Job Market Signalling of Relative Position, or Becker Married to Spence

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Abstract

We consider a matching model of the labour market where workers that differ in quality send signals to firms that are also vertically differentiated. Signals allow assortative matching in which the highest quality workers send the highest signals and are hired by the best firms. Matching is consider under both transferable and non-transferable utility. In both cases payoffs are determined by relative position - the best worker gets the best job. The standard signalling model which communicates the signaller's absolute type is a special case of the current model of signalling relative position. Second, in the relative model, equilibrium strategies and payoffs depend on the distributions of types of workers and the distribution of firms. This is in contrast with separating equilibria of the standard model which do not respond to changes in supply or demand. Surprisingly in some cases there can be inefficiently little investment in signalling.

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1 Introduction

It is more than thirty years since Spence (1973) introduced the now famous insight that investment in education could be undertaken as a signal to prospective employers. In this classic model some workers are more productive than others, but employers are not differentiated. Although Spence's work provided many important insights, one peculiarity of his model (and subsequent elaborations such as Mailath (1987)) is that in any separating equilibrium, strategies and outcomes, such as wages, do not respond to changes in the relative frequency of high and low quality workers. That is, strangely, the wages of skilled workers do not respond to changes in the supply of either skilled or unskilled labour. The same year Becker (1973) attempted to explain positive assortment in marriage, why most commonly like marries like. His formal model allowed for vertical differentiation on both sides of the marriage market. Some labour markets seem to be similar, with workers competing for high quality jobs. More recently, Cole, Mailath and Postlewaite (1992, 1995) introduced a model which one can call a "matching tournament", in which agents make an investment decision before participating in a matching market. If that investment is a signal of otherwise unobservable ability, then matching tournaments integrate aspects of both Spence's and Becker's models

This paper investigates matching tournaments under incomplete information. Workers undertake visible effort to signal underlying heterogeneous ability. Employers are also vertically differentiated, but this is observable. In a separating equilibrium, there is positive assortative matching with high quality workers sending high signals and being matched with high quality firms. When utility is non-transferable between workers and firms, equilibrium strategies and payoffs depend on the distributions of characteristics of both firms and workers. When utility is transferable and there is bargaining over wages, using the stronger assumption that workers and firms are complements in production, then also equilibrium wages depend on the distribution of types of firms and workers. That is, there is a dependence on demand and supply absent in Spence's original model. Furthermore, outcomes depend on relative position: one's wage and equilibrium payoff depend on the characteristics of others as much as they do on one's own.

There is, therefore, a fundamental difference from Spence (1973), where equilibrium strategies and payoffs depend only on the absolute characteristics of workers. Specifically, in a separating equilibrium of the classic model, employers can infer the exact productivity of workers from their level of education. As a result, the signaller's equilibrium payoffs are determined by the absolute level of her productivity, for example, she ends up being paid her marginal product. In the current model, however, a signaller's payoff will instead depend on her rank in the distribution of types in the population. For example, the best candidate will get the best job, independent of his absolute level of ability. Furthermore, how much he has to signal to communicate successfully that he is the best candidate will depend on the entire distribution of workers' characteristics. In contrast, in the classic model, equilibrium strategies depend only on an incentive compatibility condition derived from individual preferences. Therefore, they do not change in response to competitive pressures, as they do here.

One crucial aspect is that there are two potential changes in the competitive situation: changes in the distribution of workers and changes in the distribution of firms. It is possible to carry out both forms of comparative statics. For example, one can look at the effect of an improvement in the quality of workers or the quality of firms by changes in the respective distribution that satisfy first order stochastic dominance or one of its refinements. An improvement in the quality of workers or a decrease in the quality of jobs increases the competitiveness of the market and lowers workers' utility at each level of ability. Signalling increases for most types of worker, but not for all. Such a change in market conditions will induce low ability workers to reduce the effort put into signalling. When utility is transferable, that is, there is bargaining over wages, an increase in competitiveness also reduces the wage for a given ability level. Importantly, the effect of an increase in the quality of firms is equal but opposite: it lowers signalling but raises workers' utility.

There are two important implications from these comparative statics. First, there are relative effects not present in a classic model: the equilibrium outcome for any worker depends on the quality of other workers in the market. Second, since the effects are equal and opposite, if the two distributions were changed simultaneously in the same direction, there would be no net effect. In particular, the classic signalling models of Spence (1973) and Mailath (1987) can be derived as a special case of our model simply by setting the two distributions on either side of the market to be identical. This also clarifies how our work differs from that of Cole, Mailath and Postlewaite (1992, 2001) who pioneered the analysis of matching tournaments but who concentrated on this special case.[1]

An important part of Becker's (1973) analysis is the distinction between transferable (TU) and non-transferable (NTU) utility. It is assumed that any match, between husband and wife or between a worker and a firm, produces a surplus that is then divided between the partners. In the NTU case, there are exogenous limits on what divisions are possible. A labour market example is that in some European university systems wages are fixed at a national level, so there can be no bargaining over salary. One might think that this is the source of the relative effects of the present model. Imagine that the quality of workers is poor, the best of that poor bunch would get the best job even if low quality in absolute terms. In contrast, in the TU case one might wonder whether any such positional rents would be bargained away: low quality workers could be offered low wages. In the end, just as in the classical models, workers would be paid their product. It is shown in Section 4 of this paper that this is not the case. Provided that the additional assumption is made that the attributes of workers and firms are strict complements in production, equilibrium wages, signalling and welfare all depend on the distributions of characteristics of both firms and workers.

Equilibria under incomplete information are usually inefficient when compared to outcomes under complete information. Indeed, whereas Cole, Mailath and Postlewaite

[1] They also concentrate on situations of complete information, with a brief treatment of signalling only in Cole, Mailath and Postlewaite (1995).

(2001) find that a matching tournament can produce an efficient amount of investment under transferable utility and complete information. The competition for matches solves the usual hold-up problem. Here under incomplete information, the incentive to signal to achieve a better match drives up the amount of investment to an inefficiently high level, even when the signal is a productive investment such as education. However, under non-transferable utility, the question as to whether efficiency is reached is more complicated. In fact, surprisingly workers of low ability may invest too little, even under incomplete information.

This type of model may be important because many real world labour markets, particularly for professionals, have a structure that is not too far distant from Becker's marriage model. There is careful effort devoted by both sides to ensuring a good match between employer and employee, and intense competition for high ranked employers and for star candidates. Some, such as entry level markets for physicians, actually use explicit matching schemes (see, for example, Roth (2002) for a survey). Of course, the current work abstracts away from the modelling the exact matching algorithm that in practice must be used. Other labour markets without central matching are subject to greater search frictions. Shimer and Smith (2002) and Smith (2002) find conditions for approximate positive assortative matching in explicit search models, under transferable and non-transferable utility respectively. Finally, the current approach assumes that all workers have the same preferences over all jobs whereas in real markets preferences are surely more idiosyncratic. See Clark (2003) for an analysis of matching with such horizontal differentiation.

2 Matching Tournaments

In this section, I develop a model of a tournament, where a large population of contestants compete in a matching market. We have in mind three prime examples. The first is students competing for places at college. The second is the marriage market. The third is a market for jobs. For example, students in the final year of graduate school seek faculty positions at universities. We will use the terminology of this last case and talk about workers and firms. We also make the simplifying assumption that workers have a common ordering over potential jobs. That is, in the academic job market for example, all graduating students have a consensus over which would be the best university position to get, what would be the second best and so on. In contrast, while the employers all agree that they would like to hire the most able candidate, the ability of candidates is not observable. Rather potential employers must infer the ability of workers from an investment decision, for example in education, made before matching. We will look at equilibria where all employers will rank all workers in terms of this investment. In the current work, the employers have no investment decision of their own to make. Indeed, we can also consider the special case of competitive situations such as sports tournaments where the "firms" are only inanimate monetary prizes, which are assigned to candidates according to their performance.

The model can be considered as an incomplete information version of the model introduced by Cole, Mailath and Postlewaite (1992), hereafter CMP. However, we generalise their model to allow for different distributions of characteristics on the two sides of the market. This will allow both for a richer model and for comparative statics analysis of the effect of changes in those distributions. This is also how our model is differentiated from standard signalling models. As we demonstrate in Section 3, we can derive a more traditional signalling model by setting the distributions on the two sides of the market to be identical.

There are two populations of agent: workers and firms. They are differentiated in quality with a worker's type being z with z distributed on $[\underline{z}, \bar{z}]$ with $\underline{z} \geq 0$ according to the distribution $G(z)$. The distribution $G(z)$ is twice differentiable with strictly positive bounded density $g(z)$. Firms are also differentiated in their attribute s which has the twice differentiable distribution function $H(s)$ on $[\underline{s}, \bar{s}]$ and strictly positive bounded density $h(s)$ (in the case of a sports tournament $H(s)$ is just the distribution of prize money). The workers will compete amongst themselves to match with the firms. In particular, workers must choose a visible level of output or investment x from the positive real line $[0, \infty)$. Following Spence, this could be a choice of education level. An worker's type z has the general interpretation as her ability, and is positively related with the worker's productivity. After the choice of output/investment, matching will take place, with one worker matching with each firm.[2] A match between a worker of type z investing x with a firm of type s will produce output $\pi(z, s, x)$, where $\pi(\cdot)$ is an increasing function. As we will see, stable matching will be positive and assortative. That is, workers with high x will match with firms with high s.

