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Joint Propagation and Exploitation of Probabilistic and Possibilistic Information in Risk Assessment

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***Abstract*— Random variability and imprecision are two distinct facets of the uncertainty affecting parameters that influence the assessment of risk. While random variability can be represented by probability distribution functions, imprecision (or partial ignorance) is better accounted for by possibility distributions (or families of probability distributions). Because practical situations of risk computation often involve both types of uncertainty, methods are needed to combine these two modes of uncertainty representation in the propagation step. A hybrid method is presented here, which jointly propagates probabilistic and possibilistic uncertainty. It produces results in the form of a random fuzzy interval. This paper focuses on how to properly summarize this kind of information; and how to address questions pertaining to the potential violation of some tolerance threshold. While exploitation procedures proposed previously entertain a confusion between variability and imprecision, thus yielding overly conservative results, a new approach is proposed, based on the theory of evidence, and is illustrated using synthetic examples.**

***Index Terms*— (Random) Fuzzy Intervals, Probability, Possibility, Belief functions, Dependence.**

I. I

Risk assessment methods have become popular support tools in decision-making processes. In the field of contaminated soil management, for example, risk assessment is typically used to establish whether certain levels of soil contamination might represent a threat for human health. The assessment is carried out using predictive "models" that involve a certain number of parameters. Uncertainty is an unavoidable component of such a procedure. In addition to the uncertainty regarding the model itself, each model parameter is usually fraught with some degree of uncertainty. This uncertainty may have essentially two origins : randomness due to natural variability resulting from heterogeneity or stochasticity, or imprecision due to lack of information resulting, for example, from systematic measurement error or expert opinion. As suggested by Ferson and Ginzburg [27], distinct methods are needed to adequately represent random variability (often referred to as "objective uncertainty") and imprecision (often referred to as "subjective uncertainty").

In risk assessment, no distinction is traditionally made between these two types of uncertainty, both being represented by means of a single probability distribution. In case of partial ignorance, the use of a single probability measure introduces information that is in fact not available. This may seriously bias the outcome of a risk analysis in a non-conservative manner (see [23]). Let $T:\mathbb{R}^n \to \mathbb{R}$ be a function (model) of n parameters x_i ($x = (x_1, ..., x_n)$). The main issue is thus to carry the uncertainty attached to the variables over to $T(x)$ with the least possible loss of initial information. This is uncertainty propagation. It may occur in practice, that some parameters of empirical models can be represented by probability distributions (due to observed variability, and sufficient statistics) while others are better represented by possibility distributions (due to imprecision), or by belief functions of Shafer (in the case of partially observed variability and partial ignorance).

Many researchers have addressed uncertainty in risk assessments using either one or the other of these modes of representation. For example Labieniec et al. [37] used probability distribution functions to address uncertainty in the estimation of the risk of human exposure due to the presence of contaminated land. Prado et al. [40] applied probability theory in risk assessments related to the underground disposal of nuclear waste. Dou et al. [13], Bardossy et al. [1], Freissinet et al. [31] present applications of possibility theory to environmental problems. But fewer have considered combining these different modes of representation (probability, possibility, belief function) within the same computation of risk.

Kaufmann and Gupta [35] introduced hybrid numbers which simultaneously express imprecision (fuzzy number) and randomness (probability). In Guyonnet et al. [33] a method, dubbed "hybrid" method, was proposed for a joint handling of probability and possibility distributions in the computation of risk. This method is related to an earlier proposal by Ferson et al. [9] [25] who extended the approach of Kaufmann and used hybrid arithmetic to treat risk analysis [22]. The hybrid method combines random sampling of the probability distributions (Monte Carlo method [7]) with fuzzy interval analysis [14]. The result is a random fuzzy set [32]. In order to compare the random fuzzy set to a tolerance criterion, Guyonnet et al. [33] proposed to summarize the resulting random fuzzy interval under the form of a single fuzzy interval, from which two cumulative (optimistic and pessimistic) distributions can be derived for the purpose of comparison with a tolerance threshold.

We consider four important issues in risk assessment [27]: the first one is how to represent the available information faithfully [26], the second one is how to account for dependencies, correlations between the parameters in the propagation process (linear, non linear monotone dependency, interaction ...). For example the assumption of stochastic independence between parameters can generate too optimistic results [24] [29]. The third issue is the choice of the propagation technique [3] [23]. The last one is how to exploit the results of propagating variability and imprecision jointly. This paper focuses on the two last steps. We revisit the joint propagation of possibility and probability distributions through a numerical model, laying bare the underlying assumptions, and we propose a new post-processing method, suggested in [4], in the framework of belief functions. We also discuss indices quantifying the amount of variability and incompleteness of a random fuzzy set, with a view to present the results of risk analysis to a user in an understandable format.

In Section 2, we recall basic concepts of numerical possibility theory [15] and belief functions [42] in connection with imprecise probabilities [12]. In Section 3, we explain the hybrid propagation method [25] [33] in detail and we study the links with the random set approach using belief functions to propagate uncertainties [3]. In Section 4, we discuss the exploitation of the random fuzzy results and show that the postprocessing step proposed by Guyonnet et al. [33] introduces a confusion between variability and imprecision that may yield overly-conservative results. Then, we propose an alternative approach that explicitly accounts for the difference between the two components of uncertainty. Ferson's method [9] [25] [22] is also recalled. The difference between the three information summarization methods is laid bare and illustrated with synthetic numerical examples.

II. U T

The aim of this section is to recall uncertainty theories useful in the sequel, namely probability theory, possibility theory and the theory of belief functions, albeit from the standpoint of imprecise probabilities. Our purpose is to distinguish between situations where uncertainty is due to the variability of the observed phenomenon, from situations where it is due to a mere lack of knowledge. While the former is handled by means of probability theory, the latter is more naturally captured by set-valued representations whereby all that is known is that a certain value belongs to a certain set, which is possibly fuzzy. This is the idea of possibility theory. More general theories combine the two frameworks, thus yielding more general, and also more costly representations.

A. Probability theory:the frequentist view

Probability theory is taylored to the representation of precise observations tainted with variability. To consider a classical setting of dice tossing, one can see the number obtained after each toss but one does not always obtain the same outcome for each toss. All probability measures P can be defined from a sample space Ω equipped with probability mesure, defined on an algebra $\mathcal{A}$ of measurable subsets. In the discrete case, a distribution function $p : \Omega \longrightarrow [0,1]$ exists such that $\sum_{\omega\in\Omega} p(\omega) = 1$. In the continuous case, let X be a real random variable $X \longrightarrow \mathbb{R}$. A probability measure P_X on $\mathbb{R}$ is induced from the sample space, with density p_X such that $\int_{\mathbb{R}} p(x)dx = 1$ Namely for any measurable subset $A \subseteq \mathbb{R}$, called event, it holds:

$$\text{(discrete case)} : P_X(A) = \sum_{\omega:X(\omega)\in A} p(\omega), \tag{1}$$

$$\text{(continuous case)} : P_X(A) = \int_A p_X(x)dx, \tag{2}$$

The cumulative distribution function of X is $F : \mathbb{R} \to [0,1]$, defined from p_X as follows:

$$F(x) = P_X((-\infty, x]) = P(X \leq x) = \int_{-\infty}^{x} p_X(t)dt, \ \forall\ x \in \mathbb{R} \tag{3}$$

The number $p(\omega)$ represents the (limit) frequency of observing ω after many trials in the discrete case, and the density of ω in the continuous case. Probability measures P verify:

$$\forall A, B \subseteq \Omega \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{4}$$

Probability measures are self-dual, that is $P(A) = 1 - P(\overline{A})$.

B. Limitations of subjective probability

When faced with incomplete information regarding a given model parameter, e.g. the knowledge that the parameter value is located somewhere between a value *min* and a value *max*, it is common to assume a uniform probability distribution between *min* and *max*. This approach appeals to the Laplace principle of insufficient reason, according to which all that is equally plausible is equally probable. It can also be justified on the basis of the "maximum entropy" approach (see [34]). More generally, the subjective probability view (for instance, Lindley [38]) claims that any state of knowledge, however incomplete, can be represented by means of a single (a priori) probability. Such a claim is based on the theory of exchangeable bets, where the degree of probability of an event is understood as the price a player accepts to pay for buying a lottery ticket that brings one dollar to the player if this event occurs. It is assumed that the ticket seller and the player exchange roles if the former finds the price proposed by the latter unfair. In such a constrained framework, it is easy to verify that the lottery ticket prices for all events must follow the rules of probability theory, or else the player is sure to lose money. This view of probability is indeed purely subjective since two different persons may offer different prices for the lottery tickets. Based on this conceptual framework, Bayesian probabilists tend to dismiss all alternative approaches to incomplete information and belief representation as being irrational.

However, this point of view can be challenged in various ways. First, adopting uniform probabilities to express ignorance implies that degrees of probability will depend on the size of the universe of discourse. Two uniform probability distributions relative to two different frames of discernment representing the same problem may be incompatible with each other (Shafer [42]). Besides, if ignorance means not being able to tell whether one contingent event is more or less probable than any other contingent event, then uniform probabilities cannot account for this postulate because, unless the frame of discernment is Boolean: even assuming a uniform probability, some event will have a probability higher than another [20]. The subjective probability approach has also been criticized by pointing out that the exchangeable bet framework is debatable: the player may be allowed not to play. There is a maximal price (s)he is ready to pay for buying the lottery ticket, and the seller has a minimal price below which he no longer wants to sell it. This is the basis of the imprecise probability framework of Walley [43]. The bottom line of the criticism made by non-additive probability theories is that while in the exchangeable bet framework, lottery prices are induced by the belief state of the player, there is no one-to-one correspondence between lottery prices and degrees of belief. For instance, the meaning of a uniform probability distribution obtained from an expert is ambiguous: it may be that the expert knows the underlying phenomenon is really random (like a fair die), or that (s)he is totally ignorant of this phenomenon, hence sees no reason to bet more money on one outcome rather than another.

