

Jordan's principal angles in complex vector spaces

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SUMMARY

We analyse the possible recursive definitions of principal angles and vectors in complex vector spaces and give a new projector based definition. This enables us to derive important properties of the principal vectors and to generalize a result of Björck and Golub (*Math. Comput.* 1973; **27**(123):579–594), which is the basis of today's computational procedures in real vector spaces. We discuss other angle definitions and concepts in the last section. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The principal or canonical angles (and the related principal vectors) between two subspaces provide the best available characterization of the relative subspace positions. Although Jordan [1] introduced the concept of principal angles (and vectors) in 1875 (see also References [2, 3]), the principal angles were rediscovered several times. Davis and Kahan [4] include an interesting account of these works to which we can add Reference [5] as the most recent contribution.

Jordan's recursive definition of the principal angles was formalized by Hotelling [6] in his theory of canonical correlations. The importance of the matter and Hotelling's paper [6] initiated many further investigations such as References [7–9].

Definition 1

Let $x, y \in \mathbb{R}^n$, $x, y \neq 0$. Then the (real) angle $\Theta(x, y)$ between x and y is defined by

$$\cos \Theta(x, y) = \frac{x^T y}{\|x\| \|y\|} \quad (0 \leq \Theta \leq \pi) \quad (1)$$

where $\|x\| = (x^T x)^{1/2}$.

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Definition 2

Let $M_1, M_2 \subset \mathbb{R}^n$ be subspaces with $p_1 = \dim(M_1) \geq \dim(M_2) = p_2 \geq 1$. The principal angles $\theta_k \in [0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \dots, p_2$ by

$$\cos \theta_k = \max_{\substack{u \in M_1, v \in M_2, \|u\| = \|v\| = 1 \\ u_i^T u = 0, v_i^T v = 0, i = 1, \dots, k-1}} u^T v = u_k^T v_k \quad (2)$$

The vectors $\{u_1, \dots, u_{p_2}\}, \{v_1, \dots, v_{p_2}\}$ are called principal vectors of the pair of spaces.

Notice that $0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_{p_2} \leq \pi/2$. The principal angles are uniquely defined, while the principal vectors are not.

Definition 2 is based on the concept of angles between two real vectors and the standard inner product. When looking for similar definitions in complex vector spaces we encountered the following problems:

1. We did not find a generally accepted definition that includes both the principal angles and the principal vectors.
2. The available definitions give only the principal angles and use CS decomposition [3, 10] or eigenvalues (see, e.g. References [4, 5, 8, 10–13]). Hence there is some loss of the geometric character.
3. There is some ambiguity in the definition of angle between complex vectors.

Scharnhorst [14] enlists six angle concepts between complex vectors (Euclidean (imbedded) angle, complex-valued angle, Hermitian angle, real-part angle, Kasner's pseudo angle, Kähler angle) that have different geometric properties. For example, if one defines $\cos \phi = |\langle x, y \rangle| / (\|x\| \|y\|)$, then $\phi = \pi/2$ if and only if $\langle x, y \rangle = 0$, but the law of cosines does not hold. If one uses $\cos \phi = \operatorname{Re} \langle x, y \rangle / (\|x\| \|y\|)$, then the law of cosines holds, but $\phi = \pi/2$ may hold for $\langle x, y \rangle \neq 0$ (see also References [4, 15]). The variety of angle definitions and properties clearly requires a thorough extension of Jordan's original concept [1, 2] to the complex case.

Here we first analyse the possible recursive definitions in complex vector spaces and give a new projector based definition as well. The new definition enables us to derive important properties of the principal vectors and to generalize a result of Björck and Golub [9], which is the basis of today's computational procedures in real vector spaces (see, e.g. References [16–19]). We also discuss other definitions in the last section.

2. RECURSIVE DEFINITIONS IN COMPLEX VECTOR SPACES

We first consider those angle definitions between vectors that are related to the complex inner product. If not otherwise stated, a general complex valued inner product and the induced norm will be assumed in the following considerations.

Definition 3

Let $x, y \in \mathbb{C}^n$, $x, y \neq 0$. Then the complex (-valued) angle $\Theta_c(x, y)$ between x and y is defined by

$$\cos \Theta_c(x, y) = \frac{\langle x, y \rangle}{\|x\| \|y\|} \quad (3)$$

The Hermitian angle $\Theta_H(x, y)$ between vectors x and y is defined by

$$\cos \Theta_H(x, y) = \frac{|\langle x, y \rangle|}{\|x\| \|y\|} \quad (0 \leq \Theta_H(x, y) \leq \pi/2) \tag{4}$$

The real-part angle is defined by

$$\cos \Theta_r(x, y) = \frac{\operatorname{Re}\langle x, y \rangle}{\|x\| \|y\|} \tag{5}$$

Scharnhorst [14] also deals with another possibility, i.e. the imbedding of the complex n -space into a real $2n$ -space (Euclidean angle).

Since the complex numbers are not ordered, we can use only the Hermitian angle or the real-part angle to define the principal angles between subspaces.

Definition 4

Let $M_1, M_2 \subset \mathbb{C}^n$ be subspaces with $p_1 = \dim(M_1) \geq \dim(M_2) = p_2 \geq 1$. The principal angles $\theta_k \in [0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \dots, p_2$ by

$$\cos \theta_k = \max_{\substack{u \in M_1, v \in M_2, \|u\| = \|v\| = 1 \\ \langle u_i, u \rangle = 0, \langle v_i, v \rangle = 0, i = 1, \dots, k-1}} |\langle u, v \rangle| = \langle u_k, v_k \rangle \tag{6}$$

The vectors $\{u_1, \dots, u_{p_2}\}, \{v_1, \dots, v_{p_2}\}$ are called principal vectors of the pair of spaces.

Definition 5

Let $M_1, M_2 \subset \mathbb{C}^n$ be subspaces with $p_1 = \dim(M_1) \geq \dim(M_2) = p_2 \geq 1$. The principal angles $\hat{\theta}_k \in [0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \dots, p_2$ by

$$\cos \hat{\theta}_k = \max_{\substack{u \in M_1, v \in M_2, \|u\| = \|v\| = 1 \\ \langle \hat{u}_i, u \rangle = 0, \langle \hat{v}_i, v \rangle = 0, i = 1, \dots, k-1}} \operatorname{Re}\langle u, v \rangle = \langle \hat{u}_k, \hat{v}_k \rangle \tag{7}$$

The vectors $\{\hat{u}_1, \dots, \hat{u}_{p_2}\}, \{\hat{v}_1, \dots, \hat{v}_{p_2}\}$ are called principal vectors of the pair of spaces.

Definition 4 can also be found in Reference [20] without the restriction $\cos \theta_k = \langle u_k, v_k \rangle$. It also corresponds to Dixmier’s minimal angle definition [21] repeated p_2 -times (see also Reference [22]). It is easy to prove

Lemma 6

The angles $\hat{\theta}_k$ defined in (7) are the same as the angles θ_k of (6), and $\{\hat{u}_k, \hat{v}_k\}$ and $\{u_k, v_k\}$ are both corresponding principal vector pairs.

Proof

Consider the case when $k = 1$. Then

$$\cos \theta_1 = \max_{u \in M_1, v \in M_2, \|u\| = \|v\| = 1} |\langle u, v \rangle| = \langle u_1, v_1 \rangle$$

and

$$\cos \hat{\theta}_1 = \max_{u \in M_1, v \in M_2, \|u\| = \|v\| = 1} \operatorname{Re}\langle u, v \rangle = \langle \hat{u}_1, \hat{v}_1 \rangle$$

Since $\cos \theta_1, \cos \widehat{\theta}_1 \geq 0$, we have $0 \leq \langle u_1, v_1 \rangle = \operatorname{Re} \langle u_1, v_1 \rangle \leq \cos \widehat{\theta}_1$. Hence $\cos \theta_1 \leq \cos \widehat{\theta}_1$. In turn, $\cos \widehat{\theta}_1 = \langle \widehat{u}_1, \widehat{v}_1 \rangle = |\langle \widehat{u}_1, \widehat{v}_1 \rangle| \leq \cos \theta_1$. Hence $\theta_1 = \widehat{\theta}_1$ and $\langle u_1, v_1 \rangle = \cos \theta_1 = \cos \widehat{\theta}_1 = \langle \widehat{u}_1, \widehat{v}_1 \rangle$. Clearly we can apply the same argument for $k > 1$. \square

The third definition of principal angles and vectors exploits basic properties of orthogonal projections and seems to be new. We use the following notations: $M_1^{(1)} = M_1, M_2^{(1)} = M_2$ and

$$M_1^{(k)} = M_1 \cap \mathcal{R}^\perp(u_1, \dots, u_{k-1}), \quad M_2^{(k)} = M_2 \cap \mathcal{R}^\perp(v_1, \dots, v_{k-1}) \quad (8)$$

Definition 7

Let $M_1, M_2 \subset \mathbb{C}^n$ be subspaces with $p_1 = \dim(M_1) \geq \dim(M_2) = p_2 \geq 1$ and let P_1 be the orthogonal projection onto M_1 and P_2 the orthogonal projection onto M_2 . Then the principal angles $\theta_k \in [0, \pi/2]$ between M_1 and M_2 are recursively defined for $k = 1, \dots, p_2$ by

$$\cos \theta_k = \max_{\substack{v \in M_2^{(k)} \\ \|v\|=1}} \|P_1 v\| = \|P_1 v_k\| = \max_{\substack{u \in M_1^{(k)} \\ \|u\|=1}} \|P_2 u\| = \|P_2 u_k\| \quad (9)$$

The vectors $\{u_1, \dots, u_{p_2}\}, \{v_1, \dots, v_{p_2}\}$ are called principal vectors of the pair of spaces.

Theorem 8

The principal angles given by Definition 7 are identical with those of Definition 4.

Proof

First we assume that $k = 1$. For any $u \in M_1$ and $v \in M_2$ of unit norm we have

$$|\langle u, v \rangle| = |\langle P_1 u, v \rangle| = |\langle u, P_1 v \rangle| \leq \|u\| \|P_1 v\| \leq \max_{\substack{v \in M_2 \\ \|v\|=1}} \|P_1 v\| = \|P_1 v_1\|$$

There is equality if $u = \lambda P_1 v$. If $P_1 v_1 \neq 0$, then condition $\|u\| = 1$ gives $u_1 = P_1 v_1 / \|P_1 v_1\| \in M_1$ and for this u_1 we have $\cos \theta_1 = |\langle u_1, v_1 \rangle| = \langle u_1, v_1 \rangle = \|P_1 v_1\|$. Hence u_1 and v_1 are corresponding principal vectors so that $P_1 v_1 = (\cos \theta_1) u_1$. If $\|P_1 v_1\| = 0$, then $\langle u, v \rangle = 0$ holds for any $u \in M_1$. Thus $\theta_1 = \pi/2$ and we can select any $u_1 \in M_1$ ($\|u_1\| = 1$) as first principal vector. We also have the relation $P_1 v_1 = (\cos \theta_1) u_1 (= 0)$. Since the role of M_1 and M_2 is symmetric, the other relation of (9) with P_2 clearly holds. So does the equality $P_2 u_1 = (\cos \theta_1) v_1$.

For any $u \in M_1^{(2)}, \langle u, v_1 \rangle = \langle P_1 u, v_1 \rangle = \langle u, P_1 v_1 \rangle = (\cos \theta_1) \langle u, u_1 \rangle = 0$. Similarly, if $v \in M_2^{(2)}$, then $\langle v, u_1 \rangle = \langle P_2 v, u_1 \rangle = \langle v, P_2 u_1 \rangle = (\cos \theta_1) \langle v, v_1 \rangle = 0$. Hence $\mathcal{R}(u_1) \perp M_2 \cap \mathcal{R}^\perp(v_1) = M_2^{(2)}$ and $\mathcal{R}(v_1) \perp M_1 \cap \mathcal{R}^\perp(u_1) = M_1^{(2)}$. Assume now that

$$\mathcal{R}(u_1, \dots, u_{k-1}) \perp M_2^{(k)}, \quad \mathcal{R}(v_1, \dots, v_{k-1}) \perp M_1^{(k)} \quad (10)$$

and

$$P_1 v_j = (\cos \theta_j) u_j, \quad P_2 u_j = (\cos \theta_j) v_j \quad (11)$$

hold for $j \leq k - 1$.

For any $k > 1, u \in M_1^{(k)} \subset M_1$ ($\|u\| = 1$) and $v \in M_2^{(k)}$ ($\|v\| = 1$) we obtain

$$|\langle u, v \rangle| = |\langle P_1 u, v \rangle| = |\langle u, P_1 v \rangle| \leq \|u\| \|P_1 v\| \leq \max_{\substack{v \in M_2^{(k)} \\ \|v\|=1}} \|P_1 v\| = \|P_1 v_k\|$$

We show that $P_1v \in M_1^{(k)}$ for any $v \in M_2^{(k)}$. Since $P_1v \in M_1$, $P_1v \in M_1^{(k)}$ if and only if $P_1v \perp \mathcal{R}(u_1, \dots, u_{k-1})$. This however follows from $\langle P_1v, u_i \rangle = \langle v, P_1u_i \rangle = \langle v, u_i \rangle = 0$ for $i \leq k - 1$. If $P_1v_k \neq 0$, then for $u_k = P_1v_k / \|P_1v_k\|$, $\cos \theta_k = |\langle u_k, v_k \rangle| = \langle u_k, v_k \rangle = \|P_1v_k\|$ and $P_1v_k = (\cos \theta_k)u_k$ hold. If $P_1v_k = 0$, then for any $u \in M_1^{(k)}$ we have $\langle u, v \rangle = 0$. Hence $\theta_k = \pi/2$, $u_k \in M_1^{(k)}$ is any unit vector and $P_1v_k = (\cos \theta_k)u_k (= 0)$. Again applying the symmetry principle we obtain the same for P_2 and also the relation $P_2u_k = (\cos \theta_k)v_k$.

Assume that (11) holds for $j \leq k < p_2$. Let $u \in M_1^{(k+1)}$. For any v_j ($j \leq k$),

$$\langle u, v_j \rangle = \langle P_1u, v_j \rangle = \langle u, P_1v_j \rangle = (\cos \theta_j)\langle u, u_j \rangle = 0$$

Similarly, if $v \in M_2^{(k+1)}$, then

$$\langle v, u_j \rangle = \langle P_2v, u_j \rangle = \langle v, P_2u_j \rangle = (\cos \theta_j)\langle v, v_j \rangle = 0$$

Hence we proved relation (10) for $k + 1$ and the equivalence of Definitions 7 and 4. □

At the end of the process (recursive definition) we obtain the orthonormal vectors $\{u_i\}_{i=1}^{p_2} \subset M_1$ and $\{v_i\}_{i=1}^{p_2} \subset M_2$ such that

$$\mathcal{R}(v_1, \dots, v_{p_2}) = M_2 \perp M_1 \cap \mathcal{R}^\perp(u_1, \dots, u_{p_2}) \tag{12}$$

$\langle u_i, v_i \rangle = \cos \theta_i$ and

$$P_1v_i = (\cos \theta_i)u_i, \quad P_2u_i = (\cos \theta_i)v_i \quad (i = 1, \dots, p_2) \tag{13}$$

The biorthogonality relation $\langle u_i, v_j \rangle = (\cos \theta_j)\delta_{ij}$ follows from the relation

$$\langle u_i, v_j \rangle = \langle P_1u_i, v_j \rangle = \langle u_i, P_1v_j \rangle = (\cos \theta_j)\langle u_i, u_j \rangle = (\cos \theta_j)\delta_{ij}$$

Property (13) and the following consequence also appear in Reference [7] for real vector spaces. Afriat called the pairs (u_i, v_i) reciprocal.

Corollary 9

$$P_1P_2u_i = (\cos \theta_i)P_1v_i = (\cos^2 \theta_i)u_i \text{ and } P_2P_1v_i = (\cos \theta_i)P_2u_i = (\cos^2 \theta_i)v_i.$$

It is possible to give another characterization of the canonical angles using the projection definition. Recall that the distance of a vector x from the subspace M_1 can be expressed by $\|(I - P_1)x\|$, where P_1 is the orthogonal projection onto M_1 . Observe that P_1x and $(I - P_1)x$ are mutually orthogonal vectors and the distance comes from the Pythagorean theorem in the inner product norm metric. In the view of this distance function, we can interpret $\cos \theta_k$ in Definition 7 as the distance of the unit ball in $M_2^{(k)}$ from M_1^\perp . We get the same result from the dual relation: it is the distance of the unit ball in $M_1^{(k)}$ from M_2^\perp .

The given recursive definitions are quite analogous to the variational characterization of the eigenvalues of a Hermitian matrix as given by the Courant–Fischer theorem. The connection is more apparent later, when the principal angles are given by singular values.

3. THE PRINCIPAL ANGLES AND THE SVD

Here we analyse the connection of the principal angles and the singular value decomposition that was first exploited in Reference [9].

Let $X = [x_1, \dots, x_{n_1}]$, $Y = [y_1, \dots, y_{n_2}]$ and

$$G(X, Y) = [\langle y_j, x_i \rangle]_{i,j=1}^{n_1, n_2} \quad (14)$$

Clearly, $G(X, X)$ is the classical Gram matrix. For any $A \in \mathbb{C}^{n_1 \times n_1}$ and $B \in \mathbb{C}^{n_2 \times n_2}$ we have

$$G(XA, YB) = A^H G(X, Y) B \quad (15)$$

where A^H stands for the Hermitian adjoint or transpose of matrix A . By definition $G(XA, YB) = [\langle YB e_j, XA e_i \rangle]_{i,j=1}^{n_1, n_2}$ and

$$\begin{aligned} \langle YB e_j, XA e_i \rangle &= \left\langle \sum_{k=1}^{n_2} b_{kj} y_k, \sum_{l=1}^{n_1} a_{li} x_l \right\rangle \\ &= \sum_{l=1}^{n_1} \bar{a}_{li} \left(\sum_{k=1}^{n_2} b_{kj} \langle y_k, x_l \rangle \right) \\ &= e_i^T A^H G(X, Y) B e_j \end{aligned}$$

which proves the claim. Observe that the vectors $X = [x_1, \dots, x_{n_1}]$ are orthonormal if and only if $G(X, X) = I$. Similarly, $X = [x_1, \dots, x_{n_1}]$ and $Y = [y_1, \dots, y_{n_2}]$ are biorthogonal ($n_1 \geq n_2$) if and only if

$$G(X, Y) = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \quad (\Gamma = \text{diag}(\langle y_i, x_i \rangle)) \quad (16)$$

Let $U = [u_1, \dots, u_{p_2}]$ and $V = [v_1, \dots, v_{p_2}]$. Since $\mathcal{R}(V) = M_2$, any orthonormal basis of $M_1 \cap \mathcal{R}^\perp(U)$, say \widehat{U} , is orthogonal to M_2 . Then the columns of $U_1 = [U, \widehat{U}]$ are orthonormal and span M_1 . The columns of U_1 and V are biorthogonal, that is

$$G(U_1, V) = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \quad (\Gamma = \text{diag}(\cos \theta_i)) \quad (17)$$

Remark 10

It is the restriction $\cos \theta_k = \langle u_k, v_k \rangle$ of Definition 4 or 5 that guarantees that Γ has non-negative real entries. Without this restriction Γ may become complex.

From relation (17) we immediately obtain

Theorem 11 (Afriat [7, 23], Gutmann and Shepp [24])

In any pair of subspaces M_1 and M_2 there exist orthonormal bases $\{u_i\}_{i=1}^{p_1}$ and $\{v_i\}_{i=1}^{p_2}$ such that $\langle u_i, v_i \rangle \geq 0$ and $\langle u_i, v_j \rangle = 0$ if $i \neq j$.

Following Watkins [17] we show that the SVD approach of Björck and Golub [9] is a direct consequence of the recursive definitions.

Lemma 12

Let $Q_i \in \mathbb{C}^{n \times p_i}$ be any matrix having orthonormal columns that span M_i ($i = 1, 2$). Then there exists unitary matrices $F_i \in \mathbb{C}^{p_i \times p_i}$ ($i = 1, 2$) such that $Q_1 = U_1 F_1$ and $Q_2 = V F_2$.

Proof

There is a non-singular matrix $F_1 \in \mathbb{C}^{p_1 \times p_1}$ such that $Q_1 = U_1 F_1$. Since the columns of Q_1 are orthonormal,

$$G(Q_1, Q_1) = G(U_1 F_1, U_1 F_1) = F_1^H G(U_1, U_1) F_1 = F_1^H F_1 = I$$

The rest of claim follows similarly. □

We can observe now that

$$G(Q_1, Q_2) = F_1^H G(U_1, V) F_2 = F_1^H \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} F_2 \tag{18}$$

is a singular value decomposition with the singular values $\{\cos \theta_i\}_{i=1}^{p_2}$. Since Γ is independent of the choice of Q_1 and Q_2 , we can determine the principal angles and vectors between M_1 and M_2 as follows.

Theorem 13

Let the columns of $Q_1 \in \mathbb{C}^{n \times p_1}$ and $Q_2 \in \mathbb{C}^{n \times p_2}$ be orthonormal bases for M_1 and M_2 , respectively. Let

$$G(Q_1, Q_2) = Y \Sigma Z^H, \quad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_{p_2}) \in \mathbb{R}^{p_1 \times p_2} \tag{19}$$

be a singular value decomposition, where $Y \in \mathbb{C}^{p_1 \times p_1}$ and $Z \in \mathbb{C}^{p_2 \times p_2}$ are unitary. Then the principal angles and principal vectors between the subspaces M_1 and M_2 are given by

$$\cos(\theta_i) = \sigma_i, \quad u_i = Q_1 Y e_i, \quad v_i = Q_2 Z e_i \quad (i = 1, \dots, p_2) \tag{20}$$

For real vector spaces with the standard inner product and subspaces of the same dimension Watkins [17] derived the latter result from Definition 2 with a different technique.

If we take the inner product $\langle x, y \rangle_W = y^H W x$, where W is Hermitian and positive definite, then $G(Q_1, Q_2) = Q_1^H W Q_2$. This gives an easy extension of Theorem 2.8 of Argentati [18], which is already a generalization of Björck and Golub [9] (see also Reference [16]). Stability analysis of SVD based algorithms for computing the principal angles are given by Björck and Golub [9], Argentati [18], Knyazev and Argentati [19].

4. OTHER ANGLE DEFINITIONS AND CONCEPTS

There are other concepts for the angle between subspaces. The product angle (cosine) ