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Jensen Inequality Approach to Stability Analysis of Discrete-Time Systems with Time-Varying Delay

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Abstract— **This paper studies the problem of stability analysis for discrete-time delay systems. By using new Lyapunov functional and the discrete Jensen inequality, new stability criteria are presented in terms of linear matrix inequalities (LMIs) and proved to be less conservative than the existing ones. Compared with the existing results, the computational complexity of the obtained stability criteria is reduced greatly since less decision variables are involved. Numerical examples are given to illustrate the effectiveness and advantages of the proposed method.**

I. INTRODUCTION

During the last two decades, the stability problem of linear continuous-time systems with time-delay has received considerable attention [6]-[9]. The practical examples of time delay systems include engineering, communications and biological systems. The existence of delay in a practical system may induce instability, oscillation and poor performance.

Compared with continuous-time systems with time-delay, discrete-time systems with time-varying delay have strong background in engineering applications, among which network based control has been well recognized to be a typical example (see [3]-[5], [12]). One should notice that little effort has been made towards investigating the stability of discrete time-delay systems. The reason is that for linear discrete-time systems with constant time-delay, one can transform them into the delay-free systems via state augmentation approach. However, the augmentation approach cannot be applied to linear discrete-time systems with time-varying delay. Recently, there have been some works investigating the stability of discrete systems with time-varying delay via Lyapunov approaches [13], [14].

By employing the Moon's inequality [10] to estimate the cross products between two vectors, [14] proposed a stability condition which was dependent on the minimum and maximum delay bounds. By defining new Lyapunov functional and circumventing the utilization of some bounding inequalities for cross products between two vectors, [13] improved the result in [14], and the free-weighting matrix method (see [6]) was adopted to reduce the conservatism of the results. However, the introduction of the free-weighting matrices may increase the number of decision variables, then it may lead to the increase of the computational complexity inevitably.

In this paper, by defining a new Lyapunov functional and using the discrete Jensen inequality, new stability criteria are presented for discrete-time delay systems. Since the discrete Jensen inequality is adopted and no any free-weighting matrices are introduced, the computational complexity of the obtained stability criteria is reduced greatly compared with the existing results. On the other hand, it is shown that the presented stability conditions are less conservative than the corresponding ones in [13] and [14].

This paper is organized as follows. Section 2 gives the problem statement. The stability criteria for nominal systems and uncertain delay systems are presented in Section 3. Section 4 presents the comparison of the obtained stability criteria with some existing ones. Section 5 gives some examples to illustrate the effectiveness of the presented stability criteria. Section 6 concludes this paper.

II. PROBLEM STATEMENT

Consider the following discrete-time system with a time-varying state delay [13]:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d_k) \\ x(k) = \phi(k) \quad k = -d_M,\ -d_M+1,\ \cdots,\ 0, \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state vector, A and A_d are constant matrices with appropriate dimensions; d_k is a time-varying delay in the state, and it satisfies

$$d_m \le d_k \le d_M, \tag{2}$$

where d_m and d_M are constant positive integers representing the lower and upper delays, respectively.

The purpose of this paper is to find new stability criteria which are of less conservatism and less computational complexity than the existing results.

For system (1)-(2), the Moon's inequality was used in [14] to bound the inner product between two vectors, and the obtained stability condition is listed as follows:

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Lemma 1. [14] System (1)-(2) is asymptotically stable if there exist matrices $P = P^T > 0,\ Q = Q^T > 0,\ X = X^T > 0,\ Z = Z^T > 0$, and Y satisfying

$$\Upsilon = \begin{bmatrix} -P & 0 & PA & PA_d \\ * & -d_M^{-1}Z & Z(A-I) & ZA_d \\ * & * & \Upsilon_1 & -Y \\ * & * & * & -Q \end{bmatrix} < 0, \tag{3}$$

$$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \geq 0, \tag{4}$$

where

$$\Upsilon_1 = -P + d_M X + Y + Y^T + (d_M - d_m + 1)Q.$$

By using the free-weighting method, [13] presented an improved result on Lemma 1 as follows:

