JOHN'S ELLIPSOID AND THE INTEGRAL RATIO OF A LOG-CONCAVE FUNCTION

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Abstract. We extend the notion of John's ellipsoid to the setting of integrable log-concave functions. This will allow us to define the integral ratio of a log-concave function, which will extend the notion of volume ratio, and we will find the log-concave function maximizing the integral ratio. A reverse functional affine isoperimetric inequality will be given, written in terms of this integral ratio. This can be viewed as a stability version of the functional affine isoperimetric inequality.

1. Introduction and notation

Asymptotic geometric analysis is a rather new branch in mathematics, which comes from the interaction of convex geometry and local theory of Banach spaces. From its beginning, the research interests in this area have been focused in understanding the geometric properties of the unit balls of high-dimensional Banach spaces and their behavior as the dimension grows to infinity. The unit ball of a finite dimensional Banach space is a centrally symmetric convex body and some of these geometric properties include the study of sections and projections of convex bodies, which are also convex bodies. However, when the distribution of mass in a convex body is studied, a convex body K is regarded as a probability space with the uniform probability on K and then the projections of the measure on linear subspaces are not the uniform probability on a convex body anymore and the class of convex bodies is left. Nevertheless, as a consequence of Brunn-Minkowski's inequality, we remain in the class of log-concave probabilities, which are the probability measures with a log-concave density with respect to the Lebesgue measure. It is natural then, to work in the more general setting of log-concave functions rather than in the setting of convex bodies and a big part of the research in the area has gone in the direction of extending results from convex bodies to log-concave functions (see, for instance, [AKM], [FM], [AKSW], [KM], [C], [CF]), while many of the open problems in the field are nowadays stated in terms of log-concave functions rather than in terms of convex bodies.

Date: April 24, 2017.

The first named author is partially supported by Spanish grants MTM2013-42105-P, MTM2016-77710-P projects and by IUMA. The second named author is partially supported by Fundación Séneca, Science and Technology Agency of the Región de Murcia, through the Programa de Formación Postdoctoral de Personal Investigador, project reference 19769/PD/15, and the Programme in Support of Excellence Groups of the Región de Murcia, Spain, project reference 19901/GERM/15. The third named author is supported by CAPES and IMPA. The second, third, and fourth authors are partially supported by MINECO project reference MTM2015-63699-P, Spain.

In [J] John proved that, among all the ellipsoids contained in a convex body K, there exists a unique ellipsoid $\mathcal{E}(K)$ with maximum volume. This ellipsoid is called the John's ellipsoid of K. Furthermore, he characterized the cases in which the John's ellipsoid of K is the Euclidean ball B_2^n. This characterization, together with Brascamp-Lieb inequality [BL], led to many important results in the theory of convex bodies, showing that, among centrally symmetric convex bodies, the cube is an extremal convex body for many geometric parameters like the Banach-Mazur distance to the Euclidean ball, the volume ratio, the mean width, or the mean width of the polar body, see [B], [SS], [Ba]. The non-symmetric version of these problems has also been studied, see for instance [S], [Le], [Pa], [JN], [Sch1].

A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be log-concave if it is of the form $f(x)=e^{-v(x)}$, with $v:\mathbb{R}^n\to(-\infty,+\infty]$ a convex function. Note that log-concave functions are continuous on their support and, since convex functions are differentiable almost everywhere, then so are log-concave functions. In this paper we will extend John's theorem to the context of log-concave functions. We will consider ellipsoidal functions (we will sometimes simply call them ellipsoids), which will be functions of the form

$$\mathcal{E}^a(x)=a\chi_{\mathcal{E}}(x),$$

with a a positive constant and $\chi_{\mathcal{E}}$ the characteristic function of an ellipsoid $\mathcal{E}$, *i.e.*, an affine image of the Euclidean ball ($\mathcal{E}=c+TB_2^n$ with $c\in\mathbb{R}^n$ and $T\in GL(n)$, the set of linear matrices with non-zero determinant). The determinant of a matrix T will be denoted by $|T|$. The volume of a convex body K will also be denoted by $|K|$.

Given a log-concave function $f:\mathbb{R}^n\to\mathbb{R}$, we will say that an ellipsoid $\mathcal{E}^a$ is contained in f if for every $x\in\mathbb{R}^n$, $\mathcal{E}^a(x)\le f(x)$. Notice that if $\mathcal{E}^a\le f$, then necessarily $0<a\le\|f\|_\infty$ and that for any $t\in(0,1]$

$$\mathcal{E}^{t\|f\|_\infty}\le f$$

if and only if the ellipsoid $\mathcal{E}$ is contained in the convex body

$$K_t(f)=\{x\in\mathbb{R}^n\,:\,f(x)\ge t\|f\|_\infty\}.$$

If $f=\chi_K(x)$ is the characteristic function of a convex body K, then an ellipsoid $\mathcal{E}$ is contained in K if and only if $\mathcal{E}^t\le f$ for any $t\in(0,1]$. In Section 2 we will show the following:

Theorem 1.1. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be an integrable log-concave function. There exists a unique ellipsoid $\mathcal{E}(f)=\mathcal{E}^{t_0\|f\|_\infty}$ for some $t_0\in[e^{-n},1]$, such that*

- $\mathcal{E}(f)\le f$
- $\displaystyle\int_{\mathbb{R}^n}\mathcal{E}(f)(x)dx=\max\left\{\int_{\mathbb{R}^n}\mathcal{E}^a(x)dx\,:\,\mathcal{E}^a\le f\right\}.$

We will call this ellipsoid the John's ellipsoid of f.

The existence and uniqueness of the John's ellipsoid of an integrable log-concave function f will allow us to define the integral ratio of f:

Definition 1.1. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be an integrable log-concave function and $\mathcal{E}(f)$ its John's ellipsoid. We define the integral ratio of f:*

$$I.rat(f)=\left(\frac{\int_{\mathbb{R}^n}f(x)dx}{\int_{\mathbb{R}^n}\mathcal{E}(f)(x)dx}\right)^{\frac{1}{n}}.$$

Remark. This quantity is affine invariant, i.e., $I.rat(f\circ T) = I.rat(f)$ for any affine map T. When $f=\chi_K$ is the characteristic function of a convex body then $I.rat(f) = v.rat(K)$, the volume ratio of K (Recall that $v.rat(K) = \left(\frac{|K|}{|\mathcal{E}(K)|}\right)^{\frac{1}{n}}$, where $\mathcal{E}(K)$ is the John's ellipsoid of K).

