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Jensen polynomials for the Riemann zeta function and other sequences

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This contribution is part of the special series of Inaugural Articles by members of the National Academy of Sciences elected in 2017.

Edited by Kenneth A. Ribet, University of California, Berkeley, CA, and approved April 8, 2019 (received for review February 12, 2019)

In 1927, Pólya proved that the Riemann hypothesis is equivalent to the hyperbolicity of Jensen polynomials for the Riemann zeta function $\zeta(s)$ at its point of symmetry. This hyperbolicity has been proved for degrees $d \leq 3$. We obtain an asymptotic formula for the central derivatives $\zeta^{(2n)}(1/2)$ that is accurate to all orders, which allows us to prove the hyperbolicity of all but finitely many of the Jensen polynomials of each degree. Moreover, we establish hyperbolicity for all $d \leq 8$. These results follow from a general theorem which models such polynomials by Hermite polynomials. In the case of the Riemann zeta function, this proves the Gaussian unitary ensemble random matrix model prediction in derivative aspect. The general theorem also allows us to prove a conjecture of Chen, Jia, and Wang on the partition function.

Riemann hypothesis | Jensen polynomials | hyperbolic polynomials

1. Introduction and Statement of Results

Expanding on notes of Jensen, Pólya (1) proved that the Riemann hypothesis (RH) is equivalent to the hyperbolicity of the Jensen polynomials for the Riemann zeta function $\zeta(s)$ at its point of symmetry. More precisely, he showed that the RH is equivalent to the hyperbolicity of all Jensen polynomials associated with the sequence of Taylor coefficients $\{\gamma(n)\}$ defined by

$$(-1+4z^2)\,\Lambda\left(\frac{1}{2}+z\right) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!}\cdot z^{2n}, \qquad [1]$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$, where we say that a polynomial with real coefficients is hyperbolic if all of its zeros are real, and where the Jensen polynomial of degree d and shift n of an arbitrary sequence $\{\alpha(0), \alpha(1), \alpha(2), \dots\}$ of real numbers is the polynomial

$$J_\alpha^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j) X^j. \qquad [2]$$

Thus, the RH is equivalent to the hyperbolicity of the polynomials $J_\gamma^{d,n}(X)$ for all nonnegative integers d and n (1–3). Since this condition is preserved under differentiation, to prove RH, it would be enough to show hyperbolicity for the $J_\gamma^{d,0}(X)$.* Due to the difficulty of proving RH, research has focused on establishing hyperbolicity for all shifts n for small d. Previous to this paper, hyperbolicity was known for $d \leq 3$ by work† of Csordas et al. (5) and Dimitrov and Lucas (3).

Asymptotics for the $\gamma(n)$ were obtained from Coffey (6) and Pustyl'nikov (7). We improve on their results by obtaining an arbitrary precision asymptotic formula‡ (*Theorem 9*), a result that is of independent interest. We will use this strengthened result to prove the following theorem for all degrees d.

Significance

The Pólya–Jensen criterion for the Riemann hypothesis asserts that *RH* is equivalent to the hyperbolicity of certain Jensen polynomials for all degrees $d \geq 1$ and all shifts n. For each degree $d \geq 1$, we confirm this criterion for all sufficiently large shifts n. This represents a theoretical advance in the field. The method of proof is rooted in the newly discovered phenomenon that these polynomials are nicely approximated by Hermite polynomials. Furthermore, it is shown that this method applies to a large class of related problems.

Author contributions: M.G., K.O., L.R., and D.Z. wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

See Commentary on page 11085.

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Published online May 21, 2019.

*The hyperbolicity for $J_\gamma^{d,0}(X)$ has been confirmed for $d \leq 2 \cdot 10^{17}$ by Chasse (cf. theorem 1.8 of ref. 4).

†These works use a slightly different normalization for the $\gamma(n)$.

‡Our results imply the results in refs. 6 and 7 after typographical errors are corrected.

Theorem 1. *If $d \geq 1$, then $J_{\gamma}^{d,n}(X)$ is hyperbolic for all sufficiently large n.*

An effective proof of *Theorem 1* for small d gives the following theorem.

Theorem 2. *If $1 \leq d \leq 8$, then $J_{\gamma}^{d,n}(X)$ is hyperbolic for every $n \geq 0$.*

Theorem 1 follows from a general phenomenon that Jensen polynomials for a wide class of sequences α can be modeled by the Hermite polynomials $H_d(X)$, which we define (in a somewhat nonstandard normalization) as the orthogonal polynomials for the measure $\mu(X) = e^{-X^2/4}$ or more explicitly by the generating function

$$\sum_{d=0}^{\infty} H_d(X) \frac{t^d}{d!} = e^{-t^2+Xt} = 1 + X\,t + (X^2-2)\frac{t^2}{2!} + (X^3-6X)\frac{t^3}{3!} + \cdots . \qquad [3]$$

More precisely, we will prove the following general theorem describing the limiting behavior of Jensen polynomials of sequences with appropriate growth.

