

# JOINT CONTINUITY IN SEMITOPOLOGICAL SEMIGROUPS

BY  
J. D. LAWSON

The principal goal of this paper is squeezing out points of joint continuity from a separately continuous action of a semigroup on a topological space. The paper itself is a variation on a theme by R. Ellis, who showed that separate continuity on a locally compact Hausdorff group implies joint continuity for the multiplication function [6]. The paper closes with several diverse applications.

Separately continuous compact semigroups arise naturally in certain settings in analysis. De Leuw and Glicksberg obtained them via weakly almost periodic functions [5]. They also arise as the maximal ideal space of the measure algebra of a locally compact abelian group; this situation has been systematically investigated by J. Taylor and others. Perhaps the most complete treatment of compact semigroups with multiplication separately continuous appears in the treatise of Berglund and Hofmann [2].

## 1. Preliminaries

Let  $X, Y, Z$  be topological spaces,  $\pi: X \times Y \rightarrow Z$ . The function  $\pi$  is called *left continuous* if  $\pi_y: X \rightarrow Z$  defined by  $\pi_y(x) = \pi(x, y)$  is continuous for each  $y \in Y$ . In terms of nets,  $\pi$  is left continuous if  $x_\alpha \rightarrow x, y \in Y$  implies  $\pi(x_\alpha, y) \rightarrow \pi(x, y)$ . The function  $\pi$  is *right continuous* if  $\pi_x: Y \rightarrow Z$  defined by  $\pi_x(y) = \pi(x, y)$  is continuous for each  $x \in X$ . The function  $\pi$  is *separately continuous* if it is both right and left continuous.

The function  $\pi$  is (*jointly*) *continuous at*  $(x, y)$  if given any open set  $W$  with  $\pi(x, y) \in W$  there exist open sets  $U$  and  $V$  with  $x \in U, y \in V$  such that  $u \in U, v \in V$  imply  $\pi(u, v) \in W$ . If  $\pi$  is continuous at every  $(x, y) \in X \times Y$ , then  $\pi$  is said to be *continuous* (or *jointly continuous*).

A semigroup  $S$  on a topological space is a *left (right) semitopological semigroup* if the multiplication function is left (right) continuous. If the multiplication function is separately continuous, then  $S$  is a *semitopological semigroup*; if  $S$  is Hausdorff and the multiplication function is continuous, then  $S$  is a *topological semigroup*.

A semigroup  $S$  is said to *act* on (the left of) a set  $X$  if there exists a function  $\pi: S \times X \rightarrow X$  satisfying

$$(1) \quad \pi(st, x) = \pi(s, \pi(t, x))$$

for all  $s, t \in S, x \in X$ . Customarily  $\pi(s, x)$  is written  $sx$ ; in this case (1) becomes

---

Received November 6, 1972.

$$(1)^* \quad (st)x = s(tx).$$

The function  $\pi$  is called a (left) action of  $S$  on  $X$  and  $S$  is called a (left) transformation semigroup on  $X$ .

### 2. The underlying principle

A semigroup  $S$  admits several natural quasi-orders closely related to the Green's relations. The  $\mathcal{L}$ -order is defined by  $s \leq t$  if  $Ss \cup \{s\} \subset St \cup \{t\}$ . Let  $S$  and  $X$  be topological spaces,  $S$  a semigroup, and  $\pi: S \times X \rightarrow X$  an action of  $S$  on  $X$ . The basic principle of this paper is that  $\pi$  is continuous at  $(s, x)$  if there exist "enough"  $t \in S$  below  $s$  in the  $\mathcal{L}$ -order such that  $\pi$  is continuous at  $(t, x)$ .

For  $x, y \in S$ , define

$$C(x, y) = \{s \in S: sx \neq sy\}.$$

The following lemma and proposition are precise formulations of the basic principle.

LEMMA 2.1. *Let  $S$  be a right semitopological semigroup,  $X$  a Hausdorff space, and  $\pi: S \times X \rightarrow X$  a right separately continuous action. If for  $s \in S$ ,  $x, y \in X$ ,  $sx \neq y$ , there exists  $r \in C(sx, y)$  such that  $\pi$  is continuous at  $(rs, x)$ , then there exist open sets  $W, U, V$  such that  $s \in W$ ,  $x \in U$ ,  $y \in V$  and  $\pi(W \times U) \cap V = \emptyset$ .*

*Proof.* Since  $rsx \neq ry$ , there exist open sets  $A$  and  $B$  such that  $rsx \in A$ ,  $ry \in B$ , and  $A \cap B = \emptyset$ . Since  $\pi$  is continuous at  $(rs, x)$ , there exist open sets  $P$  and  $U$  such that  $rs \in P$ ,  $x \in U$ , and  $\pi(P \times U) \subset A$ . Since left translations are continuous in  $S$ , there exists an open set  $W$ ,  $s \in W$ , such that  $rW \subseteq P$ . Since the action is right separately continuous, there exists an open set  $V$ ,  $y \in V$ , such that  $\pi(r \times V) \subset B$ .