In Section 3 we will give an upper bound for the integral ratio of log-concave functions, finding the functions that maximize it. Namely, denoting by Δ_n and B_∞^n the regular simplex centered at the origin and the unit cube in $\mathbb{R}^n$, and by $\|\cdot\|_K$ the gauge function associated to a convex body K containing the origin, which is defined as

$$\|x\|_K = \inf\{\lambda>0\,:\,x\in\lambda K\},$$

we will prove the following

Theorem 1.2. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be an integrable log-concave function. Then,*

$$I.rat(f)\leq I.rat(g_c),$$

where $g_c(x) = e^{-\|x\|_{\Delta_n - c}}$ for any $c\in\Delta^n$. Furthermore, there is equality if and only if $\frac{f}{\|f\|_\infty} = g_c\circ T$ for some affine map T and some $c\in\Delta^n$. If we assume f to be even, then

$$I.rat(f)\leq I.rat(g),$$

where $g(x) = e^{-\|x\|_{B_\infty^n}}$, with equality if and only if $\frac{f}{\|f\|_\infty} = g\circ T$ for some linear map $T\in GL(n)$.

The value of the integral ratio of these functions will be computed and we obtain the following

Corollary 1.3. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be an integrable log-concave function. Then,*

$$I.rat(f)\leq \frac{e}{n}(n!)^{\frac{1}{n}}v.rat(\Delta^n)\sim c\sqrt{n}.$$

If we assume f to be even, then

$$I.rat(f)\leq \frac{e}{n}(n!)^{\frac{1}{n}}v.rat(B_\infty^n)\sim c\sqrt{n}.$$

The isoperimetric inequality states that for any convex body K the quantity $\frac{|\partial K|}{|K|^{\frac{n-1}{n}}}$ is minimized when K is a Euclidean ball. This inequality cannot be reversed in general. However, in [B], it was shown that for any symmetric convex body K, there exists an affine image TK such that the quotient $\frac{|\partial TK|}{|TK|^{\frac{n-1}{n}}}$ is bounded above by the corresponding quantity for the cube B_∞^n. If we do not impose symmetry then the regular simplex is the maximizer. This linear image is the one such that TK is in John's position, *i.e.*, the maximum volume ellipsoid contained in K is the Euclidean ball. The quantity studied in the isoperimetric inequality is not affine invariant but in [P], a stronger affine version of the isoperimetric inequality was established. Namely, it was shown that for any convex body K

$$|K|^{\frac{n-1}{n}}|\Pi^*(K)|^{\frac{1}{n}}\leq |B_2^n|^{\frac{n-1}{n}}|\Pi^*(B_2^n)|^{\frac{1}{n}},$$

where $\Pi^*(K)$, which is called the polar projection body of K, is the unit ball of the norm $\|x\|_{\Pi^*(K)} = |x||P_{x^\perp}K|$, being $P_{x^\perp}K$ the projection of K onto the hyperplane orthogonal to x. This inequality is known as Petty's projection inequality and there is equality in it if and only if K is an ellipsoid. Furthermore, following the idea in the proof of the reverse isoperimetric inequality, a stability version of it was given in [A], showing that for any convex body K

$$|K|^{\frac{n-1}{n}}|\Pi^*(K)|^{\frac{1}{n}} \geq \frac{1}{v.rat(K)}|B_2^n|^{\frac{n-1}{n}}|\Pi^*(B_2^n)|^{\frac{1}{n}}. \tag{1}$$

The isoperimetric inequality and Petty's projection inequality have their functional extensions. Namely, Sobolev's inequality, which states that for any function f in the Sobolev space

$$W^{1,1}(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) \,:\, \frac{\partial f}{\partial x_i} \in L^1(\mathbb{R}^n) \quad \forall i \right\}$$

we have

$$\||\nabla f|\|_1 \geq n|B_2^n|^{\frac{1}{n}}\|f\|_{\frac{n}{n-1}},$$

and the affine Sobolev's inequality, proved in [Z], which states that

$$\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}} \leq \frac{|B_2^n|}{2|B_2^{n-1}|}, \tag{2}$$

where $\Pi^*(f)$ is the unit ball of the norm

$$\|x\|_{\Pi^*(f)} = \int_{\mathbb{R}^n} |\langle \nabla f(y), x\rangle| dy.$$

We would like to recall here the fact that $W^{1,1}(\mathbb{R}^n)$ is the closure of $\mathcal{C}^1_{00}$, the space of $\mathcal{C}^1$ functions with compact support, [M]. These inequalities are actually equivalent to their geometric counterparts.

In Section 4 we will follow the same ideas to obtain functional versions of the reverse isoperimetric inequality and a stability version of the affine Sobolev inequality. We will prove the following extension of (1), which is a reverse form of (2) in the class of log-concave functions.

Theorem 1.4. *Let $f \in W^{1,1}(\mathbb{R}^n)$ be a log-concave function. Then*

$$\frac{\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)} \geq \frac{1}{e^{\frac{\int_{\mathbb{R}^n} f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{n\int_{\mathbb{R}^n} f(x)dx}}\|f\|_\infty^{\frac{1}{n}}\left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}} I.rat(f)}.$$

Remark. By (2) the left-hand side term is bounded above by 1. This lower bound is affine invariant, and if $f = \chi_K$ is the characteristic function of a convex body, then we recover inequality (1). Besides, since $\log\frac{f(x)}{\|f\|_\infty}$ is a concave function, if f is centered we have by Jensen's inequality that

$$\frac{\int_{\mathbb{R}^n} f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{\int_{\mathbb{R}^n} f(x)dx} \leq \log\left(\frac{f\left(\frac{\int_{\mathbb{R}^n} xf(x)dx}{\int_{\mathbb{R}^n} f(x)dx}\right)}{\|f\|_\infty}\right) = \log\left(\frac{f(0)}{\|f\|_\infty}\right)$$

and so

$$\frac{\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)} \geq \frac{1}{f(0)^{\frac{1}{n}}\left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}} I.rat(f)}.$$

Remark. Let us note that if $\int_{\mathbb{R}^n} f(x)dx = 1$ the inequality in Theorem 1.4 turns into

$$e^{\frac{-1}{n}\int f(x)\log f(x)}dx \leq \frac{I.rat(f)|\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)},$$

which along with the affine Sobolev inequality (2) provides us with a bound for the power entropy of f of the following form

$$H(f) := e^{\frac{-2}{n}\int_{\mathbb{R}^n} f(x)\log f(x)dx} \leq \left(\frac{I.rat(f)}{\|f\|_{\frac{n}{n-1}}}\right)^2.$$

For other recently studied connections between Information theory and convex geometry we refer to [BM1], [BM2] and references therein.