Lemma 2. [13] System (1)-(2) is asymptotically stable if there exist matrices $P = P^T > 0,\ Q = Q^T \geq 0,\ R = R^T \geq 0,\ Z_i = Z_i^T > 0\ (i = 1,\ 2),\ M,\ S,\ N$ satisfying

$$\Xi = \begin{bmatrix} \Xi_1 + \Xi_2 + \Xi_2^T + \Xi_3 & \Xi_4 \\ * & \Xi_5 \end{bmatrix} < 0 \tag{5}$$

where

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & A^T PA_d & 0 \\ * & A_d^T PA_d - Q & 0 \\ * & * & -R \end{bmatrix},$$
$$\Xi_{11} = A^T PA - P + (d_M - d_m + 1)Q + R,$$
$$\Xi_2 = \begin{bmatrix} M+N & S-M & -S-N \end{bmatrix},$$
$$\Xi_3 = d_M \begin{bmatrix} A-I & A_d & 0 \end{bmatrix}^T (Z_1 + Z_2) \begin{bmatrix} A-I & A_d & 0 \end{bmatrix},$$
$$\Xi_4 = \begin{bmatrix} \sqrt{d_M}M & \sqrt{d_M - d_m}S & \sqrt{d_M}N \end{bmatrix},$$
$$\Xi_5 = diag\{-Z_1,\ -Z_1,\ -Z_2\}.$$

Corresponding to the Jensen integral inequality [2], we can get the following discrete Jensen inequality which will be exploited for the stability analysis of the system (1)-(2):

Lemma 3. For any constant matrix $M \in R^{n\times n}$, $M = M^T > 0$, integers $\gamma_2 \geq \gamma_1$, vector function $\omega:\ \{\gamma_1,\ \gamma_1+1,\ \cdots,\ \gamma_2\} \to R^n$ such that the sums in the following are well defined, then

$$\begin{aligned} &-(\gamma_2 - \gamma_1 + 1)\sum_{i=\gamma_1}^{\gamma_2} \omega^T(i) M \omega(i) \\ &\leq -\Big(\sum_{i=\gamma_1}^{\gamma_2} \omega(i)\Big)^T M \Big(\sum_{i=\gamma_1}^{\gamma_2} \omega(i)\Big). \end{aligned} \tag{6}$$

Proof: Analogous to the proof of Lemma 1 in [1], it is easy to see, using the Schur complement, that

$$\begin{bmatrix} \omega^T(i) M \omega(i) & \omega^T(i) \\ \omega(i) & M^{-1} \end{bmatrix} \geq 0 \tag{7}$$

for any $i \in \{\gamma_1,\ \gamma_1+1,\ \cdots,\ \gamma_2\}$. Sum of the above inequality from γ_1 to γ_2 yields

$$\begin{bmatrix} \sum_{i=\gamma_1}^{\gamma_2} \omega^T(i) M \omega(i) & \sum_{i=\gamma_1}^{\gamma_2} \omega^T(i) \\ \sum_{i=\gamma_1}^{\gamma_2} \omega(i) & (\gamma_2 - \gamma_1 + 1) M^{-1} \end{bmatrix} \geq 0. \tag{8}$$

Using the Schur complement again, it gets that (6) holds. ■

III. MAIN RESULTS

In this section, a new stability criterion for system (1)-(2) will be presented, and extended to cope with uncertain systems.

A. New Stability Criteria

For system (1)-(2), we give the following stability condition using the discrete Jenson inequality.

Theorem 1. System (1)-(2) is asymptotically stable if there exist matrices $P = P^T > 0,\ Q_i = Q_i^T \geq 0,\ U_i = U_i^T > 0\ (i = 1,\ 2,\ 3)$ satisfying

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & U_3 & U_1 \\ * & \Lambda_{22} & U_2 & U_2 \\ * & * & -Q_3 - U_2 - U_3 & 0 \\ * & * & * & -Q_2 - U_1 - U_2 \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{aligned} \Lambda_{11} &= A^T PA - P + (d_M - d_m + 1)Q_1 + Q_2 + Q_3 - (U_1 + U_3) \\ &\quad + (A-I)^T U (A-I), \\ \Lambda_{12} &= A^T PA_d + (A-I)^T UA_d, \\ \Lambda_{22} &= A_d^T PA_d - Q_1 - 2U_2 + A_d^T UA_d, \\ U &= d_M^2 U_1 + (d_M - d_m)^2 U_2 + d_m^2 U_3. \end{aligned}$$

