TSUKUBA J. MATH.
Vol. 33 No. 1 (2009), 39–56

JACOBI OPERATORS ALONG THE STRUCTURE FLOW ON REAL HYPERSURFACES IN A NONFLAT COMPLEX SPACE FORM

By

U-Hang Ki, Hiroyuki Kurihara and Ryoichi Takagi

Abstract. Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ, ξ, η, g). In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is ξ-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterizes the homogeneous real hypersurfaces of type A in a complex projective space or a complex hyperbolic space when $R_\xi \phi S = S\phi R_\xi$ holds on M, where S denotes the Ricci tensor of type $(1,1)$ on M.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex n-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature $4c$ and let M be an orientable real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from $(J, \tilde{g})$.

It is known that there are no real hypersurface with parallel Ricci tensors in a nonflat complex space form (see [6], [8]). This result say that there does not exist locally symmetric real hypersurfaces in a nonflat complex space form. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the first author started the study on real hypersurfaces in a complex space form by using the operator R_ξ in [3], [4] and [5]. Recently Ortega, Pérez and Santos [12] have proved that there are no real hypersurfaces in $P_n\mathbf{C}$, $n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [13] due to them.

2000 Mathematics Subject Classification: 53B20, 53C15, 53C25.
Keywords: complex space form, real hypersurface, structure Jacobi operator, Ricci tensor.
Received December 4, 2007.
Revised October 1, 2008.

Now in this paper, motivated by results mentioned above we consideer the parallelism of the structure Jacobi operator R_ξ in the direction of the structure vector field, that is $\nabla_\xi R_\xi = 0$.

In 1970's, the third author [14], [15] classified the homogeneous real hypersurfaces of $P_n\mathbf{C}$ into six types. On the other hand, Cecil and Ryan [2] extensively studied a Hopf hypersurface, which is realized as tubes over certain submanifolds in $P_n\mathbf{C}$, by using its focal map. By making use of those results and the mentioned work of Takagi, Kimura [10] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbf{C}$ whose all principal curvatures are constant. For the case $H_n\mathbf{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbf{C}$ or $H_k\mathbf{C}$ $(0 \le k \le n-1)$ adding a horosphere in $H_n\mathbf{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [11] (resp. Montiel and Romero [10]) showed that a real hypersurface in $P_n\mathbf{C}$ (resp. $H_n\mathbf{C}$) is locally congruent to one of real hypersurfaces of type A if and only if the Reeb flow ξ is isometric or equivalently the structure operator ϕ commutes with the shape operator H.

Among the results related R_ξ we mention the following ones.

Theorem 1 (Cho and Ki [5]). *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi H = HR_\xi$. Then M is a Hopf hypersurface in $M_n(c)$. Further, M is locally congruent to one of the following hypersurfaces:*

(1) *In cases that $M_n(c) = P_n\mathbf{C}$ with $\eta(H\xi) \neq 0$,*
   (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
   (A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \le k \le n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$.*

(2) *In cases $M_n(c) = H_n\mathbf{C}$,*
   (A_0) *a horosphere;*
   (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$;*
   (A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \le k \le n-2)$.*

In this paper we study a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $R_\xi\phi S = S\phi R_\xi$, where S denotes the Ricci tensor of the hypersurface. We give another characterization of real hypersurfaces of type A in $M_n(c)$ by above two conditions. The main purpose of the present paper is to establish Main Thoerem stated in section 5. We note that the condition $R_\xi\phi S = S\phi R_\xi$ is a much weaker condition. Indeed, every Hopf hypersurface always satisfies this condition.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

We denote by $M_n(c)$, $c \neq 0$ be a nonflat complex space form with the Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature $4c$ and Levi-Civita connection $\tilde{\nabla}$. For an immersed $(2n-1)$-dimensional Riemannian manifold $\tau : M \to M_n(c)$, the Levi-Civita connection ∇ of induced metric and the shape operator H of the immersion are characterized

$$\tilde{\nabla}_X Y = \nabla_X Y + g(HX, Y)v, \quad \tilde{\nabla}_X v = -HX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and v a unit normal vector on M. In the sequel the indeces $i, j, k, l, \dots$ run over the range $\{1, 2, \dots, 2n-1\}$ unless otherwise stated. For a local orthonormal frame field $\{e_i\}$ of M, we denote the dual 1-forms by $\{\theta_i\}$. Then the connection forms θ_{ij} are defined by

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0.$$

Then we have

$$\nabla_{e_i} e_j = \sum_k \theta_{kj}(e_i) e_k = \sum_k \Gamma_{kij} e_k,$$

where we put $\theta_{ij} = \sum_k \Gamma_{ijk}\theta_k$. The structure tensor $\phi = \sum_i \phi_i e_i$ and the structure vector $\xi = \sum_i \xi_i e_i$ satisfy

$$\sum_k \phi_{ik}\phi_{kj} = \xi_i\xi_j - \delta_{ij}, \quad \sum_j \xi_j\phi_{ij} = 0, \quad \sum_i \xi_i^2 = 1, \quad \phi_{ij} + \phi_{ji} = 0,$$

$$d\phi_{ij} = \sum_k (\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki} - \xi_i h_{jk}\theta_k + \xi_j h_{ik}\theta_k), \tag{2.1}$$

$$d\xi_i = \sum_j \xi_j\theta_{ji} - \sum_{j,k} \phi_{ji} h_{jk}\theta_k.$$

We denote the components of the shape operator or the second fundamental tensor H of M by h_{ij}. The components $h_{ij;k}$ of the covariant derivative of H are given by $\sum_k h_{ij;k}\theta_k = dh_{ij} - \sum_k h_{ik}\theta_{kj} - \sum_k h_{jk}\theta_{ki}$. Then we have the equation of Gauss and Codazzi

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.2}$$

$$h_{ij;k} - h_{ik;j} = c(\xi_k\phi_{ij} + \xi_i\phi_{kj} - \xi_j\phi_{ik} - \xi_i\phi_{jk}), \tag{2.3}$$

respectively.