Let us introduce some more notation:

For any function $f : \mathbb{R}^n \to \mathbb{R}$ and any $\varepsilon > 0$, we will denote f_ε the function given by

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)^\varepsilon.$$

If f and g are two log-concave functions, then their Asplund product is the log-concave function

$$f \star g(z) = \max_{z=x+y} f(x)g(y) = \max_{y\in\mathbb{R}^n} f(z-y)g(y).$$

2. John's ellipsoid of a log-concave function

In this section we show the existence and uniqueness of the John's ellipsoid of an integrable log-concave function and show that the integral ratio of a function is an affine invariant.

For any ellipsoid $\mathcal{E}^a$, its integral is $a|\mathcal{E}|$. Since for any $t \in (0,1]$ the convex body $K_t(f)$ has a unique maximum volume ellipsoid $\mathcal{E}_t(f) = \mathcal{E}(K_t(f))$, then

$$\max\left\{\int_{\mathbb{R}^n} \mathcal{E}^a(x)dx \,:\, \mathcal{E}^a \leq f\right\} = \max_{t\in(0,1]} \phi_f(t),$$

where

$$\phi_f(t) = t\|f\|_\infty |\mathcal{E}_t(f)|.$$

Thus, in order to prove Theorem 1.1 we need to prove that the function $\phi_f(t)$ attains a unique maximum in the interval $(0,1]$ at some point $t_0 \geq e^{-n}$. Then the ellipsoid $\mathcal{E}(f)$ will be the function

$$\mathcal{E}(f)(x) = t_0\|f\|_\infty \chi_{\mathcal{E}_{t_0}(f)}(x) = (\mathcal{E}_{t_0}(f))^{t_0\|f\|_\infty}(x),$$

where $\mathcal{E}_{t_0}(f)$ is the John's ellipsoid of the convex body $K_{t_0}(f)$. If $f = \chi_K$ with K a convex body then the John's ellipsoid of f will be the characteristic function of the John's ellipsoid of K $\mathcal{E}(K)^1 = \chi_{\mathcal{E}(K)}$. We will prove that ϕ_f attains a unique maximum in the interval $(0,1]$. First we prove the following:

Lemma 2.1. *Let $L_1, L_2 \subseteq \mathbb{R}^n$ be two convex bodies. Then, for any $\lambda \in [0,1]$*

$$|\mathcal{E}((1-\lambda)L_1 + \lambda L_2)|^{\frac{1}{n}} \geq (1-\lambda)|\mathcal{E}(L_1)|^{\frac{1}{n}} + \lambda|\mathcal{E}(L_2)|^{\frac{1}{n}}.$$

Proof. Let $\mathcal{E}(L_i) = a_i + T_i B_2^n$ with T_i a symmetric positive definite matrix, $i = 1, 2$. Then

$$\begin{aligned} (1-\lambda)L_1 + \lambda L_2 &\supseteq (1-\lambda)\mathcal{E}(L_1) + \lambda\mathcal{E}(L_2) \\ &= (1-\lambda)a_1 + \lambda a_2 + (1-\lambda)T_1 B_2^n + \lambda T_2 B_2^n \\ &\supseteq (1-\lambda)a_1 + \lambda a_2 + ((1-\lambda)T_1 + \lambda T_2)B_2^n. \end{aligned}$$

Since by Minkowski's determinant inequality, for any two symmetric positive definite matrices A, B we have that $|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$ with equality if and only if $B = sA$ for some $s > 0$, we obtain

$$\begin{aligned} |\mathcal{E}\left((1-\lambda)L_1 + \lambda L_2\right)|^{\frac{1}{n}} &\geq |(1-\lambda)T_1 + \lambda T_2|^{\frac{1}{n}} |B_2^n|^{\frac{1}{n}} \\ &\geq ((1-\lambda)|T_1|^{\frac{1}{n}} + \lambda|T_2|^{\frac{1}{n}})|B_2^n|^{\frac{1}{n}} \\ &= (1-\lambda)|\mathcal{E}(L_1)|^{\frac{1}{n}} + \lambda|\mathcal{E}(L_2)|^{\frac{1}{n}}. \end{aligned}$$

□

Now, Theorem 1.1 will be a consequence of the following

Lemma 2.2. *Let $f : \mathbb{R}^n \to [0, +\infty)$ be an integrable log-concave function and let $\phi_f : (0, 1] \to \mathbb{R}$ defined as before. Then ϕ_f is continuous in $(0, 1)$,*

$$\lim_{t \to 0^+} \phi_f(t) = 0,$$

and ϕ_f attains its maximum value at some $t_0 \in (0, 1]$. Furthermore, such t_0 is unique.

Proof. Since f is log-concave, for any $u_1, u_2 \in [0, \infty)$ and any $\lambda \in [0, 1]$,

$$\begin{aligned} \{x \in \mathbb{R}^n : f(x) \geq e^{-((1-\lambda)u_1 + \lambda u_2)}\|f\|_\infty\} &\supseteq (1-\lambda)\{x \in \mathbb{R}^n : f(x) \geq e^{-u_1}\|f\|_\infty\} \\ &+ \lambda\{x \in \mathbb{R}^n : f(x) \geq e^{-u_2}\|f\|_\infty\} \end{aligned}$$

and then, by Lemma 2.1

$$\begin{aligned} |\mathcal{E}_{e^{-((1-\lambda)u_1+\lambda u_2)}}(f)|^{\frac{1}{n}} &\geq |\mathcal{E}\left((1-\lambda)K_{e^{-u_1}}(f) + \lambda K_{e^{-u_2}}(f)\right)|^{\frac{1}{n}} \\ &\geq (1-\lambda)|\mathcal{E}_{e^{-u_1}}(f)|^{\frac{1}{n}} + \lambda|\mathcal{E}_{e^{-u_2}}(f)|^{\frac{1}{n}} \\ &\geq |\mathcal{E}_{e^{-u_1}}(f)|^{\frac{1-\lambda}{n}} |\mathcal{E}_{e^{-u_1}}(f)|^{\frac{\lambda}{n}}, \end{aligned} \tag{4}$$

where the last inequality is the arithmetic-geometric mean inequality.