Theorem 3. *Let $\{\alpha(n)\}$, $\{A(n)\}$, and $\{\delta(n)\}$ be three sequences of positive real numbers with $\delta(n)$ tending to zero and satisfying*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) = A(n)j - \delta(n)^2 j^2 + \mathrm{o}\left(\delta(n)^d\right) \qquad \text{as } n \to \infty, \qquad [4]$$

for some integer $d \geq 1$ and all $0 \leq j \leq d$. Then, we have

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)}\, J_{\alpha}^{d,n}\left(\frac{\delta(n)\,X\,-\,1}{\exp(A(n))}\right)\right) = H_d(X), \qquad [5]$$

uniformly for X in any compact subset of $\mathbb{R}$.

Since the Hermite polynomials have distinct roots, and since this property of a polynomial with real coefficients is invariant under small deformation, we immediately deduce the following corollary.

Corollary 4. *The Jensen polynomials $J_{\alpha}^{d,n}(X)$ for a sequence $\alpha\colon \mathbb{N} \to \mathbb{R}$ satisfying the conditions in Theorem 3 are hyperbolic for all but finitely many values n.*

Theorem 1 is a special case of this corollary. Namely, we shall use *Theorem 9* to prove that the Taylor coefficients $\{\gamma(n)\}$ satisfy the required growth conditions in *Theorem 3* for every $d \geq 2$.

Theorem 3 in the case of the Riemann zeta function is the derivative aspect Gaussian unitary ensemble (GUE) random matrix model prediction for the zeros of Jensen polynomials. To make this precise, recall that Dyson (8), Montgomery (9), and Odlyzko (10) conjecture that the nontrivial zeros of the Riemann zeta function are distributed like the eigenvalues of random Hermitian matrices. These eigenvalues satisfy Wigner's Semicircular Law, as do the roots of the Hermite polynomials $H_d(X)$, when suitably normalized, as $d \to +\infty$ (see chapter 3 of ref. 11). The roots of $J_{\gamma}^{d,0}(X)$, as $d \to +\infty$, approximate the zeros of $\Lambda\left(\frac{1}{2}+z\right)$ (see ref. 1 or lemma 2.2 of ref. 12), and so GUE predicts that these roots also obey the Semicircular Law. Since the derivatives of $\Lambda\left(\frac{1}{2}+z\right)$ are also predicted to satisfy GUE, it is natural to consider the limiting behavior of $J_{\gamma}^{d,n}(X)$ as $n \to +\infty$. The work here proves that these derivative aspect limits are the Hermite polynomials $H_d(X)$, which, as mentioned above, satisfy GUE in degree aspect.

Returning to the general case of sequences with suitable growth conditions, *Theorem 3* has applications in combinatorics where the hyperbolicity of polynomials determines the log-concavity of enumerative statistics. For example, see the classic theorem by Heilmann and Leib (13), along with works by Chudnovsky and Seymour (14), Haglund (15), Haglund et al. (16), Stanley (17), and Wagner (18), to name a few. *Theorem 3* represents a criterion for establishing the hyperbolicity of polynomials in enumerative combinatorics. The theorem reduces the problem to determining whether suitable asymptotics hold. Here, we were motivated by a conjecture of Chen, Jia, and Wang concerning the Jensen polynomials $J_p^{d,n}(X)$, where $p(n)$ is the partition function. Nicolas (19) and Desalvo and Pak (20) proved that $J_p^{2,n}(X)$ is hyperbolic for $n \geq 25$, and, more recently, Chen et al. proved (21) that $J_p^{3,n}(X)$ is hyperbolic for $n \geq 94$, inspiring them to state as a conjecture the following result.

Theorem 5 (Chen–Jia–Wang Conjecture). *For every integer $d \geq 1$, there exists an integer $N(d)$ such that $J_p^{d,n}(X)$ is hyperbolic for $n \geq N(d)$.*

Table 1 gives the conjectured minimal value for $N(d)$ for $d = 2^j$ with $1 \leq j \leq 5$. More precisely, for each $d \leq 32$ it gives the smallest integer such that $J_p^{d,n}(X)$ is hyperbolic for $N(d) \leq n \leq 50{,}000$.

Remark 6: Larson and Wagner (22) have made the proof of *Theorem 5* effective by a brute-force implementation of Hermite's criterion (see theorem C of ref. 3). They showed that the values in the table are correct for $d = 4$ and $d = 5$ and that $N(d) \leq (3d)^{24d}(50d)^{3d^2}$ in general. The true values are presumably much smaller and are probably of only polynomial growth, the numbers $N(d)$ in the table being approximately of size $10\,d^2 \log d$.