We now consider preferences, under the assumption of non-transferable utility (NTU) (we go on to consider transferable utility in Section 4). That is, there are some benefits arising from the match between firm and worker that are not dividable and/or excludable. Here I assume that some aspect of a firm's type s that is attractive to workers but which cannot be divided between worker and firm. In the context of the academic job market, s could be interpreted as prestige or reputation of a university, in the marriage market, s could be a measure of attractiveness to the opposite sex. In sports tournaments, it is simply the value of a cash prize. Any cash payments from firms to to workers are not negotiable. Hence, workers in their choice of match care solely about the value of s in a firm.[3]

For the workers, we assume that each has the same utility function $U(z, s, x)$ that depends on her type, match and action respectively. This is similar to the general signalling model of Mailath (1987) that assumes signallers' (here workers') payoffs depend

[2] This model does allow for a single firm to hire multiple workers in the following limited way. The distribution of firms $H(s)$ can be relabelled the distribution of jobs, several of which may by offered by a single firm. However, the assumption that $H(s)$ is continuous means that every job is different, one firm cannot offer (a positive measure of) identical jobs.

[3] For example, in some European countries, academic wages are fixed by national agreement. Since all universities pay the same, candidates simply prefer to be employed by the most prestigious institution.

on their type, action and type as perceived by the receivers (here firms). In a separating equilibrium, by definition, the perceived type is equal to the true type, so that in the traditional model the utility a signaller receives would be of the form $U(z, z, x)$. In a matching tournament, however, even when there is a separating equilibrium and so perceived type equals true type, the exact payoff that a worker receives will depend on the matching scheme in place and will not depend solely on his true type. However, the model is still close enough to draw upon Mailath's (1987) results. To this end, I make similar assumptions on the workers' utility function as follows:

1. U is twice continuously differentiable (smoothness);
2. $U_z(z, s, x) > 0$, $U_s(z, s, x) > 0$ (monotonicity);
3. $U_{zx}(z, s, x) > 0$ and $U_{zs}(z, s, x) > 0$ (complementarity);
4. $U_x(z, s, x) = 0$ has a unique solution in x denoted $\gamma(z, s)$ that maximises $U(z, s, x)$ and $U_{xx}(z, s, x) < 0$ (concavity);

Firms in their choice of worker prefer workers of high productivity. Within that general framework, we can consider three special cases.

Story 1: Complete Information Here a worker's observable action x represents the production of an asset useful to the firms. For example, a worker's investment in human capital may make her an attractive hire. The product of a match is strictly increasing in the worker's investment: $\pi_x(z, s, x) > 0, \pi_z(z, s, x) = 0$

Story 2A: Valueless Signalling. Here the observable action x is costly to the worker, but serves no use in itself to firms. However, it may act as signal of a worker's type z and the utility of firms is increasing in the type of their match. For example, as in Spence's (1973) classic model, education may signal ability. The product of a match is strictly increasing in the worker's type: $\pi_z(z, s, x) > 0, \pi_x(z, s, x) = 0$

Story 2B: Constructive Signalling. Here the observable action x increases output. However, output also depends on a worker's unobservable type z. For example, education may both signal ability and increase human capital. The product of a match is strictly increasing in both the worker's type and her investment: $\pi_x(z, s, x) > 0, \pi_z(z, s, x) > 0$

Note that Stories 1, 2A and 2B can be similar on a formal level under NTU if in the signalling equilibrium in 2A or 2B, the equilibrium is symmetric with output x increasing in type z. Then, a high visible output will indicate a high type and lead to a good match, so that the incentives for a worker over his choice of x may be identical as in the case when a high x brings an intrinsic benefit.

Following CMP (1992, 1998), a matching is a function $\phi : [0, 1] \to [0, 1] \cup \{\emptyset\}$ that is measure-preserving and one-to-one on $\phi([0, 1])$, where $\phi(i) = j \in [0, 1]$ is i's match and $\phi(i) = \emptyset$ indicates that i is not matched. That is, for all measurable subsets $A \subset [0, 1]$,

$\phi^{-1}(A)$ is measurable and $\lambda(\phi^{-1}(A)) = \lambda(A)$, where λ denotes Lebesgue measure. A matching is stable if there does not exist $i \neq i' \in [0,1]$ such that $\phi(i')P_i\phi(i)$ and $iP_{\phi(i')}i'$, with both preferences holding strictly.

The first condition is the equivalent in a continuum of requiring exactly one worker being matched to one firm. The second is the stability condition standard in most matching problems, that requires that matches made are not subject to unravelling in the sense that it should not be possible to find a worker and a firm who would prefer to match with each other in place of their current matches. In the case of incomplete information (Stories 2A and 2B) matching is done with respect to visible output x. That is, a firm prefers a worker i over a worker j if and only if $x_i > x_j$. This can be incentive compatible with the true underlying preferences of the firms when the distribution of x in the population of workers corresponds to the distribution of z (Stories 2A and 2B). The theoretical exercise here is to find conditions for when this is the case.

In this context, an equilibrium will be a strategy $x(z)$ for the workers and an associated matching scheme that is stable given observable output and the strategy $x(z)$. Furthermore, for incentive compatibility, the matching is required to be stable *ex post*. That is, firms do not regret their match once the type of the worker has been revealed. We call such an equilibrium symmetric if all workers use the same strategy, that is, the same mapping $x(z)$ from type to output.

Suppose for the moment that the equilibrium strategy $x(z)$ is differentiable and strictly increasing (we will go on to show that such an equilibrium exists). Let us aggregate all the output decisions of the workers into a distribution summarised by a distribution function $F(x)$. A strictly increasing symmetric strategy implies that in equilibrium an agent of type z_i who produces $x(z_i)$ would have a position in the distribution of output $F(x(z_i))$ equal to his rank $G(z_i)$ in the distribution of ability. This enables the firms to infer which worker is in fact the most able. This in turn allows the matches to be made through the following assortative matching mechanism so that workers with high (respectively low) x are matched with firms with high (respectively low) s. More specifically, a worker's rank in level of output determines the rank of his match. That is, a worker making a choice x_i will achieve a match of value $s_i = H^{-1}(F(x_i))$ or $F(x_i) = H(s_i)$. Then, we can show that the assortative scheme outlined above is stable. That is, we can find no worker and firm who would both prefer each other in place of their current match.[4]

Lemma 1 *Suppose the utility of firms is strictly increasing in x (Story 1) or in z (Story 2A) or in both x and z (Story 2B). Suppose all workers adopt a symmetric strictly increasing strategy $x(z)$, then the assortative matching, such for a worker of*

[4] Results of this type go back to Becker (1973). See Cole, Mailath and Postlewaite (1995), Fernandez and Galí (1999) for a tournament approach similar to that employed here. Eeckhout (2000) and Legros and Newman (2004) find conditions for when positive assortative matching is the *only* stable matching scheme.

type z_i for any $z_i \in [\underline{z}, \bar{z}]$, with output $x_i = x(z_i)$ her match is of type s_i, where

$$G(z_i) = F(x_i) = \phi(F(x_i)) = H(s_i), \tag{1}$$

is the only stable matching.

We now derive a symmetric equilibrium strategy for the workers. Suppose all agents adopt a strictly increasing differentiable strategy $x(z)$. Then the equilibrium relationship (1) implies that we can define the function

$$S(z) = H^{-1}(G(z)) \tag{2}$$

which gives the equilibrium match of a worker of type z that depends on both G and H. Note that we have

$$S'(z) = \frac{g(z)}{h(H^{-1}(G(z)))}. \tag{3}$$

This implies an equilibrium utility of the form:

$$U(z, S(z), x(z)) \tag{4}$$

Note that utility, through $S(z)$, now depends on both the distribution $G(z)$ of workers's types and the distribution $H(s)$ of firms' characteristics.

Suppose positive assortative matching was assigned by a central planner, rather than determined by the workers' competitive choice of investment. Then, what level of investment would workers choose? Since in general, workers gain some direct utility from their own investment x, their choice will not in general be zero. This level of investment that is optimal in the absence of matching considerations will be useful as a point of comparison with the Nash equilibrium level of investment that will eventually be derived.

Definition 1 *Let $x = \gamma_N(z)$ maximise $U(z, S(z), x)$, that is the condition $U_x(z, S(z), \gamma_N(z)) = 0$ holds at every level of $z \in [\underline{z}, \bar{z}]$. The function $\gamma_N(z)$ is called the privately optimal level of investment x under NTU.*

Suppose now one agent produces $x(\hat{z})$ in place of her equilibrium choice $x(z)$ and then chooses $\hat{z}$ to maximise her payoff. Her reduced form utility is $U(z, S(\hat{z}), x(\hat{z}))$. This gives a first order condition

$$U_x(z, S(\hat{z}), x(\hat{z}))\, x'(\hat{z}) + U_s(z, S(\hat{z}), x(\hat{z}))S'(\hat{z}) = 0. \tag{5}$$

Now, in a symmetric equilibrium it must be that $\hat{z} = z$. Using this and rearranging the resulting first order condition, we have the following differential equation.

$$x'(z) = -\frac{U_s(z, S(z), x)}{U_x(z, S(z), x)} S'(z) \tag{6}$$

This differential equation will give us our equilibrium strategy, in combination with the boundary condition we now derive.

Lemma 2 *In a symmetric equilibrium of the matching tournament with positive assortative matching and continuous strictly increasing strategies,* $x(\underline{z}) = \gamma_N(\underline{z})$.