Consequently, the function $g(u) := |\mathcal{E}_{e^{-u}}(f)|^{\frac{1}{n}}$ is concave on $[0, \infty)$ and thus continuous on $(0, \infty)$ and the function $\phi_f(t) = t\|f\|_\infty g^n(-\log(t))$ is continuous on $(0, 1)$.

Let us now prove that $\lim_{t\to 0^+} \phi_f(t) = 0$. Let $\varepsilon > 0$. Since f is integrable, we can find $R(\varepsilon)$ big enough such that

$$\int_{\mathbb{R}^n \setminus R(\varepsilon)B_2^n} f(x)dx < \frac{\varepsilon}{2}.$$

Now, for any $t < \frac{\varepsilon}{2\|f\|_\infty |R(\varepsilon)B_2^n|}$ we have that

$$\begin{aligned} t\|f\|_\infty |K_t(f)| &= t\|f\|_\infty |K_t(f) \cap R(\varepsilon)B_2^n| + t\|f\|_\infty |K_t(f) \setminus R(\varepsilon)B_2^n| \\ &< t\|f\|_\infty |R(\varepsilon)B_2^n| + \int_{K_t(f)\setminus R(\varepsilon)B_2^n} f(x)dx \\ &< \frac{\varepsilon}{2} + \int_{\mathbb{R}^n \setminus R(\varepsilon)B_2^n} f(x)dx \end{aligned}$$

$$< \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then,

$$0 \leq \lim_{t \to 0^+} t \|f\|_\infty |\mathcal{E}_t(f)| \leq \lim_{t \to 0^+} t \|f\|_\infty |K_t(f)| = 0$$

and so

$$\lim_{t \to 0^+} \phi_f(t) = 0,$$

or, equivalently,

$$\lim_{u \to \infty} e^{-\frac{u}{n}} g(u) = 0.$$

Besides, since g is concave there exists $\lim_{u \to 0^+} e^{-\frac{u}{n}} g(u) \in \mathbb{R}$. Consequently, $e^{-\frac{u}{n}} g(u)$ attains its maximum for some $u_0 \in [0, \infty)$ and so ϕ_f attains its maximum for some $t_0 = e^{-u_0} \in (0, 1]$.

Let us prove that such u_0 (and thus t_0) is unique. Assume that there exist two different $u_1 < u_2$ at which $e^{-\frac{u}{n}} g(u)$ attains its maximum. Then, the function $h(u) = -\frac{u}{n} + \log g(u)$, which is concave since g is concave, attains its maximum at u_1 and u_2. Thus, for every $\lambda \in [0, 1]$

$$h((1-\lambda)u_1 + \lambda u_2) = (1-\lambda)h(u_1) + \lambda h(u_2)$$

and so

$$g((1-\lambda)u_1 + \lambda u_2) = g^{1-\lambda}(u_1) g^\lambda(u_2).$$

Consequently, all the inequalities in (4) are equalities and, since there is equality in the arithmetic-geometric mean inequality, $g(u_1) = g(u_2)$. But then $h(u_1) > h(u_2)$, which contradicts the assumption of the maximum being attained at two different points.

It is left to prove that $t_0 \geq e^{-n}$. This will be done as an observation in the proof of Lemma 3.1.

□

Now that we have established the existence and uniqueness of the John's ellipsoid of an integrable log-concave function f, we can define the integral ratio of f as

$$I.rat(f) = \left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} \mathcal{E}(f)(x)dx} \right)^{\frac{1}{n}} = \left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\max_{t \in (0,1]} \phi_f(t)} \right)^{\frac{1}{n}}.$$

The integral ratio of a function is an affine invariant, *i.e.*, for any affine map T we have that $I.rat(f \circ T) = I.rat(f)$. This is a consequence of the following lemma.

Lemma 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable log-concave function and let T be an affine map. Then for any $t \in (0, 1]$*

$$\mathcal{E}_t(f \circ T^{-1}) = T\mathcal{E}_t(f).$$

As a consequence

$$\phi_{f \circ T^{-1}}(t) = |T| \phi_f(t),$$

the maximum of $\phi_{f \circ T^{-1}}$ and ϕ_f is attained for the same t_0, and

$$\mathcal{E}(f \circ T^{-1}) = \mathcal{E}(f) \circ T^{-1}.$$

Proof. Notice that

$K_t(f\circ T^{-1})=\{x\in\mathbb{R}^n\,:\,f(T^{-1}x)\geq t\|f\|_\infty\}=T\{x\in\mathbb{R}^n\,:\,f(x)\geq t\|f\|_\infty\}=TK_t(f).$

Consequently

$$\mathcal{E}_t(f\circ T^{-1})=T\mathcal{E}_t(f).$$

□

A log-concave function will be said to be in John's position if $\mathcal{E}(f)=(B_2^n)^{t_0\|f\|_\infty}$ for some $t_0\in(0,1]$. As a consequence of the previous lemma, for any log-concave integrable function there exists an affine map T such that $f\circ T$ is in John's position.

3. Maximal value of the integral ratio of log-concave functions

In this section we will obtain an estimate for the function $\phi_f(t)$ that will allow us to give an upper bound for the integral ratio of any integrable log-concave function. In order to do that we start proving the following

Lemma 3.1. *Let f be an integrable log-concave function such that* $\max_{t\in(0,1]}\phi_f(t)=\phi_f(t_0)$*, i.e., its John's ellipsoid is $\mathcal{E}(f)=\mathcal{E}_{t_0}(f)^{t_0\|f\|_\infty}$. Then for every $t\in(0,1]$*

$$|\mathcal{E}_t(f)|\leq\left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n|\mathcal{E}_{t_0}(f)|.$$

Besides, if there is equality for every $t\in(0,1]$, then for some $c_t\in\mathbb{R}^n$

$$\mathcal{E}_t(f)=c_t+\left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)\mathcal{E}_{t_0}(f).$$

Proof. Notice that if ϕ_f attains its maximum at t_0, then the function $\widetilde{g}(u):=e^{-\frac{u}{n}}|\mathcal{E}_{e^{-u}}(f)|^{\frac{1}{n}}$, defined on $[0,\infty)$, attains its maximum at $u_0=-\log t_0$. Then, for every $u\in[0,\infty)$,

$$|\mathcal{E}_{e^{-u}}(f)|^{\frac{1}{n}}\leq|\mathcal{E}_{e^{-u_0}}(f)|^{\frac{1}{n}}e^{-\frac{u_0-u}{n}}.$$