Table 1. Conjectured minimal values of *N*(*d*)

d	1	2	4	8	16	32
N(*d*)	1	25	206	1,269	6,917	35,627

Table 2. Comparison of $\gamma(n)$ and $\widehat{\gamma}(n)$

n		$\widehat{\gamma}(n)$			$\gamma(n)$			$\gamma(n)/\widehat{\gamma}(n)$
10	$\approx$	1.6313374394	$\times 10^{-17}$	$\approx$	1.6323380490	$\times 10^{-17}$	$\approx$	1.000613367
100	$\approx$	6.5776471904	$\times 10^{-205}$	$\approx$	6.5777263785	$\times 10^{-205}$	$\approx$	1.000012038
1,000	$\approx$	3.8760333086	$\times 10^{-2567}$	$\approx$	3.8760340890	$\times 10^{-2567}$	$\approx$	1.000000201
10,000	$\approx$	3.5219798669	$\times 10^{-32265}$	$\approx$	3.5219798773	$\times 10^{-32265}$	$\approx$	1.000000002
100,000	$\approx$	6.3953905598	$\times 10^{-397097}$	$\approx$	6.3953905601	$\times 10^{-397097}$	$\approx$	1.000000000

Theorem 5 suggests a natural generalization. As is well known, the numbers $p(n)$ are the Fourier coefficients of a modular form, namely,

$$\frac{1}{\eta(\tau)} = \sum_{n=0}^{\infty} p(n)\, q^{n-\frac{1}{24}} \qquad (\Im(\tau)>0,\ q=e^{2\pi i\tau}), \tag{6}$$

where $\eta(\tau)=q^{1/24}\prod(1-q^n)$ is the Dedekind eta-function. *Theorem 5* is then an example of a more general theorem about the Jensen polynomials of the Fourier coefficients of an arbitrary weakly holomorphic modular form, which, for the purposes of this work, will mean a modular form (possibly of fractional weight and with multiplier system) with real Fourier coefficients on the full modular group $\mathrm{SL}_2(\mathbb{Z})$ that is holomorphic apart from a pole of (possibly fractional) positive order at infinity. If f is such a form, we denote its Fourier expansion by[§]

$$f(\tau) = \sum_{n\in -m+\mathbb{Z}_{\geq 0}} a_f(n)\, q^n \qquad (m\in\mathbb{Q}_{>0},\ a_f(-m)\neq 0). \tag{7}$$

Then, we will prove the following theorem, which includes *Theorem 5*.

Theorem 7. *If f is a weakly holomorphic modular form as above, then for any fixed $d\geq 1$, the Jensen polynomials $J_{a_f}^{d,n}(X)$ are hyperbolic for all sufficiently large n.*

Our results are proved by showing that each of the sequences of interest to us [the partition function, the Fourier coefficients of weakly holomorphic modular forms, and the Taylor coefficients at $s=\frac{1}{2}$ of $4s(1-s)\Lambda(s)$] satisfies the hypotheses of *Theorem 3*, which we prove in Section 2. Actually, in Section 2, we prove a more general result (*Theorem 8*) that gives the limits of suitably normalized Jensen polynomials for an even bigger class of sequences having suitable asymptotic properties (but without necessarily the corollary about hyperbolicity). *Theorem 7* giving the hyperbolicity for coefficients of modular forms (and hence also for the partition function) is proved in Section 3. In Section 4, we prove *Theorem 9*, which gives an asymptotic formula to all orders for the Taylor coefficients of $\Lambda(s)$ at $s=\frac{1}{2}$, and in Section 5, we prove *Theorems 1* and *2* for the Riemann zeta function by using these asymptotics to verify that the hypotheses of *Theorem 3* are fulfilled by the numbers $\gamma(n)$. We conclude in Section 6 with some numerical examples.

2. Proof of Theorem 3

We deduce *Theorem 3* from the following more general result.

Theorem 8. *Suppose that $\{E(n)\}$ and $\{\delta(n)\}$ are positive real sequences with $\delta(n)$ tending to 0, and that $F(t)=\sum_{i=0}^{\infty} c_i t^i$ is a formal power series with complex coefficients. For a fixed $d\geq 1$, suppose that there are real sequences $\{C_0(n)\},\ldots,\{C_d(n)\}$, with $\lim_{n\to+\infty} C_i(n)=c_i$ for $0\leq i\leq d$, such that for $0\leq j\leq d$, we have*

$$\frac{\alpha(n+j)}{\alpha(n)}\, E(n)^{-j} = \sum_{i=0}^{d} C_i(n)\,\delta(n)^i j^i + \mathrm{o}\left(\delta(n)^d\right) \qquad \text{as } n\to+\infty. \tag{8}$$

Then, the conclusion of Theorem 3 holds with $\exp(A(n))$ replaced by $E(n)$ and $H_d(X)$ replaced by $H_{F,d}(X)$, where the polynomials $H_{F,m}(X)\in\mathbb{C}[x]$ are now defined either by the generating function $F(-t)\,e^{Xt}=\sum H_{F,m}(X)\,t^m/m!$ or in closed form by $H_{F,m}(X):=m!\sum_{k=0}^{m}(-1)^{m-k}c_{m-k}\,X^k/k!$.

