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JEŚMANOWICZ' CONJECTURE WITH FERMAT NUMBERS

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Abstract. Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$. In 1956, Jeśmanowicz conjectured that for any positive integer n, the only solution of $(an)^x + (bn)^y = (cn)^z$ in positive integers is $(x, y, z) = (2, 2, 2)$. Let $k \geq 1$ be an integer and $F_k = 2^{2^k} + 1$ be k-th Fermat number. In this paper, we show that Jeśmanowicz' conjecture is true for Pythagorean triples $(a, b, c) = (F_k - 2, 2^{2^{k-1}+1}, F_k)$.

1. INTRODUCTION

Let a, b, c be relatively prime positive integers such that $a^2 + b^2 = c^2$ with b even. Clearly, for any positive integer n, the Diophantine equation

$$(na)^x + (nb)^y = (nc)^z, \quad x, y, z \in \mathbb{N} \tag{1.1}$$

has the solution $(x, y, z) = (2, 2, 2)$. In 1956, Sierpiński [8] showed there is no other solution when $n = 1$ and $(a, b, c) = (3, 4, 5)$. Jeśmanowicz [3] proved that when $n = 1$ and $(a, b, c) = (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)$, Eq.(1.1) has only the solution $(x, y, z) = (2, 2, 2)$. Moreover, he conjectured that for any positive integer n, Eq.(1.1) has no solution other than $(x, y, z) = (2, 2, 2)$. Let $k \geq 1$ be an integer and $F_k = 2^{2^k} + 1$ be k-th Fermat number. Recently, the first author of this paper and Yang [9] proved that if $1 \leq k \leq 4$, then Jeśmanowicz' conjecture is true, that is, the Diophantine equation

$$((F_k - 2)n)^x + (2^{2^{k-1}+1}n)^y = (F_k n)^z, \quad x, y, z \in \mathbb{N} \tag{1.2}$$

has no solution other than $(x, y, z) = (2, 2, 2)$. For related problems, see for example [1, 6] and [7].

In this paper, we extend this result as follows.

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Theorem 1. *For any positive integers n and k, Eq.(1.2) has only the solution* $(x, y, z) = (2, 2, 2)$.

Throughout this paper, for positive integers a and m with a prime to m, we denote by $\text{ord}_m(a)$ the least positive integer h such that $a^h \equiv 1 \pmod{m}$.

2. Lemmas

In this section, we prepare several lemmas.

Lemma 1. ([5]). *For any positive integer m, the Diophantine equation $(4m^2 - 1)^x + (4m)^y = (4m^2 + 1)^z$ has only the solution $(x, y, z) = (2, 2, 2)$.*

Lemma 2. (See [1, Lemma 2]). *Let a, b, c be positive integers such that $a^2 + b^2 = c^2$. If $z \geq max\{x, y\}$, then the Diophantine equation $a^x + b^y = c^z$ has only the positive solution $(x, y, z) = (2, 2, 2)$.*

Lemma 3. (See [4, Corollary 1]). *If Eq.(1.1) has a solution $(x, y, z) \neq (2, 2, 2)$, then x, y, z are distinct.*

Lemma 4. (See [2, Lemma 2.3]). *Let a, b, c be any primitive Pythagorean triple such that $a^2 + b^2 = c^2$. Assume that the Diophantine equation $a^x + b^y = c^z$ has only the trivial solution in positive integers x, y and z. Then Eq.(1.1) has no solution satisfying $z < y < x$ or $z < x < y$.*

Lemma 5. *Let k be a positive integer. If (x, y, z) is a solution of Eq.(1.2) with $(x, y, z) \neq (2, 2, 2)$, then $x < z < y$.*

Proof. By Lemmas 2-4, it is sufficient to prove that Eq.(1.2) has no solution (x, y, z) satisfying $y < z < x$. By Lemma 1, we may assume that $n \geq 2$. Suppose that Eq.(1.2) has a solution (x, y, z) with $y < z < x$. Then, dividing Eq.(1.2) by n^y, we find

$$2^{(2^{k-1}+1)y} = n^{z-y}\Big(F_k^z - (F_k - 2)^x n^{x-z}\Big). \tag{2.1}$$

By (2.1) we may write $n = 2^r$ with $r \geq 1$. Since the second factor on the right-hand side of (2.1) is odd, it has to be 1, that is,

$$F_k^z - (F_k - 2)^x 2^{r(x-z)} = 1. \tag{2.2}$$

Since $F_k \equiv 2 \pmod{3}$, equation (2.2) implies $2^z \equiv 1 \pmod{3}$, hence $z \equiv 0 \pmod{2}$. Write $z = 2z_1$. Then

$$\Big(\prod_{i=0}^{k-1} F_i\Big)^x 2^{r(x-z)} = (F_k^{z_1} - 1)(F_k^{z_1} + 1). \tag{2.3}$$