C. Numerical Possibility Theory

Possibility theory [15] is convenient to represent consonant imprecise knowledge. The basic notion is the possibility distribution, denoted π, here a mapping from the real line to the unit interval, unimodal and upper semi-continuous. A possibility distribution describes the more or less plausible values of some uncertain variable X. Possibility theory provides two evaluations of the likelihood of an event, for instance that the value of a real variable X should lie within a certain interval: the possibility Π and the necessity N . The normalized measure of possibility Π (respectively necessity N) is defined from the possibility distribution $\pi : \mathbb{R} \to [0,1]$ such that $\sup_{x\in\mathbb{R}} \pi(x) = 1$ as follows:

$$\Pi(A) = \sup_{x\in A} \pi(x) \tag{5}$$

$$N(A) = 1 - \Pi(\overline{A}) = \inf_{x\notin A} (1 - \pi(x)) \tag{6}$$

- The possibility measure Π verifies :

$$\forall A, B \subseteq \mathbb{R} \quad \Pi(A \cup B) = \max(\Pi(A), \Pi(B)) \tag{7}$$

- The necessity measure N verifies :

$$\forall A, B \subseteq \mathbb{R} \quad N(A \cap B) = \min(N(A), N(B)) \tag{8}$$

A possibility distribution may also be viewed as a nested set of confidence intervals, which are the α-cuts $[\underline{x}_\alpha, \overline{x}_\alpha] = \{x, \pi(x) \geq \alpha\}$ of π. The degree of certainty that $[\underline{x}_\alpha, \overline{x}_\alpha]$ contains X is $N([\underline{x}_\alpha, \overline{x}_\alpha])$ ($= 1 - \alpha$ if π is continuous). Conversely, suppose a nested set of intervals A_i with degrees of certainty λ_i that A_i contains X is available. Provided that λ_i is interpreted as $N(A_i) \geq \lambda_i$, and π is chosen as the least specific possibility distribution satisfying these inequalities [18], this is equivalent to knowing the possibility distribution

$$\pi(x) = \min_{i=1\ldots n} \{1 - \lambda_i, x \notin A_i\}$$

with convention $\min \emptyset = 1$.

A pair (interval A, necessity weight λ) supplied by an expert is interpreted as stating that the (subjective) probability $P(A)$ is at least equal to λ [18] where A is a measurable set. This definition is mathematically meaningful [8], and in particular, the α-cut of a continuous possibility distribution can be understood as the inequality $P(X \in [\underline{x}_\alpha, \overline{x}_\alpha]) \geq 1 - \alpha$. Equivalently, the probability $P(X \notin [\underline{x}_\alpha, \overline{x}_\alpha])$ is at most equal to α. Degrees of necessity are equated to lower probability bounds and degrees of possibility are then equated to upper probability bounds.

D. Evidence Theory

The theory of belief functions [42] (also called evidence theory) allows imprecision and variability to be treated separately within a single finite framework. Indeed, belief functions provide mathematical tools to process information which is at the same time of random and imprecise nature. We typically find this kind of knowledge when one uses some measurement device which has a systematic error (imprecision) and a random error (variability). We may obtain a sample of random intervals $([m_i - \delta, m_i + \delta])_{i=1\ldots K}$ supposedly containing the true value, where δ is a systematic error, m_i is the observed measurement $i = 1 \ldots K$ and K is the number of interval observations. Each interval is attached probability ν_i of observing the measured value m_i. That is, we obtain a mass distribution $(\nu_i)_{i=1\ldots K}$ on intervals. The probability mass ν_i can be freely re-allocated to points within interval $[m_i - \delta, m_i + \delta]$. However, there is not enough information to do it.

Contrary to probability theory which assigns probability weights to atoms (elements of the referential) the theory of evidence may assign such weights to any subsets, called focal sets, with the understanding that portions of these weights may move freely from one element of such subsets to another . As in possibility theory, evidence theory provides two indicators, plausibility Pl and belief Bel, to qualify the validity of a proposition stating that the value of variable X should lie within a set A (a certain interval for example). Plausibility Pl and belief Bel measures are defined from the mass distribution:

$$\nu : \mathcal{P}(\Omega) \to [0,1] \quad \text{such that} \quad \sum_{E\in\mathcal{P}(\Omega)} \nu(E) = 1 \tag{9}$$

as follows:

$$Bel(A) = \sum_{E, E\subseteq A} \nu(E) \tag{10}$$

$$Pl(A) = \sum_{E, E\cap A\neq\emptyset} \nu(E) = 1 - Bel(\bar{A}) \tag{11}$$

where $\mathcal{P}(\Omega)$ is the power set of Ω and E is called focal element of $\mathcal{P}(\Omega)$, when $\nu(E) > 0$.

$Bel(A)$ gathers the imprecise evidence that asserts A ; following Dempster [12], it is the minimal amount of probability that can be assigned to A by sharing the probability weights defined by the mass function among single values in the focal sets. $Pl(A)$ gathers the imprecise evidence that does not contradict A ; it is the maximal amount of probability that can be assigned to A in the same fashion.

Evidence theory encompasses possibility and probability theory.

- When focal elements are nested, a belief measure Bel is a necessity measure, that is $Bel = N$. A Plausibility measure Pl is a possibility measure, that is $Pl = \Pi$.
- When focal elements are some disjoint intervals, plausibility Pl and belief Bel measures are both probability measures, that is we have $Bel = P = Pl$, for unions of such intervals.

Thus, all probability distributions and all possibility distributions may be interpreted by mass functions. Hence, one may work in a common framework to treat the information of imprecise and random nature.

E. Approximate Encoding of Continuous Possibility and Probability as Belief Functions

Belief functions [42] encompass possibility and probability theories in the finite case (see Section II-D). Here we explain more precisely how we can build a mass distribution ν from a probability distribution function p or a possibility distribution π. In the continuous case, the representation will be approximate but this is how we shall make computations.

1) Probability $\rightarrow$ Belief function.

   Let X be a real random variable with a probability density p_X. By discretizing it into m intervals, we define, as focal elements, disjoint intervals $(]a_i, a_{i+1}])_{i=1...m}$ and we can build the mass distribution $(\nu_i)_{i=1...m}$ as follows $\forall i = 1...m$ [17]:

$$\nu(]a_i, a_{i+1}]) = \nu_i = P(X \in]a_i, a_{i+1}]) \tag{12}$$

2) Possibility $\rightarrow$ Belief function.

   Let Y be a possibilistic variable. We denote by π the possibility distribution of Y and π_α the α-cuts of π. Focal elements for Y corresponding to α-cuts are denoted $(\pi_{\alpha_j})_{j=1...q}$ with $\alpha_0 = \alpha_1 = 1 > \alpha_2 > \cdots > \alpha_q > \alpha_{q+1} = 0$ and are nested. We denote by $(\nu_j = \alpha_j - \alpha_{j+1})_{j=1...q}$ the mass distribution associated to $(\pi_{\alpha_j})_{j=1...q}$ (see Figure 1 for instance). Note that we thus approximate π by the discrete possibility distribution π_* such that $\pi_*(x) = \alpha_j \leq \pi(x)$ if $x \in \pi_{\alpha_j} - \pi_{\alpha_{j+1}}$. It is a lower approximation of π. Alternatively one might prefer an upper approximation π^* such that $\pi^*(x) = \alpha_j \geq \pi(x)$ if $x \in \pi_{\alpha_{j-1}} - \pi_{\alpha_j}$ (see Figure 1 for instance).

F. Imprecise Probability

Let $\mathcal{P}$ be a probability family on the referential Ω. For all $A \subseteq \Omega$ measurable, we can define:

$$\text{its upper probability} \quad \overline{P}(A) = \sup_{P \in \mathcal{P}} P(A) \tag{13}$$

$$\text{its lower probability} \quad \underline{P}(A) = \inf_{P \in \mathcal{P}} P(A). \tag{14}$$

Let $\mathcal{P}(\underline{P} < \overline{P}) = \{P, \forall A \subseteq \Omega, \underline{P}(A) \leq P(A) \leq \overline{P}(A)\}$ be the family probability induced from upper $\overline{P}$ and lower $\underline{P}$ probability induced from $\mathcal{P}$. Clearly $\mathcal{P}$ is a proper subset of $\mathcal{P}(\underline{P} < \overline{P})$ generally. The notion of cumulative distribution function becomes a pair of upper & lower cumulative distribution functions $\overline{F}$ and $\underline{F}$ defined as follows:

$$\forall x \in \mathbb{R} \quad \overline{F}(x) = \overline{P}(X \in]-\infty, x]) \tag{15}$$

$$\forall x \in \mathbb{R} \quad \underline{F}(x) = \underline{P}(X \in]-\infty, x]) \tag{16}$$

where X is a random variable associated to P. The gap between $\overline{F}$ and $\underline{F}$ reflects the incomplete nature of the knowledge, thus picturing what is unknown.

We can interpret any pair of dual functions necessity/possibility $[N, \Pi]$, or belief/plausibility $[Bel, Pl]$ as upper and lower probabilities induced from specific probability families.