Proof of Theorems 8 and 3. After replacing $\exp(A(n))$ by $E(n)$, the polynomial appearing on the left-hand side of **[5]** becomes

$$\frac{\delta(n)^{-d}}{\alpha(n)}\, J_\alpha^{d,n}\left(\frac{\delta(n)\,X-1}{E(n)}\right) = \sum_{k=0}^{d}\binom{d}{k}\left[\delta(n)^{k-d}\sum_{j=k}^{d}(-1)^{j-k}\binom{d-k}{j-k}\frac{\alpha(n+j)}{\alpha(n)E(n)^j}\right]X^k.$$

Since $0\leq j\leq d$, and since the error term in **[8]** is $\mathrm{o}(\delta(n)^d)$, we may reorder summation and find that the limiting value as $n\to+\infty$ of the quantity in square brackets satisfies

$$\lim_{n\to+\infty}\left[\sum_{i=0}^{d} C_i(n)\,\delta(n)^{k-d+i}\sum_{j=k}^{d}(-1)^{j-k}\binom{d-k}{j-k}j^i\right] = (-1)^{d-k}(d-k)!\,c_{d-k},$$

[§]Note that with these notations we have $p(n)=a_f(n-\frac{1}{24})$ for $f=1/\eta$, but making this shift of argument is irrelevant for the applicability of *Theorem 7* to *Theorem 5*, since the required asymptotic property is obviously invariant under translations of n.

Table 3. The polynomials $\widehat{J}_p^{2,n}$ and $\widehat{J}_p^{3,n}$

n	$\widehat{J}_p^{2,n}(X)$	$\widehat{J}_p^{3,n}(X)$
100	$\approx 0.9993X^2 + 0.0731X - 1.9568$	$\approx 0.9981X^3 + 0.2072X^2 - 5.9270X + 1.1420$
200	$\approx 0.9997X^2 + 0.0459X - 1.9902$	$\approx 0.9993X^3 + 0.1284X^2 - 5.9262X - 1.4818$
300	$\approx 0.9998X^2 + 0.0346X - 1.9935$	$\approx 0.9996X^3 + 0.0965X^2 - 5.9497X - 1.3790$
400	$\approx 0.9999X^2 + 0.0282X - 1.9951$	$\approx 0.9998X^3 + 0.0786X^2 - 5.9621X - 1.2747$
$\vdots$	$\vdots$	$\vdots$
10^8	$\approx 0.9999X^2 + 0.0000X - 1.9999$	$\approx 0.9999X^3 + 0.0000X^2 - 5.9999X - 0.0529$

because the inner sum, which is the $(d-k)$th difference of the polynomial $j \mapsto j^i$ evaluated at $j=0$, vanishes for $i < d-k$ and equals $(d-k)!$ for $i = d-k$. *Theorem 8* follows, and *Theorem 3* is just the special case $E(n) = e^{A(n)}$ and $F(t) = e^{-t^2}$. □

3. Proof of Theorem 7

Assume that f is a modular form of (possibly fractional) weight k on $\mathrm{SL}_2(\mathbb{Z})$ (possibly with multiplier system) and with a pole of (possibly fractional) order $m > 0$ at infinity and write its Fourier expansion at infinity as in [**7**]. It is standard, either by the circle method of Hardy–Ramanujan–Rademacher or by using Poincaré series (for example, see ref. 23), that the Fourier coefficients of f have the asymptotic form

$$a_f(n) = A_f\, n^{\frac{k-1}{2}}\, I_{k-1}(4\pi\sqrt{mn}) + \mathrm{O}\left(n^C\, e^{2\pi\sqrt{mn}}\right), \qquad [9]$$

as $n \to \infty$ for some nonzero constants A_f [an explicit multiple of $a_f(-m)$] and C, where $I_\kappa(x)$ denotes the usual I-Bessel function. In view of the expansion of Bessel functions at infinity, this implies that $a_f(n)$ has an asymptotic expansion to all orders in $1/n$ of the form

$$a_f(n) \sim e^{4\pi\sqrt{mn}}\, n^{\frac{2k-3}{4}}\, \exp\left(c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right),$$

for some constants $c_0,\, c_1,\, \ldots$ depending on f [and in fact only on m and k if we normalize the leading coefficient $a_f(-m)$ of f to be equal to 1]. This gives an asymptotic expansion

$$\log\left(\frac{a_f(n+j)}{a_f(n)}\right) \sim 4\pi\sqrt{m}\sum_{i=1}^{\infty}\binom{1/2}{i}\frac{j^i}{n^{i-\frac{1}{2}}} + \frac{2k-3}{4}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}j^i}{i\, n^i} + \sum_{i,k\geq 1} c_k\binom{-k}{i}\frac{j^i}{n^{i+k}}, \qquad [10]$$

valid to all orders in n, and it follows that the sequence $\{a_f(n)\}$ satisfies the hypotheses of *Theorem 3* with $A(n) = 2\pi\sqrt{m/n} + \mathrm{O}(1/n)$ and $\delta(n) = (\pi/2)^{1/2} m^{1/4} n^{-3/4} + \mathrm{O}(n^{-5/4})$. *Theorem 7* then follows from the corollary to *Theorem 3*.

4. Asymptotics for $\Lambda^{(n)}(\frac{1}{2})$

Previous work of Coffey (6) and Pustyl'nikov (7) offer asymptotics[¶] for the derivatives $\Lambda^{(n)}\left(\frac{1}{2}\right)$. Here, we follow a slightly different approach and obtain effective asymptotics, a result which is of independent interest. To describe our asymptotic expansion, we first give a formula for these derivatives in terms of an auxiliary function, whose asymptotic expansion we shall then determine.

