\left\{f_{n}^{1}(t)\right\},\left\{f_{n}^{2}(t)\right\}, \ldots,\left\{f_{n}^{k}(t)\right\}\right) ;\left\{f_{n}(t)\right\}=\left\{f_{n}(t)\right\}_{n=0}^{\infty} \tag{2.12}
\end{equation*}
$$

where ${ }^{2}$

$$
\begin{equation*}
f_{n}^{p_{1}}(t)=f_{n}\left(u_{p_{1}}(t)\right) \prod_{r=0}^{p_{1}-1}\left|\tilde{Z}\left(v_{p_{1}}^{r}(t)\right)\right| \tag{2.13}
\end{equation*}
$$

and every of the functions $u_{p_{1}}(t)$ defines an automorphism on $[a, a+2 l]$,
(b) the second generation of the following $L_{2}$-orthogonal systems

$$
\begin{equation*}
\left\{f_{n}^{p_{1}, p_{2}}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], p_{1}, p_{2}=1, \ldots, k \tag{2.14}
\end{equation*}
$$

i. e.

$$
\hat{G}\left[\left\{f_{n}^{p_{1}}(t)\right\}\right]=\left(\left\{f_{n}^{p_{1}, 1}(t)\right\},\left\{f_{n}^{p_{1}, 2}(t)\right\}, \ldots,\left\{f_{n}^{p_{1}, k}(t)\right\}\right),
$$

where

$$
f_{n}^{p_{1}, p_{2}}(t)=f_{n}\left(u_{p_{1}}\left(u_{p_{2}}(t)\right)\right) \prod_{r=0}^{p_{1}-1}\left|\tilde{Z}\left(v_{p_{1}}^{r}\left(u_{p_{2}}(t)\right)\right)\right| \prod_{r=0}^{p_{2}-1}\left|\tilde{Z}\left(v_{p_{2}}^{r}(t)\right)\right|
$$

and each of the functions $u_{p_{1}}\left(u_{p_{2}}(t)\right)$ defines an automorphism on $[a, a+2 l]$,
(c) and so on up to the $s^{\text {th }}$ geneation of the $L_{2}$-orthogonal systems

$$
\begin{equation*}
\left\{f_{n}^{p_{1}, p_{2}, \ldots, p_{s}}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], p_{1}, \ldots, p_{s}=1, \ldots, k \tag{2.17}
\end{equation*}
$$

i. e.

$$
\begin{equation*}
\hat{G}\left[\left\{f_{n}^{p_{1}, \ldots, p_{s-1}}(t)\right\}\right]=\left(\left\{f_{n}^{p_{1}, \ldots, p_{s-1}, 1}(t)\right\},\left\{f_{n}^{p_{1}, \ldots, p_{s-1}, 2}(t)\right\}, \ldots,\left\{f_{n}^{p_{1}, \ldots, p_{s-1}, k}(t)\right\}\right) \tag{2.18}
\end{equation*}
$$

[^2]where
\[

$$
\begin{align*}
& f_{n}^{p_{1}, p_{2}, \ldots, p_{s}}(t)=f_{n}\left(u_{p_{1}}\left(u_{p_{2}}\left(\ldots\left(u_{p_{s}}(t)\right) \ldots\right)\right)\right) \times \\
& \prod_{r=0}^{p_{1}-1} \mid \tilde{Z}\left(v_{p_{1}}^{r}\left(u_{p_{2}}\left(u_{p_{3}}\left(\ldots\left(u_{p_{s}}(t)\right) \ldots\right)\right)\right) \mid \times\right. \\
& \prod_{r=0}^{p_{2}-1} \mid \tilde{Z}\left(v_{p_{2}}^{r}\left(u_{p_{3}}\left(u_{p_{3}}\left(\ldots\left(u_{p_{s}}(t)\right) \ldots\right)\right)\right) \mid \times\right. \\
& \vdots  \tag{2.19}\\
& \prod_{r=0}^{p_{s-1}-1}\left|\tilde{Z}\left(v_{p_{s}-1}^{r}\left(u_{p_{s}}(t)\right)\right)\right| \times \\
& \prod_{r=0}^{p_{s}-1}\left|\tilde{Z}\left(v_{p_{s}}(t)\right)\right|
\end{align*}
$$
\]