Proof: Choose a Lyapunov functional candidate as:

$$\begin{aligned} V(k) &= V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) \\ &\quad + V_6(k) + V_7(k) + V_8(k), \end{aligned} \tag{10}$$

where

$$\begin{aligned} V_1(k) &= x^T(k) P x(k), \\ V_2(k) &= \sum_{i=k-d_k}^{k-1} x^T(i) Q_1 x(i), \\ V_3(k) &= \sum_{i=k-d_M}^{k-1} x^T(i) Q_2 x(i), \\ V_4(k) &= \sum_{i=k-d_m}^{k-1} x^T(i) Q_3 x(i), \\ V_5(k) &= \sum_{j=-d_M+1}^{-d_m} \sum_{i=k+j}^{k-1} x^T(i) Q_1 x(i), \\ V_6(k) &= d_M \sum_{i=-d_M}^{-1} \sum_{m=k+i}^{k-1} \eta^T(m) U_1 \eta(m), \\ V_7(k) &= (d_M - d_m) \sum_{i=-d_M}^{-d_m-1} \sum_{m=k+i}^{k-1} \eta^T(m) U_2 \eta(m), \\ V_8(k) &= d_m \sum_{i=-d_m}^{-1} \sum_{m=k+i}^{k-1} \eta^T(m) U_3 \eta(m), \\ \eta(k) &= x(k+1) - x(k), \end{aligned}$$

and $P = P^T > 0,\ Q_i = Q_i^T \geq 0,\ U_i = U_i^T > 0\ (i = 1,\ 2,\ 3)$ are matrices to be determined. From Lemma 3, it yields that

$$\begin{aligned} &-d_M \sum_{l=k-d_M}^{k-1} \eta^T(l) U_1 \eta(l) \\ &\leq -\Big(\sum_{l=k-d_M}^{k-1} \eta(l)\Big)^T U_1 \Big(\sum_{l=k-d_M}^{k-1} \eta(l)\Big) \\ &= -[x(k) - x(k-d_M)]^T U_1 [x(k) - x(k-d_M)], \end{aligned} \tag{11}$$

and

$$-(d_M-d_m)\sum_{l=k-d_M}^{k-d_m-1}\eta^T(l)U_2\eta(l)$$
$$=-(d_M-d_m)\sum_{l=k-d_k}^{k-d_m-1}\eta^T(l)U_2\eta(l)$$
$$-(d_M-d_m)\sum_{l=k-d_M}^{k-d_k-1}\eta^T(l)U_2\eta(l), \tag{12}$$

$$-(d_M-d_m)\sum_{l=k-d_k}^{k-d_m-1}\eta^T(l)U_2\eta(l)$$
$$\le -\Big(\sum_{l=k-d_k}^{k-d_m-1}\eta(l)\Big)^T U_2\Big(\sum_{l=k-d_k}^{k-d_m-1}\eta(l)\Big)$$
$$=-[x(k-d_m)-x(k-d_k)]^T U_2[x(k-d_m)-x(k-d_k)], \tag{13}$$

$$-(d_M-d_m)\sum_{l=k-d_M}^{k-d_k-1}\eta^T(l)U_2\eta(l)$$
$$\le -\Big(\sum_{l=k-d_M}^{k-d_k-1}\eta(l)\Big)^T U_2\Big(\sum_{l=k-d_M}^{k-d_k-1}\eta(l)\Big)$$
$$=-[x(k-d_k)-x(k-d_M)]^T U_2[x(k-d_k)-x(k-d_M)], \tag{14}$$

$$-d_m\sum_{l=k-d_m}^{k-1}\eta^T(l)U_3\eta(l)$$
$$\le -\Big(\sum_{l=k-d_m}^{k-1}\eta(l)\Big)^T U_3\Big(\sum_{l=k-d_m}^{k-1}\eta(l)\Big)$$
$$=-[x(k)-x(k-d_m)]^T U_3[x(k)-x(k-d_m)]. \tag{15}$$