Proof: In a symmetric equilibrium, an individual with ability $\underline{z}$ has rank 0 and utility $U(z, s, x) = U(\underline{z}, \underline{s}, x(\underline{z}))$ that does not depend on the agent's rank. Therefore, in equilibrium she chooses x to maximize $U(\underline{z}, \underline{s}, x)$. That is, she must choose $\gamma_N(\underline{z})$, or there would be a profitable deviation. ∎

The lowest ranked worker acts as though matching considerations did not matter. This reflects the equilibrium competitive response to the expectation that one is going to come last.

Proposition 1 *The unique solution to the differential equation (6) on* $(\underline{z}, \bar{z}]$ *together with the boundary condition,* $x(\underline{z}) = \gamma_N(\underline{z})$*, and the assortative matching scheme (1) constitute the unique symmetric separating equilibrium to the tournament matching game under NTU and Story 1, Story 2A or Story 2B. Equilibrium investment* $x(z)$ *is greater than the privately optimal level* $\gamma_N(z)$ *everywhere on* $(\underline{z}, \bar{z}]$.

The proof follows (see the Appendix) from the results of Mailath (1987) on the existence of separating equilibria in standard signalling models. The method is to show that any symmetric increasing equilibrium strategy $x(z)$ is continuous and then differentiable. Hence, it must constitute a solution to the differential equation (6), which has a unique solution on $(\underline{z}, \bar{z}]$. The main technical problem is that at $\underline{z}$, U_x is zero, implying that the derivative x' is unbounded. However, imposition of the boundary condition, see Lemma 2, together with boundedness of U_x is enough to rule out multiple solutions to the differential equation. The result only concerns fully separating equilibria. It is impossible to rule out other equilibria. As in standard signalling models, there also exist many pooling equilibria.

An important question will be whether separating equilibria are efficient. Compare $x(z)$ with $\gamma_N(z)$. From the point of view of workers, they are Pareto ranked. They obtain the same match in both cases, but with higher effort in the separating equilibrium. All workers (except the lowest type $\underline{z}$) would be better off under $\gamma_N(z)$. However, this privately optimal investment level is not a Nash equilibrium. To be clear, although workers would be better off under $\gamma_N(z)$, it may not be socially optimal. When investment is productive and enters into the utility of firms (Stories 1, 2B), welfare is a more complex issue. We discuss this further in Section 5 and after.

3 Signaling Relative versus Absolute Productivity

In the signalling model introduced by Spence (1973) in a separating equilibrium, the worker is paid his marginal product which is revealed by the equilibrium strategy. In the case of a continuum of types, Mailath (1987) gives the equivalent conditions. In the current notation, if all agents adopt a strictly increasing strategy $x(z)$, then if an agent of type z_i makes a choice x_i, an observer can infer that an agent's type is $z_i = x^{-1}(x_i)$. In the context of an otherwise competitive labour market, if an agent's type is her productivity, she would then be paid z_i. More generally, Mailath assumes an agent's utility is given by $U(z, \hat{z}, x)$, where $\hat{z}$ is the agent's perceived type.

Our current model differs in that the reward structure does not depend on the (inferred) type of an agent, rather it depends on his rank. That is, here equilibrium payoffs depend on $G(\hat{z})$ rather than $\hat{z}$, or from (4), $U(z, H^{-1}(G(\hat{z})), x)$. It might seem that the utility formulation used here, as it has the same basic arguments $z, \hat{z}, x$, is a special case of the Spence/Mailath absolute signalling model. However, I would argue the opposite is true, the absolute is a special case of the relative.

First, reducing utility to the form $U(z, \hat{z}, x)$ removes the dependence of an individual's utility on the distributions of workers' and firms' characteristics. In the relative model, changes in the characteristics of others can affect the utility of an agent who remains unchanged herself. Second, it is possible to reproduce the standard signalling model within the relative model. In the Spence model, the labour market is competitive given the available information. For example, since in a separating equilibrium a worker's productivity is revealed, she is paid her marginal product. In the present context, the equilibrium reward for each agent must be equal to his type, or $S(z) = z$. Note that this condition will automatically be satisfied if $G(\cdot) = H(\cdot)$, that is, the distribution of rewards from jobs is identical to the distribution of types. Let us look at the effect of this in the context of the simple signalling model considered earlier. If indeed $G(\cdot) = H(\cdot)$, then $S(z) = z$, $S'(z) = 1$ and $\underline{s} = \underline{z}$, so that the differential equation (6) reduces to

$$x'(z) = -\frac{U_s(z, z, x)}{U_x(z, z, x)}, \tag{7}$$

which is effectively the same as that given in Mailath (1987, p1353).

Notice that in contrast to the general case, the differential equation does not depend on the distribution functions $H(s)$ and $G(z)$. Consequently, unlike in the model of signalling relative position, changes in the distribution of types or jobs have no effect on the equilibrium strategy. Or rather, since as we will see later in Section 6, changes in the two distributions have opposite effects, when as here the two distributions are constrained to be equal to each other, a movement of one distribution is cancelled out by the movement of the other.

This is not to say that there is no change at all. Even though the equilibrium strategy does not change, the level of output will respond to simultaneous movements in the distribution of abilities and jobs. For example, suppose both $G(z)$ and $H(s)$ are uniform on [0,1], and the equilibrium strategy is $x(z) = z/2$. Now, if both distributions are changed so that now $G(z) = z^2$ and $H(s) = s^2$ on $[0, 1]$ and the average z rises from $1/2$ to $2/3$, the equilibrium strategy will still be $z/2$, but average output will be $1/3$ not $1/4$.

4 Transferable Utility

The argument of this paper is that there is a distinction between signalling relative and absolute position. It would seem a reasonable hypothesis, however, that the difference would melt away once utility is transferable. For example, if a particular job has high non-monetary benefits, an employer may compensate by offering a lower salary. For example, it is often said that the oldest and most prestigious universities do not pay their faculty the highest salaries. Nonetheless, we find here that we can obtain similar results to those with non-transferable utility, in that even here, equilibrium strategies and utility depend upon relative position.

Suppose in contrast to what we have assumed up to now that the surplus created by matching is continuously divisible between the two partners. As Becker (1973) discovered, in this case assortative matching is only stable if the two attributes, here z and s, are complements in a joint production process. This is in contrast with the situation with the non-transferable utility assumed up to now, where all that was required for stability was that workers' utility was increasing in s and firms' utility was increasing in z.

In this section, we assume that if a worker with attribute z is matched with a firm of type s then they will have a joint product $\pi(z, s, x)$. Assume that the function π is twice continuously differentiable with partial derivatives $\pi_z(z, s, x) > 0$ and $\pi_s(z, s, x) > 0$, but $\pi_x(z, s, x) \geq 0$. Furthermore, assume the cross partial derivative $\pi_{zs}(z, s, x)$ are strictly positive, z and s are complements in production. Let $\pi_{xs}(z, s, x) \geq 0$. Lastly, assume $\pi_{xx}(z, s, x) \leq 0$.

Denote the share of this product that goes to the worker as $w(z, x)$, and share of the firm $r = \pi(z, s, x) - w(z, x)$. We now replace the original form of the worker's utility with $U(z, w, x)$. That is, now the worker only values a match in terms of the wages she will receive from that job. Otherwise the utility function has the same properties and satisfies the same assumptions 1. $-$ 4. as in Section 2. Lastly, assume in addition a further condition

5. $U_{wx}(z, w, x) \leq 0$ for $x \geq \gamma(z, w)$.

Under transferable utility, stability demands that the wage function $w(z, x)$ gives no worker-firm pair the incentive to match with each other rather than with their current match. The first step to determining the appropriate level of wages is taken from Becker's (1973) observation that the payment to each partner should be related to her marginal productivity for a matching to be stable. Let us first fix investment at some constant level x. Then, for our positive assortative matching to be stable, where a worker of type z is matched with a firm of type $S(z) = H^{-1}(G(z))$, it must be that

$$w(z + \varepsilon, x) + \pi(z, S(z), x) - w(z, x) \geq \pi(z + \varepsilon, S(z), x) \tag{8}$$

That is, the total payoff to a worker of type $z+\varepsilon$ and a firm of type $S(z)$ must be greater under the current matching arrangements than the output from a matching between each other. Otherwise, the worker of type $z + \varepsilon$ could strike a bargain with the firm of type $S(z)$ whereby they would both be better off. Similarly, if the we fix the type of worker at z, for stability given two workers producing output levels $x + \varepsilon$ and x, it must be that

$$w(z, x + \varepsilon) + \pi(z, S(z), x) - w(z, x) \geq \pi(z, S(z), x + \varepsilon) \tag{9}$$

In an equilibrium of the game of incomplete information, what is assumed is that matches are made and wage bargains struck on the basis of the perceived type of the workers. However, in any separating equilibrium, workers' actions fully reveal their underlying type. Again, let us assume that all workers adopt the same smooth strategy $x(z)$, which as it is strictly increasing reveals their type. Later it is shown that such an equilibrium exists. For now, this assumption together with the above inequalities are enough to determine the following. Cole, Mailath and Postlewaite (2001) offer a much more complete treatment of the equivalent problem under complete information.[5]

Proposition 2 *Let workers adopt a strictly increasing smooth strategy $x(z)$, and let C be an arbitrary constant satisfying $0 \leq C \leq \pi(\underline{z}, \underline{s}, x(\underline{z}))$, then positive assortative matching satisfying the relation (1) with the following bargaining solution,*

$$w_z(z, x) = \pi_z(z, S(z), x), \;\; w_x(z, x) = \pi_x(z, S(z), x), \tag{10}$$

$$w(z) = w(z, x(z)) = \int_{\underline{z}}^{z} [\pi_x(t, S(t), x(t))x'(t) + \pi_z(t, S(t), x(t))] \, dt + C$$

is stable.