and each of the functions

$$
u_{p_{1}}\left(u_{p_{2}}\left(\ldots\left(u_{p_{s}}(t)\right) \ldots\right)\right)
$$

defines an automorphism on $[a, a+2 l]$,
(d) for every fixed $k, s \in \mathbb{N}$ the $L_{2}$-orthonormal system

$$
\begin{equation*}
\left(\prod_{i=0}^{s} \sqrt{\frac{2 l}{\frac{i}{T+2 l}-\stackrel{i}{T}}}\right) f_{n}^{p_{1}, \ldots, p_{s}}(t), t \in[a, a+2 l], p_{1}, \ldots, p_{s}=1, \ldots, k \tag{2.20}
\end{equation*}
$$

is corresponding with (2.19),
(e) finally, all these formulas are true for all sufficiently big $T>0$, that is we have the continuum set of possibilities how to construct new classes of $L_{2}$-orthogonal systems (2.17).
2.5. Let us denote the set of all $L_{2}$-orthogonal systems not containing the functions

$$
\begin{equation*}
\varphi_{1}(t), \tilde{Z}^{2}(t)=\frac{\mathrm{d} \varphi_{1}(t)}{\mathrm{d} t} \tag{2.21}
\end{equation*}
$$

as

$$
L_{2}^{0}([a, a+2 l])
$$

Now it is true by Theorem 1 that for every fixed

$$
\begin{aligned}
& \left\{f_{n}(t)\right\}_{n=0}^{\infty}=\left\{f_{n}(t)\right\} \in L_{2}^{0}([a, a+2 l]), \\
& \left\{f_{n}(t)\right\} \xrightarrow{\hat{G}}\left\{f_{n}^{p_{1}}(t)\right\} \xrightarrow{\hat{G}} \ldots \xrightarrow{\hat{G}}\left\{f_{n}^{p_{1}, \ldots, p_{s}}(t)\right\} .
\end{aligned}
$$

We instantly get the following.

## Corollary 1.

$$
\begin{equation*}
L_{2}^{0}([a, a+2 l]) \xrightarrow{\hat{G}} L_{2}^{1}([a, a+2 l]) \xrightarrow{\hat{G}} \ldots \xrightarrow{\hat{G}} L_{2}^{s}([a, a+2 l]), \tag{2.22}
\end{equation*}
$$

where

$$
L_{2}^{i}([a, a+2 l])
$$

stands for the $i^{\text {th }}$ generation of the image of $L_{2}^{0}([a, a+2 l])$. Consequently, the union

$$
\begin{equation*}
\bigcup_{i=1}^{s} L_{2}^{i}([a, a+2 l]) \tag{2.23}
\end{equation*}
$$

represents a kind of $\zeta$-extension of the $s^{\text {th }}$ order of $L_{2}^{0}([a, a+2 l])$ for every fixed segment $[a, a+2 l]$ and every fixed $k, s \in \mathbb{N}$. Finally, the union over all the segments

$$
\begin{equation*}
\bigcup_{a \in \mathbb{R}, l>0}\left\{\bigcup_{i=1}^{s} L_{2}^{i}([a, a+2 l])\right\} \tag{2.24}
\end{equation*}
$$

represents the complete $\zeta$-extension of the $s^{\text {th }}$ order of the set

$$
\begin{equation*}
\bigcup_{a \in \mathbb{R}, l>0} L_{2}^{0}([a, a+2 l]) \tag{2.25}
\end{equation*}
$$

2.6.

Remark 4. We may select another function

$$
\psi(t) \neq \varphi_{1}(t), \psi(t) \nsim \varphi_{1}(t), t \rightarrow \infty
$$

instead of the Jacob's ladder and use the pair

$$
\psi(t), \frac{\mathrm{d} \psi(t)}{\mathrm{d} t}
$$

to extend $L_{2}^{0}([a, a+2 l])$. This way however will not attain $\zeta$-extensions (2.13), (2.16), (2.19), (2.23) and (2.24) and therefore will be irrelevant.

## 3. JACOB'S LADDERS

3.1. Let us remind that the Jacob's ladder

$$
\varphi_{1}(t)=\frac{1}{2} \varphi(t)
$$

was introduced in [1], see also [3], where the function $\varphi(t)$ is an arbitrary continuous solution of the nonlinear integral equation ${ }^{3}$

$$
\begin{equation*}
\int_{0}^{\mu[x(T)]} Z^{2}(t) e^{-\frac{2}{x(T)} t} \mathrm{~d} t=\int_{0}^{T} Z^{2}(t) \mathrm{d} t \tag{3.1}
\end{equation*}
$$

where

$$
\begin{align*}
Z(t) & =e^{i \vartheta(t)} \zeta\left(\frac{1}{2}+i t\right) \\
\vartheta(t) & =-\frac{t}{2} \ln \pi+\operatorname{Im}\left\{\ln \Gamma\left(\frac{1}{4}+i \frac{t}{2}\right)\right\} \tag{3.2}
\end{align*}
$$

and the class of functions $\{\mu\}$ is specified as

$$
\mu \in C^{\infty}\left(\left[y_{0},+\infty\right)\right)
$$

being monotonically increasing, unbounded from above and obeying the inequality

$$
\begin{equation*}
\mu(y) \geq 7 y \ln y \tag{3.3}
\end{equation*}
$$

Every admissible function $\mu(y)$ generates a solution

$$
y=\varphi(T ; \mu)=\varphi(T)
$$

[^3]Remark 5. The function $\varphi_{1}(T)$ is called Jacob's ladder as an analogue of the Jacob's dream in Chumash, Bereishis, 28:12.
3.2. Let us remind that the Hardy-Littlewood integral (1918)

$$
\begin{equation*}
\int_{0}^{T}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2} \mathrm{~d} t \tag{3.4}
\end{equation*}
$$

can be expressed as follows:

$$
\begin{equation*}
\int_{0}^{T}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2} \mathrm{~d} t=T \ln T+(2 c-1-\ln 2 \pi) T+R(T) \tag{3.5}
\end{equation*}
$$

with, for example, Ingham's error term

$$
\begin{equation*}
R(T)=\mathcal{O}\left(T^{1 / 2} \ln T\right)=\mathcal{O}\left(T^{1 / 2+\delta}\right), \delta>0, T \rightarrow \infty \tag{3.6}
\end{equation*}
$$

for arbitrary small $\delta$.
Next, it is true by Good's $\Omega$-theorem (1977), that

$$
\begin{equation*}
R(T)=\Omega\left(T^{1 / 4}\right), T \rightarrow \infty \tag{3.7}
\end{equation*}
$$

Remark 6. Let

$$
\begin{equation*}
R_{a}(T)=\mathcal{O}\left(T^{1 / 4+a}\right), a \in\left[\delta, \frac{1}{4}+\delta\right], T \rightarrow \infty \tag{3.8}
\end{equation*}
$$

Then, by (3.7), one obtain for every valid estimate of the type (3.8) that

$$
\begin{equation*}
\limsup _{T \rightarrow \infty}\left|R_{a}(T)\right|=+\infty \tag{3.9}
\end{equation*}
$$

In other words, every expression of the type (3.5) and (3.9) possesses an unbounded error at infinity.
3.3. Under the circumstances (3.5) and (3.9) we have shown in our paper 1 that the Hardy-Littlewood integral (3.4) has an infinite set of almost exact representations expressed by the following formula.

## Formula1.

$$
\begin{align*}
& \int_{0}^{T}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2} \mathrm{~d} t=  \tag{3.10}\\
& \varphi_{1}(T) \ln \left\{\varphi_{1}(T)\right\}+(c-\ln 2 \pi) \varphi_{1}(T)+c_{0}+\mathcal{O}\left(\frac{\ln T}{T}\right), T \rightarrow \infty
\end{align*}
$$

( $c$ is the Euler's constant and $c_{0}$ is the constant from the Titchmarsh-KoberAtkinson formula) with the error term vanishing at infinity:

$$
\begin{equation*}
\tilde{R}(T)=\mathcal{O}\left(\frac{\ln T}{T}\right) \xrightarrow{T \rightarrow \infty} 0 \tag{3.11}
\end{equation*}
$$

Remark 7. The comparison of (3.9) and (3.11) completely characterizes the level of exactness of our representation (3.10) of the Hardy-Littlewood integral (3.4).
3.4. Now, let us remind other formulae demonstrating the power of Jacob's ladder $\varphi_{1}(t)$.

First we have obtained the following ${ }^{4}$
Formula2.

$$
\begin{align*}
& \int_{T}^{T+U}\left|\zeta\left(\frac{1}{2}+i \varphi_{1}(t)\right)\right|^{4}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2} \mathrm{~d} t \sim \frac{1}{2 \pi^{2}} U \ln ^{5} T  \tag{3.12}\\
& U=T^{7 / 8+2 \delta}, T \rightarrow \infty
\end{align*}
$$

Remark 8. The formula (3.12) is the first asymptotic formula of the sixth order in $|\zeta|$ on the critical line in the theory of the Riemann's zeta-function.

Next, let

$$
S(t)=\frac{1}{\pi} \arg \left\{\zeta\left(\frac{1}{2}+i t\right)\right\}, S_{1}(T)=\int_{0}^{T} S(t) \mathrm{d} t
$$

where the function arg is defined in the usual way. We have obtained the following two formulae concerning $S(t)$ function 5

## Formula3.

$$
\begin{align*}
& \int_{T}^{T+U}\left[\arg \left\{\zeta\left(\frac{1}{2}+i \varphi_{1}(t)\right)\right\}\right]^{2 k}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2} \mathrm{~d} t \sim  \tag{3.13}\\
& \frac{1}{2^{k}} U \ln T(\ln \ln T)^{k}, U \in\left[T^{1 / 3+\delta}, \frac{T}{\ln T}\right], T \rightarrow \infty
\end{align*}
$$

for every fixed $k \in \mathbb{N}$.

## Formula4.

$$
\begin{equation*}
\int_{T}^{T+U}\left\{S_{1}\left[\varphi_{1}(t)\right]\right\}^{2 k}\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2} \mathrm{~d} t \sim a_{k} U \ln T, T \rightarrow \infty \tag{3.14}
\end{equation*}
$$

Remark 9. New kind of the classical A. Selberg's formulae (1946) are expressed by means of our results (3.13) and (3.14).

Remark 10. Let us notice explicitly, that formulae (3.12) - (3.14) are $\zeta$-correlation formulae on the critical line. For example, (3.13) at $k=1$ describes interaction between values of the functions

$$
\left[\arg \left\{\zeta\left(\frac{1}{2}+i \varphi_{1}(t)\right)\right\}\right]^{k},\left|\zeta\left(\frac{1}{2}+i t\right)\right|^{2}
$$

## 4. Proof of Theorem 1

4.1. In [5], (7.1) (7.2), we have shown the following results concerning direct and reverse iterations:

Lemma 1. If

$$
\begin{equation*}
U=o\left(\frac{T}{\ln T}\right), T \rightarrow \infty \tag{4.1}
\end{equation*}
$$

then for every function

$$
g(t) \in L([T, T+U])
$$

[^4]the following holds true:
\[

$$
\begin{equation*}
\int_{T}^{T+U} g(t) \mathrm{d} t=\int_{T}^{\frac{p}{T+U}} g\left[\varphi_{1}^{p}(\tau)\right] \prod_{r=0}^{p-1} \tilde{Z}^{2}\left[\varphi_{1}^{r}(\tau)\right] \mathrm{d} \tau, p=1, \ldots, k \tag{4.2}
\end{equation*}
$$

\]

for every fixed $k \in \mathbb{N}$.
Remark 11. Let us notice also the subsection 2.2, (g).

