Jamming of Granular Flow in a Two-Dimensional Hopper

Kiwing To,[1] Pik-Yin Lai,[2] and H. K. Pak[3]
[1]*Institute of Physics, Academia Sinica, Taipei, Taiwan 115, Republic of China*
[2]*Department of Physics and Center for Complex Systems, National Central University, Chung-Li, Taiwan 320, Republic of China*
[3]*Department of Physics, Pusan National University, Pusan, Korea*
(Received 24 August 2000)

We study experimentally the jamming phenomenon of granular flow of monodisperse disks of $D =$ 5 mm diameter in a two-dimensional hopper with opening R. The jamming probability $J(d)$ is measured where $d \equiv R/D$. We found that $J(d)$ decreases from 1 to zero when d increases from 2 to 5. From observing the disk configurations of the arch in the jamming events, the jamming probability can be explained quantitatively by treating the arch as the trajectory of a restricted random walker.

DOI: 10.1103/PhysRevLett.86.71 PACS numbers: 45.70.Qj, 45.70.Mg, 45.70.Vn

Granular systems consist of particles which interact among themselves only by interparticle contacts [1–3]. In nature, many important phenomena such as avalanche, landslide, soil fluidization, and blood flow can be related to three-dimensional (3D) granular flow. On the other hand, two-dimensional (2D) flow phenomena can be found in the baggage flow on conveyer belts, the transport of cans and bottles in factories, and traffic jam in a city. Although there are many theoretical, experimental, and computer simulation studies in granular systems, our basic understanding of the static and dynamical properties of granular systems is far from clear. For example, in the simple problem of granular flow through a hopper, one finds that the flow is jammed after a few particles are discharged when the opening is smaller than a critical value [4]. However, very little is known about how the transition from flowing to jamming occurs. With the advance in experimental techniques and fast electronic computers, studies in laboratory experiments [5] and computer simulations [6] showed that jamming is due to arch formation at the hopper opening. Nevertheless, there is not even a quantitative description of the arch that leads to jamming. In this Letter, we report our studies on the basic mechanism of the jamming process of granular flow in a 2D hopper. We measured the jamming probability as a function of the hopper opening. Our results show that the jamming probability can be understood quantitatively by a simple geometrical model in which the arch that leads to jamming is treated as the trajectory of a restricted self-avoiding random walker. The effects of friction and the hopper angle on the jamming probability are also discussed.

Figure 1 is a schematic diagram of our experimental setup. We fabricated a 2D hopper with an aluminum base plate. The walls (FP and MP) of this hopper are 4 mm thick aluminum plates each having a cut at the opening of the hopper so that both FP and MP make an angle ϕ with the horizontal direction when the hopper is at the upright position. The hopper angle ϕ can be changed by replacing the walls. The movable wall (MP) is attached to a stepping motor (SM) controlled translation stage (B) such that the hopper opening R can be varied continuously using a stepping motor controller (SMC) through a personal computer (PC). In the hopper, we put 200 monodisperse stainless steel disks of 3 mm thick and $D = 5$ mm in diameter. To observe the disk motion in the hopper, its front plate is made of 2 mm thick transparent Plexiglas. Since MP and FP are 4 mm thick, the disks cannot flip over inside the hopper. The disk surfaces are polished to reduce the friction among the disks and that between the disks and the walls. The hopper is mounted on a vertical rotating stage such that the symmetry axis of the hopper is perpendicular to the axis of the rotation. When the hopper is rotated from the upside down to the upright position, the disks in the hopper will fall down toward the opening. Either all of the disks fall out of the hopper or some disks are left in the hopper due to jamming at the hopper opening. The motion of the disks is captured by a CCD video camera and the video images are taken by a frame grabber (FG) to the same PC that controls R. Image processing software is developed to analyze the captured video and to determine if, in each revolution, the flow in the hopper is jammed or not. Figure 2(a) shows an image of a typical jamming event captured in the experiment. For each opening R, we counted the number of jamming events N_a and

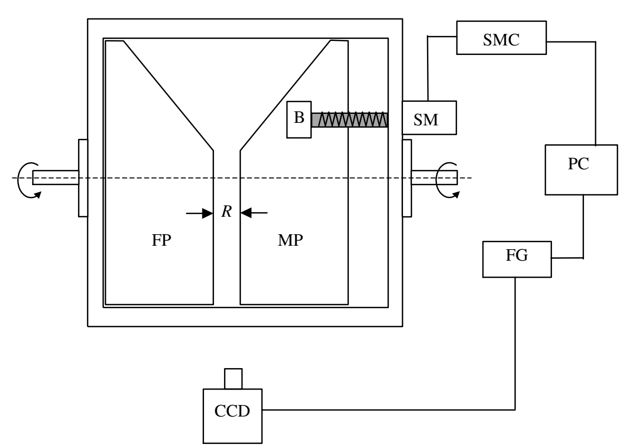

FIG. 1. Schematic diagram of the experimental setup.