From (2.2) the structure Jacobi operator $R_\xi = (\Xi_{ij})$ is given by

$$\Xi_{ij} = \sum_{k,l} h_{ik}h_{jl}\xi_k\xi_l - \sum_{k,l} h_{ij}h_{kl}\xi_k\xi_l + c\xi_i\xi_j - c\delta_{ij}, \tag{2.4}$$

From (2.2) the Ricci tensor $S = (S_{ij})$ is given by

$$S_{ij} = (2n+1)c\delta_{ij} - 3c\xi_i\xi_j + hh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.5}$$

where $h = \sum_i h_{ii}$.

First we remark

Lemma 1. *Let U be an open set in M and F a smooth function on U. We put $dF = \sum_i F_i\theta_i$. Then we have*

$$F_{ij} - F_{ji} = \sum_k F_k\Gamma_{kij} - \sum_k F_k\Gamma_{kji}.$$

Proof. Taking the exterior derivate of $dF = \sum_i F_i\theta_i$, we have the formula immediately. □

Now we retake a local orthonormal frame field e_i in such a way that (1) $e_1 = \xi$, (2) e_2 is in the direction of $\sum_{i=2}^{2n-1} h_{1i}e_i$ and (3) $e_3 = \phi e_2$. Then we have

$$\xi_1 = 1, \quad \xi_i = 0 \quad (i \geq 2), \quad h_{1j} = 0 \quad (j \geq 3) \quad \text{and} \quad \phi_{32} = 1. \tag{2.6}$$

We put $\alpha := h_{11}$, $\beta := h_{12}$, $\gamma := h_{22}$, $\varepsilon := h_{23}$ and $\delta := h_{33}$.

Promise. Hereafter the indeces $p, q, r, s, \ldots$ run over the range $\{4, 5, \ldots, 2n-1\}$ unless otherwise stated.

Since $d\xi_i = 0$, we have

$$\theta_{12} = \varepsilon\theta_2 + \delta\theta_3 + \sum_p h_{3p}\theta_p,$$

$$\theta_{13} = -\beta\theta_1 - \gamma\theta_2 - \varepsilon\theta_3 - \sum_p h_{2p}\theta_p, \tag{2.7}$$

$$\theta_{1p} = \sum_q \phi_{qp}h_{q2}\theta_2 + \sum_q \phi_{qp}h_{q3}\theta_3 + \sum_{q,r} \phi_{qp}h_{qr}\theta_r.$$

We put

$$\theta_{23} = \sum_i X_i\theta_i, \quad \theta_{2p} = \sum_i Y_{pi}\theta_i, \quad \theta_{3p} = \sum_i Z_{pi}\theta_i. \tag{2.8}$$

Then it follows from $d\phi_{2i} = 0$ that $Y_{pi} = -\sum_q \phi_{pq}Z_{qi}$ or $Z_{pi} = \sum_q \phi_{pq}Y_{qi}$. The equations (2.4) and (2.5) are rewritten as

$$\Xi_{ij} = -\alpha h_{ij} + h_{1i}h_{1j} + c\delta_{i1}\delta_{j1} - c\delta_{ij}, \tag{2.9}$$

$$S_{ij} = hh_{ij} - \sum_k h_{ik}h_{jk} - 3c\delta_{i1}\delta_{j1} + (2n+1)c\delta_{ij}, \tag{2.10}$$

respectively.

3. Real Hypersurfaces Satisfying $\nabla_\xi R_\xi = 0$ and $R_\xi\phi S = S\phi R_\xi$

First we assume that $\nabla_\xi R_\xi = 0$. The components $\Xi_{ij;k}$ of the covariant derivativation of $R_\xi = (\Xi_{ij})$ is given by

$$\sum_k \Xi_{ij;k}\theta_k = d\Xi_{ij} - \sum_k \Xi_{kj}\theta_{ki} - \sum_k \Xi_{ik}\theta_{kj}.$$

Substituting (2.9) into the above equation we have

$$\begin{aligned}\sum_k \Xi_{ij;k}\theta_k = {} & -(d\alpha)h_{ij} - \alpha dh_{ij} + (dh_{1i})h_{1j} + h_{1i}(dh_{1j}) \\ & + \alpha\sum_k h_{kj}\theta_{ki} - \alpha h_{1j}\theta_{1i} - \beta h_{1j}\theta_{2i} - c\delta_{j1}\theta_{1i} \\ & + \alpha\sum_k h_{ik}\theta_{kj} - \alpha h_{1i}\theta_{1j} - \beta h_{1i}\theta_{2j} - c\delta_{i1}\theta_{1j}.\end{aligned} \tag{3.1}$$

In the following, we assume that $\beta \neq 0$.

Our assumption $\nabla_\xi R_\xi = 0$ is equivalent to $\Xi_{ij;1} = 0$, which can be stated as follows:

(3.2) $\varepsilon = 0, \quad \alpha\delta + c = 0, \quad h_{3p} = 0,$

(3.3) $(\beta^2 - \alpha\gamma)_1 - 2\alpha \sum_p h_{2p} Y_{p1} = 0,$

(3.4) $(\beta^2 - \alpha\gamma - c)X_1 + \alpha \sum_p h_{2p} Z_{p1} = 0,$

(3.5) $(\alpha h_{2p})_1 + \alpha \sum_q h_{pq} Y_{q1} + (\beta^2 - \alpha\gamma) Y_{p1} - \alpha \sum_q h_{2q}\Gamma_{qp1} = 0,$

(3.6) $\alpha h_{2p} X_1 - \sum_q (\alpha h_{qp} + c\delta_{pq}) Z_{q1} = 0,$

(3.7) $-(\alpha h_{pq})_1 + \alpha h_{2q} Y_{p1} + \alpha \sum_r h_{rq}\Gamma_{rp1} + \alpha h_{2p} Y_{q1} + \alpha \sum_r h_{pr}\Gamma_{rq1} = 0.$

Hereafter we shall use (3.2) without quoting.

Furthermore we assume that $R_\xi \phi S = S\phi R_\xi$. Under the assumption $\nabla_\xi R_\xi = 0$, we have the following additional equations

(3.8) $(h\delta - \delta^2 + (2n+1)c) h_{2p} = 0,$

(3.9) $\widetilde{R_\xi}\tilde{\phi}A = 0,$

(3.10) $\widetilde{R_\xi}\tilde{\phi}\tilde{S} = \tilde{S}\tilde{\phi}\widetilde{R_\xi}.$

where $A = {}^t(h_{24}, h_{25}, \dots, h_{2,2n-1})$, $\widetilde{R_\xi} = (\Xi_{pq})$, $\tilde{\phi} = (\phi_{pq})$, $\tilde{S} = (S_{pq})$.

Now, properly speaking, we should denote the equation (2.3) by, e.g., $(23)_{ijk}$. In this paper we denote it by (ijk) simply. Then we have the following equations (112)–$(q1p)$.

(112) $\alpha_2 - \beta_1 = 0,$

(212) $\beta_2 - \gamma_1 - 2\sum_p h_{2p} Y_{p1} = 0,$

(312) $(\alpha - \delta)\gamma - \beta X_2 + (\gamma - \delta)X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = -c,$

(113) $\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0,$

(213) $\beta_3 - \alpha\delta + \gamma\delta + (\gamma - \delta)X_1 - \beta^2 - \sum_p h_{2p} Z_{p1} = c,$

$$(313)\quad \beta X_3 + \delta_1 = 0,$$

$$(223)\quad \gamma_3 - 2\beta\delta + 2\sum_p h_{2p}Y_{p3} + (\gamma - \delta)X_2 - \beta\gamma - \sum_p h_{2p}Z_{p2} = 0,$$

$$(323)\quad \sum_p h_{2p}Z_{p3} - \delta_2 - (\gamma - \delta)X_3 = 0,$$

$$(1p1)\quad \alpha_p + \beta Y_{p1} = 0,$$

$$(12p)\quad \beta_p + 2\sum_{q,r} h_{2q}\phi_{rq}h_{rp} + \beta Y_{p2} + \alpha\sum_q \phi_{qp}h_{2q} = 0,$$

$$(13p)\quad -2\delta h_{2p} + \beta Y_{p3} + \alpha h_{2p} - \beta X_p = 0,$$

$$(22p)\quad \gamma_p + 2\sum_q h_{2q}Y_{qp} - h_{2p2} - \sum_q h_{qp}Y_{q2} + \beta\sum_q \phi_{qp}h_{2q} + \gamma Y_{p2} + \sum_q h_{2q}\Gamma_{qp2} = 0,$$

$$(23p)\quad \delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q}Z_{qp} - h_{2p3} - \sum_q h_{qp}Y_{q3} + \gamma Y_{p3} + \sum_q h_{2q}\Gamma_{qp3} = 0,$$

$$(33p)\quad \delta_p + h_{2p}X_3 - \sum_q h_{qp}Z_{q3} + \delta Z_{p3} = 0,$$

$$(21p)\quad \beta_p + \sum_{q,r} h_{2q}\phi_{rq}h_{rp} - h_{2p1} - \sum_q h_{qp}Y_{q1} + \gamma Y_{p1} + \sum_q h_{2q}\Gamma_{qp1} = 0,$$

$$(31p)\quad -\delta h_{2p} + \alpha h_{2p} - \beta X_p + h_{2p}X_1 - \sum_q h_{qp}Z_{q1} + \delta Z_{p1} = 0,$$

$$(32p)\quad \delta X_p + \beta h_{2p} - \gamma X_p + \sum_q h_{2q}Z_{qp} + h_{2p}X_2 - \sum_q h_{pq}Z_{q2} + \delta Z_{p2} = 0,$$

$$(1pq)\quad 2\sum_{r,s} h_{rp}\phi_{sr}h_{sq} - \alpha\sum_r \phi_{rp}h_{rq} + \alpha\sum_r \phi_{rq}h_{rp} - \beta Y_{pq} + \beta Y_{qp} = -2c\phi_{pq},$$

$$(2pq)\quad h_{2pq} + \sum_r h_{rp}Y_{rq} - \beta\sum_r \phi_{rp}h_{rq} - \gamma Y_{pq} - \sum_r h_{2r}\Gamma_{rpq} - h_{2qp}$$
$$- \sum_r h_{rq}Y_{rp} + \beta\sum_r \phi_{rq}h_{rp} + \gamma Y_{qp} + \sum_r h_{2r}\Gamma_{rqp} = 0,$$

$$(q1p)\quad \sum_{r,s} h_{rq}\phi_{sr}h_{sp} - \alpha\sum_r \phi_{rq}h_{rp} - \beta Y_{qp} - h_{pq1}$$
$$+ h_{2q}Y_{p1} + \sum_r h_{rq}\Gamma_{rp1} + h_{2p}Y_{q1} + \sum_r h_{rp}\Gamma_{rq1} = c\phi_{pq},$$

$$(q3p)\quad h_{3qp} - \varepsilon Y_{qp} - \delta Z_{qp} - \sum_r h_{3r}\Gamma_{rqp} - h_{2q}X_p + \sum_r h_{qr}Z_{rp} - h_{qp3}$$
$$+ h_{q2}Y_{p3} + h_{q3}Z_{p3} + \sum_r h_{qr}\Gamma_{rp3} + h_{p2}Y_{q3} + h_{p3}Z_{q3} + \sum_r h_{pr}\Gamma_{rq3} = 0.$$

Remark. We did not write $(p2q)$, $(3pq)$ and (pqr) since we need not use them.

4. Formulas and Lemmas

Promise. In the following, we shall abbreviate the expression "take account of the coefficient of θ_i in the exterior derivative of $\cdots$" to "see θ_i of d of $\cdots$".

In this section we study the crucial case where $\beta \neq 0$. By (3.6) and $(31p)$ we have

$$\beta X_p = (\alpha - \delta) h_{2p}. \tag{4.1}$$

This and $(13p)$ imply that

$$\beta Y_{p3} = \delta h_{2p}. \tag{4.2}$$

The equation (3.9) can be rewritten as

$$\sum_{q,r} (\alpha h_{pq} + c\delta_{pq}) \phi_{qr} h_{r2} = 0, \tag{4.3}$$

which, together with (4.2), implies

$$\beta \sum_{q,r} (h_{pq} - \delta\delta_{pq}) Z_{q3} = \delta \sum_{q,r} (h_{pq} - \delta\delta_{pq}) \phi_{qr} Y_{r3} = 0.$$

Hence it follows from $(33p)$ and $(1p1)$ that

$$\delta_p = -h_{2p} X_3 \quad \text{and} \quad \alpha_p = -\beta Y_{p1}. \tag{4.4}$$

Thus since (4.4) and $\alpha_p \delta + \alpha \delta_p = 0$ obtained from (3.2) we have

$$\beta \delta Y_{p1} = -\alpha h_{2p} X_3, \tag{4.5}$$

and so $\sum_p h_{2p} Z_{p1} = 0$. By (4.2), we have

$$\sum_p h_{2p} Z_{p3} = \sum_{p,q} h_{2p} \phi_{pq} Y_{q3} = \frac{\delta}{\beta} \sum_{p,q} h_{2p} \phi_{pq} h_{2q} = 0. \tag{4.6}$$

From (3.6), (4.3) and (4.5) we have

$$h_{2p} X_1 = 0. \tag{4.7}$$

Now we shall prove the following key lemma.

Lemma 2. $H(e_2) \in \operatorname{span}\{e_1, e_2\}$.

PROOF. Suppose that $h_{2p} \neq 0$. Then from (4.7) we have $X_1 = 0$. We can select the vector e_4 so that $h_{24} \neq 0$ and $h_{25} = \cdots = h_{2,2n-1} = 0$. We put $e_5 := \phi e_4$ and $\rho := h_{24} (\neq 0)$. Note that $\phi_{54} = 1$. Then by (4.3) we have

$$h_{55} = \delta, \quad h_{p5} = 0 \quad (p \neq 5).$$

Put $p = 5$ in $(32p)$. Then by above equation and (4.1) we have $X_5 = 0$ and so $Z_{45} = 0$. Thus we have $Y_{55} = 0$. Furthermore, put $p = q = 5$ in $(q1p)$. Then, since $\Gamma_{551} = Y_{55} = 0$, we have

$$\alpha_1 = \delta_1 = 0. \tag{4.8}$$

Thus, from (313), (323), (4.6) and (112) we have

$$X_3 = 0, \tag{4.9}$$

$$\alpha_2 = \delta_2 = 0, \tag{4.10}$$

$$\beta_1 = 0. \tag{4.11}$$

By (4.4) and (4.9) we have $\alpha_p = \delta_p = 0$. Thus it follows from $(1p1)$ that

$$\alpha_p = \delta_p = Y_{p1} = Z_{p1} = 0. \tag{4.12}$$

Now we put $F = \alpha$, $i = 1$ and $j = p$ in Lemma 1. Then, from (2.7), (4.8), (4.10) and (4.12) we have

$$0 = \alpha_{1p} - \alpha_{p1} = \sum_k \alpha_k \Gamma_{k1p} - \sum_k \alpha_k \Gamma_{kp1} = \alpha_3(\Gamma_{31p} - \Gamma_{3p1}) = \alpha_3 h_{2p}.$$

Thus we have $\alpha_3 = 0$. Hence it follows from (4.8), (4.10) and (4.12) that α and δ are constant, which, together with (113), imply

$$\alpha = 3\delta. \tag{4.13}$$

On the other hand, seeing $\theta_1 \wedge \theta_3$ of d of θ_{23}, we have

$$X_2 = -2\beta. \tag{4.14}$$

Thus, from (312) and (4.13) we have

$$2\delta\gamma + \beta^2 = -c. \tag{4.15}$$

Seeing θ_1 and θ_2 of d of (4.15) and taking account of (4.8), (4.11) and (212), we have

$$\gamma_1 = 0, \quad \beta_2 = 0 \quad \text{and} \quad \gamma_2 = 0. \tag{4.16}$$

