JACOBI POLYNOMIAL EXPANSIONS WITH POSITIVE COEFFICIENTS AND IMBEDDINGS OF PROJECTIVE SPACES

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To I. J. Schoenberg on his 65th birthday

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In much of Schoenberg's work there has been a strong interconnection between analytic and geometric reasoning. Here we use a remark he made about imbeddings of metric spaces to prove part of a conjecture about when a Jacobi polynomial $P_n^{(\gamma,\delta)}(x)$ can be expanded in terms of another $P_k^{(\alpha,\beta)}(x)$ with nonnegative coefficients. Also we get from a different special case of this conjecture some nonimbedding theorems for projective spaces.

$P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree n, order (α, β), $\alpha, \beta > -1$, is defined by

$$(1)\quad (1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dx}\right)^n[(1-x)^{n+\alpha}(1+x)^{n+\beta}].$$

These polynomials are orthogonal on $(-1, 1)$ with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ and what is crucial for us is that $P_n^{(\alpha,\beta)}(1)>0$. We consider the expansion

$$(2)\qquad P_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} \alpha_k P_k^{(\alpha,\beta)}(x)$$

and ask for what values of $\alpha, \beta, \gamma, \delta$ are all the coefficients α_k, $k=0, 1, \cdots, n$, nonnegative. For $\beta=\delta$ and $\gamma>\alpha$ the α_k were computed by Szegö [8] and were found to be positive. He used this relation to solve the end point Cesàro summability problem for Jacobi series.

For $\alpha=\beta$, $\gamma=\delta$ the α_k were given by Gegenbauer [5] and again they are nonnegative for $\alpha>\gamma$. This has been used by Hua [6] and Askey and Wainger [1]. Actually this result of Gegenbauer is a special case of Szegö's result. For

$$(3)\qquad \frac{P_n^{(\alpha,-1/2)}(2x^2-1)}{P_n^{(\alpha,-1/2)}(1)} = \frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}$$

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and

$$\frac{xP_n^{(\alpha,1/2)}(2x^2-1)}{P_n^{(\alpha,1/2)}(1)} = \frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)}.$$

Thus (2) for $\beta=\delta=-\frac{1}{2}$ is equivalent to (2) for n even, $\alpha=\beta$, $\gamma=\delta$; and (2) for $\beta=\delta=\frac{1}{2}$ is equivalent to (2) for n odd, $\alpha=\beta$, $\gamma=\delta$. Since the proof of Szegö's result is easier and more natural than any proof I know of Gegenbauer's result, I like to think of Szegö's result as the more fundamental. However, it would be nice to have α_k in the general case (2) and to get the positivity for the known cases from the general case. Unfortunately I am unable to find a simple enough formula for α_k. α_k has been computed by Feldheim [3] and he gets it as a ${}_3F_2$. I haven't seen his proof, but a proof using (1) a couple of times, many integrations by parts and the binomial theorem is easy. This proof is identical with Szegö's proof for $\beta=\delta$ until the last step when $(1+x)^c$ is expanded in terms of $(1-x)^j$. Explicitly

$$a_k=\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)\Gamma(n+k+\gamma+\delta+1)\Gamma(n+\delta+1)\Gamma(n-k+\gamma-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(k+\beta+1)\Gamma(n+k+\alpha+\delta+1)\Gamma(n+\gamma+\delta+2)\Gamma(n-k+1)}$$
$$\cdot {}_3F_2(\delta-\beta,\alpha-\gamma+1,\alpha+k+1;\alpha-\gamma+k-n+1,n+k+\alpha+\delta+2;1).$$

A reasonable conjecture which includes both of the above cases is that $\alpha_k\geqq 0$ if (γ,δ) lies in the triangular region above the line $\delta=\beta$ and to the right of the line through (α,β) and $(-1,-1)$. By Szegö's result it would be sufficient to show this for (γ,δ) on the line through $(-1,-1)$ and (α,β). This is one of a number of problems that is equivalent to a certain ${}_3F_2$ being positive. It seems that a systematic study of when these and other generalized hypergeometric functions are positive would yield many interesting results.

This conjecture is false for (γ,δ) above the line through $(-1,-1)$ and (α,β). $P_1^{(\alpha,\beta)}(x)=\frac{1}{2}[(\alpha+\beta+2)x+(\alpha-\beta)]$ and $P_0^{(\alpha,\beta)}(x)=1$. A computation shows that

$$P_1^{(\gamma,\delta)}(x)=\left(\frac{\gamma+\delta+2}{\alpha+\beta+2}\right)P_1^{(\alpha,\beta)}(x)$$
$$+\frac{[(\gamma-\delta)(\alpha+\beta+2)+(\beta-\alpha)(\gamma+\delta+2)]}{2(\alpha+\beta+2)}P_0^{(\alpha,\beta)}(x)$$

and the second coefficient is nonnegative if and only if $\gamma\geqq((\alpha+1)(\delta+1)/(\beta+1))-1$, i.e. (γ,δ) lies to the right of the given line.

This remark has an interesting consequence when combined with some work on Bochner on positive definite functions on Riemannian spaces. Schoenberg defined a function f on $[0, \infty]$ as positive definite on a separable metric space X if $\sum_{i,j=0}^{n} f(\text{dist}(x_i, x_j))\rho_i\bar{\rho}_j \geqq 0$ for all $x_i \in X$ and complex ρ_i. For the sphere S^k he has found all the positive definite functions [7] and they are just $f(\theta) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\alpha)}(\cos\theta)$ with $\sum a_n P_n^{(\alpha,\alpha)}(1) < \infty$, $a_n \geqq 0$. Here $\alpha = (k-3)/2$. Since S_k can be isometrically imbedded in S_l for $k < l$, it follows that $P_n^{(\gamma,\gamma)}(\cos\theta) = \sum_{k=0}^{n} \alpha_k P_k^{(\alpha,\alpha)}(\cos\theta)$ with $\alpha_k \geqq 0$ for $\gamma > \alpha$ and γ, α half integers, as Schoenberg observed. This remark that the isometric imbedding of a metric space in a second metric space gives rise to a reverse inclusion in their positive definite functions can be used to obtain a couple of interesting results when combined with work of Bochner. For a number of Riemannian manifolds, including the real projective spaces $P^d(R)$, the complex projective spaces, $P^d(C)$, the quaternionic projective spaces $P^d(H)$, and the Cayley elliptic plane P^{16}, Bochner has found the positive definite functions [2]. Here d is the real dimension of the space. They are $\sum_{n=0}^{\infty} a_n\phi_n$, with $a_n \geqq 0$ and ϕ_n the spherical function of degree n. These spherical functions are Jacobi polynomials. For $P^d(R)$ they are given in [4] as $P_{2n}^{(\alpha,\alpha)}(\cos(\pi\theta/2L))$ where L is the diameter of the space in question. Using (3) we see that they are also $P_n^{(\alpha,-1/2)}(\cos(\pi\theta/L))$. Here $\alpha = (d-2)/2$, $d = 2, 3, \cdots$. For $P^d(C)$ the spherical functions are $P_n^{(\alpha,0)}(\cos(\pi\theta/L))$, $\alpha = (d-2)/2$, $d = 4, 6, \cdots$. For $P^d(H)$ they are $P_n^{(\alpha,1)}(\cos(\pi\theta/L))$, $\alpha = (d-2)/2$, $d = 8, 12, \cdots$, and for the Cayley elliptic plane they are $P_n^{(7,3)}(\cos(\pi\theta/L))$. See [4].

If each of these spaces has diameter equal to one we can isometrically imbed $P^d(R)$ in $P^{2d}(C)$, which can be isometrically imbedded in $P^{4d}(H)$. Also $P^8(H)$ can be isometrically imbedded in P^{16} so we have that $\alpha_k \geqq 0$ for certain values of $\alpha, \beta, \gamma, \delta$. They are the values on the lines through $(-1, -1)$ of the form $(k/2-1, -1/2)$, $(k-1, 0)$, $(2k-1, 1)$, $(7, 3)$, $k = 2, 3, \cdots$.

In the other direction since α_k is not always greater than or equal to zero for points above these lines we have that you cannot isometrically imbed $P^{d+1}(R)$ in $P^{2d}(C)$ or $P^{4d}(H)$, that $P^{2d+2}(C)$ cannot be isometrically imbedded in $P^{4d}(H)$ and that $P^3(R)$, $P^6(C)$ and $P^{12}(H)$ cannot be isometrically imbedded in P^{16} when they have the same diameter. When the space with smaller real dimension has a larger diameter you clearly cannot imbed isometrically. If the diameter is smaller, then if you could isometrically imbed one of these spaces you could also isometrically imbed a circle of the same diameter. Thus we need to consider

$$P_1^{(\gamma,\delta)}\left(\cos\frac{\theta}{L}\right) = \sum_{k=0}^{\infty} \alpha_k \cos k\theta$$

with $L > 1$, and $\gamma > \delta \geqq -\frac{1}{2}$.

$$P_1^{(\gamma,\delta)}(x) = ((\gamma - \delta)/2) + ((\gamma + \delta + 2)/2)x$$

and so

$$\alpha_k = \frac{(\gamma + \delta + 2)}{\pi} \int_0^{\pi} \cos\frac{\theta}{L} \cos k\theta d\theta, \quad k = 1, 2, \cdots.$$

A simple calculation shows that

$$\alpha_k = (\gamma + \delta + 2)(-1)^k \sin(\pi/L)/\pi L(k^2 - 1/L^2)$$

and since $L > 1$ this is not always nonnegative.

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