### 4.2. Proof of Theorem 1.

4.2.1. Let

$$
\begin{equation*}
\left\{f_{n}(t)\right\}_{n=0}^{\infty} \subset L_{2}([a, a+2 l]) \tag{4.3}
\end{equation*}
$$

is arbitrary fixed system of orthogonal functions. Then $l$ is also fixed positive number and condition (4.1) is fulfilled for $U=2 l$ for all sufficiently big and positive $T$. Now we have $(m \neq n)$

$$
\begin{equation*}
0=\int_{a}^{a+2 l} f_{m}(t) f_{n}(t) \mathrm{d} t=\int_{T}^{T+2 l} f_{m}(\tau-T+a) f_{n}(\tau-T+a) \mathrm{d} \tau= \tag{4.4}
\end{equation*}
$$

next we obtain by Lemma 1 for any sufficiently big $T$

$$
\begin{equation*}
=\int_{T}^{\frac{p}{T+2 l}} f_{m}\left[\varphi_{1}^{p}(\rho)-T+a\right] f_{n}\left[\varphi_{1}^{p}(\rho)-T+a\right] \prod_{r=0}^{p-1} \tilde{Z}^{2}\left[\varphi_{1}^{r}(\rho)\right] \mathrm{d} \rho= \tag{4.5}
\end{equation*}
$$

and next, by simple sunstitution

$$
\rho=\rho(t)=\frac{\stackrel{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}, \rho \in[\stackrel{p}{T}, \stackrel{p}{T+2 l}], t \in[a, a+2 l],
$$

where $\rho(t)$ is absolutely continuous and increasing, we obtain

$$
\begin{align*}
& =\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l} \int_{a}^{a+2 l} f_{m}\left[\varphi_{1}^{p}\left(\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}\right)-T+a\right] \times \\
& f_{n}\left[\varphi_{1}^{p}\left(\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}\right)-T+a\right] \times  \tag{4.6}\\
& \prod_{r=0}^{p-1} \tilde{Z}^{2}\left[\varphi_{1}^{r}\left(\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}\right)\right] \mathrm{d} t=
\end{align*}
$$

and, in the next step of the first cycle, we finish with

$$
\begin{equation*}
=\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l} \int_{a}^{a+2 l} f_{m}^{p}(t) f_{n}^{p}(t) \mathrm{d} t \Rightarrow \int_{a}^{a+2 l} f_{m}^{p}(t) f_{n}^{p}(t) \mathrm{d} t=0, \tag{4.7}
\end{equation*}
$$

where

$$
\begin{align*}
& f_{n}^{p}(t)=f_{n}\left[\varphi_{1}^{p}\left(\frac{\stackrel{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}\right)-T+a\right] \times \\
& \prod_{r=0}^{p-1}\left|\tilde{Z}\left[\varphi_{1}^{r}\left(\frac{\widehat{p+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}\right)\right]\right|, t \in[a, a+2 l], p=1, \ldots, k . \tag{4.8}
\end{align*}
$$

4.2.2. Now we give the following

Definition 1. The symbol $\hat{G}$ stands for vector operator defined on the set all $L_{2}$-orthogonal systems

$$
\left\{f_{n}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], a \in \mathbb{R}, l>0
$$

defined by three integral transformatrions (4.4) - (4.6). $\hat{G}$ maps an $L_{2}$-orthogonal system into $k$-tuple of new orthogonal systems

$$
\begin{aligned}
& \hat{G}\left[\left\{f_{n}(t)\right\}\right]=\left(\left\{f_{n}^{1}(t)\right\},\left\{f_{n}^{2}(t)\right\}, \ldots,\left\{f_{n}^{k}(t)\right\}\right)=\left\{f_{n}^{p}(t)\right\}_{n=0}^{\infty}, \\
& p=1, \ldots, k, t \in[a, a+2 l] ;\left\{f_{n}(t)\right\}_{n=0}^{\infty}=\left\{f_{n}(t)\right\}, \ldots
\end{aligned}
$$

for every fixed $k \in \mathbb{N}$.
4.2.3. Let us notice that the transformation $\sqrt{6}$

$$
\begin{equation*}
u_{p}(t)=\varphi_{1}^{p}\left(\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}\right)-T+a, t \in[a, a+2 l] \tag{4.9}
\end{equation*}
$$

has the following properties:
(a) By [5], subsection 6.1

$$
\begin{aligned}
& u_{p}(a)=\varphi_{1}^{p}(\stackrel{p}{T})-T+a=T-T+a=a ; \stackrel{p}{T}=\varphi_{1}^{-p}(T), \\
& u_{p}(a+2 l)=\varphi_{1}(\widehat{T+2 l})-T+a=a+2 l .
\end{aligned}
$$

(b) Since the continuous function $\varphi_{1}^{p}(t)$ is increasing and

$$
\rho=\frac{\frac{p}{T+2 l}-\stackrel{p}{T}}{2 l}(t-a)+\stackrel{p}{T}, t \in[a, a+2 l]
$$

it is evident that the composite function

$$
u_{p}(t), t \in[a, a+2 l]
$$

is also increasing and therefore

$$
t \in[a, a+2 l] \Rightarrow u_{p}(t) \in[a, a+2 l] .
$$

Remark 12. We have as a consequence of (a) and (b) that new automorphism on $[a, a+2 l]$ is defined by the one-to-one correspondence (4.9) for every fixed sufficiently big positive $T$. Of course, every function $u_{p_{i}}(t)$ defines an automorphism on $[a, a+$ 2l] too.

[^5]4.2.4. By making use of the operator $\hat{G}$ on the system
$$
\left\{f_{n}^{p_{1}}(t)\right\}_{n=0}^{\infty}, t \in[a, a+2 l], p_{1}=1, \ldots, k
$$
(the second cycle) we obtain
\[

$$
\begin{equation*}
\hat{G}\left[\left\{f_{n}^{p_{1}}(t)\right\}\right]=\left\{f_{n}^{p_{1}, p_{2}}(t)\right\}_{n=0}^{\infty}, p_{1}, p_{2}=1, \ldots, k, \tag{4.10}
\end{equation*}
$$

\]

where

$$
\begin{align*}
& f_{n}^{p_{1}, p_{2}}(t)=  \tag{4.11}\\
& f_{n}\left[\varphi_{1}^{p_{1}}\left(\frac{p_{1}}{T+2 l}-\stackrel{p_{1}}{T}\left(\varphi_{1}^{p_{2}}\left(\frac{p_{2}}{T+2 l}-\stackrel{p_{2}}{2 l}(t-a)+\stackrel{p_{2}}{T}\right)-T\right)+\stackrel{p_{1}}{T}\right)-T+a\right] \times \\
& \prod_{r=0}^{p_{1}-1} \left\lvert\, \tilde{Z}\left[\varphi_{1}^{r}\left(\left(\frac{\overparen{T+2 l}}{2 l}-\stackrel{p_{1}}{T}\left(\varphi_{1}^{p_{2}}\left(\frac{p_{1}}{T+2 l}-\stackrel{p_{2}}{T}(t-a)+\stackrel{p_{2}}{T}\right)-T\right)+\stackrel{p_{1}}{T}\right)\right)\right] \times\right. \\
& \prod_{r=0}^{p_{2}-1}\left|\tilde{Z}\left[\varphi_{1}^{r}\left(\frac{\widehat{T+2 l}-\stackrel{p_{2}}{T}}{2 l}(t-a)+\stackrel{p_{2}}{T}\right)\right]\right|
\end{align*}
$$