Now, assume all workers adopt the strategy $x(z)$, but one agent contemplates a deviation to $x(\hat{z})$. He would expect a match with a firm of type $S(\hat{z})$ and a payment of $w(\hat{z}, x(\hat{z}))$, even though the actual product of the match will be $\pi(z, S(\hat{z}), x(\hat{z}))$. This gives first order conditions

$$U_x(z, w(\cdot), x(\hat{z}))x'(\hat{z}) + U_w(z, w(\cdot), x(\hat{z}))w_x(\hat{z}, x(\hat{z}))x'(z) + U_w(z, w(\cdot), x(\hat{z}))w_z(\hat{z}, x(\hat{z})) = 0.$$

In a symmetric equilibrium $\hat{z} = z$. Then, substituting from (10), we obtain the following differential equation.

$$x'(z) = \frac{-U_w(z, w, x)\pi_z(z, S(z), x)}{U_x(z, w, x) + \pi_x(z, S(z), x)U_w(z, w, x)}. \tag{11}$$

We need to define a level of investment x which is privately optimal, that is independent of matching considerations. Assume that that the positive assortative matching scheme $S(z)$ is exogenously imposed. This implies that an increase in x can only increase wages by increasing output not by a more favourable match. Or in other words, in the absence of matching considerations we need only consider the partial derivative of wages with respect to output $w_x(z, x) = \pi_x$. This enables the following definition.

[5] In particular, they show that the bargaining solution here $w(z)$ can have a finite number of discontinuities or jumps, though equally, completely continuous solutions are not excluded. Here, I concentrate on continuous solutions for reasons of simplicity.

Definition 2 *Let $x = \gamma_T(z)$ maximise $U(z, w(z), x)$, that is,*

$$U_x(z, w(z), \gamma_T(z)) + U_w(z, w(z), \gamma_T(z))\pi_x(z, S(z), \gamma_T(z)) = 0,$$

at every level of $z \in [\underline{z}, \bar{z}]$. The function $\gamma_T(z)$ is called the privately optimal level of investment x under TU.

This privately optimal level of investment will give us the appropriate boundary condition for the equilibrium differential equation.

$$x(\underline{z}) = \gamma_T(\underline{z}). \tag{12}$$

This, together with the earlier Proposition 1, leads to the next result.

Proposition 3 *The unique solution to the differential equation (11) on $(\underline{z}, \bar{z}]$ together with the boundary condition (12), the assortative matching scheme (1) and the wage function (10) constitute a symmetric equilibrium to the tournament matching game under Story 2A or 2B with transferable utility.*

Our equilibrium differential equation (11), while clearly not identical to the differential equation (6) that arose in the NTU case, does depend on the distributions $G(z)$ and $H(s)$ in a way the differential equation (7) in the standard model does not. Hence, both equilibrium payments $w(z)$ and the equilibrium strategy $x(z)$ will respond to changes in either in the distribution of ability $G(z)$ or of jobs $H(z)$.

We conclude this section with a couple of examples.

Example 1 *Workers are distributed according to $G(z) = z$ on [0,1], firms according to $H(s) = s^2$ on [0,1]. The production function is $\pi(z, s) = zs$ (Story 2A), which together with the matching assumption that $H(s) = G(z)$ implies that a worker of type z_i matches with a firm of type $s_i = \sqrt{z_i}$ and together they produce $z_i^{3/2}$. From the above analysis, $w'(z) = S(z) = \sqrt{z}$, and given $w(0) = 0$, $w(z) = (2/3)z^{3/2}$: workers get a bigger share, despite s being higher on average than z! This is because a worker's marginal product is determined by his match, i.e. the type of his employer, which is higher than her marginal product, which is determined by the type of her employee.*

The next example illustrates that relative effects occur in a strictly smaller set of cases under NTU than under TU. Suppose we take a production function where $\pi_{zs} = 0$, there are not strict complementarities, we find that the payment to the worker is determined by her absolute type, even though her equilibrium outcome would be determined by her relative position under NTU.

Example 2 *Assume now the production function is $\pi(z, s) = z + s$, so that a worker of type z matches with a firm of type $s = S(z)$ together they will produce $z + S(z)$. From the above analysis, $w'(z) = 1$, and given $w(0) = 0$, $w(z) = z$. Each worker gets his paid his type irrespective of the particular form of two distributions $G(z)$ and $H(s)$.*

5 Welfare

In this section, I consider whether in matching tournaments investment in visible output x is socially optimal, or whether it over or under supplied. Which will be the case is not obvious as there are two factors that work in opposite directions. First, workers may not internalise the benefit of the effect of additional investment on the profits of firms, leading to too little investment. Second, competition between workers for matching opportunities can push investment up, possibly to excessive levels. In the case of complete information, Cole et al. (2001) find that as investment raises one's marginal product, which in a TU framework leads to higher wages, this solves the first problem. Thus, efficient investment is possible in a non-cooperative equilibrium. However, Peters (2004) finds that, in a NTU framework, again under perfect information, the second factor is weaker than the first, and investment is inefficiently low.

Incomplete information offers different results and also some different questions. Here, I show that investment will be excessive even under TU and even when investment is productive and not a pure signal. In contrast, under NTU, investment can be either excessive or insufficient. Rege (2001) shows that if signalling is instrumental to matching, as it is here, then a separating equilibrium can generate higher welfare than completely random matching. When there are complements in production, there is a trade off between the costs of signalling and the benefits of assortative matching that it permits. Or, in other words, a more realistic point of comparison for a signalling equilibrium is with a situation of zero information, rather than one of complete information.

In the subsequent analysis, I take the more traditional route of comparing the non-cooperative outcome with first best. If there are complementarities between firms and workers, then from the results of Becker (1973), the maximisation of total output demands the positive assortative matching scheme $S(z)$. For this reason, in this section I extend the assumption that there are at least weak complementarities between workers and firms to the NTU case. Since matching is efficient, this allows us to concentrate on a different issue: whether, for each pair formed under this scheme, the worker chooses a level of investment that is optimal from the point of view of joint welfare. Often, as we will see, she invests too much. Sometimes, she invests too little.

5.1 NTU

Assume that the utility of workers is $U = U(z, s, x)$. The utility of a firm is simply the productivity of the worker it hires $\pi(z, s, x)$. Total welfare of a match between a firm and worker is given by

$$W = U(z, s, x) + \pi(z, s, x). \tag{13}$$

Then the first order conditions for a socially optimal level of investment are

$$\frac{dW}{dx} = U_x(z, S(z), x) + \pi_x(z, s, x) = 0 \tag{14}$$

Note that if π_x is zero, so that x is non-productive, the social optimum requires x to be equal to the privately optimal level γ_N. If $\pi_x > 0$, further conditions are needed for the above condition to be sufficient for a social optimum (see Proposition 4 below).

The non-cooperative first order conditions are

$$\frac{dU}{dx} = U_x(z, S(z), x) + U_s(z, S(z), x)\frac{S'(z)}{x'(z)} = 0 \tag{15}$$

If we make the assumption that investment is always productive or $\pi_x > 0$, then comparison of (14) and (15) leads directly to the next result.

Proposition 4 *Suppose that $\pi_x(z, s, x) > 0$ and that $\pi_{xx}(z, s, x) \leq 0$, then there exists a unique solution $\gamma_N^*(z)$ to the equation (14) at each level of z. Furthermore, for low types the non-cooperative level of investment $x(z)$ is less than the social optimum $\gamma_N^*(z)$. That is, there is an $z_1 > \underline{z}$ such that $x(z) < \gamma_N^*(z)$ on $[\underline{z}, z_1)$.*

Proof: The concavity of U in x and the concavity of p together ensure the first order conditions (14) define a maximum. We have, for the lowest type, $x(\underline{z}) = \gamma_N(\underline{z})$ by Lemma 2. However, at $\underline{z}$, as $\pi_x > 0$, for a social optimum from (14), the lowest type should produce more than $\gamma_N(\underline{z})$. ∎

That is, low types invest too little as their low prospects give no incentive to do more than which is privately optimal. However, one can also see that there is no fundamental reason why high types should also invest too little. We imagine that typically they will invest too much. Particularly, if the production function is concave, then as the marginal product of investment falls, the socially optimal investment will approach the cooperative level for high z. Concavity is not necessary for high types to overinvest, as is now shown by the following example.

Example 3 *Let $U = x(z-x)+s$ and $\pi(z, s, x) = x+z+s$. Then $\gamma_N^*(z) = (1+z)/2$. But if $S(z) = z^2$ on [0,1], then the noncooperative solution is $x(z) = 1.28z$. The solutions cross at $z_1 = 0.641$ and the noncooperative investment is higher than the socially optimal level for higher levels of z.*

That is, in this example, low ability workers invest too little and high ability invest too much.