Since the function $h(u)=|\mathcal{E}_{e^{-u_0}}(f)|^{\frac{1}{n}}e^{-\frac{u_0-u}{n}}$ is convex in $\mathbb{R}$ and the function $g(u)=|\mathcal{E}_{e^{-u}}(f)|^{\frac{1}{n}}$ is concave in $[0,\infty)$, as we have seen in Lemma 2.2 the graph of g is under the tangent at u_0 to the graph of h. Thus, for every $u\in[0,\infty)$

$$g(u)\leq g(u_0)\left(1+\frac{u-u_0}{n}\right).$$

Observe that since $g(0)\geq 0$, it must be $u_0\leq n$ and thus $t_0\geq e^{-n}$. Setting $u=-\log t$ we obtain that for every $t\in(0,1]$

$$|\mathcal{E}_t(f)|\leq\left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n|\mathcal{E}_{t_0}(f)|.$$

If there is equality for every $t\in(0,1]$, then for every $u\in[0,\infty)$

$$g(u)=g(u_0)\left(1+\frac{u-u_0}{n}\right)$$

and the function g is an affine function. Thus, for every $u_1,u_2\in[0,\infty)$ and any $\lambda\in[0,1]$ all the inequalities in (4) are equalities and, by the equality cases in

Minkowski's determinant inequality, $\mathcal{E}_{e^{-u_1}}(f)$ and $\mathcal{E}_{e^{-u_2}}(f)$ are homothetic for every $u_1, u_2 \in [0,\infty)$ and so, for every $t \in (0,1]$

$$\mathcal{E}_t(f) = c_t + \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)\mathcal{E}_{t_0}(f)$$

for some $c_t \in \mathbb{R}^n$. □

The maximizers of the integral ratio will be log-concave functions like the ones defined in the following lemma. Let us study some of their properties

Lemma 3.2. *For any $t_0 \geq e^{-n}$ and convex body $K \subseteq \mathbb{R}^n$ with $0 \in K$, let*

$$f_{K,t_0}(x) = e^{-\max\{\|x\|_K - (n+\log t_0), 0\}}.$$

Then

- $K_t(f_{K,t_0}) = \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right) K_{t_0}(f_{K,t_0})$
- $\mathcal{E}_t(f_{K,t_0}) = \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right) \mathcal{E}_{t_0}(f_{K,t_0})$
- $\max_{t\in(0,1]} \phi_{f_{K,t_0}}(t) = \phi_{f_{K,t_0}}(t_0)$
- $I.rat(f_{K,t_0}) = \dfrac{v.rat(K)}{t_0^{\frac{1}{n}}} \left(\int_0^1 \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n dt\right)^{\frac{1}{n}}$
- *$I.rat(f_{K,t_0})$ is decreasing in t_0 in the interval $[e^{-n}, 1]$.*

Proof. Notice that $\|f_{K,t_0}\|_\infty = 1$. Then, by definition of $K_t(f_{K,t_0})$

$$\begin{aligned} K_t(f_{K,t_0}) &= \{x \in \mathbb{R}^n \,:\, \max\{\|x\|_K - (n+\log t_0), 0\} \leq -\log t\} \\ &= \left(n - \log\left(\frac{t}{t_0}\right)\right) K = n\left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right) K \end{aligned}$$

Consequently, for any $t \in (0,1]$

$$K_t(f_{K,t_0}) = \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right) K_{t_0}(f_{K,t_0}).$$

Then

$$\mathcal{E}_t(f_{K,t_0}) = \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right) \mathcal{E}_{t_0}(f_{K,t_0})$$

and

$$\phi_{f_{K,t_0}}(t) = \frac{t}{t_0}\left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n \phi_{f_{K,t_0}}(t_0).$$

Since the function $g(x) = x(1-\log x)$ attains its maximum at $x = 1$, $\phi_{f_{K,t_0}}(t)$ attains its maximum at $t = t_0$. Consequently

$$\begin{aligned} I.rat(f_{K,t_0})^n &= \frac{1}{t_0|\mathcal{E}_{t_0}(f_{K,t_0})|}\int_{\mathbb{R}^n} f_{K,t_0}(x)dx \\ &= \frac{1}{t_0|\mathcal{E}_{t_0}(f_{K,t_0})|}\int_0^1 |K_t(f_{K,t_0})|dt \\ &= \frac{|K_{t_0}(f_{K,t_0})|}{t_0|\mathcal{E}_{t_0}(f_{K,t_0})|}\int_0^1 \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n dt \\ &= \frac{v.rat(K)^n}{t_0}\int_0^1 \left(1 - \frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n dt. \end{aligned}$$

Changing variables $t = t_0 e^{-s}$ we have

$$I.rat(f_{K,t_0})^n = v.rat(K)\int_{\log t_0}^{+\infty}\left(1+\frac{1}{n}s\right)^n e^{-s}ds,$$

which is clearly decreasing in $t_0 \in [e^{-n}, 1]$. □

Now, we have the following, which in particular, since $I.rat(f_{B_\infty^n,t_0})$ and $I.rat(f_{\Delta^n,t_0})$ decrease in t_0, implies Theorem 1.2.

Theorem 3.3. *Let $t_0 \in (0,1]$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable log-concave such that $\max_{t\in(0,1]}\phi_f(t) = \phi_f(t_0)$, i.e., its John's ellipsoid is $\mathcal{E}(f) = \mathcal{E}_{t_0}(f)^{t_0\|f\|_\infty}$. Then we have that*

$$I.rat(f) \le I.rat(f_{\Delta^n,t_0})$$

with equality if and only if $\frac{f}{\|f\|_\infty} = f_{\Delta_n - c,t_0}\circ T$ for some affine map T and some $c \in \Delta^n$. If f is even

$$I.rat(f) \le I.rat(f_{B_\infty^n,t_0})$$

with equality if and only if $\frac{f}{\|f\|_\infty} = f_{B_\infty^n,t_0}\circ T$ for some $T \in GL(n)$.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\max_{t\in(0,1]}\phi_f(t) = \phi_f(t_0)$. Then

$$\begin{aligned}
I.rat(f)^n &= \frac{1}{t_0\|f\|_\infty|\mathcal{E}_f(t_0)|}\int_{\mathbb{R}^n} f(x)dx\\
&= \frac{1}{t_0|\mathcal{E}_f(t_0)|}\int_0^1 |K_t(f)|dt\\
&= \frac{1}{t_0|\mathcal{E}_f(t_0)|}\int_0^1 v.rat(K_t)^n|\mathcal{E}_f(t)|dt\\
&\le \frac{v.rat(\Delta^n)^n}{t_0|\mathcal{E}_f(t_0)|}\int_0^1 |\mathcal{E}_f(t)|dt\\
&\le \frac{v.rat(\Delta^n)^n}{t_0}\int_0^1\left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n dt\\
&= I.rat(f_{\Delta^n,t_0})^n.
\end{aligned}$$