Define $\Delta V(k)=V(k+1)-V(k)$, then along the solution of (1) we have

$$\Delta V_1(k)=x^T(k+1)Px(k+1)-x^T(k)Px(k)$$
$$=[Ax(k)+A_d(k-d_k)]^T P[Ax(k)+A_d(k-d_k)]$$
$$-x^T(k)Px(k), \tag{16}$$
$$\Delta V_2(k)\le x^T(k)Q_1x(k)-x^T(k-d_k)Q_1x(k-d_k)$$
$$+\sum_{i=k-d_M+1}^{k-d_m}x^T(i)Q_1x(i), \tag{17}$$
$$\Delta V_3(k)=x^T(k)Q_2x(k)-x^T(k-d_M)Q_2x(k-d_M), \tag{18}$$
$$\Delta V_4(k)=x^T(k)Q_3x(k)-x^T(k-d_m)Q_3x(k-d_m), \tag{19}$$
$$\Delta V_5(k)=(d_M-d_m)x^T(k)Q_1x(k)-\sum_{i=k-d_M+1}^{k-d_m}x^T(i)Q_1x(i), \tag{20}$$
$$\Delta V_6(k)=d_M^2\eta^T(k)U_1\eta(k)-d_M\sum_{m=k-d_M}^{k-1}\eta^T(m)U_1\eta(m)$$
$$\le d_M^2[(A-I)x(k)+A_d(k-d_k)]^T U_1$$
$$\times[(A-I)x(k)+A_d(k-d_k)]$$
$$-[x(k)-x(k-d_M)]^T U_1[x(k)-x(k-d_M)], \tag{21}$$

$$\Delta V_7(k)=(d_M-d_m)\sum_{i=-d_M}^{-d_m-1}[\eta^T(k)U_2\eta(k)$$
$$-\eta^T(k+i)U_2\eta(k+i)]$$
$$=(d_M-d_m)^2\eta^T(k)U_2\eta(k)$$
$$-(d_M-d_m)\sum_{m=k-d_M}^{k-d_m-1}\eta^T(m)U_2\eta(m)$$
$$\le(d_M-d_m)^2[(A-I)x(k)+A_dx(k-d_k)]^T U_2$$
$$\times[(A-I)x(k)+A_dx(k-d_k)]$$
$$-[x(k-d_m)-x(k-d_k)]^T U_2[x(k-d_m)-x(k-d_k)]$$
$$-[x(k-d_k)-x(k-d_M)]^T U_2[x(k-d_k)-x(k-d_M)], \tag{22}$$

$$\Delta V_8(k)=d_m\sum_{i=-d_m}^{-1}[\eta^T(k)U_3\eta(k)-\eta^T(k+i)U_3\eta(k+i)]$$
$$=d_m^2\eta^T(k)U_3\eta(k)-d_m\sum_{m=k-d_m}^{k-1}\eta^T(m)U_3\eta(m)$$
$$\le d_m^2[(A-I)x(k)+A_d(k-d_k)]^T U_3$$
$$\times[(A-I)x(k)+A_d(k-d_k)]$$
$$-[x(k)-x(k-d_m)]^T U_3[x(k)-x(k-d_m)]. \tag{23}$$

Thus, it follows

$$\Delta V(k)\le \zeta^T(k)\Lambda\zeta(k), \tag{24}$$

where

$$\zeta(k)=\begin{bmatrix} x^T(k) & x^T(k-d_k) & x^T(k-d_m) & x^T(k-d_M)\end{bmatrix}^T,$$

and Λ is defined in (9). Therefore, from (9) the asymptotic stability of system (1)-(2) is established. ■

Remark 1. By using the discrete Jensen inequality, Theorem 1 presents a new LMI-based stability criterion for discrete system (1)-(2). It is obvious that the number of decision variables contained in (9) is less than the one in Lemma 2. In addition, the stability condition in Theorem 1 is also less conservative than those in Lemma 1 and Lemma 2, which will be proved in the next section.