5.2 TU

We continue with the general formulation that allows for x to be productive. A worker of type z choosing investment x matching with a firm of type s produces output $\pi(z, x, s)$. Profits are the residual output or $r(z, x) = \pi(z, x(z), S(z)) - w(z, x)$. Assume that the utility of workers is $U = U(z, w, x)$. The utility of a firm is simply its profit from hiring a worker r. The total payoff of a match is given by

$$W = U(z, w, x) + r(z, x) \tag{16}$$

Assume for a social optimum that wages reflect only marginal physical product and not matching considerations so that $w_x(z, x) = \pi_x(z, S(z), x)$. Note that this implies that $\partial r/\partial x = 0$. As the worker appropriates her marginal product from increased investment, it has no effect on profit. Then the condition for a social optimum is

$$\frac{dW}{dx} = U_x(z, w, x) + U_w(z, w, x)\pi_x(z, S(z), x) = 0. \tag{17}$$

That is, the social optimum equates the marginal cost of investment to the worker U_x and its marginal product $U_w \pi_x$. Note that under TU, this condition is the same as for the privately optimal level of investment $\gamma_T(z)$. This reflects the results of CMP (2001), who find that with complete information, a matching tournament can induce the efficient amount of investment.

However, under incomplete information there is a gap between private incentives and the social optimum. For an individual, an increase in investment x both may raise the output once matched and improve the match achieved. Once matching considerations are included, one has $dw(z, x)/dz = \pi_x(z, S(z), x)x'(z) + \pi_z(z, S(z), x)$. The non-cooperative first order conditions are

$$\frac{dU}{dx} = U_x(z, w, x) + U_w(z, w, x)\pi_x(z, S(z), x) + \frac{U_w(z, w, x)\pi_z(z, S(z), x)}{x'(z)} = 0 \tag{18}$$

Clearly, comparing (17) and (18), there is an additional positive term in (18). This is because each individual has an additional private return from increasing output as it permits a better match.

Proposition 5 *In the matching tournament with incomplete information and under TU, the equilibrium level of output $x(z)$ exceeds the socially optimal level almost everywhere.*

Proof: This follows directly from (17) and (18). ∎

6 Comparative Statics

We will now consider the effect on equilibrium utility and strategies of changes in the distribution of workers $G(z)$ and changes in the distribution of firms or jobs $H(s)$. In doing this, we consider only separating equilibria. We saw in Section 2 that equilibrium behaviour depends on the matching function S which is jointly determined by G and H. Our first question is what are the effects of changes in the underlying distributions on the matching function $S(z)$. We will then be better placed to answer questions about changes in equilibrium behaviour. In what follows we assume two economies A,B that are identical apart from having different distributions of workers or different distributions of jobs.[6]

Regime G: Change in the Distribution of Workers. In regime G, we assume that the economies have identical distributions of jobs, i.e. $H_A = H_B = H$, but differ in the distributions of workers, i.e. $G_A \neq G_B$. We also assume that G_A and G_B have the same support $[\underline{z}, \bar{z}]$. Different distributions of workers' abilities imply that the two societies have different matching functions, i.e. $S_A(z) = H^{-1}(G_A(z))$ and $S_B(z) = H^{-1}(G_B(z))$.

Regime H: Change in the Distribution of Jobs. In regime H, we assume that the economies have identical distributions of workers, i.e. $G_A = G_B = G$, but differ in the distributions of jobs, i.e. $H_A \neq H_B$. We again assume that H_A and H_B have the same support $[\underline{s}, \bar{s}]$. Again, different distributions of jobs imply that the two economies have different matching functions, i.e. $S_A(z) = H_A^{-1}(G(z))$ and $S_B(z) = H_B^{-1}(G(z))$.

We use stochastic dominance to order different distributions. One says one distribution G_A is stochastically higher or stochastically dominates another distribution G_B if $G_A(z) \leq G_B(z)$ for all z.

Proposition 6 *Regime G: if G_A first order stochastically dominates G_B, then $S_A(z) \leq S_B(z)$ for all $z \in [\underline{z}, \bar{z}]$.*

Proof: The first claim follows as since $H(\cdot)$ is an increasing function so is $H^{-1}(\cdot)$. Therefore, if for any z, $G_A(z) \leq G_B(z)$ then $S_A(z) \leq S_B(z)$. ∎

This is illustrated in Figure 2. We can now prove corresponding but very different results for changes in the distribution of jobs.

Proposition 7 *Regime H: if H_A first order stochastically dominates H_B, then $S_A(z) \geq S_B(z)$ for all $z \in [\underline{z}, \bar{z}]$.*

Proof: First, if H_A first order stochastically dominates H_B, then we have $H_A(s) \leq H_B(s)$ for all $s \in [\underline{s}, \bar{s}]$. This implies that if $G(z) = H_A(s^+) = H_B(s^-)$, then $s^+ \geq s^-$. But then $s^+ = H_A^{-1}(G(z)) \geq s^- = H_B^{-1}(G(z))$. ∎

That is, it seems that the comparative statics from changes in H are the reverse to those from changes in G. See Figures 1 and 2.

[6] For investigation of the effect of changes in the degree of inequality amongst workers in a similar framework, see Hopkins and Kornienko (2004a, 2005).

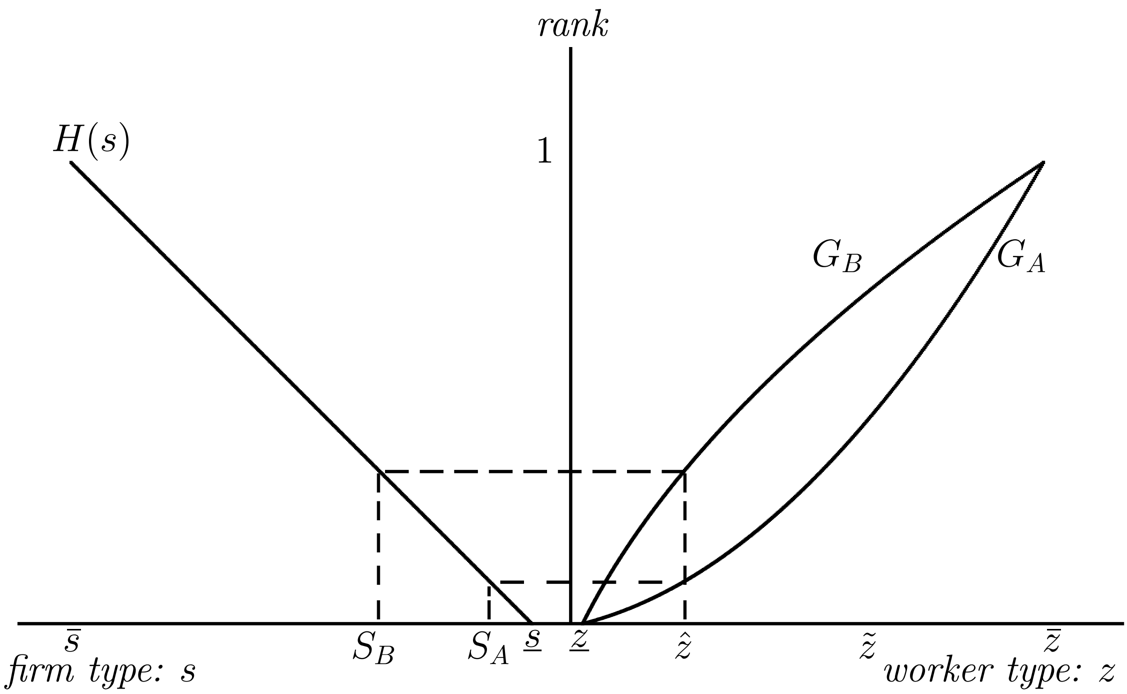

Figure 1: Regime G: a worker with given ability $\hat{z}$ has a match S_A under the stochastically higher distribution of ability G_A that is worse than the match S_B under the lower distribution of ability G_B.

6.1 NTU

We now apply the above results to see how equilibrium investment and utility respond to changes in the distribution of ability $G(z)$ and the distribution of jobs $H(s)$. Let $U^*(z, S(z), x(z)) = U^*(z)$ be workers' equilibrium utility under NTU. We have by the envelope theorem

$$\frac{dU^*(z, S(z), x(z))}{dz} = U^{*\prime}(z) = U_z(z, S(z), x(z)) \tag{19}$$

We first show that an increase in relative competition, in the sense of an increase in the quality of workers or a decrease in the quality of jobs available reduces equilibrium utility at every level of ability. In what follows, the assumption that $U_{zs} > 0$ and $U_{zx} > 0$ is crucial.

Proposition 8 *Suppose that* **either** *$G_A(z)$ first order stochastically dominates $G_B(z)$,* **or** *$H_B(z)$ first order stochastically dominates $H_A(z)$. Then, $U_A^*(z) \leq U_B^*(z)$ for all z in $[\underline{z}, \bar{z}]$.*

Proof: Note that the function $U^*(z)$ is continuously differentiable as $x(z)$ and $S(z)$ are continuously differentiable. Given the common boundary conditions (see Lemma 2) we have $U_A^*(\underline{z}) = U_B^*(\underline{z})$. In equilibrium, $x^*(z) > \gamma_N(z)$ (except perhaps at $\underline{z}$). It follows that $U^*(z, S(z), x(z))$ is strictly decreasing in x.

