JOINT SPECTRA AND JOINT NUMERICAL RANGES FOR PAIRWISE COMMUTING OPERATORS IN BANACH SPACES

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In a recent paper M. Cho [**5**] asked whether Taylor's joint spectrum $\sigma(a_1, \ldots, a_n; X)$ of a commuting n-tuple $(a_1, \ldots, a_n)$ of continuous linear operators in a Banach space X is contained in the closure $V(a_1, \ldots, a_n; X)^-$ of the joint spatial numerical range of $(a_1, \ldots, a_n)$. Among other things we prove that even the convex hull of the classical joint spectrum $Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)$, considered in the Banach algebra $\langle a_1, \ldots, a_n \rangle$, generated by $a_1, \ldots, a_n$, is contained in $V(a_1, \ldots, a_n; X)^-$.

0. Notation. Throughout this paper X will always denote a Banach space over the complex numbers $\mathbb{C}$, and $L(X)$ will denote the Banach algebra of all continuous linear operators on X. Operator will always mean continuous linear operator. X' denotes the dual space of X and for $a \in L(X)$ we let a' denote the dual operator. Given a subset $B \subset X$ we let B^- denote the closure of B.

1. Joint spectra. Let $(a_1, \ldots, a_n) \in L(X)^n$ denote an n-tuple of pairwise commuting operators, and let A denote a closed unital subalgebra containing $a_1, \ldots, a_n$. In accordance with Bonsall and Duncan [**1**, p. 24] the *joint spectrum* $Sp(a_1, \ldots, a_n; A)$ *of* $a_1, \ldots, a_n$ *with respect to* A consists of those points $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ such that either

$$\sum_{i=1}^{n} (z_i - a_i)A$$

is a proper right ideal in A or

$$\sum_{i=1}^{n} A(z_i - a_i)$$

is a proper left ideal in A. Suitable choices for A are $L(X)$ [**8**], the commutant algebra $\{a_1, \ldots, a_n\}^c$ of $a_1, \ldots, a_n$ in $L(X)$ [**10**], the bicommutant algebra $\{a_1, \ldots, a_n\}^{cc}$ [**7**] or the Banach algebra $\langle a_1, \ldots, a_n \rangle$ generated by $a_1, \ldots, a_n$ (Gelfand).

J. L. Taylor [**10**] considers a spatial version of joint spectrum denoted by $\sigma(a_1, \ldots, a_n; X)$ throughout.

Moreover we consider the following concepts. By definition a point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ belongs to the *joint point spectrum* $P\sigma(a_1, \ldots, a_n; X)$ or the *joint approximate point spectrum* $AP\sigma(a_1, \ldots, a_n; X)$, if there exists a vector $x \neq 0$ or a sequence $(x_n)_{n \in \mathbb{N}}$ with $\|x_n\| = 1$ such that $a_i x = z_i x$ for $1 \leq i \leq n$ and

$$\|a_i x_n - z_i x_k\| \to 0 \quad \text{as} \quad k \to \infty \quad \text{for} \quad 1 \leq i \leq n,$$

respectively. Finally, $C\sigma(a_1, \dots, a_n; X) := P\sigma(a_1', \dots, a_n'; X')$ and $AC\sigma(a_1, \dots, a_n; X) := AP\sigma(a_1', \dots, a_n'; X')$ denote the *joint compression spectrum* and the *joint approximate compression spectrum*, respectively.

Obviously, we have

$$AP\sigma(a_1, \dots, a_n; X) \cup AC\sigma(a_1, \dots, a_n; X) \subseteq Sp(a_1, \dots, a_n; L(X)) \cap \sigma(a_1, \dots, a_n; X).$$

Moreover, by **[10]**

$$\sigma(a_1, \dots, a_n; X) \subseteq Sp(a_1, \dots, a_n; \{a_1, \dots, a_n\}^c)$$

with proper inclusion in general.

We start with a summary of polynomial spectral mapping theorems.