- Let π be a possibility distribution inducing a pair of functions $[N, \Pi]$. We define the probability family $\mathcal{P}(\pi) = \{P, \forall A \text{ measurable}, N(A) \leq P(A)\} = \{P, \forall A \text{ measurable}, P(A) \leq \Pi(A)\}$. In this case, $\sup_{P \in \mathcal{P}(\pi)} P(A) = \Pi(A)$ and $\inf_{P \in \mathcal{P}(\pi)} P(A) = N(A)$ (see [8], [18]).
- Conversely, given $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ some measurable subsets of Ω with their confidence degrees $1 - \alpha_1 \leq ... \leq 1 - \alpha_n$ ($1 - \alpha_i$ probabilities given by experts for example), we define the probability family as follows: $\mathcal{P} = \{P, \forall A_i, 1 - \alpha_i \leq P(A_i)\}$. We thus know that $\overline{P} = \Pi$ and $\underline{P} = N$ (see [18], and in the infinite case [8]). We hence can define upper $\overline{F}$ and lower $\underline{F}$ cumulative distribution functions such that $\forall x \in \mathbb{R} \quad \underline{F}(x) \leq F(x) \leq \overline{F}(x)$ with :

$$\overline{F}(x) = \Pi(X \in]-\infty, x]) \tag{17}$$

$$\underline{F}(x) = N(X \in]-\infty, x]) \tag{18}$$

- A mass distribution ν may encode probability family $\mathcal{P} = \{P, \forall A \text{ measurable}, Bel(A) \leq P(A)\} = \{P, \forall A \text{ measurable}, P(A) \leq Pl(A)\}$ [12]. In this case we have: $\overline{P} = Pl$ and $\underline{P} = Bel$, so that:

$$\forall P \in \mathcal{P}, Bel \leq P \leq Pl \tag{19}$$

  Hence, We can define upper $\overline{F}$ and lower $\underline{F}$ cumulative distribution functions such that $\forall x \in \mathbb{R} \quad \underline{F}(x) \leq F(x) \leq \overline{F}(x)$ with :

$$\overline{F}(x) = Pl(X \in]-\infty, x]) \tag{20}$$

$$\underline{F}(x) = Bel(X \in]-\infty, x]) \tag{21}$$

So, we may cast possibility theory and evidence theories into a probabilistic framework respectful of the incompleteness of the available information. Possibility distributions π and mass distributions ν then encode probability families $\mathcal{P}(\pi)$ and thus allow to represent incomplete probabilistic knowledge. The intervals $[N, \Pi]$ induced from π and $[Bel, Pl]$ induced from ν thus provide some bracketing of ill-known probabilities [5] [21] [19] [26]. Note that this is not at all the view of belief functions advocated by Shafer, nor Smets [41] [1]. Moreover, while a unique probability measure can be reconstructed from the cumulative distribution F, there are several mass functions yielding a given pair of upper and lower cumulative distribution functions.

III. J

Let us assume $k < n$ random variables $(X_1, ..., X_k)$ taking values $(x_1, ..., x_k)$ and $n-k$ possibilistic variables $(X_{k+1}, ..., X_n)$ taking values $(x_{k+1}, ..., x_n)$ represented by possibility distributions $(\pi^{X_{k+1}}, ..., \pi^{X_n})$. This section explains how to propagate heterogeneous uncertainties pervading the parameters $(X_i)_{i=1...n}$ through a function T by means of an hybrid probabilistic/possibilistic method.

[1]These authors systematically refrain from referring to probability bounds, and rather view $Bel(A)$ as a degree of belief per se.

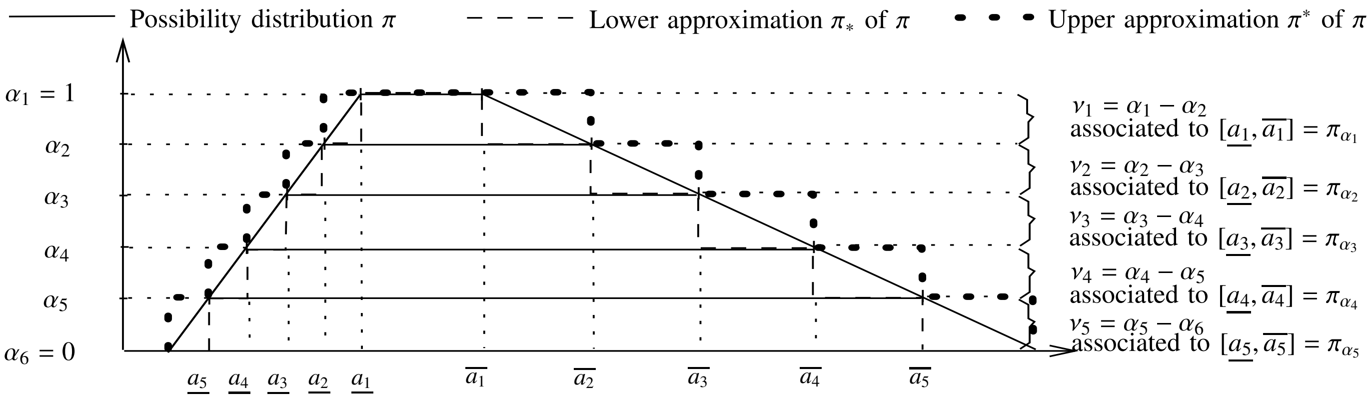


Fig. 1. Transformation possibility→belief function

A. The hybrid propagation method

The hybrid propagation method, variants of which were proposed in [9] [25] [33] involves two main steps (see Figure 2). It combines a Monte Carlo technique (Random Sampling [7]) with the extension principle of fuzzy set theory [14]. We first perform a Monte Carlo sampling of the random variables, taking into account dependencies (if known), thus processing variability (probability). Values thus obtained form prescribed k-tuples ($X_1 = x_1, ..., X_k = x_k$) and fuzzy interval analysis is used to estimate T. The knowledge on the value of $T(X)$ becomes a fuzzy subset, for each k-tuple. Random sampling is resumed and the process is performed in an iterative fashion in order to obtain a sample $(\pi_1^T, ..., \pi_m^T)$ of fuzzy subsets where m is the realization number of the k random variables. $T(X)$ then becomes a fuzzy random variable (or a random possibility distribution) in the sense of [32] [36].

The hybrid procedure is summarized as follows [33]

1) Generate k random numbers $(p_1, ..., p_k)$ from a uniform distribution on $[0, 1]$ taking account dependencies (if known) and sample the k probability distribution functions to obtain a realization of the k random variables: $(x_1, ..., x_k)$ (see Figure 2.a)
2) Select a possibility value α and the corresponding cut as the selected interval.
3) Interval calculation : calculate the Inf (smallest) and Sup (largest) values of $T(x_1, ..., x_k, X_{k+1}, ..., X_n)$, considering all values located within the α-cuts for each possibility distribution (see Figure 2.b).
4) Assign these Inf and Sup values to the lower and upper limits of the α-cut of $T(x_1, ..., x_k, X_{k+1}, ..., X_n)$.
5) Return to step 2 and repeat steps 3 and 4 for another α-cut. The fuzzy result of $T(x_1, ..., x_k, X_{k+1}, ..., X_n)$ is obtained from the Inf and Sup values of $T(x_1, ..., x_k, X_{k+1}, ..., X_n)$ for each α-cut.
6) Return to step 1 to generate a new realization of the random variables. A family of fuzzy numbers $(\pi_1^T, ..., \pi_m^T)$ is obtained (see Figure 2.c).

B. Underlying independence assumptions

The classical Monte Carlo method has been criticized by Ferson [23] because it presupposes stochastic independence between random variables. In the case where we know that random variables are independent, the Monte Carlo method is correct. It is worthwile noticing that within a Monte Carlo approach the rank correlation (non linear monotone dependency) between the random variables [6] can be taken into consideration (if known). Even if we can account for some dependencies between random variables with Monte-Carlo, it is necessary to be aware that the Monte Carlo method cannot account for all forms of dependence.

Similarly, we must be careful with the extension principle because it underlies a meta-dependence assumption on possibilistic variables. In fact the presence of imprecision on $X_{k+1}, ..., X_n$ potentially generates two levels of dependencies. The first one is a (meta-)dependence between information sources attached to variables and the second one is a dependence between variables themselves. The extension principle [14], $\forall u \in \mathbb{R}$ defines the resulting possibility distribution as:

$$\pi^T(u) = \sup_{x_{k+1},...,x_n, T(x_1,...,x_n)=u} \min(\pi^{X_{k+1}}(x_{k+1}), ..., \pi^{X_n}(x_n)). \quad (22)$$

It is equivalent to performing interval analysis on $\alpha - cuts$ and hence assumes strong dependence between information sources (observers) supplying the input possibility distributions, since the same confidence level is chosen to build these α-cuts [17]. Namely, one expert interprets fuzzy intervals π^X and π^Y for two possibilistic variables X and Y as α-cuts π_α^X and π_α^Y with the same confidence degree $1 - \alpha$. This suggests that if the source informing on X is rather precise then the one informing on Y is also precise (for instance it is the same source). It induces a dependence between the knowledge of X and the knowledge of Y since for instance pairs of values in $\pi_1^X \times \pi_1^Y$ are supposed to be the most plausible. However, this form of meta-dependence does not presuppose any genuine functional (objective) dependence between possibilistic variables inside the domain $\pi_\alpha^X \times \pi_\alpha^Y$ (the observed phenomenona). The use of "minimum" assumes the non-interaction of $X_{k+1}, ..., X_n$, which expresses a lack of knowledge about the links between the actual values of $X_{k+1}, ..., X_n$, hence a lack of commitment as to whether $X_{k+1}, ..., X_n$ are linked or not. Indeed, the least specific joint possibility distribution whose projections on the X and Y axes is precisely $\pi^{X,Y} = \min(\pi^X, \pi^Y)$.