Besides, if there is equality, all the inequalities are equalities and so $v.rat(K_t) = v.rat(\Delta^n)$, which implies that $K_t = T_t\Delta^n$, for some affine map T_t and $|\mathcal{E}_f(t)| = \left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)^n|\mathcal{E}_f(t_0)|$, which by Corollary 3.1 implies that the John's ellipsoid of every level set $\mathcal{E}_f(t) = c_t + \left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)\mathcal{E}_f(t_0)$ and so $T_t = c_t + \left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)T$ for every $t \in (0,1]$. Thus, we have that

$$K_t = c_t + \left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)T\Delta^n.$$

By Lemma 2.3 we can assume without loss of generality that $K_{t_0} = n\Delta^n$. In such case $c_{t_0} = 0$. Then, calling $t = e^{-s}$ and $t_0 = e^{-s_0}$ we have that

$$K_{e^{-s}} = c_{e^{-s}} + \left(1+\frac{s}{n}-\frac{s_0}{n}\right)K_{e^{-s_0}}.$$

By log-concavity, we have that for every $s \in [0, s_0]$

$$K_{e^{-s}} \supseteq \frac{s}{s_0}K_{e^{-s_0}} + \left(1-\frac{s}{s_0}\right)K_1$$

$$= \left(1-\frac{s}{s_0}\right)c_1+\left(1+\frac{s}{n}-\frac{s_0}{n}\right)K_{e^{-s_0}}$$

and then $c_{e^{-s}}=\left(1-\frac{s}{s_0}\right)c_1$. If $s\geq s_0$ we have that

$$\begin{aligned} K_{e^{-s_0}} &\supseteq \frac{s_0}{s}K_{e^{-s}}+\left(1-\frac{s_0}{s}\right)K_1\\ &= \frac{s_0}{s}c_{e^{-s}}+\left(1-\frac{s_0}{s}\right)c_1+K_{e^{-s_0}}\end{aligned}$$

and also in this case $c_{e^{-s}}=\left(1-\frac{s}{s_0}\right)c_1$. Thus, for any $s\geq 0$

$$K_{e^{-s}}=\left(1-\frac{s}{s_0}\right)c_1+\left(1+\frac{s}{n}-\frac{s_0}{n}\right)K_{e^{-s_0}}.$$

Consequently, $\frac{f}{\|f\|_\infty}=e^{-v(\cdot)}\circ T$ with T an affine map and

$$\begin{aligned} v(x) &= \inf\{s : x\in K_{e^{-s}}\}\\ &= \inf\left\{s : x\in\left(1-\frac{s}{s_0}\right)c_1+\left(1+\frac{s}{n}-\frac{s_0}{n}\right)K_{e^{-s_0}}\right\}\\ &= \inf\left\{s : x\in\frac{n}{s_0}c_1+(s+n-s_0)\left(\frac{K_{e^{-s_0}}}{n}-\frac{1}{s_0}c_1\right)\right\}\\ &= \inf\left\{s : x-\frac{n}{s_0}c_1\in+(s+n-s_0)\left(\frac{K_{e^{-s_0}}}{n}-\frac{1}{s_0}c_1\right)\right\}\\ &= \max\left\{\left\|x-\frac{n}{s_0}c_1\right\|_{\left(\frac{1}{n}K_{e^{-s_0}}-\frac{1}{s_0}c_1\right)}-(n-s_0),0\right\}\\ &= \max\left\{\left\|x+\frac{n}{\log t_0}c_1\right\|_{\left(\frac{1}{n}K_{t_0}+\frac{1}{\log t_0}c_1\right)}-(n+\log t_0),0\right\}\\ &= \max\left\{\left\|x+\frac{n}{\log t_0}c_1\right\|_{\left(\Delta^n+\frac{1}{\log t_0}c_1\right)}-(n+\log t_0),0\right\}.\end{aligned}$$

Notice that for v to be well defined necessarily $c=\frac{1}{-\log t_0}c_1\in\Delta^n$ and then there exists $c\in\Delta^n$ such that

$$\frac{f}{\|f\|_\infty}=e^{-\max\left\{\|\cdot-nc\|_{(\Delta^n-c)}-(n+\log t_0),0\right\}}\circ T$$

or, equivalently,

$$\frac{f}{\|f\|_\infty}=e^{-\max\left\{\|\cdot\|_{(\Delta^n-c)}-(n+\log t_0),0\right\}}\circ T$$

The same proof works in the even case. In the even case we know that $c_t=0$ for any t and then $T_t=\left(1-\frac{1}{n}\log\left(\frac{t}{t_0}\right)\right)T$. Thus, we can assume without loss of generality that $K_{t_0}=nB_\infty^n$ and then it implies that $\frac{f}{\|f\|_\infty}=f_{B_\infty^n,t_0}\circ T$. □

Finally, we will compute the integral ratio of this maximizing function in the following lemma. We will do it for a whole class of functions that include the maximizing one.