1.1. THEOREM. *Let $(a_1, \dots, a_n) \in L(X)^n$ denote an n-tuple of pairwise commuting operators, and let Q denote a polynomial in n variables. Then the following spectral mapping theorems hold*:

(i) $Q(Sp(a_1, \dots, a_n; A)) = Sp(Q(a_1, \dots, a_n); A)$,

where A denotes a unital Banach subalgebra of $L(X)$ containing $a_1, \dots, a_n$;

(ii) $Q(\sigma(a_1, \dots, a_n; X)) = \sigma(Q(a_1, \dots, a_n); X) = Sp(Q(a_1, \dots, a_n); L(X))$;

(iii) $Q(AP\sigma(a_1, \dots, a_n; X)) = AP\sigma(Q(a_1, \dots, a_n); X)$.

For (i) *see* **[8]**, *for* (ii) *see* **[11]** *and for* (iii) *see* **[6]**.

Given a compact subset $K \subset \mathbb{C}^n$ let

$$\text{p.c.h.}(K) := \{z \in \mathbb{C}^n : |Q(z)| \leq \max_{t \in K} |Q(t)| \text{ for all polynomials } Q\}$$

denote the *polynomially convex hull* of K. If $K = \text{p.c.h.}(K)$, then K is said to be *polynomially convex*. It is a well-known fact in classical Banach algebra theory, that $Sp(a_1, \dots, a_n; \langle a_1, \dots, a_n \rangle)$ is a polynomially convex compact set. (See [**3**, p. 101] or [**12**, p. 44]).

Observe that if $n = 1$ then polynomially convex means having a connected complement. But if $n > 1$ there is no topological description of polynomially convex sets. Indeed let us note the following fact which gives a general context of Wermer's remark and example in [**12**, p. 36].

1.2. REMARK. Each compact subset of $\mathbb{C}^n$ is homeomorphic to a polynomially convex set in $\mathbb{C}^{2n}$. More precisely: given a compact subset K of $\mathbb{C}^n$, the set

$$\tilde{K} := \{(z, \bar{z}) \in \mathbb{C}^{2n} : z \in K\}$$

is polynomially convex, where "bar" denotes complex conjugation.

Proof. 1° Observe that given an n-tuple $(a_1, \ldots, a_n) \in L(H)^n$ of pairwise commuting normal operators on a Hilbert space H, we have from [7]

$$Sp(a_1, \ldots, a_n; \{a_1, \ldots, a_n\}^{cc}) = AP\sigma(a_1, \ldots, a_n; H)$$
$$= Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n, a_1^*, \ldots, a_n^* \rangle),$$

a^* denoting the Hilbert space adjoint of a. To see the last non-trivial inclusion, let $z = (z_1, \ldots, z_n) \notin AP\sigma(a_1, \ldots, a_n; H)$. Then the positive operator $\sum_{i=1}^{n} (a_i - z_i)^*(a_i - z_i)$ is a topological monomorphism and therefore a bijection. This proves that

$$z \notin Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n, a_1^*, \ldots, a_n^* \rangle).$$

2° Next recall that $Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n, a_1^*, \ldots, a_n^* \rangle)$ is the projection of $Sp(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*; \langle a_1, \ldots, a_n, a_1^*, \ldots, a_n^* \rangle)$ onto the first n coordinates (compare Zelasko [15]).

3° Note that for an n-tuple $(a_1, \ldots, a_n)$ of diagonal operators

$$a_k := \operatorname{diag}((\alpha_{jk})_{j \in \mathbb{N}}) \ (1 \le k \le n) \quad \text{on} \quad l^2$$

we have

$$AP\sigma(a_1, \ldots, a_n; l^2) = \{\alpha^{(i)} : i \in \mathbb{N}\}^-,$$

where $\alpha^{(i)} = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in})$ $(i \in \mathbb{N})$. Especially given a compact subset K of $\mathbb{C}^n$ let $\{\alpha^{(i)} : i \in \mathbb{N}\}$ denote a dense subset of K and define diagonal operators as above. Then $K = AP\sigma(a_1, \ldots, a_n; l^2)$ [4, 5.1].

4° Finally putting together 1°–3° we get the desired result.

The following lemma, which also has been used in [13] especially states as its main consequence that the polynomially convex hulls of almost all joint spectra coincide with $Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)$.