As a consequence of the dependence between the choice of confidence levels, one cannot interpret the calculus of possibilistic variables as a conservative counterpart to the

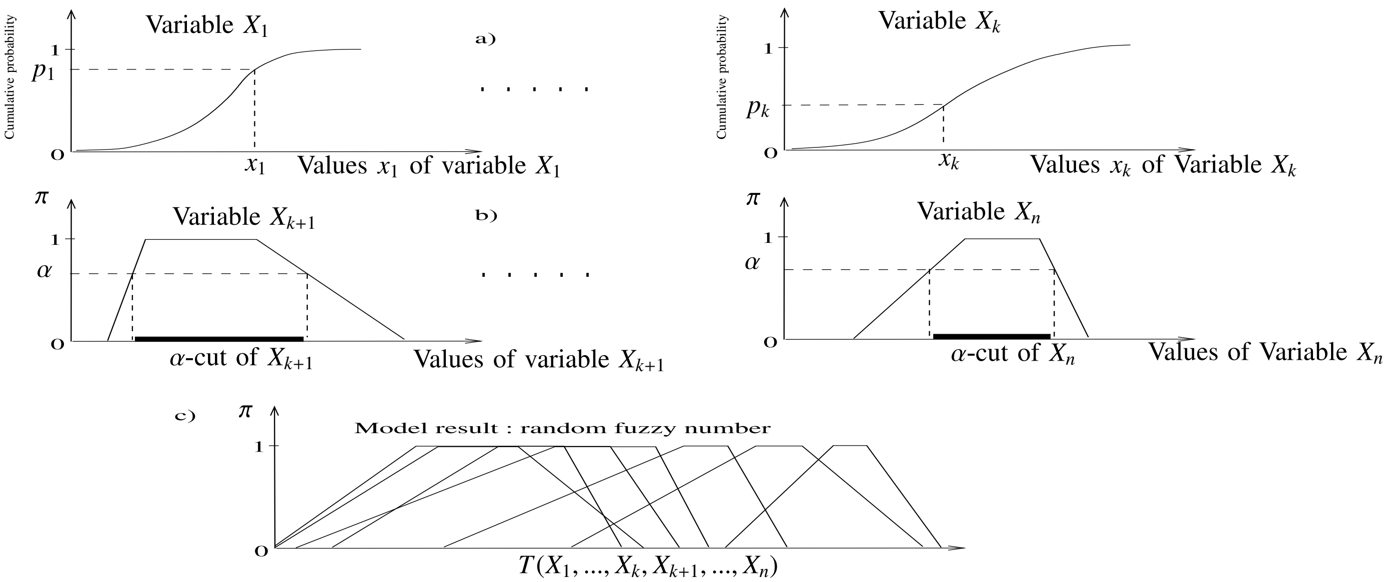


Fig. 2. Schematic illustration of the "hybrid" method

calculus of probabilistic variables under stochastic independence. Namely if P^X and P^Y are probability measures assigned to X and Y such that $P^X \in \mathcal{P}(\pi^X)$ and $P^Y \in \mathcal{P}(\pi^Y)$, it does not imply that the joint probability $P^{X,Y} = P^X \cdot P^Y$ is contained in $\mathcal{P}(\min(\pi^X, \pi^Y))$. To wit, assume X and Y have uniform probability densities on $[0, 1]$, and that π^X and π^Y are linear decreasing on $[0, 1]$, so that $\pi^X(u) = \pi^Y(u) = 1 - u$. Clearly, $P^X([u, 1]) = \Pi^X([u, 1]) = 1 - u$. So $P^X \in \mathcal{P}(\pi^X)$, $P^Y \in \mathcal{P}(\pi^Y)$. Yet, let $C(u) = \{(x, y) : x + y \geq u\}$. It is clear that $P^{X,Y}(C(u)) = 1 - \frac{u^2}{2}$ if $u \leq 1$ and $\frac{(2-u)^2}{2}$ otherwise; while $\Pi^{X,Y}(C(u)) = \sup_x \min(1 - x, 1 - u + x) = 1 - \frac{u}{2}$. So, $\Pi^{X,Y}(C(u)) < P^{X,Y}(C(u))$ whenever $u \leq 1$. It means that if $Z = X + Y$, then P^Z does not belong to $\mathcal{P}(\pi^Z)$.

Besides, the hybrid propagation method clearly assumes stochastic independence between the group of probabilistic variables and the group of possibilistic ones, the latter viewed as forming a random Cartesian product on the space of possibilistic variables, as explained in the next section. Being aware of the underlying assumptions, we can use this methodology in risk assessment. We will see in the next sections how we can estimate for example $P(T(X) \in] - \infty, t])$ (where t can be a threshold) from this hybrid result (random fuzzy number).

C. Casting uncertainty propagation in the setting of random sets

Belief functions [42] encompass possibility and probability theory. In this section the hybrid method is cast in this enlarged setting so as to better lay bare the underlying independence assumptions and illustrate the links between the propagation results obtained with the hybrid approach and what could be a pure random set approach. For the sake of clarity, consider a continuous function T of two variables. Let X be a discrete random variable with $\Omega_X = \{x_1, ..., x_m\}$ and $p_i^X = P(X = x_i)$, Y a possibilistic variable. with possibility distribution π^Y. Focal elements for X are singletons $(\{x_i\})_{i=1...m}$ and the mass distribution is equal to $(p_i^X)_{i=1..m}$ because X is discrete. We choose a discrete probability for the sake of clarity. Focal elements for Y corresponding to α-cuts are denoted $(\pi^Y_{\alpha_j})_{j=1...q}$ with $\alpha_q > 0$ and are nested. We denote by $(\nu_j^Y = \alpha_j - \alpha_{j+1})_{j=1...q}$ the mass distribution associated to $(\pi^Y_{\alpha_j})_{j=1...q}$. We thus encode probabilistic and possibilistic variables as belief functions, in the spirit of section 2.4.

Under the hybrid method, $T(X, Y)$ is a discrete random fuzzy subset. That is, we obtain m fuzzy numbers $(\pi_i^T)_{i=1...m}$ with corresponding probabilities $(p_i^X)_{i=1..m}$. Under the random set approach [3] we interpret this random fuzzy set as $m \times q$ focal elements (intervals) with mass distributions $(p_i^X \nu_j^Y)_{i=1...m, j=1...q}$ and focal elements $\pi_{ij}^T = T(x_i, \pi_{\alpha_j})$. We have the following result:

Proposition: The plausibility Pl^T and belief function Bel^T associated to focal elements π_{ij}^T and mass distribution $(p_i^X \nu_j^Y)_{i=1...m, j=1...q}$ are such that $\forall A$ measurable set,

$$Pl^T(A) = \sum_{i=1}^{m} p_i^X \Pi_i^T(A)$$

$$Bel^T(A) = \sum_{i=1}^{m} p_i^X N_i^T(A)$$

where Π_i^T and N_i^T are the possibility and necessity measures associated to fuzzy numbers π_i^T.

Proof : The calculation of Pl^T reads as follows:

$$Pl^T(A) = \sum_{(i,j),\ A \cap \pi_{ij}^T \neq \emptyset} p_i^X \nu_j^Y = \sum_{i=1}^{m} p_i^X \sum_{j=1...q,\ A \cap \pi_{ij}^T \neq \emptyset} \nu_j^Y.$$

So, $Pl^T(A) = \sum_{i=1}^{m} p_i Pl_i^T(A)$, for each i varying from 1 to n, we have $\pi_{ij}^T \subseteq ... \subseteq \pi_{ik}^T\ \forall j \geq k$. Thus $Pl_i^T(A) = \Pi_i^T(A)$.□

These results still hold when several independent probabilistic variables are involved. These results do not directly apply with more than one possibilistic variable. Indeed recall that fuzzy arithmetic presupposes total dependence between α-cuts.

The hybrid method can be cast in the random set setting, when there are several (say two) possibilistic and probabilistic variables. Consider X, Y, two possibilistic variables encoded as belief functions, their focal elements being $(\pi^X_{\alpha_i})_{i=1...q}$, $(\pi^Y_{\alpha_j})_{j=1...q}$ and the mass distributions $(\nu^X_i)_{i=1...q}$, $(\nu^Y_j)_{j=1...q}$. Let Z, W be two discrete probabilistic variables encoded by their focal elements $(\{z_k\})_{k=1...m}$, $(\{w_l\})_{l=1...m}$ and the mass distributions $(p^Z_k)_{k=1...m}$, $(p^W_l)_{l=1...m}$. If independence between focal sets is assumed, we can define the joint mass distribution (denoted ν_{ijkl}), associated to focal elements $\pi^T_{ijkl} = T(\pi^X_{\alpha_i}, \pi^Y_{\alpha_j}, \{z_k\}, \{w_l\})$ of $T(X, Y, Z, W)$, by:

$$\forall i, j, k, l \quad \nu_{ijkl} = \nu^X_i \nu^Y_j p^Z_k p^W_l.$$

It corresponds to applying a Monte-Carlo method to all variables. For each possibility distribution, an α-cut (here $\pi^X_{\alpha_i}$, and $\pi^Y_{\alpha_j}$) is independently selected. We thus assume the stochastic independence of the focal elements pertaining to different variables.