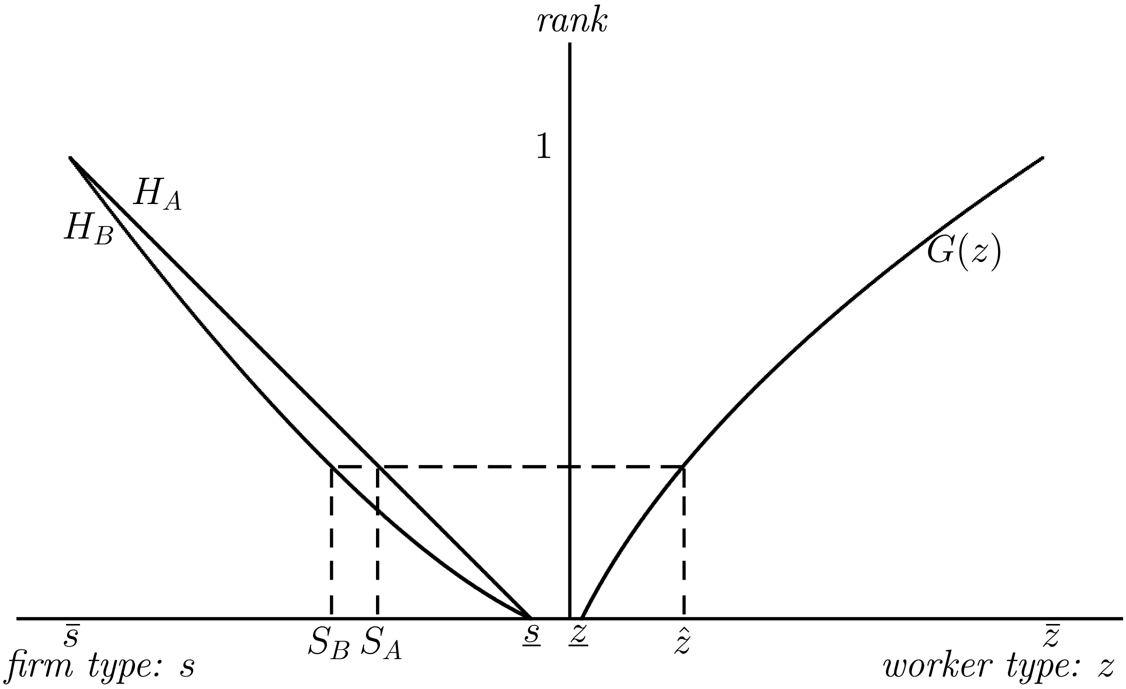

Figure 2: Regime H: a worker with given ability $\hat{z}$ has a match S_A under the stochastically lower distribution of jobs H_A that is worse than the match S_B under the higher distribution of jobs H_B.

Suppose the claim is false, and there exists at least one interval on $(\underline{z}, \tilde{z}]$ where $U_A^*(z) \geq U_B^*(z)$. Let us denote the set of points as $I_U = \{z \in (\underline{z}, \bar{z}] : U_A^*(z) > U_B^*(z)\}$ (possibly disjoint), and let $z_1 = \inf I_U \geq \underline{z}$. We can find a $z_2 \in I_U$ such that $U_A^*(z) > U_B^*(z)$ for all z in $(z_1, z_2]$. Note that since, by the common boundary condition, $U_A^*(\underline{z}) = U_B^*(\underline{z})$, we can rule out the case where $U_A^*(z_1) > U_B^*(z_1)$, so that $U_A^*(z_1) = U_B^*(z_1)$.[7] As $U_A^*(z) > U_B^*(z)$ and $S_A(z) \leq S_B(z)$ for all $z \in I_U$, it must be that $x_A(z) < x_B(z)$ for all $z \in I_U$. But then as $U^{*\prime}$ is increasing in $x(z)$ and increasing in $S(z)$, we have $U_A^{*\prime}(z) \leq U_B^{*\prime}(z)$ on I_U. This, together with $U_A^*(z_1) = U_B^*(z_1)$, implies $U_A^*(z) \leq U_B^*(z)$ for all $z \in (z_1, z_2]$, which is a contradiction. ∎

The next result shows that an increase in relative competition reduces investment by low ability types, but stimulates greater investment by high types. For this result, I make the further assumption that $S_A'(\underline{z}) < S_B'(\underline{z})$. This is a weak refinement to stochastic dominance. Note that if, for example, $G_A(z)$ first order stochastically dominates $G_B(z)$ then $S_A'(\underline{z}) > S_B'(\underline{z})$ is not possible and that $S_A'(\underline{z}) = S_B'(\underline{z})$ is not generic.

Proposition 9 *Suppose that $S_A'(\underline{z}) < S_B'(\underline{z})$ and* **either** *$G_A(z)$ first order stochastically dominates $G_B(z)$,* **or** *$H_B(z)$ first order stochastically dominates $H_A(z)$. Let x_A and x_B be the solutions to the differential equation (6) under $S_A(z)$ and $S_B(z)$ respectively. Then, $x_B(z) > x_A(z)$ on $(\underline{z}, \tilde{z})$ for some $\tilde{z} > \underline{z}$; there is then at least one crossing of $x_B(z)$ and $x_A(z)$ on $(\tilde{z}, \bar{z})$ so that $x_A(\bar{z}) \geq x_B(\bar{z})$.*

[7]For example, it is not possible that $U_A^*(z) > U_B^*(z)$ on $[\underline{z}, z_2]$.

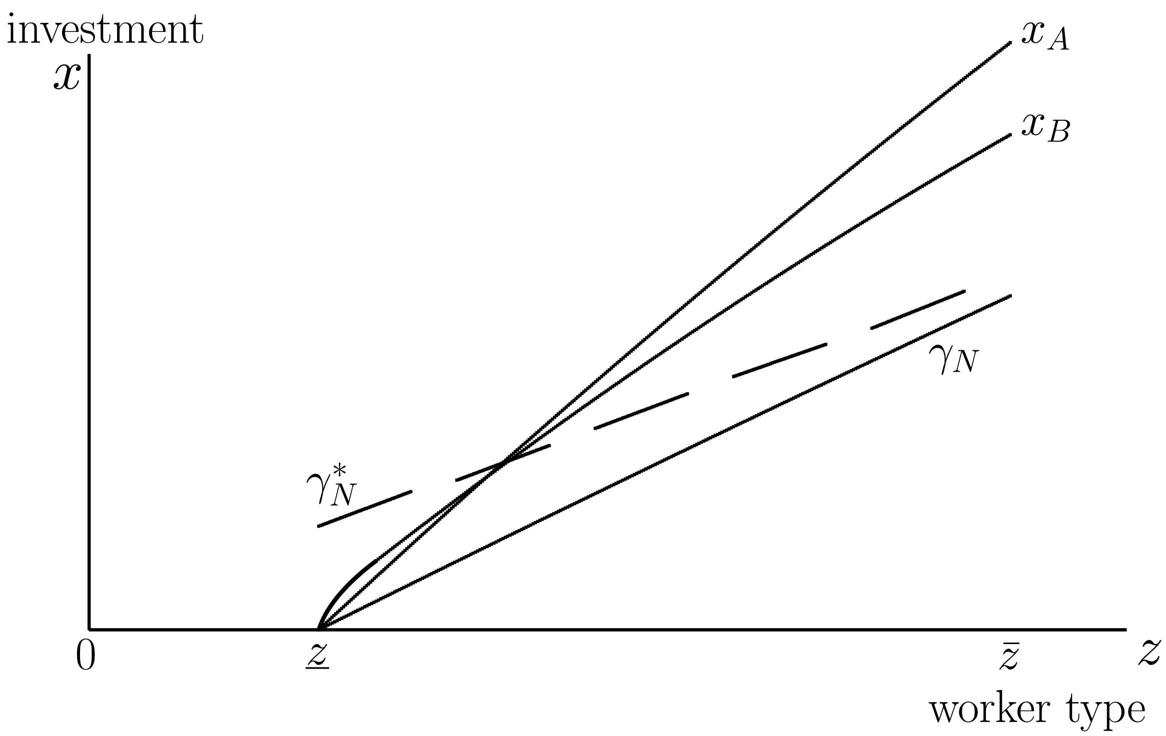

Figure 3: Under NTU, investment in the more competitive environment x_A is further from the social optimum γ_N^* for low and high ability workers than investment in the less competitive environment x_B.

Proof: First, as $S'_B(\underline{z}) > S'_A(\underline{z})$, then $x'_B(\underline{z}) > x'(\underline{z})$. So, $x_B(z) > x_A(z)$ immediately to the right of $\underline{z}$. Suppose there is no crossing on $(\underline{z}, \bar{z})$, so that $x_A(\bar{z}) < x_B(\bar{z})$ which implies that, as $S_A(\bar{z}) = S_B(\bar{z}) = \underline{s}$, the utility for the highest type must be ranked $U_A^*(\bar{z}) > U_B^*(\bar{z})$, which is a contradiction to our earlier result, Proposition 8. ∎

The comparative statics results on equilibrium investment are less precise than those on equilibrium utility. It is possible to obtain stronger results by making stronger assumptions. See Hopkins and Kornienko (2005) for examples of such results.

All the same, the results on investment do have a striking conclusion, as illustrated in Figure 3. In the more competitive environment, which has a matching function $S_A(z)$ from the point of view of workers, distortions from the socially optimal are larger. In particular, the low type workers who in any case invest too little will invest even less. And the high ability workers who put in too much effort will do even more.