Moreover, seeing θ_5 of d of (4.15), we have

$$\delta\gamma_5 + \beta\beta_5 = 0. \tag{4.17}$$

From (3.5) and (4.12) we have

$$h_{2p1} - \sum_q h_{2q}\Gamma_{qp1} = 0.$$

This, together with $(21p)$ and $(12p)$, implies

$$\beta_p + \rho h_{5p} = 0,$$

$$\beta_p + 2\rho h_{5p} + \alpha\rho\phi_{4p} + \beta Y_{p2} = 0.$$

Put $p = 4, 5, 6, \dots, 2n-1$ in above two equations to get

$$\beta_p = \begin{cases} 0 & (p \neq 5) \\ -\rho\delta & (p = 5) \end{cases}, \quad Y_{p2} = \begin{cases} 0 & (p \neq 5) \\ \rho(\alpha - \delta)/\beta & (p = 5) \end{cases},$$
$$Z_{p2} = \begin{cases} 0 & (p \neq 4) \\ -\rho(\alpha - \delta)/\beta & (p = 4) \end{cases}. \tag{4.18}$$

Hence from (4.1), (4.2), (4.17) and (4.18) we have

$$X_p = \begin{cases} 0 & (p \neq 4) \\ \rho(\alpha - \delta)/\beta & (p = 4) \end{cases}, \quad Y_{p3} = \begin{cases} 0 & (p \neq 4) \\ -\rho\delta/\beta & (p = 4) \end{cases},$$
$$Z_{p3} = \begin{cases} 0 & (p \neq 5) \\ -\rho\delta/\beta & (p = 5) \end{cases}, \quad \gamma_p = \begin{cases} 0 & (p \neq 5) \\ -\rho\beta & (p = 5) \end{cases}. \tag{4.19}$$

Now, by (213), (223), (4.15) and (4.19) we have

$$\beta_3 = \beta^2 - \gamma\delta = -\alpha\gamma - c = 3\delta(\delta - \gamma),$$
$$\gamma_3 = 3\beta\gamma - 4\rho^2\delta/\beta. \tag{4.20}$$

On the other hand, if we put $F = \beta$ and γ in Lemma 1, then from (4.11), (4.12), (4.15), (4.16), (4.18) and (4.19) we have

$$\gamma\beta_3 + \rho\beta_5 = 0,$$
$$\gamma\gamma_3 + \rho\gamma_5 = 0. \tag{4.21}$$

Eliminating β_3, β_5, γ_3, γ_5, ρ and β from (4.17), (4.18), (4.20) and (4.21), we have

$$4\gamma^2 - 6\gamma\delta - c = 0.$$

Consequently, γ is constant, which contradicts $\gamma_5 = \rho\beta$. □

Owing to Lemma 2 the matrix (h_{pq}) is diagonalizable, that is, for a suitable choice of a orthonormal frame field $\{e_p\}$ we can set

$$h_{pq} = \lambda_p \delta_{pq}.$$

Then it is easy to see

$$\tilde{R}_\xi = -((\alpha\lambda_p + c)\delta_{pq}),\qquad \tilde{S} = (\{h\lambda_p - (\lambda_p)^2 + K\}\delta_{pq}), \tag{4.22}$$

where we put $K = (2n+1)c$.

Here we shall sum up all equations obtained from Lemma 2.

From (4.1), (4.2) and (4.4) we have

$$X_p = Y_{p1} = Z_{p1} = Y_{p3} = Z_{p3} = 0, \quad \alpha_p = \delta_p = 0. \tag{4.23}$$

This, together with (3.3) and (3.4), imply

$$(\beta^2 - \alpha\gamma)_1 = 0, \tag{4.24}$$

$$(\beta^2 - \alpha\gamma - c)X_1 = 0. \tag{4.25}$$

Put $p = q$ in (3.7). Then we have

$$(\alpha\lambda_p)_1 = 0. \tag{4.26}$$

Moreover, from (112)–(32p) we have

$$\alpha_2 - \beta_1 = 0, \tag{4.27}$$

$$\beta_2 - \gamma_1 = 0, \tag{4.28}$$

$$(\alpha - \delta)\gamma - \beta X_2 + (\gamma - \delta)X_1 - \beta^2 = -c, \tag{4.29}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{4.30}$$

$$\beta_3 - \alpha\delta + \gamma\delta + (\gamma - \delta)X_1 - \beta^2 = c, \tag{4.31}$$

$$\delta_1 + \beta X_3 = 0, \tag{4.32}$$

$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 - \beta\gamma = 0, \tag{4.33}$$

$$\delta_2 + (\gamma - \delta)X_3 = 0, \tag{4.34}$$

$$\beta_p = 0, \tag{4.35}$$

$$Y_{p2} = 0, \quad Z_{p2} = 0, \tag{4.36}$$

$$\gamma_p = 0. \tag{4.37}$$

It follows from $(q1p)$ and (3.7) that

$$\alpha\beta Y_{qp} = \alpha\lambda_p\lambda_q\phi_{pq} - \alpha^2\lambda_p\phi_{pq} + \alpha_1\lambda_p\delta_{pq} - c\alpha\phi_{pq}. \tag{4.38}$$

From this, $(2pq)$ and $(q3p)$ we have

$$\beta^2(\lambda_p + \lambda_q)\phi_{pq} - (\lambda_p - \gamma)(\lambda_p\lambda_q - \alpha\lambda_q - c)\phi_{pq} - (\lambda_q - \gamma)(\lambda_p\lambda_q - \alpha\lambda_p - c)\phi_{pq} = 0, \tag{4.39}$$

$$(\lambda_q - \delta)[\alpha\{(\lambda_q)^2 - \alpha\lambda_q - c\}\delta_{pq} + \alpha_1\lambda_q\phi_{pq}] - \alpha\beta\{h_{qp3} + (\lambda_p - \lambda_q)\Gamma_{qp3}\} = 0. \tag{4.40}$$