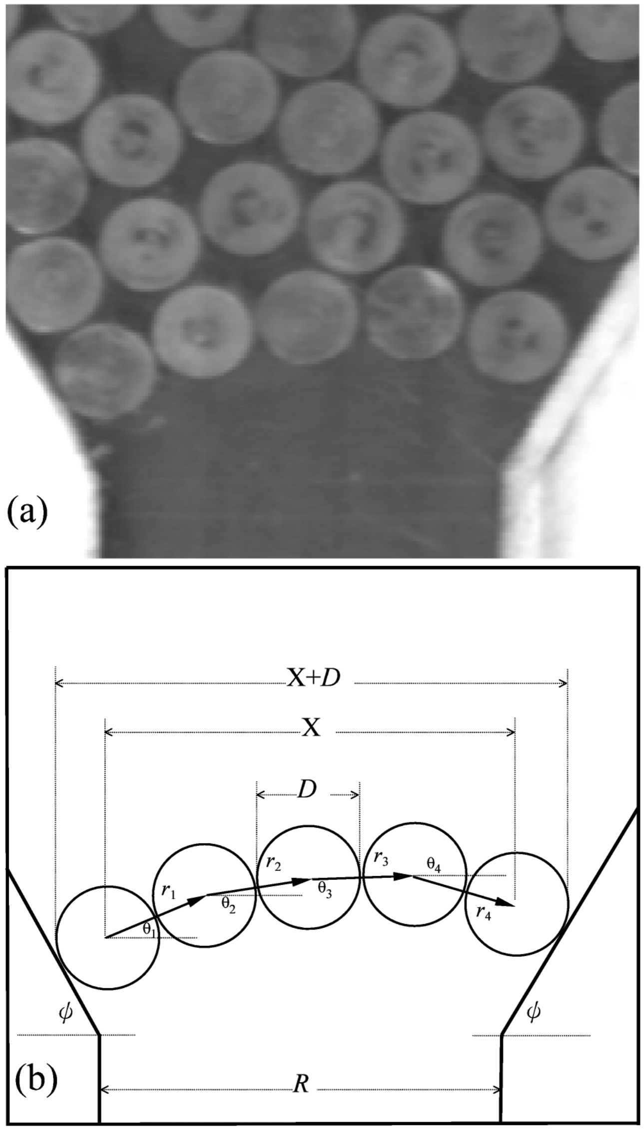

FIG. 2. (a) Image of a typical jamming event. (b) Configuration of the arch of (a).

obtained the jamming probability $J(d)$ which is defined as N_a/N_t. Here N_t is the number of the trial for each opening and $d \equiv R/D$ is the opening in units of disk diameter. Note that the relevant length scale in this problem is the disk diameter D. Hence, we express all physical length in units of D.

Figure 3 shows the jamming probabilities obtained in hoppers of $\phi = 34°$ (○) and $60°$ (△) while N_t varies between 100 and 1000. The error bars of these data are 99% confident intervals. One can see that $J(d)$ for $\phi = 34°$ and $60°$ are the same within experimental uncertainty. In general, when $d \approx 1$, i.e., $R \approx D$, $J(d)$ are close to unity. Then $J(d)$ decrease sharply from 0.9 to 0.1 in the transition region of $3.3 < d < 4.3$. Finally when $d > 5.5$, $J(d)$ are practically zero. On the other hand, when we repeated the experiment using the hopper of $\phi = 75°$, $J(d)$ (□ in Fig. 3) starts to drop at a smaller opening and the transition region is much narrower than those for $\phi =$ 34° and 60°. We shall discuss this point later in this Letter. The line through the data for $\phi = 34°$ and 60° is the theoretical prediction from a simple self-avoiding random walk model to be described below.