Following Riemann (cf. chapter 8 of ref. 25), we have

$$\Lambda(s) = \int_0^\infty t^{\frac{s}{2}-1}\,\theta_0(t)\, dt = \frac{1}{s(s-1)} + \int_1^\infty \left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right)\theta_0(t)\,\frac{dt}{t},$$

where $\theta_0(t) = \sum_{k=1}^\infty e^{-\pi k^2 t} = \frac{1}{2}(t^{-1/2} - 1) + t^{-1/2}\theta_0(1/t)$. It follows that

$$\Lambda^{(n)}\left(\tfrac{1}{2}\right) = -\,2^{n+2}\, n! + \frac{F(n)}{2^{n-1}}, \qquad [11]$$

for $n > 0$ (both are of course zero for n odd), where $F(n)$ is defined for any real $n \geq 0$ by

$$F(n) = \int_1^\infty (\log t)^n\, t^{-3/4}\,\theta_0(t)\, dt. \qquad [12]$$

In particular, if n is a positive integer, then the Taylor coefficients $\gamma(n)$ defined in [**1**] satisfy

$$\gamma(n) = \frac{n!}{(2n)!}\cdot\left(8\binom{2n}{2}\Lambda^{(2n-2)}\left(\tfrac{1}{2}\right) - \Lambda^{(2n)}\left(\tfrac{1}{2}\right)\right) = \frac{n!}{(2n)!}\cdot\frac{32\binom{2n}{2}F(2n-2) - F(2n)}{2^{2n-1}}. \qquad [13]$$

¶ It is interesting to note that Hadamard obtained rough estimates for these derivatives in 1893. His formulas are correctly reprinted on p. 125 of ref. 24.

Theorem 9. *If $n>0$, then the function $F(n)$ defined by* **[12]** *is given to all orders in n by the asymptotic expansion*

$$F(n)\sim\sqrt{2\pi}\,\frac{L^{n+1}}{\sqrt{(1+L)n-\frac{3}{4}L^2}}\,e^{L/4-n/L+3/4}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right)\qquad(n\to\infty),$$

where $L=L(n)\approx\log\left(\frac{n}{\log n}\right)$ is the unique positive solution of the equation $n=L(\pi e^L+\frac{3}{4})$ and each coefficient b_k belongs to $\mathbb{Q}(L)$, the first value being $b_1=\frac{2L^4+9L^3+16L^2+6L+2}{24\,(L+1)^3}$.

Example 10: Here, we illustrate *Theorem 9*. The two-term approximation

$$F(n)\approx\sqrt{2\pi}\,\frac{L^{n+1}}{\sqrt{(1+L)n-\frac{3}{4}L^2}}\,e^{L/4-n/L+3/4}\left(1+\frac{b_1}{n}\right)=:\widehat{F}(n),$$

is sufficiently strong for the proof of *Theorem 1*. In particular, *Theorem 9* and **[13]** imply

$$\widehat{\gamma}(n):=\frac{n!}{(2n)!}2^{6-2n}\binom{2n}{2}\widehat{F}(2n-2)=\gamma(n)\left(1+O\left(\frac{1}{n^{2-\varepsilon}}\right)\right). \qquad [14]$$

Here are some approximations $\widehat{\gamma}(n)$ obtained from this expression by numerically computing L using its defining equation above. Table 2 illustrates the high precision of this formula.

Proof of Theorem 9: We approximate the integrand in **[12]** by $f(t)=(\log t)^n t^{-3/4}e^{-\pi t}$ (from now on, we consider n as fixed and omit it from the notations). We have $t\frac{d}{dt}\log f(t)=\frac{n}{\log t}-\pi t-\frac{3}{4}$, so $f(t)$ assumes its unique maximum at $t=a$, where $a=e^L$ is the solution in $(1,\infty)$ of

$$n=\left(\pi a+\frac{3}{4}\right)\log a.$$

We can then apply the usual saddle point method. The Taylor expansion of $f(t)$ around $t=a$ is given by

$$\frac{f((1+\lambda)a)}{f(a)}=\left(1+\frac{\log(1+\lambda)}{\log a}\right)^n(1+\lambda)^{-3/4}e^{-\pi\lambda a}=e^{-C\lambda^2/2}\left(1+A_3\lambda^3+A_4\lambda^4+\cdots\right),$$

where $C=(\varepsilon+\varepsilon^2)n-\frac{3}{4}$ (here we have set $\varepsilon=\frac{1}{\log a}=L^{-1}$) and the A_i ($i\geq 3$) are polynomials of degree $\lfloor i/3\rfloor$ in n with coefficients in $\mathbb{Q}[\varepsilon]$. This expansion is found by expanding $\log(f((1+\lambda)a))-\log(f(a))$ in λ. The linear term vanishes by the choice of a, the quadratic term is $-C\lambda^2/2$, and the coefficients of the higher powers of λ are all linear expressions in n with coefficients in $\mathbb{Q}[\varepsilon]$. Exponentiating this expansion gives the claimed expression for $f((1+\lambda)a)/f(a)$, where the dominant term of each A_i is governed primarily by the exponential of the cubic term of the logarithmic expansion. The first few A_i are