Let $F_{k-1} = \prod_{i=1}^{t} p_i^{\alpha_i}$ be the standard prime factorization of F_{k-1} with $p_1 < \ldots < p_t$. By the known Fermat primes, we know that there is the possibility of $t = 1$. Moreover,

$$\mathrm{ord}_{p_i}(2) = 2^k, \quad i = 1, \ldots, t. \tag{2.4}$$

Since $\gcd(F_k^{z_1} - 1, F_k^{z_1} + 1) = 2$, by (2.3) we know that p_t divides only one of $F_k^{z_1} - 1$ and $F_k^{z_1} + 1$.

Case 1. $p_t \mid F_k^{z_1} - 1$. Then $2^{z_1} - 1 \equiv F_k^{z_1} - 1 \equiv 0 \pmod{p_t}$. Hence, we have $z_1 \equiv 0 \pmod{2^k}$ by (2.4). It follows from (2.4) that

$$F_k^{z_1} - 1 \equiv 2^{z_1} - 1 \equiv 0 \pmod{p_i}, \quad i = 1, \ldots, t.$$

Since $\gcd(F_k^{z_1} - 1, F_k^{z_1} + 1) = 2$, by (2.3) we have

$$F_k^{z_1} - 1 \equiv 0 \pmod{p_i^{\alpha_i x}}, \quad i = 1, \ldots, t.$$

Hence F_{k-1}^x divides $F_k^{z_1} - 1$.

Case 2. $p_t \mid F_k^{z_1} + 1$. Then $2^{z_1} + 1 \equiv F_k^{z_1} + 1 \equiv 0 \pmod{p_t}$, so $2^{2z_1} \equiv 1 \pmod{p_t}$. Hence, $z_1 \equiv 0 \pmod{2^{k-1}}$, but $z_1 \not\equiv 0 \pmod{2^k}$. By (2.4), for $i = 1, \ldots, t$, we have

$$2^{z_1} - 1 \not\equiv 0 \pmod{p_i},$$

$$(2^{z_1} + 1)(2^{z_1} - 1) = 2^{2z_1} - 1 \equiv 0 \pmod{p_i}.$$

Thus

$$F_k^{z_1} + 1 \equiv 0 \pmod{p_i}, \quad i = 1, \ldots, t.$$

Similarly to the preceding case, the above yields F_{k-1}^x divides $F_k^{z_1} + 1$.

However, by the assumption $z < x$, we have

$$F_{k-1}^x = \left(2^{2^{k-1}} + 1\right)^x > \left(2^{2^{k-1}} + 1\right)^{2z_1} > F_k^{z_1} + 1,$$

which is absurd. This completes the proof of Lemma 5. ∎

3. Proof of Theorem 1

By Lemma 1, we may assume that $n \geq 2$. Suppose that there exists a solution of Eq.(1.2) with $(x, y, z) \neq (2, 2, 2)$. It suffices to observe that this leads to a contradiction. By Lemma 5, we may assume $x < z < y$. Then, dividing Eq.(1.2) by n^x, we find

$$\Big(\prod_{i=0}^{k-1} F_i\Big)^x = n^{z-x}\Big(F_k^z - 2^{(2^{k-1}+1)y} n^{y-z}\Big). \tag{3.1}$$

It is clear from (3.1) that n is prime to the second factor of the right-hand side of (3.1). Let $\prod_{i=0}^{k-1} F_i = \prod_{i=1}^{t} p_i^{\alpha_i}$ be the standard prime factorization of $\prod_{i=0}^{k-1} F_i$ and write $n = \prod_{j\in S} p_j^{\beta_j}$, where $\beta_j \geq 1$, $S \subseteq \{1,\ldots,t\}$. Let $T = \{1,\ldots,t\} \setminus S$. If $T = \emptyset$, then let $P(k,n) = 1$. If $T \neq \emptyset$, then let

$$P(k,n) = \prod_{i\in T} p_i^{\alpha_i}.$$

By (3.1), we have

$$P(k,n)^x = F_k^z - 2^{(2^{k-1}+1)y} \prod_{j\in S} p_j^{\beta_j(y-z)}. \tag{3.2}$$