It is clear that there is need for simplification of our formulae (4.8) and (4.11). For this purpose we use functions of $(2.3)]^{7}$ that provides us with the results

$$
\begin{align*}
& f_{n}^{p_{1}}(t)=f_{n}\left(u_{p_{1}}(t)\right) \prod_{r=0}^{p_{1}-1}\left|\tilde{Z}\left(v_{p_{1}}^{r}(t)\right)\right|,  \tag{4.12}\\
& f_{n}^{p_{1}, p_{2}}(t)=f_{n}\left(u_{p_{1}}\left(u_{p_{2}}(t)\right)\right) \prod_{r=0}^{p_{1}-1}\left|v_{p_{1}}^{r}\left(u_{p_{2}}(t)\right)\right| \prod_{r=0}^{p_{2}-1}\left|\tilde{Z}\left(v_{p_{2}}^{r}(t)\right)\right|,
\end{align*}
$$

where, for example,

$$
\left\{u_{p_{1}}\left(t_{1}\right), t_{1}=u_{p_{2}}\left(t_{2}\right), t_{1}, t_{2} \in[a, a+2 l]\right\} \Rightarrow u_{p_{1}}\left(u_{p_{2}}\left(t_{2}\right)\right), t_{2}=t
$$

i. e. we have formulae (2.13) and (2.16).

[^6]4.2.5. Next, in the $(s-1)^{\text {th }}$-cycle we get
\[

$$
\begin{align*}
& f_{n}^{p_{1}, p_{2}, \ldots, p_{s-1}}(t)=f_{n}\left(u_{p_{1}}\left(u_{p_{2}}\left(\ldots\left(u_{p_{s-1}}(t)\right) \ldots\right)\right)\right) \times \\
& \prod_{r=0}^{p_{1}-1}\left|\tilde{Z}\left(v_{p_{1}}^{r}\left(u_{p_{2}}\left(u_{p_{3}}\left(\ldots\left(u_{p_{s-1}}(t)\right) \ldots\right)\right)\right)\right)\right| \times \\
& \prod_{r=0}^{p_{2}-1} \mid \tilde{Z}\left(v_{p_{2}}^{r}\left(u_{p_{3}}\left(u_{p_{4}}\left(\ldots\left(u_{p_{s-1}}(t)\right) \ldots\right)\right)\right) \mid \times\right.  \tag{4.13}\\
& \vdots \\
& \prod_{r=0}^{p_{s-2}-1}\left|\tilde{Z}\left(v_{p_{s}-2}^{r}\left(u_{p_{s-1}}(t)\right)\right)\right| \times \\
& \prod_{r=0}^{p_{s-1}-1}\left|\tilde{Z}\left(v_{p_{s-1}}(t)\right)\right|, s>2,
\end{align*}
$$
\]

then, if we use the operator $\hat{G}$ on (4.13) to obtain the $s^{\text {th }}$ cycle, we get the set of $k^{s}$ formulas (2.19). That means the formula (2.15) holds true for every $s \in \mathbb{N}$.

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[^0]:    Key words and phrases. Riemann zeta-function.

[^1]:    ${ }^{1}$ [5], (2.5), (2.6)

[^2]:    ${ }^{2}$ Comp. (2.3).

[^3]:    ${ }^{3}$ Also introduced in 1].

[^4]:    ${ }^{4}$ See [3], (8.3).
    ${ }^{5}$ See 2], (5.4), (5.5).

[^5]:    ${ }^{6}$ Comp. (2.3), 4.8).

[^6]:    ${ }^{7}$ See also Remark 11.