$$A_3=\left(\frac{\varepsilon}{3}+\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{3}\right)n-\frac{1}{4},\qquad A_4=-\left(\frac{\varepsilon}{4}+\frac{11\varepsilon^2}{24}+\frac{\varepsilon^3}{2}+\frac{\varepsilon^4}{4}\right)n+\frac{3}{16},$$

$$A_5=\left(\frac{\varepsilon}{5}+\frac{5\varepsilon^2}{12}+\frac{7\varepsilon^3}{12}+\frac{\varepsilon^4}{2}+\frac{\varepsilon^5}{5}\right)n-\frac{3}{20},$$

$$A_6=\left(\frac{\varepsilon^2}{18}+\frac{\varepsilon^3}{6}+\frac{17\varepsilon^4}{72}+\frac{\varepsilon^5}{6}+\frac{\varepsilon^6}{18}\right)n^2-\left(\frac{\varepsilon}{4}+\frac{91\varepsilon^2}{180}+\frac{17\varepsilon^3}{24}+\frac{17\varepsilon^4}{24}+\frac{\varepsilon^5}{2}+\frac{\varepsilon^6}{6}\right)n+\frac{5}{32}.$$

Table 4. The polynomials $\widehat{J}_{\gamma}^{2,n}$ and $\widehat{J}_{\gamma}^{3,n}$

n	$\widehat{J}_{\gamma}^{2,n}(X)$	$\widehat{J}_{\gamma}^{3,n}(X)$
100	$\approx 0.9896X^2+0.3083X-2.0199$	$\approx 0.9769X^3+0.7570X^2-5.8690X-1.2661$
200	$\approx 0.9943X^2+0.2271X-2.0061$	$\approx 0.9872X^3+0.5625X^2-5.9153X-0.9159$
300	$\approx 0.9960X^2+0.1894X-2.0029$	$\approx 0.9911X^3+0.4705X^2-5.9374X-0.7580$
400	$\approx 0.9969X^2+0.1663X-2.0016$	$\approx 0.9931X^3+0.4136X^2-5.9501X-0.6623$
$\vdots$	$\vdots$	$\vdots$
10^8	$\approx 0.9999X^2+0.0003X-2.0000$	$\approx 0.9999X^3+0.0009X^2-5.9999X-0.0014$

Plugging in $t=(1+\lambda)a$ immediately gives the asymptotic expansion

$$\int_1^\infty f(t)\,dt = a\,f(a)\int_{-1+1/a}^{\infty} e^{-C\lambda^2/2}\left(1+A_3\lambda^3+A_4\lambda^4+\cdots\right)d\lambda$$
$$= a\,f(a)\sqrt{\frac{2\pi}{C}}\left(1+\frac{3A_4}{C^2}+\frac{15A_6}{C^3}+\cdots+\frac{(2i-1)!!A_{2i}}{C^i}+\cdots\right).$$

(Here, only the part of the integral with $C\lambda^2 < B\log n$, where B is any function of n going to infinity as n does, contributes.) This equality and the expression in *Theorem 9* are interpreted as asymptotic expansions. Although these series themselves may not converge for a fixed n, we may truncate the resulting approximation at $O(n^{-A})$ for some $A>0$, and as $n\to+\infty$, this approximation becomes true to the specified precision. Substituting into this expansion the formulas for C and A_i in terms of n, we obtain the statement of the theorem with $F(n)$ replaced by the integral over $f(t)$, with only A_{2i} $(i\le 3k)$ contributing to b_k. But then the same asymptotic formula holds also for $F(n)$, since the ratio $f(t)/\theta_0(t)=1+e^{-3\pi t}+\cdots$ is equal to $1+\mathrm{O}(n^{-K})$ for any $K>0$ for t near a. □

5. Proof of Theorems 1 and 2

A. Proof of Theorem 1. For each $d\ge 1$, we use *Theorem 3* with sequences $\{A(n)\}$ and $\{\delta(n)\}$ for which

$$\log\left(\frac{\gamma(n+j)}{\gamma(n)}\right) = A(n)j - j^2\delta(n)^2 + \sum_{i=3}^{d} g_i(n)j^i + \mathrm{o}\left(\delta(n)^d\right), \quad [15]$$

for all $0\le j\le d$, where $g_i(n)=\mathrm{o}\left(\delta(n)^i\right)$. Stirling's formula, **[13]**, and **[14]** give

$$\gamma(n) = \frac{e^{n-2}n^{n+\frac{1}{2}}(1+\frac{1}{12n})L^{\widehat{n}}}{2^{\widehat{n}-3}\widehat{n}^{\widehat{n}+\frac{1}{2}}(1+\frac{1}{12\widehat{n}})}\sqrt{\frac{2\pi}{K}}\cdot\exp\left(\frac{L}{4}-\frac{\widehat{n}}{L}+\frac{3}{4}\right)\left(1+\frac{b_1(\widehat{n})}{\widehat{n}}\right)\left(1+O\left(\frac{1}{n^{2-\varepsilon}}\right)\right), \quad [16]$$