B. Uncertain Systems

In this subsection, we will extend Theorem 1 to the case of uncertain systems. If the matrices A, A_d in the dynamic equation (1) have the following form:

$$A=A_0+\Delta A,\quad A_d=A_{d0}+\Delta A_d \tag{25}$$
$$\begin{bmatrix}\Delta A & \Delta A_d\end{bmatrix}=G\Delta(k)\begin{bmatrix}H_1 & H_2\end{bmatrix} \tag{26}$$

where A_0, A_{d0}, G, H_1, H_2 are known constant matrices of appropriate dimensions. $\Delta(k)$ is a real uncertain matrix function with Lebesgue measurable elements satisfying

$$\Delta^T(k)\Delta(k)\le I, \tag{27}$$

then we have the following robust stability criterion for uncertain systems. The proof follows similar lines as in [14], so it is omitted here.

Theorem 2. System (1)-(2) with (25)-(27) is robustly asymptotically stable if there exist matrices $P = P^T > 0$, $Q_i = Q_i^T \geq 0$, $U_i = U_i^T > 0$ $(i = 1,\ 2,\ 3)$ and scalar $\varepsilon > 0$ satisfying

$$\begin{bmatrix} \Theta_1 & \varepsilon H_1^T H_2 & U_3 & U_1 & A_0^T P & (A_0 - I)^T U & 0 \\ * & \Theta_2 & U_2 & U_2 & A_{d0}^T P & A_{d0}^T U & 0 \\ * & * & \Theta_3 & 0 & 0 & 0 & 0 \\ * & * & * & \Theta_4 & 0 & 0 & 0 \\ * & * & * & * & -P & 0 & PG \\ * & * & * & * & * & -U & UG \\ * & * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0, \tag{28}$$

where

$$\begin{aligned} \Theta_1 &= -P + (d_M - d_m + 1)Q_1 + Q_2 + Q_3 - (U_1 + U_3) \\ &\quad + \varepsilon H_1^T H_1, \\ \Theta_2 &= -Q_1 - 2U_2 + \varepsilon H_2^T H_2, \\ \Theta_3 &= -Q_3 - U_2 - U_3, \\ \Theta_4 &= -Q_2 - U_1 - U_2, \\ U &= d_M^2 U_1 + (d_M - d_m)^2 U_2 + d_m^2 U_3. \end{aligned}$$

IV. COMPARISON WITH THE EXISTING RESULTS

In this section, we will prove that the stability condition in Theorem 1 is less conservative that those in Lemma 1 and Lemma 2. At first, we prove that Lemma 2 is less conservative than Lemma 1.

Theorem 3. If inequalities (3) and (4) in Lemma 1 are feasible, then inequality (5) in Lemma 2 is also feasible.

Proof: If $\Upsilon < 0$ in Lemma 1 is true, then

$$\Delta \Upsilon \Delta^T = \begin{bmatrix} \Upsilon_1 & -Y & A^T P & d_M (A - I)^T Z \\ * & -Q & A_d^T P & d_M A_d^T Z \\ * & * & -P & 0 \\ * & * & * & -d_M Z \end{bmatrix} < 0, \tag{29}$$

where

$$\Delta = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & d_M I & 0 & 0 \end{bmatrix},$$

and Υ, Υ_1 are defined in (3). By the Schur complement, it follows that

$$\begin{bmatrix} \Upsilon_1 + A^T P A & A^T P A_d - Y & d_M (A - I)^T Z \\ * & A_d^T P A_d - Q & d_M A_d^T Z \\ * & * & -d_M Z \end{bmatrix} < 0. \tag{30}$$

From (4) and by using the Schur complement, it is obvious that

$$X \geq Y Z^{-1} Y^T, \tag{31}$$

so, from (30) it yields

$$\Upsilon_2 = \begin{bmatrix} \Upsilon_3 & A^T P A_d - Y + d_M (A - I)^T Z A_d & \sqrt{d_M} Y \\ * & A_d^T P A_d - Q + d_M A_d^T Z A_d & 0 \\ * & * & -Z \end{bmatrix} < 0, \tag{32}$$

where

$$\begin{aligned} \Upsilon_3 &= -P + Y + Y^T + (d_M - d_m + 1)Q + A^T P A \\ &\quad + d_M (A - I)^T Z (A - I). \end{aligned}$$

Thus, there exists a small enough positive scalar ε, such that

$$\Upsilon_2 + \varepsilon \begin{bmatrix} I + d_M (A - I)^T (A - I) & d_M (A - I)^T A_d & 0 \\ * & d_M A_d^T A_d & 0 \\ * & * & 0 \end{bmatrix} < 0, \tag{33}$$

and this implies that $\Xi < 0$ in Lemma 2 is also true by setting

$$R = \varepsilon I,\ Z_1 = Z,\ Z_2 = \varepsilon I,\ M = \begin{bmatrix} Y \\ 0 \\ 0 \end{bmatrix},\ S = N = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and this completes the proof. ■

In the following, we prove that Theorem 1 is less conservative than Lemma 2.