Suppose there exists  $w \in W$ ,  $u \in U$ , such that  $wu \in V$ . Then  $r(wu) \in \pi(r \times V) \subset B$ . But  $(rw)u \in \pi(P \times U) \subset A$ . Hence  $wu \notin V$ .

PROPOSITION 2.2. *Let  $S$  and  $X$  be as in 2.1 with the additional assumption that  $X$  is compact. Let  $(s, x) \in S \times X$ . If for each  $y \neq sx$ , there exists  $r \in C(sx, y)$  such that  $\pi$  is continuous at  $(rs, x)$ , then  $\pi$  is continuous at  $(s, x)$ .*

*Proof.* Let  $P$  be an open set containing  $sx$ . Then  $X \setminus P$  is compact. By Lemma 2.1 for each  $y \in X \setminus P$ , there exist open sets  $W_y, U_y, V_y$  such that  $s \in W_y$ ,  $x \in U_y$ ,  $y \in V_y$  and  $\pi(W_y \times U_y) \cap V_y = \emptyset$ . A finite number of  $\{V_y: y \in X \setminus P\}$  cover  $X \setminus P$ . Let  $W$  be the intersection of the corresponding  $W_y$  and  $U$  the intersection of the corresponding  $U_y$ . Then  $W$  and  $U$  are open,  $s \in W$ ,  $x \in U$ , and  $\pi(W \times U) \subset P$ .

### 3. Transformation semigroups on metric spaces

To apply Proposition 2.2, one must have already points of joint continuity. These points of joint continuity may either be postulated or their existence

established via topological considerations. In this section we adopt the latter approach. The topological machinery employed is category theory.

A subspace  $A$  of a space  $X$  is *nowhere dense* if  $(A^*)^\circ = \emptyset$ . A subspace  $A$  is of *first category* if  $A$  is the countable union of nowhere dense sets. A subspace  $B$  is *residual* if  $X \setminus B$  is of first category. A space is a *Baire space* if every residual subspace is dense. It is well-known that a locally compact Hausdorff space is a Baire space. The following proposition from Bourbaki [4, Exercise IX, 5, 23] shows that the set of points at which a separately continuous function fails to be jointly continuous is “small” under certain topological restrictions.

**PROPOSITION 3.1.** *Let  $\pi: X \times Y \rightarrow Z$  be separately continuous. Assume  $X$  is a Baire space,  $y_0 \in Y$  has a countable basis of neighborhoods, and  $Z$  is metric. Then the set of  $x \in X$  such that  $\pi$  fails to be jointly continuous at  $(x, y_0)$  is of first category.*

**THEOREM 3.2.** *Let  $S$  be a locally compact Hausdorff (or, more generally, a Baire) right semitopological semigroup,  $X$  a compact metric space, and  $\pi: S \times X \rightarrow X$  a separately continuous action. If there exists  $s \in S$  such that  $Ss = S$  and  $\pi(s \times X) = X$ , then  $\pi$  is continuous at  $(s, x)$  for each  $x \in X$ .*

*Proof.* Let  $x \in X$ . The proof consists of showing that the hypotheses of Proposition 2.2 are satisfied at  $(s, x)$ .

Suppose  $y \neq sx$ . Pick open sets  $A$  and  $B$  such that  $y \in A$ ,  $sx \in B$ , and  $A \cap B = \emptyset$ . By hypothesis there exists  $z \in X$  such that  $sz = y$ . Hence there exists  $U$  open,  $s \in U$ , such that  $\pi(U \times z) \subset A$ ,  $\pi(U \times x) \subset B$ .

By Proposition 3.1 there exists  $t \in U$  such that  $\pi$  is continuous at  $(t, x)$ . By hypothesis there exists  $r \in S$  such that  $rs = t$ . Then  $\pi$  is continuous at  $(rs, x)$ .

Finally  $rsx = tx \in \pi(U \times x) \subset B$  and  $ry = rsz = tz \in \pi(U \times z) \subset A$ . Hence  $rsx \neq ry$ . Thus by 2.2,  $\pi$  is continuous at  $(s, x)$ .

**COROLLARY 3.3.** *Let  $S$  be a locally compact Hausdorff right semitopological semigroup with right identity 1,  $X$  a compact metric space, and  $\pi: S \times X \rightarrow X$  a separately continuous action such that  $1x = x$  for all  $x \in X$ . Then  $\pi$  is continuous at  $(1, x)$  for all  $x \in X$ .*

#### 4. Quotients of topological actions

Some elementary properties of quotients of topological actions are developed in this section. These are needed in generalizing the results of the preceding section to the non-metric case.

Let  $(S, X)$  and  $(T, Y)$  be algebraic transformation semigroups where  $S$  acts on  $X$  and  $T$  acts on  $Y$ . A homomorphism of transformation semigroups, i.e., a morphism in the category of transformation semigroups, is a pair  $(\alpha, \beta)$  where  $\alpha$  is a homomorphism from  $S$  to  $T$ ,  $\beta$  is a function from  $X$  to  $Y$ , and  $\beta(sx) = \alpha(s)\beta(x)$  for all  $s \in S, x \in X$ .

A congruence on  $(S, X)$  is a pair  $(\rho, \sigma)$  where  $\rho$  is a congruence on  $S$  and  $\sigma$  is an equivalence on  $X$  so that if  $s\rho t$  and  $x\sigma y$  then  $sx\sigma ty$ . A quotient trans-

formation semigroup  $(S/\rho, X/\sigma)$  can be defined in a natural manner which is a homomorphic image of  $(S, X)$  with respect to the natural mappings of  $S$  and  $X$  to  $S/\rho$  and  $X/\sigma$  resp.

The following topological result allows the formation of quotients of transformation semigroups in certain topological categories.

PROPOSITION 4.1. *Suppose  $A, B, C, X, Y, Z$  are topological spaces,*

$$f:A \rightarrow X, \quad g:B \rightarrow Y, \quad h:C \rightarrow Z$$

*are quotient mappings which are onto, and  $p:A \times B \rightarrow C, \pi:X \times Y \rightarrow Z$  functions such that the following square is commutative:*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi} & Z \\ \uparrow f \times g & & \uparrow h \\ A \times B & \xrightarrow{p} & C \end{array}$$

*If  $p$  is left (right) separately continuous, then  $\pi$  is left (right) separately continuous.*

*Proof.* Suppose  $p$  is left continuous. Fix  $y \in Y$ . Pick  $b \in B$  such that  $g(b) = y$ . Then the following square is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\pi_y} & Z \\ \uparrow f & & \uparrow h \\ A & \xrightarrow{p_b} & C \end{array}$$

where  $\pi_y(x) = \pi(x, y)$  and  $p_b(a) = p(a, b)$ . By supposition  $p_b$  is continuous. Hence  $hp_b$  is continuous. Thus  $\pi_y f$  is continuous and since  $f$  is a quotient mapping,  $\pi_y$  is continuous. Hence  $\pi$  is left separately continuous.

The proof for right continuous is analogous. The separately continuous case follows from the other two.

COROLLARY 4.2. *Let  $(S, X)$  be a separately continuous transformation semigroup,  $(\rho, \sigma)$  a congruence on  $(S, X)$  and  $(T, Y)$  the quotient transformation semigroup where  $T = S/\rho$  and  $Y = X/\sigma$  are equipped with the quotient topology. Then  $(T, Y)$  is a separately continuous transformation semigroup.*

*Proof.* Let  $p:S \times X \rightarrow X$  be the action of  $S$  on  $X$  and  $\pi:T \times Y \rightarrow Y$  be the induced action of  $T$  on  $Y$ . By definition of the induced action  $\pi$  the following square is commutative:

$$\begin{array}{ccc} T \times Y & \xrightarrow{\pi} & Y \\ \uparrow & & \uparrow \\ S \times X & \xrightarrow{p} & X \end{array}$$

where the vertical mappings are naturally induced. By 4.1,  $\pi$  is continuous.

**COROLLARY 4.3.** *Let  $S$  be a right semitopological semigroup,  $\sigma$  a congruence on  $S$ , and  $T$  be the quotient semigroup  $S/\sigma$  endowed with the quotient topology. Then  $T$  is a right semitopological semigroup.*

*Proof.* The square

$$\begin{array}{ccc} T \times T & \longrightarrow & T \\ \uparrow & & \uparrow \\ S \times S & \longrightarrow & S \end{array}$$

is commutative where the vertical maps are naturally induced and the horizontal maps are multiplication. Hence by 4.1, multiplication on  $T$  is right continuous.

### 5. The central theorem

We now accumulate our results to deduce the central theorem of this paper.

**THEOREM 5.1.** *Let  $S$  be a compact Hausdorff right semitopological semigroup with right identity  $1$ ,  $X$  a compact Hausdorff space, and  $\pi: S \times X \rightarrow X$  a separately continuous action such that  $\pi(1, x) = x$  for all  $x \in X$ . Then  $\pi$  is continuous at  $(1, x)$  for all  $x \in X$ .*

*Proof.* Fix  $x \in X$ . Assume  $\pi$  is not continuous at  $(1, x)$ . Then there exists an open set  $U$ ,  $x \in U$ , such that for any open sets  $V$  and  $W$  with  $1 \in V$ ,  $x \in W$ , there exist  $v \in V$ ,  $w \in W$  such that  $vw \notin U$ . Since  $X$  is completely regular, there exists a continuous function  $f$  from  $X$  into  $[0, 1]$  such that  $f(x) = 1$  and  $f(X \setminus U) = 0$ .

We inductively construct sequences  $\{s_n\}_{n=1}^\infty$  in  $S$  and  $\{x_n\}_{n=1}^\infty$  in  $X$  satisfying the following conditions for all  $n = 1, 2, \dots$ :

- (a)  $f(s_n x_n) = 0$ ,
- (b)  $f(x_n) > 1 - 1/n$ ,
- (c) For  $m < n$ ,  $|f(ss_n t y) - f(sty)| < 1/n$  for all  $s \in (S_m)^m, y \in X_m$  where

$$S_m = \{s_1, \dots, s_m, 1\}, \quad (S_m)^m = \{t_1 t_2 \dots t_m : t_i \in S_m, 1 \leq i \leq m\}$$

and  $X_m = \{x_1, \dots, x_m\}$ .

The construction is begun by choosing an open neighborhood  $Q$  of  $x$  such that  $f(Q) \subset (0, 1]$ . By supposition there exist  $s_1 \in S, x_1 \in Q$  such that  $s_1 x_1 \notin U$ , i.e.,  $f(s_1 x_1) = 0$ .

Suppose  $\{s_n\}_{n=1}^k$  and  $\{x_n\}_{n=1}^k$  have been constructed satisfying (a)-(c) for  $n = 1, \dots, k$ . There is an open neighborhood  $Q$  of  $x$  such that

$$f(Q) \subset (1 - 1/(k + 1), 1].$$

Since  $s_1 t y = sty$  for each  $(s, t, y) \in (S_k)^k \times (S_k)^k \times X_k$ , there exists an open set  $A_{(s,t,y)}$  containing  $1$  such that  $r \in A_{(s,t,y)}$  implies

$$|f(srt y) - f(sty)| < 1/(k + 1).$$

Since

$$(S_k)^k \times (S_k)^k \times X_k \supset (S_m)^m \times (S_m)^m \times X_m \text{ for } m < k,$$

the preceding inequality holds for all  $m < k + 1$ . Since there are only finitely many  $(s, t, y)$ ,  $A = \bigcap \{A_{(s,t,y)} : s, t \in (S_k)^k, y \in X_k\}$  is open. By supposition there exist  $s_{k+1} \in A$ ,  $x_{k+1} \in Q$  such that  $f(s_{k+1} x_{k+1}) = 0$ . This completes the inductive step.

Let  $T = \bigcup_n (S_n)^n$  and  $Y = \pi(T \times (\bigcup_n X_n))$ . Then  $T$  is a countable sub-semigroup of  $S$  containing 1 and  $\pi(T \times Y) \subset Y$ . Hence  $\pi(T \times Y^*) \subset Y^*$  and thus  $\pi(T^* \times Y^*) \subset Y^*$ .

Define a relation  $\sigma$  on  $Y^*$  by  $y_1 \sigma y_2$  if  $f(ty_1) = f(ty_2)$  for all  $t \in T^*$  and define a relation  $\rho$  on  $T^*$  by  $t_1 \rho t_2$  if  $f(st_1 y) = f(st_2 y)$  for all  $s \in T^*$  and for all  $y \in Y^*$ . These relations satisfy the following.

(i)  $\sigma$  is an equivalence relation. Immediate.

(ii)  $\sigma$  is a closed subset of  $Y^* \times Y^*$ . This follows easily from the right continuity of the action, and the continuity of  $f$ .

(iii)  $\rho$  is a congruence relation. Reflexivity, symmetry, and transitivity are straightforward. Suppose  $s_1 \rho s_2$  and let  $r, t \in T^*$ . Then for  $s \in T^*$  and  $y \in Y^*$ ,

$$f(s(rs_1 t)y) = f(sr(s_1)(ty)) = f(sr(s_2)(ty)) = f(s(rs_2 t)y).$$

Hence  $(rs_1 t) \rho (rs_2 t)$  and  $\rho$  is a congruence.

(iv)  $\rho$  is a closed subset of  $T^* \times T^*$ . This follows from the right continuity of multiplication, the left continuity of the action, and the continuity of  $f$ .

(v)  $(\rho, \sigma)$  is a congruence on  $(T^*, Y^*)$ . Let  $t_1 \rho t_2$  and  $y_1 \sigma y_2$ . Then for  $t \in T^*$ ,  $f(t(t_1 y_1)) = f(t(t_2 y_1)) = f((tt_2)y_1) = f((tt_2)y_2) = f(t(t_2 y_2))$ . Hence  $t_1 y_1 \sigma t_2 y_2$ .

Let  $T' = T^*/\rho$ ,  $Y' = Y^*/\sigma$ , each endowed with the quotient topology, and  $\pi'$  be the induced action of  $T'$  on  $Y'$ . Let  $\alpha: T^* \rightarrow T'$  and  $\beta: Y^* \rightarrow Y'$  be the quotient maps.

The transformation semigroup  $(T', Y')$  satisfies the following.

(1)  $T'$  and  $Y'$  are compact Hausdorff. This follows from the closure of  $\rho$  and  $\sigma$ .

(2)  $\pi'$  is separately continuous. Corollary 4.2.

(3)  $T'$  is a right semitopological semigroup. Corollary 4.3.

(4) [1], the image of 1 under  $\alpha$ , is a right identity for  $T'$ , and  $[1][y] = [y]$  for each  $[y] \in Y'$ . This is a straightforward consequence of the fact that  $(\rho, \sigma)$  is a congruence and  $(T', Y')$  is equipped with the induced structure.

(5) There exists  $f': Y' \rightarrow [0, 1]$  such that  $f'\beta(y) = f(y)$  for all  $y \in Y^*$ . Suppose  $y_1, y_2 \in Y^*$  and  $\beta(y_1) = \beta(y_2)$ . Then  $y_1 \sigma y_2$ , and hence  $f(y_1) = f(1y_1) = f(1y_2) = f(y_2)$ . Thus there exists  $f': Y' \rightarrow [0, 1]$  satisfying  $f'\beta(y) = f(y)$  for all  $y \in Y^*$ . Since  $Y'$  has the quotient topology,  $f'$  is continuous.

(6)  $Y'$  is metric. Suppose  $\beta(y_1), \beta(y_2) \in Y'$  and  $\beta(y_1) \neq \beta(y_2)$ . Then  $(y_1, y_2) \notin \sigma$ ; hence there exists  $t \in T^*$  such that  $f(ty_1) \neq f(ty_2)$ . Since  $T$  is dense in  $T^*$  and  $\pi$  is left continuous, there exists  $s$  in  $T$  such that

$f(sy_1) \neq f(sy_2)$ . Define

$$f'_{\alpha(s)}: Y' \rightarrow [0, 1]$$

by  $f'_{\alpha(s)}(\beta(y)) = f'(\alpha(s)\beta(y))$ . Then

$$f'_{\alpha(s)}(\beta(y_1)) = f'(\alpha(s)\beta(y_1)) = f'\beta(sy_1) = f(sy_1)$$

and similarly

$$f'_{\alpha(s)}(\beta(y_2)) = f(sy_2) \neq f(sy_1).$$

Hence the countable collection of continuous functions  $\{f'_{\alpha(s)}: s \in T\}$  separate points of  $Y'$ . Since  $Y'$  is compact, the product of these functions into a countable product of intervals is a homeomorphism. Thus  $Y'$  is metrizable.

(7) Let  $(\bar{s}, \bar{x})$  be a cluster point of the sequence  $(s_n, x_n)$  in the compact space  $T^* \times Y^*$ . Then  $\alpha(\bar{s}) = \alpha(1)$ . To see this, let  $y \in Y, s \in T$ . Then there exists  $n$  such that  $y = tx$  for some  $t \in (S_n)^n, x \in X_n$ . Now since  $\{s_n\}$  clusters to  $\bar{s}, f(ss_n tx)$  clusters to  $f(\bar{s}tx)$ . By condition (c),  $f(ss_n tx)$  converges to  $f(stx)$ . Hence  $f(\bar{s}y) = f(\bar{s}tx) = f(stx) = f(sy)$  for all  $s \in T, y \in Y$ . Since  $Y$  is dense in  $Y^*$  and  $y \rightarrow f(\bar{s}y)$  is continuous,  $f(\bar{s}y) = f(sy)$  for all  $s \in T, y \in Y^*$ . Since  $T$  is dense in  $T^*$  and  $s \rightarrow f(\bar{s}y)$  is continuous,  $f(\bar{s}y) = f(sy)$  for all  $s \in T^*, y \in Y^*$ . Hence  $\bar{s} \rho 1$ , i.e.,  $\alpha(1) = \alpha(\bar{s})$ .

(8)  $\pi'$  is continuous at  $(\alpha(\bar{s}), \beta(\bar{x}))$ . By (7)  $\alpha(\bar{s}) = \alpha(1)$ . By (1),  $T'$  is compact Hausdorff and by (1) and (6),  $Y'$  is compact metric. By (2),  $\pi'$  is separately continuous and by (3),  $T'$  is a right semitopological semigroup. Hence by Corollary 3.3,  $\pi'$  is continuous  $(\alpha(\bar{s}), \beta(\bar{x}))$ .

(9)  $f'\pi'(\alpha(\bar{s}), \beta(\bar{x})) = 1$ . Since  $\pi'(\alpha(\bar{s}), \beta(\bar{x})) = \pi'(\alpha(1), \beta(\bar{x})) = \beta(\bar{x})$ , we have  $f'\pi'(\alpha(\bar{s}), \beta(\bar{x})) = f'\beta(\bar{x}) = f(\bar{x})$ . Since  $\bar{x}$  is a cluster point of  $\{x_n\}$  and by condition (b) of the construction of the sequence  $\{x_n\}, f(x_n)$  converges to 1, we have  $f(\bar{x}) = 1$ .

(10)  $f'\pi'(\alpha(s_n), \beta(x_n)) = 0$ . Since  $\pi'$  is induced by  $\pi$ ,

$$f'\pi'(\alpha(s_n), \beta(x_n)) = f'(\beta(s_n x_n)) = f(s_n x_n) = 0$$

by condition (a) of the construction of the sequences  $\{s_n\}$  and  $\{x_n\}$ .

Statements (8), (9) and (10) provide the grand finale for the proof. Since  $(s_n, x_n)$  clusters to  $(\bar{s}, \bar{x})$  and since  $\pi'$  is continuous at  $(\alpha(\bar{s}), \beta(\bar{x}))$ , we have  $f'\pi'(\alpha(s_n), \beta(x_n))$  clusters to  $f'\pi'(\alpha(\bar{s}), \beta(\bar{x}))$ , i.e., the constant sequence of 0 clusters to 1, a contradiction.

**COROLLARY 5.2.** *Let  $S$  be a compact Hausdorff right semitopological semigroup with identity  $u, X$  a compact Hausdorff space, and  $\pi: S \times X \rightarrow X$  a separately continuous action. If  $g$  is a unit in  $S$ , then  $\pi$  is continuous at  $(g, x)$  for all  $x \in X$ .*

*Proof.* By hypothesis there exists  $g^{-1} \in S$  such that  $gg^{-1} = g^{-1}g = u$ . Note that  $\pi(S \times uX) \subseteq uX$  since  $sux = (us)ux = u(sux) \in uX$ . By Theorem 5.1,  $\pi|S \times uX$  is continuous at  $(u, y)$  for all  $y \in uX$ . Now the composition  $S \times X \xrightarrow{\pi} S \times uX \rightarrow X \rightarrow X$  defined by

$$(s, x) \rightarrow (g^{-1}s, ux) \rightarrow g^{-1}sux \rightarrow gg^{-1}sux$$

is the mapping  $\pi$  since  $gg^{-1}su = (usu)x = sx$ . Hence  $\pi$  is continuous at  $(g, x)$  since the composition

$$(g, x) \rightarrow (g^{-1}g, ux) = (u, ux) \rightarrow uux = ux \rightarrow gux = gx$$

is continuous at the specified points at each stage.

*Remarks.* Theorem 5.1 remains valid in a form analogous to Theorem 3.2, i.e., in Theorem 3.2 replace locally compact  $S$  by compact  $S$  and compact metric  $X$  by compact  $X$ . In order to show this more general version, the inductive constructions of the sequences  $\{s_n\}$  and  $\{x_n\}$  in the proof of 5.1 must be appropriately modified. However, I felt the slight extra generality was not worth introducing added complication to an already cumbersome proof.

Ellis proved Corollary 5.2 for the case  $S$  is all units, i.e., a group, and  $S$  is only locally compact. What one needs is that the quotient semigroup  $T'$  constructed in the proof of 5.1 is a Baire space. In the locally compact group case this will be true since the quotient mapping will be open and hence the image will again be locally compact. However, the quotient of a locally compact semitopological semigroup with respect to a closed congruence is not necessarily locally compact. Hence one is restricted to the compact case.

## 6. Applications to semitopological semigroups

In the remaining portions of the paper we turn to applications of the preceding results.

**PROPOSITION 6.1.** *Let  $S$  be a compact Hausdorff semitopological semigroup with identity 1. Then the multiplication function is continuous at any point of  $g \times S \cup S \times g$  for any unit  $g$ .*

*Proof.* This proposition is an immediate consequence of Corollary 5.2 and its analogue for right actions; simply observe that the multiplication function is an action of  $S$  on  $S$ .

**PROPOSITION 6.2.** *Let  $G$  be a subgroup of a compact Hausdorff semitopological semigroup  $S$ . Then multiplication restricted to  $G \times S$  is continuous. (Note: This is different than saying multiplication is continuous at points of  $G \times S$ .)*

*Proof.* Let  $e$  be the identity of  $G$ , and  $H = G^*$ . Then  $e$  is an identity for  $H$ . Applying Corollary 5.2 to the multiplication function restricted to  $H \times S$ , we see that the restriction is continuous at  $(g, s)$  for each  $g \in G$  and  $s \in S$ .

**COROLLARY 6.3.** *Let  $G$  be a subgroup of a compact Hausdorff semitopological semigroup. Then  $G$  is a topological group.*

*Proof.* By Proposition 6.2, multiplication is jointly continuous. Let  $H$  be  $G^*$ . Then  $H$  is compact, and  $e$ , the identity of  $G$ , is an identity for  $H$ .

Let  $\{g_\alpha\}$  be a net in  $G$  which converges to  $g \in G$ . Since  $H$  is compact, the net  $\{g_\alpha^{-1}\}$  clusters to some  $h \in H$ . Corollary 5.2 applied to  $H$  implies continuity



at  $(g, h)$ , and the right analogue of 5.2 implies continuity at  $(h, g)$ . Hence  $\{g_\alpha g_\alpha^{-1}\}$  clusters to  $gh$  and the net  $\{g_\alpha^{-1}g_\alpha\}$  clusters to  $hg$ . Thus  $hg = gh = e$ , i.e.,  $h = g^{-1}$ .

A well-known result of Mostert and Shields [9] is that a topological semigroup with 1 on a manifold has an open group of units. Joint continuity at 1 allows an analogous result for semitopological semigroups.

**PROPOSITION 6.4.** *Let  $S$  be a semitopological semigroup with identity 1 on a compact Euclidean manifold. Then the group of units is open.*

*Proof.* Let  $U$  be an open set containing 1 which is homeomorphic to  $R^n$ , the Cartesian product of  $n$  copies of the reals. Identify  $U$  with  $R^n$ , metric  $d$ . By joint continuity at  $(1, 1)$  (Theorem 5.1), there exists an open set  $V$  such that  $1 \in V$ ,  $V^2 \subset U$ . Fix  $\varepsilon > 0$  such that  $N_\varepsilon(1) \subset V$ , and let  $Q = N_\varepsilon(1)^*$ . Since multiplication is jointly continuous at  $(1, x)$  for each  $x \in Q$ , there exists a  $\delta > 0$  such that  $y \in N_\delta(1)$  implies  $d(x, yx) < \varepsilon$  for each  $x \in Q$ . Hence for each  $y \in N_\delta(1)$ , left translation by  $y$  takes  $Q$  into  $R^n$ . As a consequence of the Brouwer fixed point theorem (see [7, Lemma 1.1]), there exists  $q \in Q$  such that  $yq = 1$ . Hence each  $y \in N_\delta(1)$  has a right inverse. Similarly there exists a neighborhood of 1 consisting of elements having left inverses. The intersection is a neighborhood of 1 consisting of units. Since translation by a unit is a homeomorphism, it follows the group of units is open.

Proposition 6.4 provides an affirmative answer to a problem posed to me by John Berglund. My attempts to solve this problem formed the starting point for this paper. I am indebted to Professor Berglund for arousing my interest in this type of problem.

## 7. Joint continuity from separate continuity

In this section we are concerned with the following question: In what classes of semigroups does separate continuity imply joint? Berglund has shown that semigroups on an arc with endpoints 0 and 1 are such a class [1]. It would be of interest to know whether the results of this paper could be employed to formulate an alternate proof. Ellis has shown that locally compact topological groups are also such a class. The results of this paper give an alternate proof (although in many ways parallel).

(A) Groups. The following theorem of Ellis appears in [6].

**PROPOSITION 7.1.** *Let  $G$  be a locally compact Hausdorff group with separately continuous multiplication. Then  $G$  is a topological group.*

*Proof.* Let  $G^0$  denote the one-point compactification of  $G$ ; define multiplication on  $G^0$  by  $0x = x0 = 0$  for all  $x \in G^0$ . Then  $G^0$  is a compact Hausdorff semitopological semigroup. The proposition now follows from Corollary 6.3.

(B) Semilattices. A semilattice is a commutative, idempotent semigroup. An order can be defined on a semilattice  $S$  by  $x \leq y$  if  $xy = x$ . The order is closed if the set  $\{(x, y): x \leq y\}$  is a closed subset in the topology of  $S \times S$ .

If  $A \subset S$ , define

$$L(A) = \{y: y \leq x \text{ for some } x \in A\}$$

and  $M(A) = \{w: x \leq w \text{ for some } x \in A\}$ . A set is *decreasing (increasing)* if  $L(A) = A$  ( $M(A) = A$ ).

The multiplication function on  $S$  is said to be *lower semicontinuous* if given  $(x, y) \in S \times S$  and  $W$  an open, decreasing set containing  $L(xy)$ , there exist open sets  $U$  and  $V$  such that  $x \in U$ ,  $y \in V$ , and  $UV \subset W$ .

**PROPOSITION 7.2.** *Let  $S$  be a compact Hausdorff semilattice with closed order. Then multiplication is lower semicontinuous.*

*Proof.* Let  $W$  be an open decreasing set containing  $xy$ . Let  $\{x_\alpha\}, \{y_\beta\}$  be nets converging to  $x$  and  $y$  resp. Then  $x_\alpha \wedge y_\beta \leq x_\alpha$  for each  $\alpha, \beta$ . Hence for any cluster point  $z$  of the net  $\{x_\alpha \wedge y_\beta\}$ , we have  $z \leq x$  since the order is closed. Similarly  $z \leq y$ , and hence  $z \leq x \wedge y$ . Thus every cluster point is in  $L(x \wedge y) \subset W$ . Since  $S$  is compact and since no cluster point lies outside of  $W$ , the conclusion follows by standard arguments involving nets and continuity.

The multiplication function is *upper semicontinuous* if given  $(x, y) \in S \times S$  and  $W$  an open increasing set containing  $(x, y)$ , there exist open sets  $U, V$  with  $x \in U, y \in V$  such that  $UV \subset W$ .

**PROPOSITION 7.3.** *Let  $S$  be a compact Hausdorff semilattice in which multiplication is separately continuous. Then it is upper semicontinuous.*

*Proof.* Let  $W$  be an open increasing set containing  $xy$ . By Corollary 5.2, multiplication restricted to  $L(x) \times S$  is continuous at  $(x, y)$ . Hence there exist  $P'$  open in  $L(x)$ ,  $V$  open in  $S$ , with  $x \in P', y \in V$  and  $P'V \subset W$ . Pick  $P$  open in  $S$  such that  $P \cap L(x) = P'$ . By separate continuity there exists  $U$  open,  $x \in U$ , such that  $Ux \subset P$ . Since  $Ux \subset L(x)$ , we have  $Ux \subset P'$ .

Let  $u \in U, v \in V$ . Then  $w \geq xw = uv \in P'V \subset W$ . Hence  $w \in M(W) = W$ .

**PROPOSITION 7.4.** *Let  $S$  be a compact Hausdorff semilattice with closed order and separately continuous multiplication. Then  $S$  is a topological semilattice.*

*Proof.* By a result of Nachbin, a point in a compact Hausdorff space with closed order has a basis of neighborhoods which are the intersection of an open increasing set and an open decreasing set [10]. It follows then from Propositions 7.2 and 7.3 that multiplication is jointly continuous.

Conditions which imply joint continuity in semilattices have been considered by Borrego [3]. A modification of his results was used in [8] to help show continuity in a certain example constructed there. Proposition 7.4 appears to be a stronger and more useful formulation of conditions which give joint continuity.

(C) Dense Ideals of Continuity. A semigroup  $S$  is said to be *weakly reductive* if given  $x, y \in S, x \neq y$ , there exist  $r, s \in S$  such that  $rx \neq ry$  and

$xs \neq ys$ . The next proposition illustrates the principle that continuity "below" implies continuity "at the top."

**PROPOSITION 7.5.** *Let  $S$  be a compact Hausdorff weakly reductive semitopological semigroup. If  $I$  is a dense ideal, and if multiplication restricted to  $I \times I$  is jointly continuous, then  $S$  is a topological semigroup.*

*Proof.* Let  $(a, b) \in I \times I$  and let  $W$  be an open set containing  $ab$ . Pick an open set  $W_1$  such that  $ab \in W_1 \subset W_1^* \subset W$ . By continuity there exist sets  $U, V$  open in  $I$  such that  $U \cdot V \subset W_1$ . Then by separate continuity  $U^* \cdot V \subset W_1^*$ , and applying separate continuity again  $U^* \cdot V^* \subset W_1^* \subset W$ . Since  $I$  is dense  $U^*$  and  $V^*$  are neighborhoods of  $a$  and  $b$  resp. in  $S$ . Hence multiplication is continuous in  $S \times S$  at any point  $(a, b) \in I \times I$ .

Now let  $c \in S, d \in I$ , and  $y \in S, y \neq cd$ . Then by weak reductivity there exists  $t \in S$  such that  $tc d \neq ty$ . Since  $I$  is dense, there exists  $r \in I$  such that  $rc d \neq ry$ . Since  $I$  is an ideal,  $rc \in I$ . Hence multiplication is continuous at  $(rc, d)$ . Thus by Proposition 2.2, multiplication is continuous at  $(c, d)$ .

Now employing the right analogue of Proposition 2.2, a procedure on the right analogous to the preceding paragraph, and the conclusion of the preceding paragraph, we deduce continuity at any point  $(p, q) \in S \times S$ .

Professor K. H. Hofmann pointed out the following corollary.

**COROLLARY 7.6.** *If  $S$  is a compact semitopological semigroup with identity, the group of units  $H$  of  $S$  is nowhere dense, and  $S \setminus H$  is an ideal and a topological semigroup, then  $S$  is a topological semigroup and hence  $H$  is compact.*

#### REFERENCES

1. J. BERGLUND, *Compact connected ordered semitopological semigroups*, J. London Math. Soc., vol. 4 (1972), pp. 533-540.
2. J. BERGLUND AND K. H. HOFMANN, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture notes in Mathematics, vol. 42, Springer-Verlag, Berlin-New York, 1967.
3. J. T. BORREGO, *Continuity of the operation in a semilattice*, Colloq. Math., vol. 21 (1970), pp. 49-52.
4. N. BOURBAKI, *General topology, part 2*, Addison-Wesley, Reading, Mass., 1966.
5. K. DELEEuw AND I. GLICKSBERG, *Applications of almost periodic compactifications*, Acta Math., vol. 105 (1961), pp. 63-97.
6. R. ELLIS, *Locally compact transformation groups*, Duke Math. J., vol. 24 (1957), pp. 119-125.
7. W. HUREWICZ AND H. WALLMAN, *Dimension theory*, Princeton University Press, Princeton, N. J., 1940.
8. J. D. LAWSON, *Lattices with no interval homomorphisms*, Pacific J. Math., vol. 32 (1970), pp. 459-465.
9. P. S. MOSTERT AND A. L. SHIELDS, *Semigroups with identity on a manifold*, Trans. Amer. Math. Soc., vol. 91 (1959), pp. 380-389.
10. L. NACHBIN, *Topology and order*, Van Nostrand, Princeton, N. J., 1965.

LOUISIANA STATE UNIVERSITY  
BATON ROUGE, LOUISIANA