Lemma 3.4. *Let $\alpha \geq 1$ and $f(x) = e^{-\|x\|_K^\alpha}$. Then*

$$I.rat(f) = \left(\frac{e\alpha\Gamma\left(1+\frac{n}{\alpha}\right)^{\frac{\alpha}{n}}}{n}\right)^{\frac{1}{\alpha}} v.rat(K) \sim v.rat(K).$$

Proof. On one hand

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\|x\|_K^\alpha} dx &= \int_{\mathbb{R}^n}\int_{\|x\|_K^\alpha}^{+\infty} e^{-t} dt dx = \int_0^{+\infty}\int_{t^{\frac{1}{\alpha}}K} e^{-t} dx dt \\ &= |K|\int_0^{+\infty} t^{\frac{n}{\alpha}} e^{-t} dt = |K|\Gamma\left(1+\frac{n}{\alpha}\right).\end{aligned}$$

On the other hand, for any $t \in (0,1]$

$$K_t = (-\log t)^{\frac{1}{\alpha}} K$$

and then,

$$\mathcal{E}_t(f) = (-\log t)^{\frac{1}{\alpha}} \mathcal{E}(K),$$

where $\mathcal{E}(K)$ is the John ellipsoid of K. Thus,

$$\phi_f(t) = t(-\log t)^{\frac{n}{\alpha}} |\mathcal{E}(K)|.$$

Let us find $\max_{t\in(0,1]} t(-\log t)^{\frac{n}{\alpha}} |\mathcal{E}(K)| = \max_{s\in[0,+\infty)} e^{-s} s^{\frac{n}{\alpha}} |\mathcal{E}(K)|$. Taking derivatives we obtain that this maximum is attained at $s = \frac{n}{\alpha}$ and so

$$\max_{t\in(0,1]} t(-\log t)^{\frac{n}{\alpha}} |\mathcal{E}(K)| = \left(\frac{n}{\alpha}\right)^{\frac{n}{\alpha}} e^{-\frac{n}{\alpha}} |\mathcal{E}(K)|.$$

Consequently

$$I.rat(f) = \left(\frac{e\alpha\Gamma\left(1+\frac{n}{\alpha}\right)^{\frac{\alpha}{n}}}{n}\right)^{\frac{1}{\alpha}} v.rat(K).$$

□

4. Reverse Sobolev-type inequalities

In this section we will prove Theorem 1.4. First we will define the polar projection body of a function

Proposition 4.1. *Let $f : \mathbb{R}^n \to [0,+\infty)$ be a log-concave integrable function. If the following quantity is finite for every $x \in \mathbb{R}^n$ then it defines a norm*

$$\|x\| = 2|x| \int_{x^\perp} \max_{s\in\mathbb{R}} f\left(y + s\frac{x}{|x|}\right) dy.$$

Besides, if $f \in W^{1,1}(\mathbb{R}^n)$ this norm equals

$$\|x\| = \int_{\mathbb{R}^n} |\langle \nabla f(y), x\rangle| dy.$$

The unit ball of this norm is the polar projection body of f, which will be denoted by $\Pi^*(f)$.

Proof. Notice that

$$\begin{aligned}\|x\| &= 2|x|\int_{x^\perp}\max_{s\in\mathbb{R}} f\left(y+s\frac{x}{|x|}\right)dy\\ &= 2|x|\int_0^{+\infty}\left|\left\{y\in x^\perp\,:\,\max_{s\in\mathbb{R}} f\left(y+s\frac{x}{|x|}\right)\geq t\right\}\right|dt\\ &= 2|x|\|f\|_\infty\int_0^1 |P_{x^\perp}K_t|\,dt\\ &= 2\|f\|_\infty\int_0^1 \|x\|_{\Pi^*(K_t)}dt\end{aligned}$$

and it is clear that it is a norm.

If $f\in W^{1,1}(\mathbb{R}^n)$, for almost every t the boundary of K_t is $\{x\in\mathbb{R}^n\,:\,f(x)=t\|f\|_\infty\}$ and we have

$$\begin{aligned}\|x\|_{\Pi^*(f)} &= 2|x|\|f\|_\infty\int_0^1 |P_{x^\perp}K_t|\,dt\\ &= |x|\int_0^{\|f\|_\infty}\int_{\{f(x)=t\}}\left|\left\langle \nu(y),\frac{x}{|x|}\right\rangle\right|dH_{n-1}(y)dt\end{aligned}$$

where $\nu(y)$ is the outer normal unit vector to $\{x\in\mathbb{R}^n\,:\,f(x)\geq t\}$ and dH_{n-1} is the Haussdorff measure on the boundary of it. Since $\nu(y)=\frac{\nabla f(y)}{|\nabla f(y)|}$ almost everywhere the above expression is

$$\int_0^{\|f\|_\infty}\int_{\{f(x)=t\}}\left|\left\langle\frac{\nabla f(y)}{|\nabla f(y)|},x\right\rangle\right|dH_{n-1}(y)dt$$

which, by the co-area formula, equals

$$\int_{\mathbb{R}^n}|\langle\nabla f(y),x\rangle|dy.$$

□

We will use the following lemma to prove Theorem 1.4.

Lemma 4.2. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a log-concave function and $g(x)=(B_2^n)^a(x)$. Then*

$$\lim_{\varepsilon\to0^+} f\star g_\varepsilon(z)=f(z)$$

and

$$\lim_{\varepsilon\to0^+}\frac{f\star g_\varepsilon(z)-f(z)}{\varepsilon}=|\nabla f(z)|+f(z)\log a\quad\text{almost everywhere.}$$

Proof. By definition of the Asplund product, since f is continuous,

$$\lim_{\varepsilon\to0^+} f\star g_\varepsilon(z)=\lim_{\varepsilon\to0^+}\sup_{z=x+y} f(x)a^\varepsilon\chi_{B_2^n}\left(\frac{y}{\varepsilon}\right)=\lim_{\varepsilon\to0^+}\sup_{y\in B_2^n} f(z-\varepsilon y)a^\varepsilon=f(z).$$

Besides, if f is differentiable in z,

$$\begin{aligned}\lim_{\varepsilon\to0^+}\frac{f\star g_\varepsilon(z)-f(z)}{\varepsilon} &= \lim_{\varepsilon\to0^+}\sup_{y\in B_2^n}\frac{f(z-\varepsilon y)a^\varepsilon-f(z)a^\varepsilon+f(z)a^\varepsilon-f(z)}{\varepsilon}\\ &= \lim_{\varepsilon\to0^+}\sup_{y\in B_2^n}\frac{f(z-\varepsilon y)a^\varepsilon-f(z)a^\varepsilon}{\varepsilon}+f(z)\lim_{\varepsilon\to0^+}\frac{a^\varepsilon-1}{\varepsilon}.\end{aligned}$$

Since

$$\lim_{\varepsilon\to 0^+}\sup_{y\in B_2^n}\frac{f(z-\varepsilon y)-f(z)}{\varepsilon}=|\nabla f(z)|,$$

the previous limit equals $|\nabla f(z)|+f(z)\log a$. □

The following lemma was proved in [CF]. We reproduce it here for the sake of completeness:

Lemma 4.3. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be an integrable log-concave function. Then*

$$\lim_{\varepsilon\to 0^+}\frac{\int_{\mathbb{R}^n} f\star f_\varepsilon(x)dx-\int_{\mathbb{R}^n} f(x)dx}{\varepsilon} = n\int_{\mathbb{R}^n} f(x)dx+\int_{\mathbb{R}^n} f(x)\log f(x)dx$$