Theorem 4. If inequality (5) in Lemma 2 is feasible, then inequality (9) in Theorem 1 is also feasible.

Proof: Since

$$\Pi \Xi \Pi^T = \begin{bmatrix} \Gamma & \Xi_6 \\ * & \Xi_5 \end{bmatrix}, \tag{34}$$

where

$$\Pi = \begin{bmatrix} I & 0 & 0 & -\frac{1}{\sqrt{d_M}} I & 0 & -\frac{1}{\sqrt{d_M}} I \\ 0 & I & 0 & \frac{1}{\sqrt{d_M}} I & -\frac{1}{\sqrt{d_M - d_m}} I & 0 \\ 0 & 0 & I & 0 & \frac{1}{\sqrt{d_M - d_m}} I & \frac{1}{\sqrt{d_M}} I \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$\Xi_6 = \begin{bmatrix} \sqrt{d_M} M_1 + \frac{1}{\sqrt{d_M}} Z_1 & \Delta_1 & \sqrt{d_M} N_1 + \frac{1}{\sqrt{d_M}} Z_2 \\ \sqrt{d_M} M_2 - \frac{1}{\sqrt{d_M}} Z_1 & \Delta_2 & \sqrt{d_M} N_2 \\ \sqrt{d_M} M_3 & \Delta_3 & \sqrt{d_M} N_3 - \frac{1}{\sqrt{d_M}} Z_2 \end{bmatrix},$$

$$\begin{aligned} \Delta_1 &= \sqrt{d_M - d_m} S_1, \\ \Delta_2 &= \sqrt{d_M - d_m} S_2 + \tfrac{1}{\sqrt{d_M - d_m}} Z_1, \\ \Delta_3 &= \sqrt{d_M - d_m} S_3 - \tfrac{1}{\sqrt{d_M - d_m}} Z_1, \end{aligned}$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & d_M^{-1} Z_2 \\ * & \Gamma_{22} & (d_M - d_m)^{-1} Z_1 \\ * & * & \Gamma_{33} \end{bmatrix},$$

$$\begin{aligned} \Gamma_{11} &= A^T P A - P + (d_M - d_m + 1)Q + R - d_M^{-1}(Z_1 + Z_2) \\ &\quad + d_M (A - I)^T (Z_1 + Z_2)(A - I), \\ \Gamma_{12} &= A^T P A_d + d_M^{-1} Z_1 + d_M (A - I)^T (Z_1 + Z_2) A_d, \\ \Gamma_{22} &= A_d^T P A_d - Q - d_M^{-1} Z_1 - (d_M - d_m)^{-1} Z_1 \\ &\quad + d_M A_d^T (Z_1 + Z_2) A_d, \\ \Gamma_{33} &= -R - (d_M - d_m)^{-1} Z_1 - d_M^{-1} Z_2, \end{aligned}$$

Ξ and Ξ_5 are defined in Lemma 2, it is very easy to see that $\Gamma < 0$ holds if $\Xi < 0$.

By the Schur complement, it is obvious that $\Lambda < 0$ is equivalent to $\bar{\Lambda} < 0$, where

$$\bar{\Lambda} = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} & U_1 \\ * & \bar{\Lambda}_{22} & U_2 \\ * & * & -Q_2 - U_1 - U_2 \end{bmatrix},$$

$$\begin{aligned} \bar{\Lambda}_{11} &= \Lambda_{11} + U_3 (Q_3 + U_2 + U_3)^{-1} U_3, \\ \bar{\Lambda}_{12} &= \Lambda_{12} + U_3 (Q_3 + U_2 + U_3)^{-1} U_2, \\ \bar{\Lambda}_{22} &= \Lambda_{22} + U_2 (Q_3 + U_2 + U_3)^{-1} U_2, \end{aligned}$$

and Λ, Λ_{11}, Λ_{12} and Λ_{22} are defined in Theorem 1.