1.3. Lemma. *Let $(a_1, \ldots, a_n) \in L(X)^n$ denote an n-tuple of pairwise commuting operators. Let $K \subseteq Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)$ denote a nonempty compact set such that for every polynomial Q in n variables we have*

$$\max_{t \in K} |Q(t)| = r(Q(a_1, \ldots, a_n); L(X))$$
$$:= \max\{|z| : z \in Sp(Q(a_1, \ldots, a_n); L(X))\}.$$

Then

$$\text{p.c.h.}(K) = Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle).$$

Proof. Obviously we have

$$\text{p.c.h.}(K) \subseteq Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle).$$

On the other hand let $z = (z_1, \ldots, z_n) \notin \text{p.c.h.}(K)$. By definition there exists a polynomial Q such that

$$|Q(z)| > \max_{t \in K} |Q(t)| = r(Q(a_1, \ldots, a_n); L(X)).$$

Therefore

$$b := \sum_{j=0}^{\infty} Q(a_1, \ldots, a_n)^j Q(z)^{-j-1} \in \langle a_1, \ldots, a_n \rangle$$

is the inverse of $Q(z) - Q(a_1, \ldots, a_n)$. On the other hand

$$Q(z) - Q(a_1, \ldots, a_n) = \sum_{j=1}^{n} (z_j - a_j) Q_j(a_1, \ldots, a_n),$$

with suitable polynomials Q_j. Multiplication with b gives $z \notin Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)$ and hence the desired result.

Given a compact subset $K \subset \mathbb{C}^n$ let $\operatorname{conv}(K)$ denote the convex hull, and let $\operatorname{ext}(K)$ denote the extreme points of $\operatorname{conv}(K)$. Since $\text{p.c.h.}(K) \subseteq \operatorname{conv}(K)$, we have $\text{p.c.h.}(\operatorname{conv}(K)) = \operatorname{conv}(\text{p.c.h.}(K))$, and consequently the following result is an immediate consequence of 1.1 and 1.3.

1.4. Corollary. *Let $(a_1, \ldots, a_n) \in L(X)^n$ denote an n-tuple of pairwise commuting operators. Then we have*

$$\begin{aligned} &\operatorname{ext}(Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)) = \operatorname{ext}(\sigma(a_1, \ldots, a_n; X)) \\ &= \operatorname{ext}(Sp(a_1, \ldots, a_n; L(X))) = \operatorname{ext}(AP\sigma(a_1, \ldots, a_n; X)) \\ &= \operatorname{ext}(AC\sigma(a_1, \ldots, a_n; X)). \qquad (*) \end{aligned}$$

Especially,

$$\operatorname{ext}(\sigma(a_1, \ldots, a_n; X)) = \operatorname{ext}(AP\sigma(a_1, \ldots, a_n; X) \cap AC\sigma(a_1, \ldots, a_n; X)). \qquad (**)$$

Remark. If $n = 1$, then $\partial\sigma(a_1;\ X) \subseteq AP\sigma(a_1; X) \cap AC\sigma(a_1; X)$ (∂ denoting the boundary of .), whereas for $n = 2$ neither $\partial\sigma(a_1, a_2; X) \subseteq AP\sigma(a_1, a_2; X)$ nor $\partial\sigma(a_1, a_2, X) \subseteq AC\sigma(a_1, a_2; X)$ is true in general ([**13**, 2.5]). Therefore 1.3(**) may be regarded as a substitute for $\partial\sigma \subseteq AP\sigma \cap AC\sigma$ in the case $n \geq 2$. (**) also generalizes a previous result [**13**, 2.8] and simplifies its proof considerably.

In section 2 we shall need a concept of joint spectral radius for an n-tuple $(a_1, \ldots, a_n)$ of pairwise commuting operators.

Given a compact set $K \in \mathbb{C}^n$ let

$$\|K\|_2 := \max\left\{\left(\sum_{j=1}^{n} |z_j|^2\right)^{1/2} : (z_1, \ldots, z_n) \in K\right\}.$$

Since a continuous convex function takes its maximum on a compact set K in $\operatorname{ext}(K)$, we obtain a notion of joint spectral radius $r(a_1, \ldots, a_n)$ which is independent of the underlying concept of joint spectrum as far as the assumptions of 1.3 are fulfilled.

1.5. Definition. Let $(a_1, \ldots, a_n) \in L(X)^n$ denote an n-tuple of pairwise commuting operators. Then

$$r(a_1, \ldots, a_n) := \|K\|_2$$

is called *joint spectral radius* of $(a_1, \ldots, a_n)$, where K is one of the joint spectra considered in 1.4.