Suppose now the same value of α is selected for all possibilistic variables. In the hybrid method, the joint possibility distribution $\pi^{X,Y}$ is characterized by $\min(\pi^X, \pi^Y)$ which corresponds to nested Cartesian products of α-cuts and let $\nu^{X,Y}_i$ be the mass associated to the Cartesian product $\pi^X_{\alpha_i} \times \pi^Y_{\alpha_i}$. Then:

$$\forall i, j, k, l, \ i = j : \quad \nu_{iikl} = \nu^{X,Y}_i p^Z_k p^W_l$$
$$\forall i, j, k, l, \ i \neq j : \quad \nu_{ijkl} = 0$$

Here we assume total dependence between focal elements associated to possibilistic variables. Hence, if we want to estimate $Pl^T(A)$, for all measurable set A, using the last definition of ν_{ijkl}, we still have:

$$Pl^T(A) = \sum_{i,k,l;\ A\cap\pi^T_{iikl}\neq\emptyset} \nu^{X,Y}_i p^Z_k p^W_l = \sum_{k,l} p^Z_k p^W_l \Pi^T_{kl}(A)$$

where Π^T_{kl} are the possibility measures associated to the output possibility distributions π^T_{kl} obtained by the hybrid method.

IV. E

The results of the hybrid method are not easy to interpret by a user and need to be summarized in some way so as to be properly exploited. In this section we devise tools for evaluating how much variability and how much imprecision are contained in the output of the hybrid method, in the form of separate indices. Then, we consider the problem of checking to what extent the value of the quantity calculated by the propagation step is likely to pass a given threshold. To this end, we discuss methods for deriving cumulative distributions from random fuzzy intervals. A method previously proposed by some of the authors is criticized as being too conservative and failing to separate variability from imprecision. A new technique is proposed, based on averaging a random fuzzy interval across α-cuts, in line with the previous discussion on the hybrid method. This new technique is also compared to a proposal by Ferson. Consider $(\pi_i)_{i=1...m}$ being the sample of the random fuzzy number $T(X)$ obtained from the hybrid method for the remainder of this section.

A. Measuring variability and imprecision separately

Since the output of the propagation technique is more complex than a probability distribution, we cannot summarize it by a mean value and a variance. Not only is the result tainted with variability, but it also reflects the incompleteness of the data via the presence of fuzzy intervals. It is possible to summarize the imprecision contained in a fuzzy interval, for instance using the mean interval (Dubois and Prade [16]). The position of its middle-point (proposed a long time ago by Yager [45]) reflects the average location of the fuzzy interval. The width of the mean interval reflects the imprecision of the fuzzy interval and is precisely equal to the surface under the possibility distribution. Other evaluations like the degree of fuzziness can be envisaged (see Delgado et al.[11])

Here we propose to combine probabilistic and possibilistic summarized evaluations, with a view to process variability and imprecision separely. To evaluate the average imprecision of $(\pi_i)_i$, we can compute the average fuzzy interval (Kruse and Meyer [36]) π^{mean}_d:

$$\forall z,\ \pi^{mean}_d(z) = \sup_{\frac{1}{m}\sum_{i=1}^m x_i = z} \min(\pi_1(x_1), \ldots, \pi_m(x_m))$$

The average imprecision is measured by the area I under π^{mean}_d, that is $I = \int_{-\infty}^{+\infty} \pi^{mean}_d(u)du$.

To estimate the locational variability of T, we can work with a representative value h^r_i of each fuzzy interval π_i. Then we can estimate a standard variance V of the form:

$$V = \frac{1}{m}\sum_{i=1}^{m}(h^r_i)^2 - \frac{2}{m(m-1)}\sum_{j<i} h^r_j h^r_i$$

where h^r_i is a representative value of π_i. As the representative value h^r_i we can choose the middle point of the mean interval (of each fuzzy interval π_i). It is also equal to the average of the midpoints of the α-cuts of π_i, proposed by Yager [45] very early :

$$h^r_i = \int_0^1 \frac{(\sup \pi_{i\alpha} + \inf \pi_{i\alpha})}{2} d\alpha$$

V appears only as an indicator of result variability. For instance, on Figure 3a, the variability V is small but the imprecision I is high. In contrast, on Figure 3b, the variability is high, but the imprecision is small. We could also try to define the variability of the imprecision in the sample, considering the variance of the surface under the fuzzy numbers π_i.
We could also estimate a fuzzy variance $\pi^{variance}_d$. Let f be the function which estimates the variance:

$$f : (x_1, ..., x_m) \mapsto \frac{1}{m}\sum_{i=1}^{m} {x_i}^2 - \frac{2}{m(m-1)}\sum_{j<i} x_j x_i$$

To obtain $\pi^{variance}_d$, we can work with α-cuts, and build nested intervals $\pi^{variance}_{d,\alpha} = [\underline{V_{d,\alpha}}, \overline{V_{d,\alpha}}]$, solving:

$$\underline{V_{d,\alpha}} = \inf_{x_i\in\pi_{i,\alpha}} f(x_1, ..., x_m)$$

$$\overline{V_{d,\alpha}} = \sup_{x_i\in\pi_{i,\alpha}} f(x_1, ..., x_m)$$

This fuzzy variance describes a potential variability, because it scans variances of all probability functions compatible with the random fuzzy data. Ferson et al. [28] propose an algorithm of quadratic complexity for computing the exact lower bound $\underline{V_{d,\alpha}}$ of the sample variance for interval valued data. However, they show that computing the exact upper bound $\overline{V_{d,\alpha}}$ is NP-hard. There exists an algorithm that computes $\pi_{d,\alpha}^{variance}$ but it is exponential in the sample size. They propose an algorithm of quadratic complexity, but it presupposes all the interval midpoints are away from each other. Computing a fuzzy variance is not straightforward, as we must apply these algorithms for all α-cuts. See Dubois et al. ([30]) for recent results on this problem.

B. The fuzzy prediction interval method

Let p_i be the probability associated to fuzzy number π_i resulting from the hybrid method. If the Monte Carlo method yields distinct fuzzy numbers, then $p_i = 1/m$. Guyonnet et al. in [33] propose to synthesize this random fuzzy result into a single fuzzy subset denoted π_d, discarding outliers. For a given membership grade α, consider the intervals $\pi_{i\alpha} = [\underline{u}_{i\alpha}, \overline{u}_{i\alpha}]$. The distribution of the greatest lower bounds $\{\underline{u}_{i\alpha}\}_{i=1,m}$ and that of the least upper bounds $\{\overline{u}_{i\alpha}\}_{i=1,m}$ are built up. The interval $\pi_{d\alpha} = [\underline{u}_{d\alpha}, \overline{u}_{d\alpha}]$ is computed as $\underline{u}_{d\alpha} = \sum_{i=1}^{m} \underline{u}_{i\alpha}\delta_{]\frac{i-1}{m},\frac{i}{m}]}(1-d\%)$, $\overline{u}_{d\alpha} = \sum_{i=1}^{m} \overline{u}_{i\alpha}\delta_{]\frac{i-1}{m},\frac{i}{m}]}(d\%)$ where $\delta_{]\frac{i-1}{m},\frac{i}{m}]}(a) = 1$ if $a \in]\frac{i-1}{m},\frac{i}{m}]$, 0 otherwise. Varying $\alpha \in [0,1]$, a fuzzy interval π_d is thus built. The standard value $d = 95$ is chosen. That is, they eliminate 5% on the left and on the right side and perform the pointwise union of the remaining fuzzy intervals, thus generalizing the computation of an empirical prediction interval. Starting from this fuzzy interval π_d, we now can try to estimate the probability of events such that: $]-\infty, e]$, $]e, +\infty]$, $]e_1, e_2]$ according to whether we are interested in checking the probability that the output value lies under a threshold e, crosses this threshold, or remains in between two prescribed values.

Unfortunately, there are caveats with this postprocessing method. First, and most importantly, this method confuses variability and imprecision. It does not account for the probabilities generated by the random variables and it thus forgets the frequency of each output fuzzy number. This may put excessive emphasis on randomly generated fuzzy numbers located on the extreme right-handside and left-handside parts of the result π_d. Next, one may obtain the same result whether the π_i's have large imprecision and small variability as in Fig.3a, or they are more precise with a great variability as in Fig.3b.

We can illustrate these problems more clearly when combining intervals and probability. For instance, let A, B be two independent random variables such that: $P(A = 1) = P(A = 2) = 0.5$, $P(B = 4) = 1/3$, $P(B = 6) = 2/3$ and $C = [1,2]$. We compute $T = (A + B)/C$. With the hybrid method, we obtain a random interval: $T_1 = [2.5, 5]$ with probability 1/6, $T_2 = [3.5, 7]$ with probability 2/6; $T_3 = [3, 6]$ with probability 1/6 and $T_4 = [4, 8]$ with probability 2/6. Putting $d = 20\%$, with this postprocessing we obtain $T_d = [3, 7]$, and we assign to it a mass equal to 1, which is debatable. Indeed we eliminate the knowledge (frequency) brought by A and B, i.e. variability.

Lastly, we get false estimates of probabilities such as $P_{T(X)}([e_1, e_2])$. Indeed the method independently processes the left-hand and the right-hand sides of the fuzzy intervals π_i while they are not independent, since any α-cut is generated as a whole. The postprocessing proposed by Guyonnet et al. [33] is thus debatable. Better alternative methods can separately assess variability and imprecision.