(I) For the case of $d_m > 0$, taking

$$Q_1 = Q,\ Q_2 = R,\ Q_3 = 0,\ d_M U_1 = Z_2,$$
$$(d_M - d_m)U_2 = Z_1,\ d_m U_3 = Z_1,$$

then it follows that

$$\begin{aligned} U_2 + U_3 &= (d_M - d_m)^{-1} Z_1 + d_m^{-1} Z_1 \\ &= d_M [d_m (d_M - d_m)]^{-1} Z_1, \end{aligned} \tag{35}$$

and

$$d_M^{-1} Z_1 - U_3 + U_3 (U_2 + U_3)^{-1} U_3 = 0, \tag{36}$$
$$U_3 (U_2 + U_3)^{-1} U_2 - d_M^{-1} Z_1 = 0, \tag{37}$$
$$d_M^{-1} Z_1 + U_2 (U_2 + U_3)^{-1} U_2 - U_2 = 0, \tag{38}$$

so we have

$$\begin{aligned} \bar{\Lambda} - \Gamma &= \begin{bmatrix} \Delta_4 & U_3 (U_2 + U_3)^{-1} U_2 - d_M^{-1} Z_1 & 0 \\ * & d_M^{-1} Z_1 + U_2 (U_2 + U_3)^{-1} U_2 - U_2 & 0 \\ * & * & 0 \end{bmatrix} \\ &= 0, \end{aligned} \tag{39}$$

where

$$\Delta_4 = d_M^{-1} Z_1 - U_3 + U_3 (U_2 + U_3)^{-1} U_3.$$

Therefore, $\bar{\Lambda} < 0$ is true if $\Gamma < 0$ holds, this implies that $\Lambda < 0$ in Theorem 1 is feasible if $\Xi < 0$ in Lemma 2 holds.

(II) For the case of $d_m = 0$, if $\Gamma < 0$ holds, then there exists a small enough positive scalar $0 < \varepsilon < 1$ such that

$$\Gamma_\varepsilon = \Gamma + \varepsilon \begin{bmatrix} U_2 & 0 & 0 \\ * & U_2 & 0 \\ * & * & U_2 \end{bmatrix} < 0. \tag{40}$$

So, by taking

$$Q_1 = Q,\ Q_2 = R,\ Q_3 = 0,\ d_M U_1 = Z_2,\ d_M U_2 = Z_1,$$

we have

$$\bar{\Lambda} - \Gamma_\varepsilon = \begin{bmatrix} \Delta_5 & U_3 (U_2 + U_3)^{-1} U_2 - U_2 & 0 \\ * & U_2 (U_2 + U_3)^{-1} U_2 - \varepsilon U_2 & 0 \\ * & * & -\varepsilon U_2 \end{bmatrix}, \tag{41}$$

where

$$\Delta_5 = U_3 (U_2 + U_3)^{-1} U_3 - U_3 + U_2 - \varepsilon U_2.$$

Noting that

$$U_3 (U_2 + U_3)^{-1} U_3 = U_3 - U_2 + U_2 (U_2 + U_3)^{-1} U_2, \tag{42}$$

and for any positive definite matrix Z, the following inequality

$$\begin{bmatrix} -Z & -Z \\ * & -Z \end{bmatrix} \leq 0 \tag{43}$$

is true, so if taking

$$U_3 = (4\varepsilon^{-1} - 1) U_2, \tag{44}$$

then it yields that

$$U_2 + U_3 = 4\varepsilon^{-1} U_2, \tag{45}$$
$$U_2 (U_2 + U_3)^{-1} U_2 = \frac{1}{4} \varepsilon U_2, \tag{46}$$