6.2 TU

In this section, it is assumed that output depends only on the types involved in the match (Story 2A) or $\pi(z, s)$. This implies that the equilibrium differential equations reduce to

$$x'(z) = -\frac{U_w(z, w(z), x)}{U_x(z, w(z), x)} w'(z) = \psi(z, w(z), x)\pi_z(z, S(z)) \tag{20}$$

and

$$w'(z) = \pi_z(z, S(z)). \tag{21}$$

Proposition 10 *Let $\pi_{zs}(z, s) > 0$. Then* **either** *if $G_A(z)$ first order stochastically dominates $G_B(z)$,* **or** *if $H_B(z)$ first order stochastically dominates $H_A(z)$, it follows that $w_A(z) \leq w_B(z)$ for all $z \in [\underline{z}, \bar{z}]$.*

Proof: If $G_A(z)$ first order stochastically dominates $G_B(z)$, or if $H_B(z)$ first order stochastically dominates $H_A(z)$, then $S_A(z) \leq S_B(z)$ for all $z \in [\underline{z}, \bar{z}]$ (see Propositions 6 and 7). Hence, from the relationship (10) and the assumption $\pi_{zs} > 0$, it must be that $w'_A(z) \leq w'_B(z)$ for all $z \in [\underline{z}, \bar{z}]$ and, given a common boundary condition $w(\underline{z}) = C$, the result follows. ∎

This in turn implies a similar result on workers' utility. Let $U(z, w(z), x(z)) = U^*(z)$ be workers' equilibrium utility under TU.

Proposition 11 *Suppose* **either** *$G_A(z)$ first order stochastically dominates $G_B(z)$,* **or** *$H_B(z)$ first order stochastically dominates $H_A(z)$. Then, $U^*_A(z) \leq U^*_B(z)$ for all z in $[\underline{z}, \bar{z}]$.*

Proof: By the above Proposition 10, $w_A(z) \leq w_B(z)$ for all $z \in [\underline{z}, \bar{z}]$. We have by the envelope theorem $dU^*(z, w(z), x(z))/dz = U_z(z, w(z), x(z))$. Then, the proof is readily adaptable from the proof to the earlier result, Proposition 8. ∎

We can also find a similar result on the behaviour of investment.

Proposition 12 *Suppose that $S'_A(\underline{z}) < S'_B(\underline{z})$ and* **either** *$G_A(z)$ first order stochastically dominates $G_B(z)$,* **or** *$H_B(z)$ first order stochastically dominates $H_A(z)$. Let x_A and x_B be the solutions to the differential equation (20) under $S_A(z)$ and $S_B(z)$ respectively. Then, $x_B(z) > x_A(z)$ on $(\underline{z}, \tilde{z})$ for some $\tilde{z} > \underline{z}$.*

Proof: Given the common boundary condition that $x_A(\underline{z}) = x_B(\underline{z}) = \gamma_T(\underline{z})$ and that $S_A(\underline{z}) = S_B(\underline{z}) = \underline{s}$, evaluating the differential equation (20) at $\underline{z}$, we find that $x'_A(\underline{z}) = x'_B(\underline{z})$. However,

$$x''_A(\underline{z}) - x''_B(\underline{z}) = -\psi(z, \underline{s}, x(\underline{z}))\pi_{zs}(\underline{z}, \underline{s})\left(S'_A(\underline{z}) - S'_B(\underline{z})\right) < 0.$$

This implies that $x'_A(z) < x'_B(z)$ immediately to the right of $\underline{z}$ and the result follows. Suppose there is no crossing ∎

7 Unemployment

Up to now, it has been assumed that all workers are matched to jobs. Obviously, it is a characteristic of many real world labour markets that the least successful candidates fail to attract any offers as there are more candidates than there are job openings. It is relatively easy to modify the basic matching tournament model to allow for this. We find again that the model delivers sensible comparative statics. For example, a decrease in the number of jobs available relative to the number of workers will, in the TU case, lower wages at every level of ability.

Assume now that that the measure of firms relative to that of workers is $1-\mu$ so that a proportion $\mu > 0$ of workers will not find employment. Under assortative matching, these will be the least able, so that those having ability on the range $[\underline{z}, \hat{z})$, where $G(\hat{z}) = \mu$, will be unemployed. The utility from unemployment we take to be s_0, where $0 \leq s_0 \leq \underline{s}$. Together this implies the following assortative matching scheme

$$S(z) = \begin{array}{ll} s_0 & \text{for } z \in [\underline{z}, \hat{z}) \\ H^{-1}\left(\dfrac{G(z)-\mu}{1-\mu}\right) & \text{for } z \in [\hat{z}, \bar{z}]. \end{array} \tag{22}$$

This implies that $S'(z)$ is equal to zero on $(\underline{z}, \hat{z})$ and to $g(z)/(h(S(z))(1-\mu))$ on $[\hat{z}, \bar{z}]$.

7.1 NTU

Again it is possible to construct a symmetric separating equilibrium based on assortative matching. Those workers who anticipate unemployment will not invest any more than the cooperative level. However, if the cooperative solution is increasing in ability, this will still be separating. A greater problem is that if the worst job is strictly better than unemployment, there must be a jump in the equilibrium strategy $x(z)$ at $\hat{z}$ to prevent unemployed workers imitating the investment levels of those who are successful.

Proposition 13 *Let $x(z) = \gamma_N(z)$ on $[\underline{z}, \hat{z})$ where $\mu = G(\hat{z})$. Let $\hat{x} \geq \gamma_N(\hat{z})$ solve $U(\hat{z}, \hat{x}, \underline{s}) = U(\hat{z}, \gamma_N(\hat{z}), s_0)$. Let $x(z)$ be the solution to (6) on $[\hat{z}, \bar{z}]$ with boundary condition $x(\hat{z}) = \hat{x}$. Then, $x(z)$, together with the matching scheme (22), is a symmetric equilibrium strategy of the matching tournament under NTU.*

Proof: First, note that, in the proposed equilibrium, investment levels on the interval $(\gamma_N(\hat{z}), \hat{x})$ are off the equilibrium path. Assume that if any worker deviates and chooses x on that interval, firms believe with probability 1 that her type z is strictly less than $\hat{z}$. Then any deviation by any unemployed worker to any level of x in $[0, \hat{x})$ will not result in a job offer. There is, therefore, no incentive to make such a deviation. Deviation to a level of x above $\hat{x}$ is unprofitable by the definition of $\hat{x}$. For workers of type $z \in [\hat{z}, \bar{z}]$, the equilibrium is the same as in the case of full employment. ∎

The obvious question is what happens if the ratio of workers to jobs increases. Clearly, unemployment goes up, but we can also show that worker utility falls as the job market becomes more competitive.

Proposition 14 *Suppose $\mu_A > \mu_B$, let $G(\hat{z}_i) = \mu_i$ for $i = A, B$, and let $x_A(z), U_A^*(z)$ and $x_B(z), U_B^*(z)$ be the equilibrium strategy and utility respectively under the two respective values of μ. Then, $U_A^*(z) < U_B^*(z)$ for all $z \in (\hat{z}_B, \bar{z}]$. Further, $x_A(z) < x_B(z)$ on $(\hat{z}_B, \tilde{z})$ for some $\tilde{z} > \hat{z}_B$, but there is at least one crossing so that $x_A(\bar{z}) \geq x_B(\bar{z})$.*

Proof: A proof of first part is readily derivable from the proof of Proposition 8, simply replacing $\underline{z}$ with $\hat{z}_B$ at each point of the proof. Proof of the second part, concerning $x_A(z)$ and $x_B(z)$ similarly follows from Proposition 9. Note that $x_B(\hat{z}_B) \geq x_A(\hat{z}_B)$. If $s_0 = \underline{s}$, then $x_B(\hat{z}_B) = x_A(\hat{z}_B) = \gamma_N(\hat{z}_B)$ but $x_B'(\hat{z}_B) > x_A'(\hat{z}_B) = \gamma_N'(\hat{z}_B)$. If $x_B(\hat{z}_B) > x_A(\hat{z}_B)$, then the result follows from the continuity of x and that $U_A^*(\bar{z}) \leq U_B^*(\bar{z})$. ∎

7.2 TU

Under TU, it is simplest to work under Story 2A and assume that output is not effected by investment, or $\pi(z, s)$. Assume also that the μ unmatched workers are paid a fixed wage or benefit $\underline{w}$, whose level is exogenously fixed. Then, this return to unemployment provides a lower bound for wage bargaining, or

$$w(z) = \int_{\hat{z}}^{z} \pi_z(t, S(t))\, dt + C, \tag{23}$$

where $\underline{w} \leq C \leq \pi(\hat{z}, \underline{s})$. Then, we have the following equilibrium.

Proposition 15 *Let $x(z) = \gamma_T(z)$ on $[\underline{z}, \hat{z})$ where $\mu = G(\hat{z})$. Let $\hat{x} \geq \gamma_T(\hat{z})$ solve $U(\hat{z}, \hat{x}, C) = U(\hat{z}, \gamma_T(\hat{z}), \underline{w})$. Let $w(z) = \underline{w}$ on $[\underline{z}, \hat{z}]$ and be given by (23) on $(\hat{z}, \bar{z}]$. Let $x(z)$ be the solution to (11) on $[\hat{z}, \bar{z}]$ with boundary condition $x(\hat{z}) = \hat{x}$. Then, $x(z)$ is a symmetric equilibrium strategy of the matching tournament under TU.*

It is also possible to show that an increase in unemployment will lower wages.