If $p = q$ in above equation, then we have

$$(\lambda_p - \delta)\{(\lambda_p)^2 - \alpha\lambda_p - c\} - \beta(\lambda_p)_3 = 0. \tag{4.41}$$

5. Proof of Main Theorem

In this section we prove

Main Theorem. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then M holds $R_\xi\phi S = S\phi R_\xi$ if and only if M is locally congruent to one of the following*:

- (I) *in case that $M_n(c) = P_n\mathbf{C}$ with $\eta(H\xi) \neq 0$,*
  - (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
  - (A_2) *a tube of radius r over a totally geodesic $P_k\mathbf{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (II) *in case that $M_n(c) = H_n\mathbf{C}$,*
  - (A_0) *a horosphere,*
  - (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbf{C}$,*
  - (A_2) *a tube over a totally geodesic $H_k\mathbf{C}$ $(1 \leq k \leq n-2)$.*

Proof. First step. We prove $\beta = 0$.

Suppose that $\beta \neq 0$. It follows from (4.22) that (3.10) is equivalent to

$$(\rho_p\sigma_q - \sigma_p\rho_q)\phi_{pq} = 0,$$

where $\rho_p = \alpha\lambda_p + c$, $\sigma_p = h\lambda_p - (\lambda_p)^2 + K$. Therefore if $\phi_{qp} \neq 0$, then we have

$$(\lambda_p - \lambda_q)\{-ch + \alpha\lambda_p\lambda_q + c(\lambda_p + \lambda_q) + \alpha K\} = 0. \tag{5.1}$$

Here we assert that if $\phi_{pq} \neq 0$, then $\lambda_p = \lambda_q$. To prove this, we assume that there exist indices p and q such that

$$\phi_{pq} \neq 0, \quad \lambda_p - \lambda_q \neq 0.$$

First we prepare three Lemmas.

Lemma 3. $(K\alpha^2 - c\alpha h)_1 = 0$.

Proof. From (5.1) we have

$$(\alpha^2 K - \alpha hc) + (\alpha\lambda_p)(\alpha\lambda_q) + c(\alpha\lambda_p + \alpha\lambda_q) = 0.$$

Lemma 3 follows from this and (4.26). □

Lemma 4. $4n\alpha\alpha_1 - (\alpha\gamma)_1 = 0$.

Proof. From (4.26) we have $(\alpha \sum_p \lambda_p)_1 = 0$. Combining this equation with $h = \alpha + \gamma + \delta + \sum_p \lambda_p$, we have

$$(\alpha(h - \alpha - \gamma - \delta))_1 = 0.$$

Eliminate h from this and Lemma 3. □

Lemma 5. $(\gamma - \delta - 2n\alpha)\alpha_1 = 0$ *and* $(\gamma - \delta - 2n\alpha)\beta_1 = 0$.

Proof. From (4.24) we have $2\beta\beta_1 - (\alpha\gamma)_1 = 0$. Hence it follows from Lemma 4 that

$$2n\alpha\alpha_1 - \beta\beta_1 = 0. \tag{5.2}$$

On the other hand, by (4.32) and (4.34) we have $(\gamma - \delta)\delta_1 - \beta\delta_2 = 0$, and therefore $(\gamma - \delta)\alpha_1 - \beta\alpha_2 = 0$. Thus Lemma 5 follows from (4.27) and (5.2). □

We need to consider four cases.

Case I. Suppose that $\alpha_1 \neq 0$ and $X_1 = 0$. Owing to Lemma 5, we have $\gamma - \delta - 2n\alpha = 0$. Seeing θ_3 of d of this equation and making use of (4.29), (4.30) and (4.33), we have

$$2n\alpha^2(2n\alpha^2 - \delta^2 + 2nc) + \beta^2\{3\delta^2 + (6n + 4)c - 2n\alpha^2\} = 0. \tag{5.3}$$

Seeing θ_1 of d of (5.3) and taking account of (3.2) and (5.2), we have

$$4n^2\alpha^4 + 2n\alpha^2\{3\delta^2 + (8n+4)c\} - \beta^2(3\delta^2 + 2n\alpha^2) = 0. \tag{5.4}$$

Eliminating β from (5.3) and (5.4), we have a polynomial of degree four with respect to δ containing the term $12n\alpha^2\delta^4 \neq 0$. This shows that δ is constant since $\alpha\delta + c = 0$, which contradicts the assumption of Case I.

Case II. Suppose that $\alpha_1 \neq 0$ and $X_1 \neq 0$. By (4.25) we have

$$\beta^2 - \alpha\gamma - c = 0.$$

Then from (4.39) we have

$$(-\lambda_p\lambda_q + 2c)(\lambda_p + \lambda_q) + 2(\alpha+\gamma)\lambda_p\lambda_q - 2c\gamma = 0.$$

Multiply above equation by α^3 and see θ_1 of d of this equation. Then, from Lemma 4 and (4.26) we have

$$c(\alpha\lambda_p + \alpha\lambda_q - \alpha\gamma) + (2n+1)(\alpha\lambda_p)(\alpha\lambda_q) - 2cn\alpha^2 = 0.$$

Again, seeing θ_1 of d of above equation, we have $cn\alpha\alpha_1 = 0$, which is a contradiction.