C. Computing average upper and lower cumulative distributions

Recall $(\pi_i, p_i)_{i=1...m}$ is the sample of random fuzzy numbers resulting from the hybrid method. Let us encode each π_i with focal elements corresponding to α-cuts $(\pi_{i\alpha})_\alpha$ and the associated mass distribution is $(\nu_\alpha p_i)_\alpha$ (see Section II-E). We obtain a weighted random sampling of intervals defining a belief function. Then, we can estimate, for all measurable sets A, $Pl(A)$ and $Bel(A)$ such that:

$$Pl(A) = \sum_{(i,\alpha);\ \pi_{i\alpha}\cap A\neq\emptyset} \nu_\alpha p_i; \qquad Bel(A) = \sum_{(i,\alpha);\ \pi_{i\alpha}\subseteq A} \nu_\alpha p_i$$

We can dub it "homogeneous postprocessing". This technique again yields as in section 3.2:

$$Pl(A) = \sum_i p_i \Pi_i(A); \qquad Bel(A) = \sum_i p_i N_i(A)$$

This technique thus computes the eventwise weighted average of the possibility measures associated with each output fuzzy interval. It applies to any event, not just to the crossing of a threshold .

Let us compare the previous postprocessing method of Guyonnet et al. [33] with the pair of average cumulative distributions $\overline{F}(e) = Pl(T(X) \in (-\infty, e])$ and $\underline{F}(e) = Bel(T(X) \in (-\infty, e])$ defined from the above method. Note that the fuzzy confidence interval computed by the former method can also be expressed by a pair of cumulative distributions $\Pi_d(T(X) \in (-\infty, e])$ and $N_d(T(X) \in (-\infty, e])$ defined from π_d (see Section IV-B).

With the homogeneous postprocessing, we get $Pl(T(X) \in (-\infty, t]) = 1$ if and only if $\forall i = 1...m$, $\Pi_i(T(X) \in]-\infty, t]) = 1$. That is $t \geq t^* = \max_i\{\inf(core(\pi_i))\}$. We also have $Pl(T(X) \in (-\infty, t]) = 0$ if and only if $\forall i = 1...m$ $\Pi_i(T(X) \in (-\infty, t]) = 0$. That is $t \leq t_* = \min_i\{\inf(support(\pi_i))\}$ (see Figure 3). We study only the left-hand side part (for simplicity), that is $\Pi_d(T(X) \in (-\infty, e])$, with the Guyonnet et al. method of Section IV-B. If $d = 100\%$, the construction of $\underline{u}_{d\alpha}$ will necessarily imply:

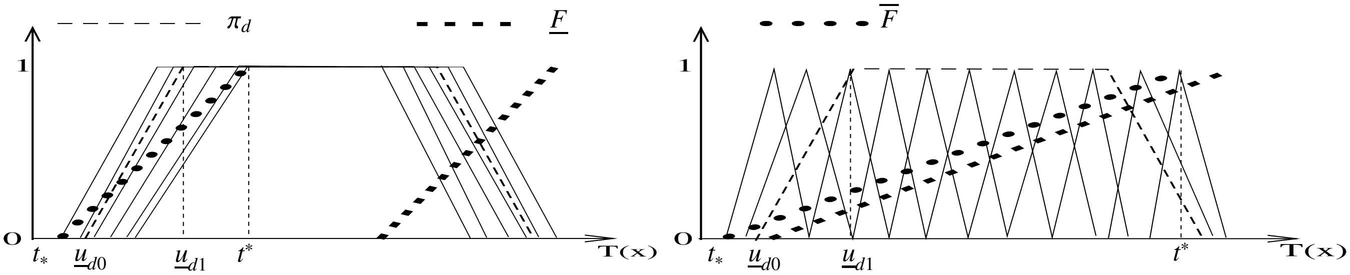
$$\text{for} \quad \alpha = 0, \quad \underline{u}_{d\alpha} = t_*$$

$$\text{for} \quad \alpha = 1, \quad \underline{u}_{d\alpha} = \min_i\{\inf(core(\pi_i))\} < t^*$$

Thus, we can say that for $d = 100\%$, the cumulative distribution $\Pi_d(T(X) \in (-\infty, e]) = 1 \ \forall \ e \geq \underline{u}_{d1}$, so it is more conservative than $Pl(T(X) \in]-\infty, e])$ obtained by the homogeneous postprocessing (because of not accounting for frequencies). If $d \neq 100\%$, the construction of $\underline{u}_{d\alpha}$ will necessarily imply (see Figure 3):

$$\text{for} \quad \alpha = 0, \quad \underline{u}_{d\alpha} \geq t_*$$

$$\text{for} \quad \alpha = 1, \quad \underline{u}_{d\alpha} \leq t^*$$



3a. Small variability of the sample
Large imprecision of each fuzzy number π_i

3b. Great variability of the sample
Little imprecision of each fuzzy number π_i

Fig. 3. Guyonnet et al. [33] postprocessing : the same possibility distribution π_d is obtained for different scenarii of variability and imprecision

That is $\Pi_d(T(X) \in (-\infty, e]) = 0 \ \forall \ e \in [t_*, \underline{u}_{d0}[$ and $\Pi_d(T(X) \in (-\infty, e]) = 1 \ \forall \ e \geq \underline{u}_{d1}[$. So, the fuzzy prediction interval method according to [33], is essentially more conservative (because of not accounting for frequencies) than the method proposed in this section, and the reason why it may appear less conservative for low plausibility values is rhe use of outlier elimination.

Finally, the homogeneous post processing method separates imprecision from variability. Namely the cumulative distributions $\overline{F}(e) = Pl(T(X) \in (-\infty, e])$ and $\underline{F}(e) = Bel(T(X) \in (-\infty, e])$ will be close to each other and their associated variances will be large in Figure 3a, thus preserving the high variability and precision of the output. On the contrary, $\overline{F}(e) = Pl(T(X) \in (-\infty, e])$ and $\underline{F}(e) = Bel(T(X) \in (-\infty, e])$ will be very steepy and far away from one another in Figure 3b, reflecting the fact that the obtained fuzzy intervals are imprecise but very close to one another. However, upper and lower cumulative distributions cannot be used for predicting if the output lies between two thresholds e_1 and e_2. This is because neither $Pl(T(X) \in [e_1, e_2])$ nor $Bel(T(X) \in [e_1, e_2])$ can be expressed in terms of upper and lower cumulative functions $\overline{F}$ and $\underline{F}$ respectively.

D. Comparison with Ferson method

Ferson et al. [9] [22] [25] also treats variability and imprecision separately in his technique for handling random fuzzy numbers. In fact, they prescribe a degree of confidence (thus a value α) and compute the upper and lower cumulative probability distributions induced by the α-cuts of the fuzzy intervals, weighted by probabilities (p_i). The upper (noted $\overline{F}_\alpha$) and lower (noted $\underline{F}_\alpha$) cumulative distributions for a prescribed α are $\forall x \in \mathbb{R}$,:

$$\overline{F}_\alpha(x) = card\{\underline{u}_{i\alpha} \leq x\}/m$$

and

$$\forall x \in \mathbb{R}, \underline{F}_\alpha(x) = card\{\overline{u}_{i\alpha} \leq x\}/m$$

The gap between $\overline{F}_\alpha$ and $\underline{F}_\alpha$ represents the imprecision due to possibilistic variables and the choice of α. The slopes of $\overline{F}_\alpha$ and $\underline{F}_\alpha$ characterize the variability of the results. Thus with this kind of representation, Ferson captures the variability and imprecision of a random fuzzy interval in a parameterized way and displays extreme pairs of cumulative distributions, respectively outer, $(\overline{F}_0, \underline{F}_0)$, and inner ones $(\overline{F}_1, \underline{F}_1)$ (see Figure 4). Thus if the user is optimistic and assumes high precision ($\alpha = 1$), he works with the cores of the fuzzy intervals, but, if cautious, he may choose $\alpha = 0$ and use their supports.

Let us compare this postprocessing technique with the average cumulative distributions in Section IV-C. In the latter approach, each interval $\pi_{i\alpha}$ is associated to mass $\nu_\alpha p_i$. With

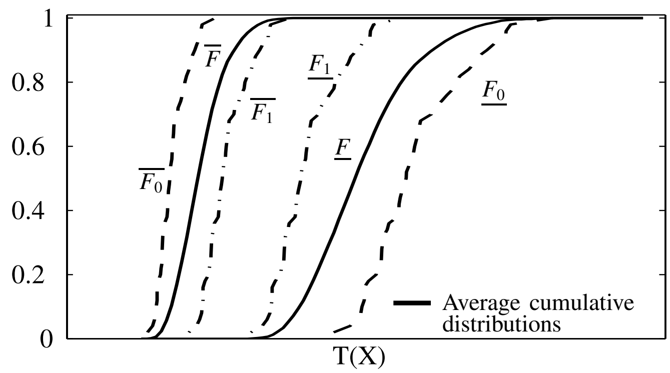


Fig. 4. Postprocessing of Ferson and comparison with our homogeneous postprocessing results.