Proposition 16 *Suppose $\mu_A > \mu_B$, let $G(\hat{z}_i) = \mu_i$ for $i = A, B$, and let $w_A(z), U_A^*(z)$ and $w_B(z), U_B^*(z)$ be the equilibrium wage, strategy and utility respectively under the two respective values of μ. Then, $w_A(z) < w_B(z)$ on $(\hat{z}_B, \bar{z})$ and $U_A^*(z) < U_B^*(z)$ on $(\hat{z}_B, \bar{z})$.*

Proof: Note that, while $S_A(\hat{z}_B) = S_B(\hat{z}_B)$, it holds that $S_A(z) < S_B(z)$ on $(\hat{z}_B, \underline{z}]$, and the result then follows from Propositions 10 and 11, replacing each instance of $\underline{z}$ in the proof with $\underline{z}_B$. ∎

8 Conclusions

This paper has introduced a model of relative signalling in a tournament-like labour market. By allowing for vertical differentiation amongst employers as well as workers, it generalises the classic model of Spence (1973). Competition for good jobs generates competition for relative position, implying that the outcome for any individual worker depend on the distribution characteristics of all firms and all workers. Changes in either distribution, representing changes in the demand and supply of labour respectively, affect equilibrium strategies and welfare.

In some research in incomplete information, lack of dependence on the distribution of types is taken to be an advantage. However, this is in the context of a different type of signalling model. Take for example a classic industrial organisation model of predation where an incumbent monopolist signals unobservable costs by its output choices. Note that in this case the distribution of types is the potential entrant's subjective beliefs about the unknown costs of the incumbent. The probability distribution in this case is *subjective* and largely unobservable as it is in the mind of the entrant. In contrast, in the labour market model considered here, the approach to beliefs is in effect *frequentist* as the distribution of types is simply the empirical distribution of workers' (or firms') qualities. The dependence of the type distribution is more natural in this context, where the distribution is observable and measurable.

The equilibria in this model, as is common under imperfect information, are not efficient. Typically, workers overinvest in education as education as well as increasing productivity also serves as a signal of ability. This offers the unusual prospect of labour taxes increasing rather than decreasing labour market efficiency. However, it was also found that low ability workers could potentially underinvest in developing useful skills, giving some rationale for a progressive tax and subsidy scheme.

Appendix

Proof of Lemma 1: In a symmetric equilibrium with a strictly increasing strategy $x(z)$, for an agent of type z_i we have $F(x(z_i)) = \Pr[x(z_i) < x(z)] = \Pr[x^{-1}(x(z_i)) < z] = G(z_i)$. Then the matching that assigns an agent with output x_i to an firm of type $s_i = H^{-1}(F(x_i)) = H^{-1}(G(z_i))$ is clearly stable as while any worker with rank $F(x_i)$ would prefer a match with any firm with $s > H^{-1}(F(x_i))$, such a firm would prefer her current match whose x, say $\hat{x}$, would be greater than x_i (and as $x(z)$ is strictly increasing, $\hat{z} = x^{-1}(\hat{x}) > z_i$). Suppose there is another matching $\tilde{\phi}$, such that a set of workers X with positive measure are matched differently than under the positive assortive matching ϕ. Then, there must exist $\hat{x} \in X$, such that $\tilde{\phi}(F(\hat{x})) > F(\hat{x})$, that is, there must be a positive measure of workers who are matched strictly higher than under ϕ. For this matching to be stable, all workers with output higher than $\hat{x}$ must be matched with firms whose s is greater than $\tilde{\phi}(F(\hat{x}))$. If not, then firm $s = \tilde{\phi}(F(\hat{x}))$

could propose a match with a worker of type $\tilde{x}$ where $\tilde{x} > \hat{x}$ and the worker $\tilde{x}$ would find it acceptable. But the measure of workers with x higher than $\hat{x}$, $\lambda(x \geq \hat{x})$, is strictly larger than the measure of firms with s greater than $\tilde{\phi}(F(\hat{x}))$, $\lambda(s \geq \tilde{\phi}(F(\hat{x})))$. But this implies that $\tilde{\phi}$ is not measure-preserving. ∎

Proof of Proposition 1: This follows from Theorems 1 and 2 of Mailath (1987, p1353). The only substantial difference is that in that work, the signaller's utility is of the form (in current notation) $V(z, \hat{z}, x)$ where V is a smooth utility function and $\hat{z}$ is the perceived type, so that in a separating equilibrium the signaller has utility $V(z, z, x)$. Now, clearly, one can find a smooth utility function $V(\cdot)$ such that $U(z, s, x) = V(z, \hat{z}, x)$ everywhere on $[\underline{z}, \bar{z}] \times [\underline{s}, \bar{s}] \times \mathbb{R}_+$. In particular, fix $G(z)$ and $H(s)$, and then $U(z, S(z), x) = V(z, z, x)$. One can then verify that the conditions 1-4 imposed on $U(\cdot)$ imply conditions 1-5 of Mailath (1987, p1352) on V.[8]

It also follows by Proposition 3 of Mailath (1987, p1362) that $x(z) \neq \gamma_N(z)$ on $(\underline{z}, \bar{z})$. Since $\hat{z}$ maximises $U(z, S(\hat{z}), x(\hat{z}))$, we have a first order condition $U_s(z, S(z), x(z)) + U_x(z, S(z), x(z))x'(z)$. By assumption 4. on $U(\cdot)$ and the definition of γ_N, $U_x(z, S(z), x) < 0$ for $x > \gamma_N(z)$, and as $U_s > 0$ everywhere by assumption, it follows that $x(z) > \gamma_N(z)$. ∎

Proof of Proposition 2: Taking Becker's (1973) hint that in the continuous case, the exact payment should be determinable, we obtain from (8),

$$w(z + \varepsilon, x) - w(z, x) \geq \pi(z + \varepsilon, S(z), x) - \pi(z, S(z), x).$$

Dividing both sides by ϵ and taking the limit of ϵ to zero, one finds that

$$w_z(z, x) \geq \pi_z(z, S(z), x). \tag{24}$$

Similarly from (9), one obtains

$$w_x(z, x) \geq \pi_x(z, S(z), x). \tag{25}$$

This also give us a bound on the total derivative $dw(z, x)/dz \geq \pi_z + \pi_x x'$. A similar analysis finds that the share of the firm satisfies

$$dr(z, x)/dz \geq \pi_s(z, S(z), x)S'(z) \tag{26}$$

But since $dw(z, x)/dz + dr(z, x)/dz = d\pi(z, S(z), x)/dz = \pi_x x'(z) + \pi_z + S'(z)\pi_s$, the above conditions hold with equality. The choice of the boundary condition $C = w(\underline{z})$ is arbitrary, except that it must be feasible, i.e. $0 \leq w(\underline{z}) \leq \pi(\underline{z}, \underline{s}, x(\underline{z}))$.

These marginal conditions imply general stability. Take any two types of worker z_1, z_2 with $z_2 > z_1$. The stability condition (8) using the formula (10) can be rewritten

[8] Mailath, in proving the intermediate result Proposiition 5 (1987, p1364), also assumes that $\partial V/\partial \hat{z}$ is bounded. Here, if we assume that both U_s and $S'(z)$ are bounded (the latter requires $g(\cdot)$ is bounded and $h(\cdot)$ is non-zero), this result will also hold.

as

$$\int_{z_1}^{z_2} \pi_z(z,S(z),x(z)) + \pi_x(z,S(z),x(z))x'(z)\ dz \geq \tag{27}$$
$$\int_{z_1}^{z_2} \pi_z(z,S(z_1),x(z)) + \pi_x(z,S(z_1),x(z))x'(z)\ dz.$$

Now, as matching is positive and assortative, the matching function $S(z)$ is increasing and $S(z) > S(z_1)$ for any $z \in (z_1, z_2]$. If, as assumed, $\pi_{zs} > 0$ and $\pi_{xs} \geq 0$ then the above equality must hold for any pair $z_2 > z_1$. ∎

Proof of Proposition 3: This follows the proof of Proposition 1. Fix $S(z)$. Fix $w(z,x)$ as a smooth increasing function $[\underline{z},\bar{z}] \times I\!R :\mapsto I\!R$, with partial derivatives as given in (10). Then, it is possible to find a smooth utility function $V(\cdot)$ such that $U(z,w(z,x),x) = V(z,\hat{z},x)$ everywhere on $[\underline{z},\bar{z}] \times I\!R^2_+$. It is then possible to verify that the conditions 1.-5. imposed on U imply Mailath's (1987, p1352) conditions 1.-5. on V. In particular, note that $V_3 = U_x + \pi_x U_w$. Mailath's condition (4) requires that $V_3(z,z,x) = 0$ has a unique solution. This here follows from the assumptions that $\pi_{xx} < 0$, and assumptions 4 and 5 on U.[9] Existence of an incentive compatible signalling equilibrium then follows from Theorems 1 and 2 of Mailath. However, for assortative matching to be stable, the wage function must satisfy (10). That is, solutions to the simultaneous differential equations (11) and

$$w'(z) = \pi_x(z,S(z),x)x'(z) + \pi_z(z,S(z),x) = \frac{\pi_z(z,S(z),x)U_x(z,w,x)}{U_x(z,w,x) + \pi_x(z,S(z),x)U_w(z,w,x)}$$

give the equilibrium investment and wage functions. ∎

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[9] Mailath also assumes that $\partial V/\partial\hat{z}$ is bounded above (see the previous footnote). This here is ensured if U_w and π_z are bounded above.

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