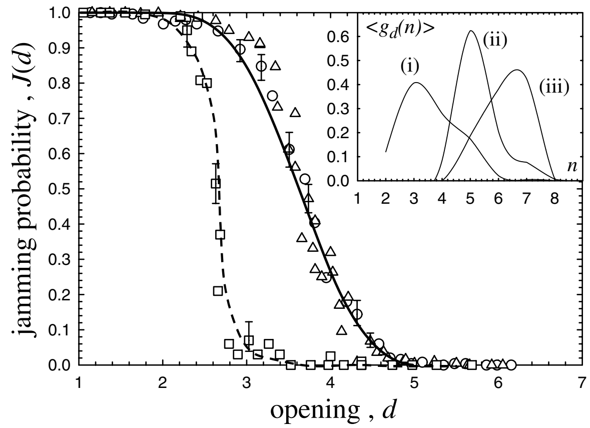

FIG. 3. Jamming probability $J(d)$ for $\phi = 34°$ (○), 60° (△), and 75° (□). The solid line is the approximation from the restricted random walk model and the dashed line is a guide to the eye for the $\phi = 75°$ data. The inset shows the statistics $\langle g_d(n) \rangle$ of the number of disks in the arch, averaged in the ranges (i) $d < 3.3$, (ii) $3.3 < d < 4.3$, and (iii) $d > 4.3$, respectively.

When we examine the disk configurations of the jamming events (such as that shown in Fig. 2), we find that jamming is the result of an arch formed at the opening of the hopper as mentioned before. From the arch configuration of the jamming event, it is obvious that the horizontal span $X + D$ of the arch is always greater than the opening R. Furthermore, the arch is everywhere convex. This is a necessary condition for static equilibrium for each disk in the arch if friction is neglected. From these observations, we assume that the jamming probability is proportional to the probability of forming an arch above the opening of the hopper.

If we transverse the arch of n disks from left to right and denote the displacement vector from the center of the ith disk to that of the $(i+1)$th disk by r_i, we can consider the vectors $(r_1, \ldots, r_{n-1})$ to form a trajectory of a random walker going from left to right in $n - 1$ steps with the following constraints:

$$\pi/2 > \theta_i > -\pi/2, \tag{1}$$

$$\theta_1 > \cdots > \theta_i > \cdots > \theta_{n-1}, \tag{2}$$

$$\forall\ i \neq j, \left| \sum_{k=1}^{i} r_k - \sum_{k=1}^{j} r_k \right| \geq D, \tag{3}$$

where θ_i is the angle between r_i and the horizontal direction. Constraint (1) keeps the random walker going from left to right. Constraint (2) is the condition of convexity of the arch. Constraint (3) is the consequence of finite disk volume, i.e., the disks cannot penetrate each other.

Note that constraint (3) can be simplified to $|\theta_i - \theta_{i-1}| < 2\pi/3$ due to constraints (1) and (2). Finally, to keep the arch, which satisfies constraints (1) to (3), in the hopper of opening R, we must have $X + D > R$ where X is the horizontal component of the displacement vector from the first disk to the last disk. Otherwise, the arch will flow out through the opening. In dimensionless variables, this can be written as

$$x + 1 > d, \tag{4}$$

with $x \equiv X/D$. To summarize, the jamming probability for a hopper of opening R will be proportional to the number of possible arch configurations that satisfy constraints (1) to (4).

To proceed further, we assume that θ_i is uniformly distributed, i.e., the probability distribution function for θ_1 is $f_1(\theta) = 1/\pi$ due to (1) and those for θ_i with $i > 1$ are $f_i(\theta) = 3/(4\pi)$ due to (3). Then the probability distribution function for an arch of n disks to have a horizontal displacement x can be written as

$$\begin{aligned} a_n(x) &= A_n \int_{-\pi/2}^{\pi/2} f_1(\theta_1)\, d\theta_1 \cdots \int_{\beta_{n-1}}^{\theta_{n-2}} f_{n-1}(\theta_{n-1})\, d\theta_{n-1}\, \delta\left(x - \sum_{i=1}^{n-1} \cos\theta_i\right) \\ &= B_n \int_{-\pi/2}^{\pi/2} d\theta_1 \cdots \int_{\beta_{n-1}}^{\theta_{n-2}} d\theta_{n-1}\, \delta\left(x - \sum_{i=1}^{n-1} \cos\theta_i\right), \end{aligned} \tag{5}$$