2. Joint numerical ranges. We consider joint spatial numerical ranges for operators on a Banach space X. For that purpose let

$$\Pi(X) := \{(x, f) \in X \times X' : 1 = \|x\| = \|f\| = f(x)\}.$$

Given $a_1, \ldots, a_n \in L(X)$ let

$$V(a_1, \ldots, a_n; X) := \{(f(a_1x), \ldots, f(a_nx)) : (x, f) \in \Pi(X)\}$$

denote the *joint spatial numerical range* of $a_1, \ldots, a_n$. Obviously $V(a_1, \ldots, a_n; X)$ is a nonempty and bounded subset of $\mathbb{C}^n$.

Our main result (2.2) will state that the convex hull of the joint approximate point spectrum is contained in the closure of the joint numerical range.

A main ingredient in the proof of this result will be the following theorem of Zenger **[16]** (see **[2**, p. 20]).

2.1. THEOREM **[16]**. *Let Y be a normed vector space over $\mathbb{C}$, let $y_1, \ldots, y_n$ be linearly independent vectors in Y, and let $\alpha_k \geq 0$ $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} \alpha_k = 1$. Then there exist $(y, f) \in \Pi(Y)$, and complex numbers $z_1, \ldots, z_n$ such that $y = \sum_{k=1}^{n} z_k y_k$ and $f(z_k y_k) = \alpha_k$ $(1 \leq k \leq n)$.*

The following is our main result.

2.2. THEOREM. *Let $(a_1, \ldots, a_n) \in L(X)^n$ denote an n-tuple of operators (not necessarily commuting!). Then*

$$\operatorname{conv}(AP\sigma(a_1, \ldots, a_n; X)) \subseteq V(a_1, \ldots, a_n; X)^-.$$

Proof. Obviously, $AP\sigma(a_1, \ldots, a_n; X) \subseteq V(a_1, \ldots, a_n; X)^-$. We next imitate the proof of Crabb's theorem in **[2**, p. 22]. Let $z^{(j)} = (z_1^j, \ldots, z_n^j) \in AP\sigma(a_1, \ldots, a_n; X)$ $(1 \leq j \leq m)$ and

$$\delta := \min\left\{\sum_{k=1}^{n} |z_k^j - z_k^i| : 1 \leq i < j \leq m\right\}, \quad 0 < \varepsilon < (4mn)^{-1}\delta.$$

By definition of the joint approximate point spectrum we find vectors $x_1, \ldots, x_m \in X$ such that $\|x_k\| = 1$ and

$$\|a_j x_k - z_j^k x_k\| < \varepsilon \quad (1 \leq j \leq n, 1 \leq k \leq m).$$

Without loss of generality (by reordering otherwise), we may assume that $\{x_1, \ldots, x_{m_0}\}$ is a maximal linearly indepenent subset of $\{x_1, \ldots, x_m\}$. Using the Hahn–Banach theorem we find $f_j \in X'$ such that $1 = \|f_j\|$ and $f_j(x_i) = \delta_{ij}$ $(1 \leq i, j \leq m_0)$. If $m_0 < m$, then

$$x_{m_0+1} = \sum_{i=1}^{m_0} f_i(x_{m_0+1}) x_i$$

and thus

$$1 = \|x_{m_0+1}\| \le \sum_{j=1}^{m_0} |f_j(x_{m_0+1})| \le m_0. \qquad (+)$$

Therefore

$$\begin{aligned}\delta(4m\,.\,n)^{-1} &> \|a_1x_{m_0+1} - z_1^{m_0+1}x_{m_0+1}\| \\ &= \left\|\sum_{k=1}^{m_0} f_k(x_{m_0+1})(a_1x_k - z_1^{m_0+1}x_k)\right\| \\ &= \left\|\sum_{k=1}^{m_0} f_k(x_{m_0+1})(a_1x_k - z_1^k x_k) + \sum_{k=1}^{m_0} f_k(z_1^k - z_1^{m_0+1})x_k\right\| \\ &\ge \left\|\sum_{k=1}^{m_0} f_k(x_{m_0+1})(z_1^k - z_1^{m_0+1})x_k\right\| - m_0\delta(4m\,.\,n)^{-1} \quad (\text{by } (+)).\end{aligned}$$