Case III. Suppose that $\alpha_1 = 0$ and $\beta^2 - \alpha\gamma - c \neq 0$. From (4.24), (4.25), (4.27), (4.28), (4.32) and (4.34) we have

$$\delta_1 = \alpha_2 = \delta_2 = X_3 = \beta_1 = \gamma_1 = \beta_2 = X_1 = 0. \tag{5.5}$$

Seeing $\theta_2 \wedge \theta_3$ of d of θ_{23} we have $\beta_3 - 2\beta^2 = \gamma\delta + 2c$, which, together with (4.31) and (5.5), imply

$$\alpha\delta - \gamma\delta - \beta^2 = \gamma\delta + c.$$

Substituting of (4.14) and (5.5) into (4.29) we have

$$\alpha\gamma - \gamma\delta + \beta^2 = -c. \tag{5.6}$$

Eliminating β from above two equations, we have

$$\alpha\delta - 3\gamma\delta + \alpha\gamma = 0. \tag{5.7}$$

Seeing θ_2 of d of (5.6) and (5.7), we have $(\alpha - \delta)\gamma_2 = 0$ and $(\alpha - 3\delta)\gamma_2 = 0$. Hence we have $\gamma_2 = 0$.

Now put $F = \alpha, \beta, \gamma$ and $i = 1$, $j = 2$ in Lemma 1. Then, we have

$$\alpha_3\gamma = \beta_3\gamma = \gamma_3\gamma = 0.$$

If $\gamma \neq 0$, then from (4.14) and (4.33) we have a contradiction. Thus $\gamma = 0$, which contradicts (5.7).

Case IV. Suppose that

$$\alpha_1 = 0, \tag{5.8}$$

$$\beta^2 - \alpha\gamma - c = 0. \tag{5.9}$$

Seeing θ_2 of d of (5.9), we have

$$(\beta^2 - \alpha\gamma)_3 = 2\beta\beta_3 - \gamma\alpha_3 - \alpha\gamma_3 = 0. \tag{5.10}$$

From (4.29)–(4.31), (4.33) and (5.9) we have the following:

$$-\delta\gamma - \beta X_2 + (\gamma - \delta)X_1 = 0, \tag{5.11}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{5.12}$$

$$\beta_3 + (\gamma - \delta)X_1 + \gamma\delta - \alpha\gamma - c = 0, \tag{5.13}$$

$$\gamma_3 - 2\beta\delta + (\gamma - \delta)X_2 + \beta\gamma = 0. \tag{5.14}$$

Substituting of (5.12)–(5.14) into (5.10) we have

$$(\delta - \gamma)(X_1 - 4\alpha) = 0,$$

by virtue of (5.11). If $\delta = \gamma$, then by (5.9) we have a contradiction. Thus

$$X_1 = 4\alpha. \tag{5.15}$$

Substituting of this equation into (5.11)–(5.13) we have

$$\beta X_2 = 4\alpha(\gamma - \delta) - \delta\gamma, \tag{5.16}$$

$$\alpha_3 + 3\beta\delta + 3\alpha\beta = 0, \tag{5.17}$$

$$\beta_3 + 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0. \tag{5.18}$$

It follows from (4.33), (5.9) and (5.16) that

$$\alpha\gamma_3 + \beta(3\alpha\gamma - 6\alpha\delta - \gamma\delta) = 0. \tag{5.19}$$

From (4.32), (5.2) and (5.8) we have $X_3 = 0$ and $\beta_1 = 0$ and therefore $\alpha_2 = \delta_2 = 0$ because of (4.27). Hence, seeing θ_1 of d of (5.9), we have $\gamma_1 = 0$, and so $\beta_2 = 0$.

Now put $F = \alpha$ and β in Lemma 1. Then we have

$$\alpha_3(\gamma + X_1) = 0, \quad \beta_3(\gamma + X_1) = 0.$$

If $\gamma + X_1 \neq 0$, then we have $\alpha_3 = \beta_3 = 0$. It follows from (4.23) and (4.35) that α, β and δ are constant and that $\alpha_i = \beta_i = 0$ for $i = 1, 2$. Furthermore, by (5.9) we see that γ is constant. Thus from (5.17)–(5.19) we have

$$\alpha + \delta = 0,$$

$$3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0,$$

$$3\alpha\gamma - 6\alpha\delta - \gamma\delta = 0.$$

Hence, by (3.2) and (5.9) we have $\alpha^2 - c = 0$ and $2\beta^2 + c = 0$, which is a contradiction. Therefore $X_1 = -\gamma$, which, together with (5.15), implies $\gamma = -X_1 = -4\alpha$. Thus it follows from (5.17) that $\gamma_3 = -4\alpha_3 = 12\beta(\delta + \alpha)$. Hence from (5.19) we have a contradiction $\alpha\delta = 0$.

Consequently, for all p, q such that $\phi_{pq} \neq 0$, we have $\lambda_p = \lambda_q$. We take p, q such that $\phi_{pq} \neq 0$. Then by (4.39) we have

$$\beta^2\lambda_p - (\lambda_p - \gamma)\{(\lambda_p)^2 - \alpha\lambda_p - c\} = 0. \tag{5.20}$$