where $\widehat{n}:=2n-2$, $L:=L(\widehat{n})$, and $K:=K(\widehat{n}):=\left(L(\widehat{n})^{-1}+L(\widehat{n})^{-2}\right)\widehat{n}-3/4$. The $L(\widehat{n})$ are values of a nonvanishing holomorphic function for $\Re(n)>1$, and so for $|j|<n-1$, we have the Taylor expansion

$$\mathcal{L}(j;n) := \frac{L(\widehat{n}+2j)}{L(\widehat{n})} = 1+\sum_{m\ge 1}\ell_m(n)\frac{j^m}{m!}.$$

If $J=\lambda(n-1)$ with $-1<\lambda<1$, then the asymptotic $L(n)\approx\log(\frac{n}{\log n})$ implies

$$\lim_{n\to+\infty}\mathcal{L}(J,n) = \lim_{n\to+\infty}\frac{L\left(\widehat{n}(\lambda+1)\right)}{L(\widehat{n})} = 1.$$

In particular, we have $\ell_1(n)=\frac{2}{K\cdot L^2}$ and $\ell_2(n)=\frac{-8(\widehat{n}-3/4L)(1+L/2)}{K^3\cdot L^5}$ and $\ell_m(n)=o\left(\frac{1}{(n-1)^m}\right)$. By a similar argument applied to

$$\mathcal{K}(j;n) := \frac{K(\widehat{n}+2j)}{K(\widehat{n})} = 1+\sum_{m\ge 1}k_m(n)\frac{j^m}{m!} \quad\text{and}\quad \mathcal{B}(j;n) := \frac{1+\frac{b_1(\widehat{n}+2j)}{\widehat{n}+2j}}{1+\frac{b_1(\widehat{n})}{\widehat{n}}} = 1+\sum_{m\ge 1}\beta_m(n)\frac{j^m}{m!},$$

we find that $\beta_m(n)=o\left(\frac{1}{(n-1)^{m+1}}\right)$, $k_1(n)=\frac{2(L+1)}{K\cdot L^2}-\frac{2\widehat{n}(L+2)}{K^2L^4}$, and $k_m(n)=o\left(\frac{1}{(n-1)^m}\right)$ for $m\ge 2$.

Table 5. The polynomials $\widehat{J}_{\gamma}^{6,\,n}$

n	$\widehat{J}_{\gamma}^{6,\,n}(X)$
100	$\approx 0.912X^6+3.086X^5-24.114X^4-55.652X^3+133.109X^2+151.696X-85.419$
200	$\approx 0.950X^6+2.374X^5-26.625X^4-42.824X^3+153.246X^2+115.849X-100.510$
300	$\approx 0.965X^6+2.011X^5-27.608X^4-36.282X^3+161.084X^2+97.843X-106.295$
400	$\approx 0.973X^6+1.780X^5-28.139X^4-32.111X^3+165.303X^2+86.428X-109.388$
⋮	⋮
10^{10}	$\approx 0.999X^6+0.000X^5-29.999X^4-0.008X^3+179.999X^2+0.020X-119.999$

Let $R(j;n)$ be the approximation for $\gamma(n+j)/\gamma(n)$ obtained from [16]. We then expand $\log R(j;n) =: \sum_{m\geq 1} g_m(n) j^m$, with the idea that we will choose $A(n) \sim g_1(n)$ and $\delta(n) \sim \sqrt{-g_2(n)}$. To this end, if $J = \lambda(n-1)$ for $-1 < \lambda < 1$, then a calculation reveals that

$$-(1+\lambda)\log(1+\lambda) = \lim_{n\to+\infty} \frac{\log R(J;n) - J\log\left(\frac{nL^2}{4\widehat{n}^2}\right) - J}{n-1}. \quad [17]$$

Therefore, $g_m(n) = O\left((n-1)^{1-m}\right)$, and algebraic manipulations give

$$g_1(n) = \log\left(\frac{nL^2}{4\widehat{n}^2}\right) + \widehat{n}\ell_1(n)\frac{L+1}{L} - \frac{2}{L} + \frac{\ell_1(n)\cdot L}{4} - \frac{k_1(n)}{2} + O\left(\frac{1}{n^{2-\varepsilon}}\right),$$

$$g_2(n) = -\frac{1}{\widehat{n}} + (4\ell_1(n) + \widehat{n}\ell_2(n))\frac{L+1}{2L} - \widehat{n}\ell_1(n)^2\frac{L+2}{2L} + O\left(\frac{1}{n^{2-\varepsilon}}\right).$$

Using the formulas for $\ell_1(n),\ \ell_2(n)$, and $k_1(n)$ above, we define

$$\delta(n) := \sqrt{\frac{1}{\widehat{n}} - \frac{2}{L^2\cdot K}} \quad \text{and} \quad A(n) := \log\left(\frac{nL^2}{4\widehat{n}^2}\right) + \frac{L-1}{L^2\cdot K} + \frac{\widehat{n}(L+2)}{L^4\cdot K^2}. \quad [18]$$

The bounds for the $g_m(n)$ and the asymptotics above imply the $o(1)$ error term in [15], and also that for sufficiently large n we have $0 < \delta(n) \to 0$. Therefore, *Theorem 3* applies, and its corollary gives *Theorem 1*.

B. Sketch of the Proof of Theorem 2. Let $A(n)$ and $\delta(n)$ be as in [18]. If we let

$$\widehat{J}_\gamma^{d,n}(X) := \frac{\delta(n)^{-d}}{\gamma(n)} \cdot J_\gamma^{d,n}\left(\frac{\delta(n)\,X\ -1}{\exp(A(n))}\right) = \sum_{k=0}^{d} \beta_k^{d,n} X^k,$$

then *Theorem 1* implies that $\lim_{n\to+\infty} \widehat{J}_\gamma^{d,n}(X) = H_d(X) =: \sum_{k=0}^{d} h_k X^k$. We have confirmed the hyperbolicity of the $\widehat{J}_\gamma^{d,n}(X)$ for $n \leq 10^6$ and $4 \leq d \leq 8$ using Hermite's criterion (see theorem C of ref. 3).

Using this criterion, we also chose vectors $\varepsilon_d := (\varepsilon_d(d), \varepsilon_d(d-1), \ldots, \varepsilon_d(0))$ of positive numbers and signs $s_d, s_{d-1}, \ldots, s_0 \in \{\pm 1\}$ for which $\widehat{J}_\gamma^{d,n}(X)$ is hyperbolic if $0 \leq s_k(\beta_k^{d,n} - h_k) < \varepsilon_d(k)$ for all k. To make use of these inequalities, for positive integers n and $1 \leq j \leq 8$, define real numbers $C(n,j)$ by

$$\frac{\gamma(n+j)}{\gamma(n)e^{A(n)j}} \cdot e^{\delta(n)^2 j^2} = 1 + \frac{C(n,j)}{n^{3/2}}. \quad [19]$$

Using an effective form of [16], it can be shown[‖] that $0 < C(n,j) < 14.25$ for all $n \geq 7$ and $1 \leq j \leq 8$. Finally, we determined numbers M_{ε_d} for which the required inequalities hold for $n \geq M_{\varepsilon_d}$. The proof follows from the fact that we found suitable choices for which $M_{\varepsilon_d} < 10^6$.

Example 11: We illustrate the case of $d = 4$ using $\varepsilon_4 := (0.041, 1.384, 0.813, 7.313, 0.804)$. For $n \geq 100$ the odd degree coefficients satisfy

$$0 < \beta_3^{4,n} < 28\,\delta(n) \quad \text{and} \quad -145.70\delta(n) < \beta_1^{4,n} < 0,$$

while the even degree coefficients satisfy

$$1 - 16.05\,\delta(n)^2 < \beta_4^{4,n} < 1, \quad -12 < \beta_2^{4,n} < -12 + 16.20\,\delta(n), \quad 12 - 16.01\,\delta(n) < \beta_0^{4,n} < 12.$$

It turns out that $M_{\varepsilon_4} := 104 < 10^6$.

6. Examples

For convenience, we let the $\widehat{J}_\alpha^{d,n}(X)$ denote the polynomials which converge to $H_d(X)$ in [5]. We now illustrate *Theorem 7* with [6], where $m = 1/24$ and $k = -1/2$. Using [10], we may choose $A(n) = \frac{2\pi}{\sqrt{24n-1}} - \frac{24}{24n-1}$ and $\delta(n) = \sqrt{\frac{12\pi}{(24n-1)^{3/2}} - \frac{288}{(24n-1)^2}}$. Although the one-term approximations of [10] given at the end of Section 3 also satisfy *Theorem 3*, the two-term approximations converge more quickly and better illustrate the result. With these data, we observe in Table 3 indeed that the degree 2 and 3 partition Jensen polynomials are modeled by $H_2(X) = X^2 - 2$ and $H_3(X) = X^3 - 6X$.

Table 4 illustrates Theorem 1 for the Riemann zeta function using (18) in the case of degrees 2 and 3.

Finally, we conclude in Table 5 with data for the degree 6 renormalized Jensen polynomials $J_\gamma^{6,n}(X)$ which converge to $H_6(X) = X^6 - 30X^4 + 180X^2 - 120$.

ACKNOWLEDGMENTS. We thank the Max Planck Institute for Mathematics in Bonn for its support and hospitality; William Y. C. Chen, Rick Kreminski, Hannah Larson, Steffen Löbrich, Peter Sarnak, and Ian Wagner for discussions related to this work; and Jacques Gélinas for bringing their attention to old work of Hadamard cited as a footnote in Section 4. M.G. and K.O. were supported by NSF Grants DMS-1502390 and DMS-1601306. K.O. was supported by Asa Griggs Candler Fund.

[‖] It turns out that $\delta(6)$ is not real.

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