Consequently,

$$\begin{aligned}\delta n^{-1}|f_j(x_{m_0+1})| &\le |f_j(x_{m_0+1})(z_1^j - z_1^{m_0+1})| \\ &= \left|f_j\left(\sum_{k=1}^{m_0} f_k(x_{m_0+1})(z_1^k - z_1^{m_0+1})x_k\right)\right| \\ &\le \left\|\sum_{k=1}^{m_0} f_k(x_{m_0+1})(z_1^k - z_1^{m_0+1})x_k\right\| \\ &< \delta(4mn)^{-1} + \delta(4n)^{-1} \le \delta(2n)^{-1},\end{aligned}$$

contradicting $(+)$. Therefore $m_0 = m$.

Next let $\alpha_k \ge 0$ $(1 \le k \le m)$, $\sum_{k=1}^{m} \alpha_k = 1$. We apply Zenger's theorem 2.1. This gives us $(x, f) \in \Pi(X)$ with $x = \sum_{k=1}^{m} t_k x_k$ such that $f(t_k x_k) = \alpha_k$ $(1 \le k \le m)$. Let $z := (z_1, \dots, z_n)$, where

$$z_i = \sum_{k=1}^{m} \alpha_k z_i^{(k)};$$

i.e. z is a convex combination of $z^{(1)}, \dots, z^{(m)} \in AP\sigma(a_1, \dots, a_n; X)$. Then

$$\begin{aligned}|f(a_j x - z_j)| &= \left|\sum_{k=1}^{m} (f(t_k a_j x_k) - f(t_k x_k) z_j^k\right| \\ &\le \sum_{k=1}^{m} |t_k|\,\|a_j x_k - z_j^k x_k\| < m\,.\,\varepsilon\end{aligned}$$

with the same argument as above using $t_k = f_k(x)$, $\|f_k\| = 1$ $(1 \le k \le m)$. This proves the theorem.

Using this and 1.4, we obtain the following result which especially gives a positive answer to the problem of Cho [**5**] whether $\sigma(a_1, \ldots, a_n; X) \subseteq V(a_1, \ldots, a_n; X)^-$ is true for an n-tuple of pairwise commuting operators.

2.3. Corollary. *Let $(a_1, \ldots, a_n) \in L(X)^n$ denote an n-tuple of pairwise commuting operators. Then*

$$\operatorname{conv}(Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)) \subseteq V(a_1, \ldots, a_n; X)^-.$$

For $n = 1$ the following result is proved in [**2**, §19 Corollary 5].

2.4. Corollary. *Let $(X, \|\cdot\|)$ denote a complex Banach space and let $N(X)$ denote the set of all norms on X equivalent to $\|\cdot\|$. For an n-tuple $(a_1, \ldots, a_n) \in L(X)^n$ of pairwise commuting operators we have*

$$\operatorname{conv}(Sp(a_1, \ldots, a_n; \langle a_1, \ldots, a_n \rangle)) = \bigcap \{V(a_1, \ldots, a_n; (X, p))^- : p \in N(X)\}.$$

Proof. Following Bonsall and Duncan [**1**, p. 14] let

$$D(L(X), Id_X) := \{f \in L(X)' : 1 = f(Id_X) = \|f\|\},$$
$$V(L(X); a_1, \ldots, a_n) := \{(f(a_1), \ldots, f(a_n)) : f \in D(L(X), Id_X)\}.$$

Obviously

$$V(a_1, \ldots, a_n; (X, p))^- \subseteq V(L((X, p)); a_1, \ldots, a_n).$$

Moreover each $p \in N(X)$ induces an operator-norm on $L(X)$ which is equivalent to the original one. On the other hand equivalent norms q on $L(X)$ such that $q(Id_X) = 1$ induce equivalent norms on X: Let $f_0 \in X'$ such that $\|f_0\| = 1$ and look at the embedding $x \to f_0 \otimes x$ from X into $L(X)$. The proof now is an immediate consequence of 2.3, the above considerations and [**1**, §2 Theorem 13].