where $\beta_{n-1} = \max\{-\pi/2, \theta_{n-2} - 2\pi/3\}$, A_n and $B_n = \frac{A_n}{\pi}(\frac{3}{4\pi})^{n-2}$ are normalization constants such that $\int_0^\infty a_n(x)\, dx = 1$. It is not difficult to check that the domain of integration in the above expression does satisfy constraints (1) to (3). Note that $a_n(x)$ is defined only for $n > 1$ since the case for $n = 1$ would correspond to an unphysical "arch" of one disk. When the arch is composed of two disks (i.e., $n = 2$), which corresponds to a one-step random walk, we have $\theta_{n-1} = \theta_1$. Hence

$$a_2(x) = B_2 \int_{-\pi/2}^{\pi/2} d\theta_1\, \delta(x - \cos\theta_1) = \frac{2B_2}{\sqrt{1 - x^2}}. \tag{6}$$

This gives $B_2 = \pi^{-1}$. For $n \geq 3$, we calculated $a_n(x)$ numerically according to expression (5). Figure 4 shows $a_n(x)$ together with the cumulative probability

$$j_n(d) \equiv \int_{d-1}^{\infty} a_n(x)\, dx \tag{7}$$

which is the probability that an arch of n disk has a horizontal component $x \geq d - 1$.

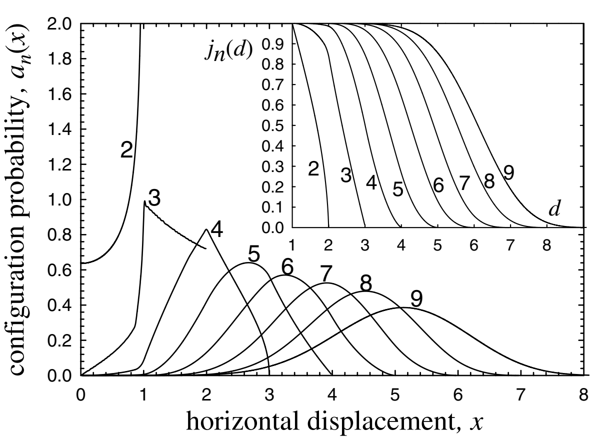

FIG. 4. Configuration probability $a_n(x)$ and cumulative probability $j_n(d)$ for $n = 2$ to $n = 9$ (from left to right) obtained by numerical integration of expressions (5) and (7).

Now jamming will take place for an arch in the hopper when inequality (4) is satisfied, i.e., $x > d - 1$. Thus, the jamming probability can be written as

$$J(d) = \sum_{n=2}^{\infty} g_d(n) j_n(d), \tag{8}$$

where $g_d(n)$ is the fraction of arches with n disks with respect to all the arches with the horizontal component x greater than $d - 1$. Hence, we classified the observed jamming events and obtained the statistics in n at different d for $\phi = 60°$. The inset in Fig. 3 shows the mean $\langle g_d(n) \rangle$ for (i) $d < 3.3$, (ii) $3.3 < d < 4.3$, and (iii) $d > 4.3$. For those with $d < 3.3$, when the jamming probability is greater than 0.9, most jamming events have an arch that consists of three disks. On the other hand, for those with $d > 4.3$, when the jamming probability is smaller than 0.1, most jamming events have an arch that consists of seven disks. In the transition region ($3.3 < d < 4.3$) when the jamming probability undergoes significant changes, the mean $\langle g_d(n) \rangle$ peaks at $n = 5$. Therefore we can approximate $J(d)$ in (8) by

$$J(d) \approx j_n(d) = \int_{d-1}^{\infty} a_n(x)\, dx, \tag{9}$$

with $n = 5$ in this range of d. The solid line in Fig. 3 shows the comparison between the experimental data and the theoretical result from the self-avoiding walk model. One can see that this simple self-avoiding random walk model, which carries only geometrical information of the arch, fits very well with the experimental jamming probabilities for $\phi = 34°$ and $60°$.