Proof. First of all, notice that if $f(x)=e^{-u(x)}$ with u a convex function, then

$$f\star f_\varepsilon(z)=e^{-(1+\varepsilon)u\left(\frac{z}{1+\varepsilon}\right)},$$

since, as u is convex, its epigraph $\operatorname{epi} u$ is a convex set and then

$$\begin{aligned}\inf_{z=x+y} u(x)+\varepsilon u\left(\frac{y}{\varepsilon}\right) &= \inf_{z=x+\varepsilon y} u(x)+\varepsilon u(y)\\ &= \inf\{\mu\,:\,(z,\mu)\in(1+\varepsilon)\operatorname{epi} u\}\\ &= (1+\varepsilon)u\left(\frac{z}{1+\varepsilon}\right).\end{aligned}$$

Then,

$$\begin{aligned}\frac{\int_{\mathbb{R}^n} f\star f_\varepsilon(x)dx-\int_{\mathbb{R}^n} f(x)dx}{\varepsilon} &= \frac{1}{\varepsilon}\left((1+\varepsilon)^n\int_{\mathbb{R}^n} e^{-(1+\varepsilon)u(x)}dx-\int_{\mathbb{R}^n} e^{-u(x)}dx\right)\\ &= \left(\frac{(1+\varepsilon)^n-1}{\varepsilon}\right)\int_{\mathbb{R}^n} e^{-(1+\varepsilon)u(x)}dx\\ &+ \int_{\mathbb{R}^n} e^{-u(x)}\left(\frac{e^{-\varepsilon u(x)}-1}{\varepsilon}\right)dx.\end{aligned}$$

Now, taking limit when ε tends to 0 we obtain the result. The monotone convergence theorem and possibly a translation of the function u allows us to interchange limits. □

Now we are able to prove Theorem 1.4:

Proof. Since all the quantities in the statement of the theorem are affine invariant, *i.e.*, they take the same value for f and for $f\circ T$, we can assume that f is in John's position. That is, $\mathcal{E}(f)=(B_2^n)^{t_0\|f\|_\infty}$. On the one hand, by Jensen's inequality

$$\begin{aligned}|\Pi^*(f)|^{\frac1n} &= |B_2^n|^{\frac1n}\left(\int_{S^{n-1}}\left(\int_{\mathbb{R}^n}|\langle\nabla f(z),\theta\rangle|dz\right)^{-n}d\sigma(\theta)\right)^{\frac1n}\\ &\geq |B_2^n|^{\frac1n}\left(\int_{S^{n-1}}\int_{\mathbb{R}^n}|\langle\nabla f(z),\theta\rangle|dzd\sigma(\theta)\right)^{-1}\\ &= |B_2^n|^{\frac1n}\left(\frac{2}{n}\frac{|B_2^{n-1}|}{|B_2^n|}\int_{\mathbb{R}^n}|\nabla f(z)|dz\right)^{-1}.\end{aligned}$$

On the other hand, let $g(x)=\mathcal{E}(f)(x)$. By Lemma 4.2, we have that

$$|\nabla f(z)|+f(z)\log(t_0\|f\|_\infty) = \lim_{\varepsilon\to 0^+}\frac{f\star g_\varepsilon(z)-f(z)}{\varepsilon}$$

$$\leq \lim_{\varepsilon\to 0^+}\frac{f\star f_\varepsilon(z)-f(z)}{\varepsilon}$$

By Lemma 4.3, integrating in $z\in\mathbb{R}^n$ we have that

$$\int_{\mathbb{R}^n}|\nabla f(z)|dz + \int_{\mathbb{R}^n} f(z)dz\log(t_0\|f\|_\infty)\leq n\int_{\mathbb{R}^n} f(z)dz+\int_{\mathbb{R}^n} f(z)\log f(z)dz.$$

Then

$$\int_{\mathbb{R}^n}|\nabla f(z)|dz\leq n\int_{\mathbb{R}^n} f(z)dz+\int_{\mathbb{R}^n} f(z)\log\frac{f(z)}{t_0\|f\|_\infty}dz.$$

Consequently, $\frac{\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)}$ is bounded below by

$$\left((t_0\|f\|_\infty)^{\frac{1}{n}}I.rat(f)\left[\left(\frac{\int_{\mathbb{R}^n}f(x)dx}{\int_{\mathbb{R}^n}f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}}+\frac{\int_{\mathbb{R}^n}f(x)\log\left(\frac{f(x)}{t_0\|f\|_\infty}\right)dx}{n\|f\|_{\frac{n}{n-1}}\|f\|_1^{\frac{1}{n}}}\right]\right)^{-1}.$$

Since $t_0\geq e^{-n}$, we can write $t_0=e^{-s_0 n}$ for some $s_0\in[0,1]$ and then

$$\begin{aligned} & t_0^{\frac{1}{n}}\left[\left(\frac{\int_{\mathbb{R}^n}f(x)dx}{\int_{\mathbb{R}^n}f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}}+\frac{\int_{\mathbb{R}^n}f(x)\log\left(\frac{f(x)}{t_0\|f\|_\infty}\right)dx}{n\|f\|_{\frac{n}{n-1}}\|f\|_1^{\frac{1}{n}}}\right]\\ = \ & e^{-s_0}\left[(1+s_0)\left(\frac{\int_{\mathbb{R}^n}f(x)dx}{\int_{\mathbb{R}^n}f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}}+\frac{\int_{\mathbb{R}^n}f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{n\|f\|_{\frac{n}{n-1}}\|f\|_1^{\frac{1}{n}}}\right].\end{aligned}$$

Since the maximum of $g(s)=e^{-s}\left[(1+s)A+B\right]$ with $A\geq 0$, $B\leq 0$ and $s\geq 0$ is attained when $s=\frac{-B}{A}$ we have that $\frac{\|f\|_{\frac{n}{n-1}}|\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)}$ is bounded below by

$$\left(e^{\frac{\int_{\mathbb{R}^n}f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{n\int_{\mathbb{R}^n}f(x)dx}}\|f\|_\infty^{\frac{1}{n}}\left(\frac{\int_{\mathbb{R}^n}f(x)dx}{\int_{\mathbb{R}^n}f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}}I.rat(f)\right)^{-1}.$$

□

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