Concluding remarks. For $(a_1, \ldots, a_n) \in L(X)^n$ and $p \in N(X)$ (see 2.4) let

$$p(a_1, \ldots, a_n) := \sup\left\{\left(\sum_{i=1}^n p(a_i x)^2\right)^{1/2} : p(x) = 1\right\}$$

and

$$v_p(a_1, \ldots, a_n) := \sup\left\{\left(\sum_{i=1}^n f(a_i x)^2\right)^{1/2} : (x, f) \in \Pi((X, p))\right\}$$

denote the *joint operator norm* and the *joint numerical radius* of $(a_1, \ldots, a_n)$ in the Banach space (X, p).

1° For a commuting n-tuple $(a_1, \ldots, a_n) \in L(X)^n$ we have by 2.4

$$r(a_1, \ldots, a_n) = \inf\{v_p(a_1, \ldots, a_n) : p \in N(X)\}$$

and the infimum is not attained in general. We do not know whether

$$r(a_1, \ldots, a_n) = \inf\{p(a_1, \ldots, a_n) : p \in N(X)\}$$

is true for $n>1$. For $n=1$ equality follows by considering the Neumann series expansion of the resolvent function.

2° It follows from 1.4 and 2.3 that

$$\mathrm{ext}(V(a_1,\dots,a_n;X)^-) \subset (\mathbb{C}^n\backslash Sp(a_1,\dots,a_n;\langle a_1,\dots,a_n\rangle))\cup \mathrm{ext}(AP\sigma(a_1,\dots,a_n;X)). \quad (*)$$

Consequently $V(a_1,\dots,a_n;X)^-$ is convex, if $(a_1,\dots,a_n)$ is *jointly convexoid* in the sense of Cho and Takagushi [**4**]. From the results 3.4 and 4.5 in [**4**] n-tuples of doubly commuting hyponormal operators are seen to be jointly convexoid. This is a somewhat weaker result than Dash's [**7**] which states that $V(a_1,\dots,a_n;X)$ itself is convex for an n-tuple of pairwise commuting normal operators $(a_1,\dots,a_n)$ on a Hilbert space X.

3° If (X,p) is uniformly convex, (*) can be improved for the "peripheral part" of $V(a_1,\dots,a_n;X)^-$. More precisely, we have

$$V(a_1,\dots,a_n;X)^-\cap\{z\in\mathbb{C}^n:|z|=p(a_1,\dots,a_n)\} \subset AP\sigma(a_1,\dots,a_n;X);$$

(see Lumer [**9**] ($n=1$) and Cho [**5**] ($n\geq 1$)).

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REFERENCES

1. F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras I* (Cambridge University Press, 1971).

2. F. F. Bonsall and J. Duncan, *Numerical ranges II* (Cambridge University Press, 1973).

3. F. F. Bonsall and J. Duncan, *Complete normed algebras* (Springer Verlag, 1973).

4. M. Cho and M. Takaguchi, Some classes of commuting n-tuples of operators, *Studia Math.* **80** (1984), 245–259.

5. M. Cho, Joint spectra of operators on Banach spaces, *Glasgow Math. J.* **28** (1986), 69–72.

6. M. Choi and C. Davis, The spectral mapping theorem for joint approximate point spectrum, *Bull. Amer. Math. Soc.* **80** (1974), 317–321.

7. A. T. Dash, Joint numerical range, *Glasnik Mat. Ser. III* **7** (1972), 75–81.

8. R. Harte, The spectral mapping theorem in several variables, *Bull. Amer. Math. Soc.* **78** (1972), 871–875.

9. G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.* **100** (1961), 29–43.

10. J. L. Taylor, A joint spectrum for several commuting operators, *J. Functional Analysis* **6** (1970), 172–191.

11. J. L. Taylor, The analytic functional calculs for several commuting operators, *Acta Math.* **125** (1970), 1–38.

12. J. Wermer, *Banach algebras and several complex variables,* (Springer Verlag, 1976).

13. V. Wrobel, The boundary of Taylor's joint spectrum for two commuting Banach space operators, *Studia Math.* **84** (1986) 105–111.

14. V. Wrobel, Joint spectra of meromorphic operators, Preprint (Kiel, 1986).

15. W. Zelasko, An axiomatic approach to joint spectra I, *Studia Math.* **64** (1979), 249–261.

16. C. Zenger, On convexity properties of the Bauer field of values of matrices, *Numer. Math.* **12** (1968), 96–105.

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