Surprisingly, when we repeated our experiments using the hopper of $\phi = 75°$, we found that $J(d)$ is very different from those for $\phi = 34°$ and $60°$ as shown in Fig. 3. To understand this we observed that in most of the jamming events, some disks do fall out of the hopper before jamming actually happens. Hence, the jamming probability should depend on the flow pattern before jamming and the number of disks that flow out of the hopper should be related to the jamming probability. On the other hand, Rose and Tanaka [7] found that due to the presence of a stagnant zone near the opening, the flow rate in a hopper does not depend on ϕ if ϕ is less than a critical value ϕ_c related to the angle of repose of the grains. In our jamming experiments, the flow pattern before jamming may have the resemblance of the stagnant zone and the number of disks out of the hopper before jamming may be treated as the flow rate. Then it is possible that in our experiment ϕ_c is between 60° and 75° so that the jamming probability is the same for the two different hoppers ($\phi = 60°$ and 34°) of our experiment. The data $J(d)$ in the experiment with the hopper of $\phi = 75°$ are indeed very different from those of $\phi = 60°$ and 34°. It drops rapidly to zero at $d = 2.5$ implying that the disks can flow out more easily. This is reasonable since at the extreme case when $\phi = 90°$ jamming should not occur when $d > 1$ and $J(d)$ should become a step function. Currently experiments are being carried out for further study of the effect of the hopper angle on the jamming probability at high hopper angles.

Friction, which has been neglected in the simple model, will increase the jamming probability. Because of finite friction present at the contact between two disks and that between the disk and the walls, θ_i may be greater than θ_{i-1}. We indeed observed a small number of jamming events in which the arch at the opening is not everywhere convex. To see more clearly how friction plays a role in the jamming probability, we repeated our experiment using disks with 25 v-shaped grooves of 0.2 mm deep at their circular edge [see drawing (b) in Fig. 5]. With these disks, we observed a significant number of jamming events in which the arch is not everywhere convex. The jamming probability for these disks in the 60° hopper is shown in Fig. 5. One can see that $J(d)$ for the rough disks is always bigger than that for the smooth disks. This is consistent with the fact that friction will increase the jamming probability and this feature can be reproduced when we compare the $j_5(d)$ obtained with and without constraint (2) as shown in the inset of Fig. 5. In addition, a shoulder appears in $J(d)$ for the rough disks at $d \approx 4.5$ in $J(d)$ which finally vanish at $d \approx 5.5$. Apparently, the presence of friction not only increases the jamming probability, it also affects the statistics in the number of disks in the arch. Indeed, when we examined the jamming events for the rough disks, we found a significant number of jamming events with an arch consisting of more than ten disks.

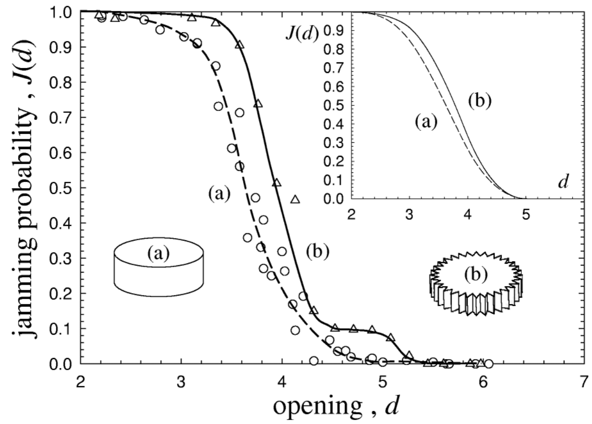

FIG. 5. Comparison of $J(d)$ for (a) disks with smooth edges and (b) disks with rough edges. The lines are guides to the eye only. The inset shows the comparison between the $j_5(d)$ obtained by numerical method (a) with the convexity constraint (dashed line) and (b) without (solid line).

To conclude, we have performed experiments on the jamming phenomenon of the granular system of monodisperse disks in 2D hoppers. When we consider the arch at the hopper opening as a trajectory of a restricted self-avoiding random walker and measure the statistics of the number of disks in the arch, the observed jamming probabilities can be accounted for quantitatively when the hopper angle is less than 75°.

The authors would like to thank Dr. C. K. Chan and Dr. K. T. Leung for valuable discussions. This work is supported by Grants No. NSC-89-2112-M-001-054 and No. NSC 89-2112-M008-052 from the National Science Council of the Republic of China.

[1] G. Ristow, *Pattern Formation in Granular Materials* (Springer, New York, 2000).
[2] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Phys. Today **49**, No. 4, 32 (1996).
[3] D. Bideau and A. Hansen, *Disorder and Granular Media* (Elsevier Science Publishers, North-Holland, 1993).
[4] W. A. Beverloo *et al.,* J. Chem. Eng. Sci. **15**, 260 (1961).
[5] R. L. Brown and J. C. Richards, Trans. Inst. Chem. Eng. **38**, 243 (1960).
[6] D. C. Hong and J. A. McLennan, Physica (Amsterdam) **187A**, 159 (1992).
[7] H. E. Rose and T. Tanaka, Engineer (London) **208**, 465 (1959).