Furthermore, from $(q3p)$, (4.38) and (4.26) we have

$$(\lambda_p)_1(\lambda_p - \delta) = 0.$$

If $(\lambda_p)_1 = 0$, then (4.26) implies $\alpha_1 = \delta_1 = 0$. Thus it follows from (4.32), (4.34) and (4.27) that $X_3 = \alpha_2 = \delta_2 = \beta_1 = 0$. Seeing θ_1 of d of (5.20), we have $\{(\lambda_p)^2 - \alpha\lambda_p - c\}\gamma_1 = 0$. If $(\lambda_p)^2 - \alpha\lambda_p - c = 0$, then from (5.20), we have $\lambda_p = 0$, which contradicts the assumption. Hence we have $\gamma_1 = 0$. Thus, from (4.28) we have $\beta_2 = 0$. If $X_1 = 0$, then by the same argument as that in Case III, we have a contradiction. Thus we have $X_1 \neq 0$ and therefore $\beta^2 - \alpha\gamma - c = 0$ because of (4.25). By the same argument as that in Case IV, we have contradiction. Hence we have $\lambda_p = \delta$. From (4.41) and (113) we have $(\lambda_p)_3 = \delta_3 = \alpha_3 = 0$ and $X_1 = \alpha - 3\delta_p$. Thus by (4.25) we have $(\beta^2 - \alpha\gamma - c)(\alpha - 3\delta) = 0$. If $\alpha - 3\delta = 0$, then α and δ are constant and therefore by the argument as above, we have a contradiction. Thus $\beta^2 - \alpha\gamma - c = 0$. From (5.20) we have $(\alpha + \delta)(\delta - \gamma) = 0$. If $\alpha + \delta = 0$, then α and δ are constant, which is also a contradiction. Hence $\delta - \gamma = 0$. However from (5.20) we have $\beta = 0$, which is a contradiction. Consequantly we proved $\beta = 0$.

Second step. Since (2.6) and $\beta = 0$, we see that α is constant in M (see [7]). Thus from (3.1) our assumption $\Xi_{ij;1} = 0$ is equivalent to $\alpha h_{ij;1} = 0$. Put $j = 1$ in

(2.3). Then by above equation we have $\alpha h_{i1;k} = -c\alpha\phi_{ik}$. Therefore since (2.1) and $d\xi_i = 0$, we have

$$\alpha \sum_{k,l} h_{ik}\phi_{lk}h_{kj} + \alpha^2 \sum_{k} \phi_{ki}h_{kj} = -\alpha h_{i1;j} = c\alpha\phi_{ij},$$

which implies that $\alpha^2(\phi H - H\phi) = 0$.

Here, we note the case $\alpha = 0$ corresponds to the case of tube of radius $\pi/4$ in $P_n\mathbf{C}$ (see [2]). However, in the case of $H_n\mathbf{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf. [1]). Owing to Okumura's work or Montiel and Romero's work stated in the Introduction, we complete the proof of our Main Theorem. □

References

[1] Berndt, J., Real hypersurfaces with constant principal curvatures in complex hyperblic spaces, J. Reine Angew. Math. **395** (1989), 132–141.

[2] Cecil, T. E. and Ryan, P. J., Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[3] Cho, J. T. and Ki, U-H., Real hypersurfaces in complex projective spaces in terms of Jacobi operators, Acta Math. Hungar. **80** (1998), 155–167.

[4] Cho, J. T. and Ki, U-H., Jacobi operators on real hypersurfaces in complex projective spaces, Tsukuba J. Math. **22** (1998), 145–156.

[5] Cho, J. T. and Ki, U-H., Real hypersurfaces in complex space forms with Reeb flow symmetric Jacobi operator, Canadian Math. Bull. **51** (2008), 359–371.

[6] Ki, U-H., Real hypersurfaces with pararell Ricci tensor of complex space form, Tsukuba J. Math. **13** (1989), 73–81.

[7] Ki, U-H. and Suh, Y. J., On real hypersurfaces of a complex space form, Math. J. Okayama Univ. **32** (1990), 207–221.

[8] Kim, U. K., Nonexistence of Ricci-parallel real hypersurfaces in $P_2(\mathbf{C})$ or $H_2(\mathbf{C})$, Bull. Korean. Math. Soc. **41** (2004), 699–708.

[9] Kimura, M., Real hypersurfaces and complex submanifolds in complex projective space, Trans. Amer. Math. Soc. **296** (1986), 137–149.

[10] Montiel, S. and Romero, A., On some real hypersurfaces of a complex hyperblic space, Geom. Dedicata **20** (1986), 245–261.

[11] Okumura, M., On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212** (1975), 355–364.

[12] Ortega, M., Pérez, J. D. and Santos, F. G., Non-existence of real hypersurfaces with parallel structure Jacobi operator in nonflat complex space forms, Rocky Mountain J. Math. **36** (2006), 1603–1613.

[13] Pérez, J. D., Santos, F. G. and Suh, Y. J., Real hypersurfaces in complex projective spaces whose structure Jacobi operator is D-parallel, Bull. Belg. Math. Soc. Simon Stevin **13** (2006), 459–469.

[14] Takagi, R., On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. **19** (1973), 495–506.

[15] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures I, II, J. Math. Soc. Japan **15** (1975), 43–53, 507–516.

U-Hang Ki
Department of Mathematics
Kyungpook National University
Daegu 702-701, Korea
E-mail address: uhangki2005@yahoo.co.kr

Hiroyuki Kurihara
Department of Computer and Media Science
Saitama Junior College
Hanasaki-ebashi, Kazo, Saitama 347-8503, Japan
E-mail address: kurihara@sjc.ac.jp
Current address
Department of Liberal Arts and Engineering Sciences
Hachinohe National College of Technology
Hachinohe, Aomori 039-1192, Japan
E-mail address: kurihara-g@hachinohe-ct.ac.jp

Ryoichi Takagi
Department of Mathematics and Informatics
Chiba University
Chiba 263-8522, Japan
E-mail address: takagi@math.s.chiba-u.ac.jp
Current address
4-2859-51 Ohnomachi, Ichikawa, Chiba 272-0872, Japan