TABLE I

ALLOWABLE UPPER BOUND OF d_M FOR GIVEN d_m

Methods	$d_m = 2$	$d_m = 4$	$d_m = 6$	$d_m = 10$	$d_m = 12$
Lemma 1	7	8	9	12	13
Lemma 2	13	13	14	15	16
Theorem 1	13	13	14	17	18

thus,

$$\begin{aligned} \bar{\Lambda} - \Gamma_\varepsilon &= \begin{bmatrix} -\frac{3}{4}\varepsilon U_2 & -\frac{1}{4}\varepsilon U_2 & 0 \\ * & -\frac{3}{4}\varepsilon U_2 & 0 \\ * & * & -\varepsilon U_2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}\varepsilon U_2 & 0 & 0 \\ * & -\frac{1}{2}\varepsilon U_2 & 0 \\ * & * & -\varepsilon U_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} -\frac{1}{4}\varepsilon U_2 & -\frac{1}{4}\varepsilon U_2 & 0 \\ * & -\frac{1}{4}\varepsilon U_2 & 0 \\ * & * & 0 \end{bmatrix} \\ &\leq \begin{bmatrix} -\frac{1}{2}\varepsilon U_2 & 0 & 0 \\ * & -\frac{1}{2}\varepsilon U_2 & 0 \\ * & * & -\varepsilon U_2 \end{bmatrix} \\ &< 0. \end{aligned} \tag{47}$$

This means that $\bar{\Lambda} < 0$ is feasible if $\Gamma < 0$ holds. Since $\Lambda < 0$ is equivalent to $\bar{\Lambda} < 0$, then $\Lambda < 0$ in Theorem 1 is also feasible if $\Xi < 0$ in Lemma 2 holds. ■

Remark 2. From the proof of Theorem 4, it is easy to see that $\Gamma < 0$ is equivalent to $\Xi < 0$, and it is worth to point out that Γ is simpler than Ξ.

Remark 3. From Theorem 3 and Theorem 4, it is easy to see that Theorem 1 in this paper is an improvement on Lemma 1 and Lemma 2, respectively. Similarly, it can be shown that the robust stability condition in Theorem 2 is less conservative than the corresponding results in [13] and [14], the proof is omitted here.

V. ILLUSTRATIVE EXAMPLES

In this section, two examples are provided to illustrate the advantage of the proposed stability results. Example 1 is used to show the merits of Theorem 1 for nominal systems, and Example 2 is used to show the merits of Theorem 2 for uncertain systems.

Example 1. [13] Consider the following system

$$x(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} x(k - d_k), \tag{48}$$

where d_k represents a time-varying state delay. The upper bounds on the time delay, d_M, which guarantee the stability of the system (1) for given lower bounds, d_m, are shown in Table 1. It is clear that the results obtained by Theorem 1 are less conservative than the ones obtained in [13] and in [14].

TABLE II

ALLOWABLE UPPER BOUND OF $\bar{\alpha}$ FOR GIVEN d_m AND d_M

Methods	$2 \le d_k \le 7$	$5 \le d_k \le 8$	$8 \le d_k \le 13$
Theorem 5 [14]	0.0661	0.5195	*infeasible*
Corollary 2 [13]	4.0365	4.0958	0.5420
Theorem 2	4.1372	4.7270	1.3145

Example 2. Consider the following uncertain discrete-time system with a time-varying delay in the state:

$$\begin{aligned} x(k+1) = &\left(\begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix} + \alpha(k) \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix} \right) x(k) \\ &+ \left(\begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} + \alpha(k) \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \right) x(k-d_k), \end{aligned} \tag{49}$$

where $|\alpha(k)| \le \bar{\alpha}$. The system matrices can be written in the form of (25)-(27) with matrices given by

$$\begin{aligned} A_0 &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad & A_{d0} &= \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \\ H_1 &= \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, & H_2 &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \\ G &= \bar{\alpha} I, & \Delta(k) &= \alpha(k)/\bar{\alpha}. \end{aligned}$$

For given d_m and d_M, our purpose is to determine the upper bound of $\bar{\alpha}$ such that the above system is asymptotically stable. The detailed comparison is given in Table 2. From Table 2, we can see that the robust stability condition presented in this note is much less conservative than that in [13] and [14].

VI. CONCLUSION

This paper studies the problem of stability for discrete-time delay systems. By defining new Lyapunov functional and using the discrete Jensen inequality, LMI-based stability conditions are derived. The presented stability conditions are proved to be less conservative than the existing ones. Meanwhile, the computational complexity of the obtained results is reduced greatly since less decision variables are involved in the stability conditions. Numerical examples have illustrated the merits and effectiveness of the proposed method.

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