the Ferson approach, level α is fixed and it computes:

$$\overline{F_\alpha}(x) = \sum_i p_i \Pi_{i\alpha}(T(X) \in (-\infty, x])$$

and

$$\underline{F_\alpha}(x) = \sum_i p_i N_{i\alpha}(T(X) \in (-\infty, x])$$

where $\Pi_{i\alpha}(T(X) \in (-\infty, x]) = 1$ if $x \geq \underline{a}_{i\alpha}$ and 0 otherwise, $N_{i\alpha}(T(X) \in (-\infty, x]) = 1$ if $x > \overline{a}_{i\alpha}$ and 0 otherwise. It is obvious, since $\pi_{i1} \subseteq \pi_i \subseteq \pi_{i0}$, that $\Pi_{i0}(T(X) \in (-\infty, x]) \geq \Pi_i(T(X) \in (-\infty, x]) \geq \Pi_{i1}(T(X) \in (-\infty, x])$ and $N_{i1}(T(X) \in (-\infty, x]) \geq N_i(T(X) \in (-\infty, x]) \geq N_{i0}(T(X) \in (-\infty, x])$. Hence (see Fig.4)

$$\overline{F_1} \leq Pl(T(X) \in (-\infty, e]) \leq \overline{F_0}$$

and (see Fig.4)

$$\underline{F_0} \leq Bel(T(X) \in (-\infty, e]) \leq \underline{F_1}$$

Hence our homogeneous method produces average upper and lower cumulative distributions which span the ranges between

$\overline{F}_0$ and $\overline{F}_1$ on the one hand, $\underline{F}_0$ and $\underline{F}_1$ on the other hand; moreover it holds [16]

$$\overline{F}(e) = \int_0^1 \overline{F_\alpha}(t)d\alpha \text{ and } \underline{F}(e) = \int_0^1 \underline{F_\alpha}(t)d\alpha$$

V. I

In order to illustrate the application and implications of these post-treatment methods, we consider a generic "model" T that is a simple function of four parameters A, B, C and D:

$$T = \frac{A+B}{D+C}$$

For the purpose of the illustration we will consider two cases:

1) One with small variability and large imprecision.
2) Another with large variability and small imprecision.

A. Case 1. Small Variability, Large Imprecision.

Both A and C are represented by normal probability distributions (of average 100, resp. 10, and standard deviation 2, resp. 0.5; see Figures 5 & 6), while B and D are represented by possibility distributions (of core=[2, 10], support=[0, 20] resp. core=[10, 50], support=[1, 90] see Figures 7 & 8). The application of the hybrid method (see III-A) yields many possibility distributions for T, ten of which are depicted in Figure 9 (1000 fuzzy realizations of the probability distribution functions were obtained). We represent the result of the hybrid method by

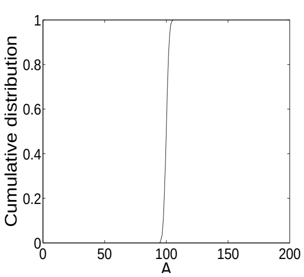


Fig. 5. Cumulative normal distribution function with mean 100 and standard deviation 2.

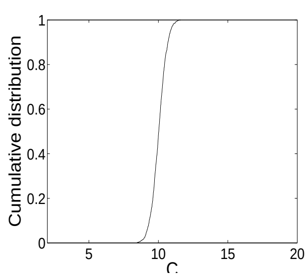


Fig. 6. Cumulative normal distribution function with mean 10 and standard deviation 0.5.

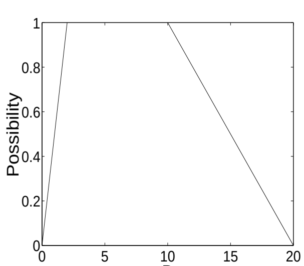


Fig. 7. Possibility distribution for B.

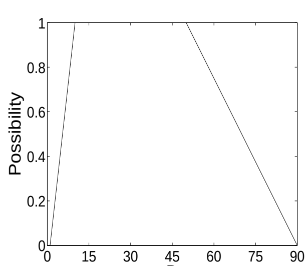


Fig. 8. Possibility distribution for D.

showing, on the same picture, imprecision and variability by means of a three-dimensional image of the random fuzzy set T. Figure 10 displays the envelope of the hybrid result. We can see on Figure 11 that a projection on the possibility space $[0, 1] \times \mathbb{R}$ provides a two-dimensional view of the random

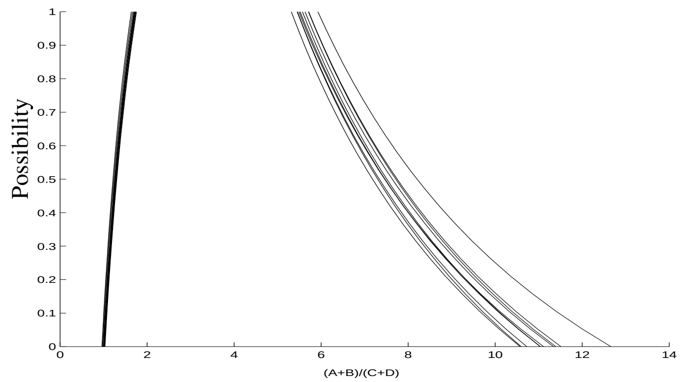


Fig. 9. 10 samples of the random fuzzy set of T.

fuzzy result (the envelope of fuzzy numbers in Figure 9). Now using a projection on the probability space $[0, 1] \times \mathbb{R}$, we obtain Ferson's view on Figure 12 (see Section IV-D). Outer (resp. inner) cumulative distribution functions correspond to α-cut=0 (resp. α-cut=1) and represent the most likely cumulative distributions (resp. the least likely cumulative distribution).

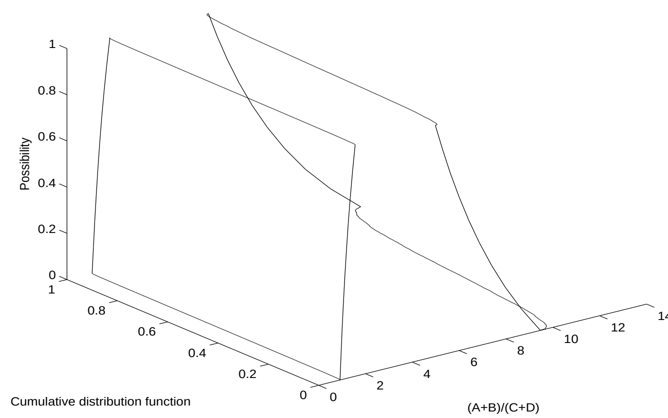


Fig. 10. Three-dimensional image of random fuzzy numbers induced by hybrid method.

Figure 13 displays upper and lower probabilities of the proposition $\frac{A+B}{C+D} \leq t$ deduced from different postprocessings presented in Sections IV-B, IV-C, IV-D where t is any value. $(\Pi, \Diamond)$ and $(N, \blacklozenge)$ are the upper resp. lower distributions obtained from the post-treatment by Guyonnet et al. (see Section IV-B). $Pl(T \leq t)$ and $Bel(T \leq t)$ are the upper resp. lower distributions obtained from our homogeneous postprocessing method (see Section IV-C). We also represent the pairs of inner and outer distributions of Ferson method. We recall Guyonnet et al. postprocessing is debatable because it forgets the variability of random variables A and C. We can see on Figure 13 that $Bel(T \in]-\infty, 10.96]) = 95\%$ whereas $N(T \in]-\infty, 11.4]) = 95\%$, namely Guyonnet et al. do over-estimate the result by 4% compared to our approach. This error is small here because variabilities of A and B are small, but it will be more important in Case 2. The post-treatment of Guyonnet et al. therefore appears more conservative. The method of Ferson in section IV-D presents some disadvantages. Indeed,

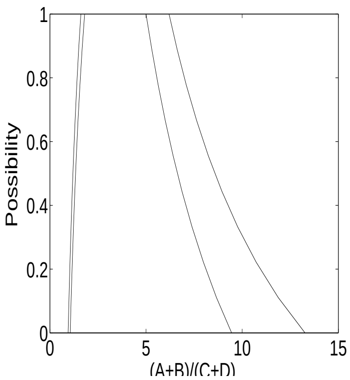


Fig. 11. The projection of Three-dimensional image T on the possibility space $[0,1] \times \mathbb{R}$.

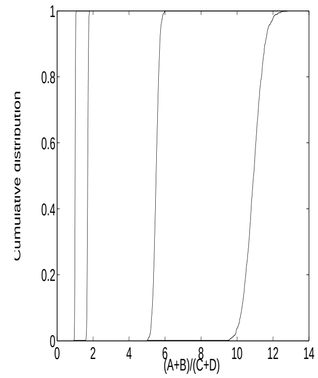


Fig. 12. The projection of Three-dimensional image T on the probability space $[0,1] \times \mathbb{R}$.

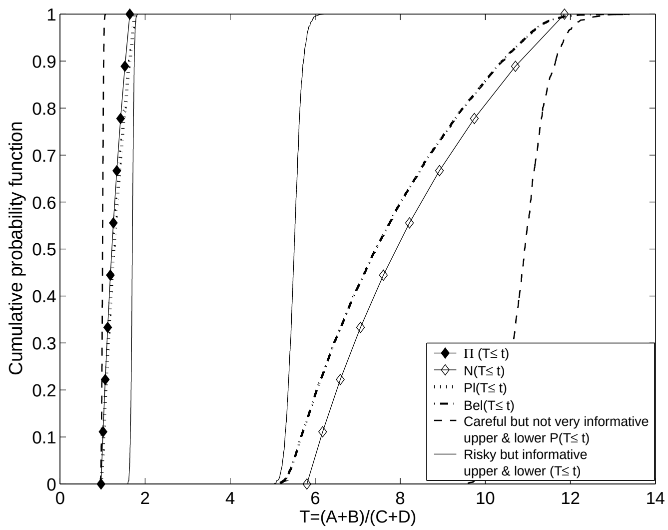


Fig. 13. Comparison with three postprocessing on indicators of the veracity of the proposition "$\frac{A+B}{C+D} \leq t$".

there is a lower fractile at 95%(resp. upper probability) for $\alpha = 0$ equal to 5.8 (resp. upper probability for $\alpha = 1$ equal to 11.9). That is, the value 11.9 is prudent but not very informative and 5.8 is a risky value but maybe not sufficiently conservative. The question is: what would be a reasonable value for α. Our homogeneous postprocessing in Section IV-C seems to be a good trade-off between the inner and the outer distribution pairs, that also discriminates between variability and imprecision. The imprecision due to B and D is reflected in the gap between Pl and Bel measures, the variability due to A and C is pictured in the slope of Pl and Bel. One way to estimate the total uncertainty (imprecision+variability) on T is to provide a confidence 90% interval (for example) whose lower bound is computed from $Pl(T \leq t)$ and upper bound from $Bel(T \leq t)$(here $[1.7, 10.76]$). Being aware of the dependency assumptions between parameters in the hybrid method (see Section III-A) we are not sure that the actual probability $P(T \in]-\infty, t])$ lies between $Bel(T \in]-\infty, t])$ and $Pl(T \in]-\infty, t])$ if special forms of unpredicted dependence are present [3].

B. Case 2. Large Variability, Small Imprecision.

As previously, both A and C have normal probability distributions (of average 100, resp. 10, and standard deviation 30, resp. 3; see Figures 14 & 15), while B and D are represented by possibility distributions (of core=$\{10\}$, support=$[8, 12]$ resp. core=$\{45\}$, support=$[42, 48]$; see Figures 16 & 17). Figure 18 proposes samples of outputs. Figure 22 presents the same distributions as in Case 1. Contrary to Case 1, this example highlights the defects of Guyonnet et al., postprocessing. Indeed we obtain for example $\Pi(T \in]-\infty, 1.1]) = 95\%$ whereas $Pl(T \in]-\infty, 2.87]) = 95\%$, i.e. a underestimation of 62% compared to 2.87. Assuming a tolerance criterion of 2.5, the post-treatment of Guyonnet et al. yields a lower cumulative probability of 0%, (that is, $\frac{A+B}{C+D} > t$ is true), while from our homogeneous postprocessing, it is close to 75%. The larger discrepancy between the two postprocessing methods for Case 2 is due to the fact that Case 2 is dominated by variability rather than by imprecision.

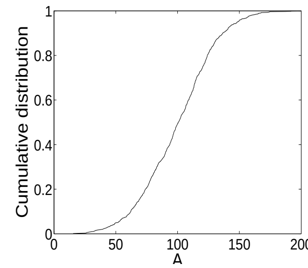


Fig. 14. Cumulative normal distribution function of mean 100 and standard deviation 30.

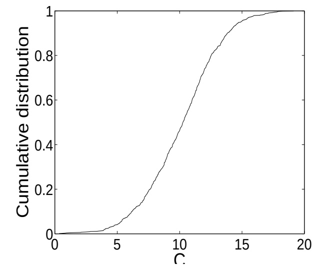


Fig. 15. Cumulative normal distribution function of mean 10 and standard deviation 3.

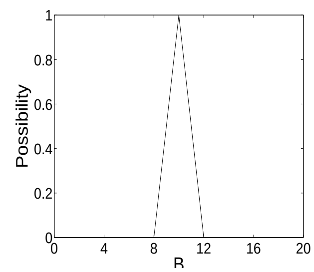


Fig. 16. Possibility distribution for B.

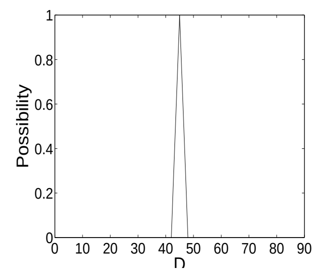


Fig. 17. Possibility distribution for D.

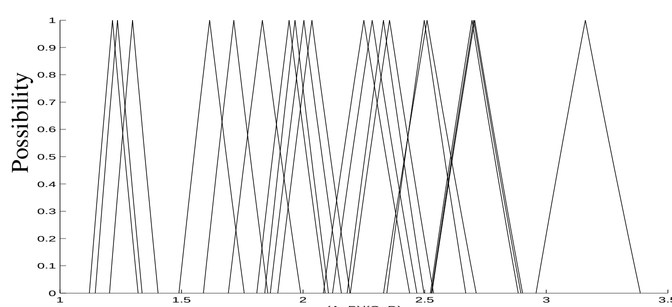


Fig. 18. 20 samples of the random fuzzy set of T.

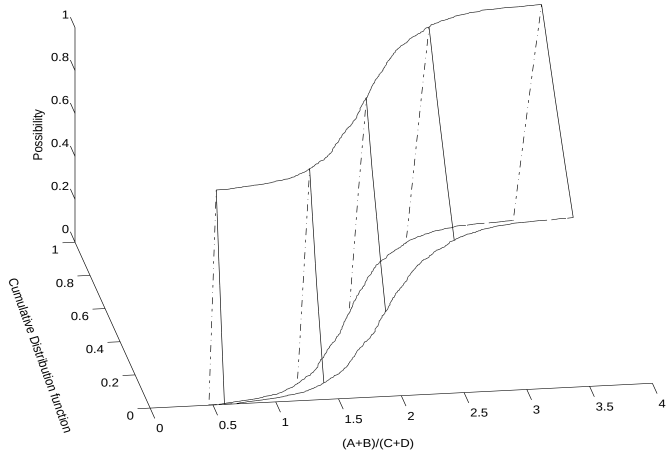


Fig. 19. Three-dimensional image of random fuzzy numbers induced by hybrid method.

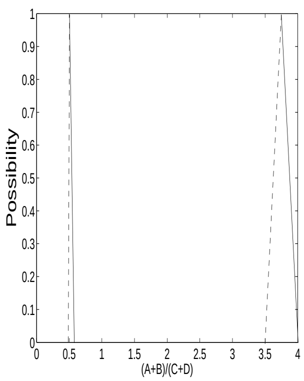


Fig. 20. The projection of Three-dimensional image T on the possibility space $[0, 1] \times \mathbb{R}$..

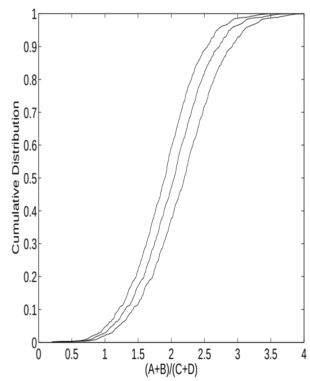


Fig. 21. The projection of Three-dimensional image T on the probability space $[0, 1] \times \mathbb{R}$.

VI. C

This paper proposes an approach to jointly propagate probabilistic and possibilistic uncertainty in deterministic mathematical models. It provides a computational device for generating fuzzy random variables. Dependence and independence assumptions underlying the approach have been laid bare, and a postprocessing method based on belief functions has been devised so as to extract useful information. This method assesses the imprecision and the variability of the results separately, and extracts average upper and lower cumulative distributions for checking the positioning of the output variable with respect to a threshold. Our proposal improves on previous works. This methodology is currently experimented on environmental pollution prediction problems where some parameters are ill-informed and while statistical data are available on other ones (see [2], [3]). The most common pitfall in risk assessment is to assume variability in the face of partial ignorance, thereby conveying a level of confidence in the outcome of the analysis that is not consistent with the knowledge that is truly available. An important message to be delivered to decision-makers, or other stakeholders, is that a risk specialist should be equipped with a formal language where the lack of knowledge on model parameters is encoded in a specific way,

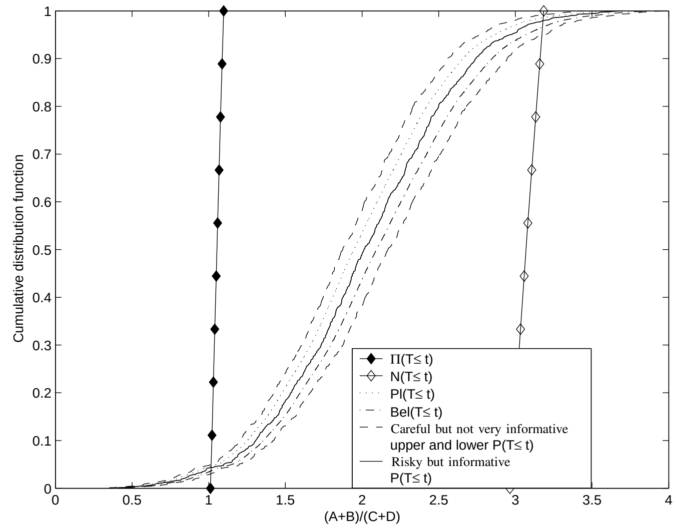


Fig. 22. Comparison with three postprocessing on indicators of the veracity of the proposal "$\frac{A+B}{C+D} \leq t$".

different from observed data stemming from acknowledgedly random phenomena.

Further research is needed for representing knowledge on dependence. The hybrid propagation scheme presented here does account for partial prior knowledge on distributions, not so much on dependence. Accounting for dependence between variables in the propagation process is a very difficult problem, let alone partial knowledge on dependence. Using ideas of rank correlations [6], copulas [39] and the general framework of upper and lower probabilities introduced by Couso et al. [10] we may try to take into consideration some links or dependencies which could exist between possibilistic variables. Current work [3] is devoted to the computation of conservative bounds that avoid making independence or dependence assumptions. Such bounds can be obtained, even if tediously so in the common framework of random sets [12] outlined above, improving on Williamson and Downs [44].

A

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R

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