If $P(k,n) = 1$, then $S = T = \{1,\ldots,t\}$, and $p_1 = 3$. So, as seen in the proof of Lemma 5, taking the equation in (3.2) modulo 3 implies that z is even. Write $z = 2z_1$. By (3.2), we have

$$2^{(2^{k-1}+1)y} \prod_{j\in S} p_j^{\beta_j(y-z)} = (F_k^{z_1} - 1)(F_k^{z_1} + 1).$$

Since $\gcd(F_k^{z_1}-1, F_k^{z_1}+1) = 2$, we find that $2^{(2^{k-1}+1)y-1}$ divides only one of $F_k^{z_1}+1$ and $F_k^{z_1} - 1$. Thus $2^{(2^{k-1}+1)y-1} \leq F_k^{z_1} + 1$. However, by the assumption $z < y$, we have

$$2^{(2^{k-1}+1)y-1} \geq 2^{(2^{k-1}+1)(z+1)-1} > 2^{(2^{k-1}+1)2z_1} > (F_k + F_k - 2)^{z_1} \geq F_k^{z_1} + 1,$$

which is a contradiction.

Now we assume that $P(k,n) > 1$. First, we shall show that x is even.

Since $y \geq 2$, it follows from (3.2) that

$$P(k,n)^x \equiv 1 \pmod{2^{2^k}}. \tag{3.3}$$

If $3 \mid P(k,n)$, then $P(k,n) \equiv -1 \pmod 4$. This together with (3.3) implies that x is even. Hence, we may assume $P(k,n) \not\equiv 0 \pmod 3$. Then $P(k,n) \equiv 1 \pmod 4$. We can write $P(k,n) = 1 + 2^v W$, where v, W are positive integers such that $v \geq 2$ and W is odd. Suppose that x is odd, then

$$P(k,n)^x = 1 + 2^v W', \quad 2 \nmid W'.$$

Thus $v \geq 2^k$ by (3.3), and so $P(k,n) \geq F_k$, which is a contradiction with

$$P(k,n) < \prod_{i=0}^{k-1} F_i = F_k - 2.$$

Therefore, x is even. We can write $x = 2^u N$, where u, N are positive integers such that N is odd.

Second, we shall prove that z is even.

Case 1. $P(k,n) \equiv -1 \pmod 4$. We can write $P(k,n) = 2^d M - 1$, where d, M are positive integers such that $d \geq 2$ and M is odd. Then

$$P(k,n)^x = 1 + 2^{u+d} V, \quad 2 \nmid V.$$

By (3.3) we have $u + d \geq 2^k$.

Since $S \neq \emptyset$, we can choose a $\nu \in S$, and we put $p_\nu = 2^r t' + 1$ with $r \geq 1$, $2 \nmid t'$. Then

$$2^{d+r-1} < (2^d M - 1)(2^r t' + 1) = P(k,n) \cdot p_\nu \leq \prod_{i=0}^{k-1} F_i = 2^{2^k} - 1.$$

Thus $d + r \leq 2^k$. Hence $u \geq r$. By (3.2) we have

$$P(k,n)^x \equiv 2^z \pmod{p_\nu}.$$

Noting that $p_\nu - 1 \mid 2^u t'$, we have

$$2^{t'z} \equiv P(k,n)^{2^u t' N} \equiv 1 \pmod{p_\nu}.$$

Since $\mathrm{ord}_{p_\nu}(2)$ is even and t' is odd, we have $z \equiv 0 \pmod 2$.

Case 2. $P(k,n) \equiv 1 \pmod 4$. Similarly to the preceding case, we can show that z is even.

Write $z = 2z_1, x = 2x_1$. By (3.2), we have

$$2^{(2^{k-1}+1)y} \prod_{j \in S} p_j^{\beta_j (y-z)} = \left(F_k^{z_1} - P(k,n)^{x_1}\right)\left(F_k^{z_1} + P(k,n)^{x_1}\right). \tag{3.4}$$

Since

$$\gcd\left(F_k^{z_1} - P(k,n)^{x_1}, F_k^{z_1} + P(k,n)^{x_1}\right) = 2,$$

we find from (3.4) that $2^{(2^{k-1}+1)y-1}$ divides only one of $F_k^{z_1} + P(k,n)^{x_1}$ and $F_k^{z_1} - P(k,n)^{x_1}$. Thus $2^{(2^{k-1}+1)y-1} \leq F_k^{z_1} + P(k,n)^{x_1}$. However, by the assumption $x < z < y$, we have

$$2^{(2^{k-1}+1)y-1} > (F_k + F_k - 2)^{z_1} > F_k^{z_1} + P(k,n)^{x_1},$$

which is a contradiction. This completes the proof of Theorem 1.

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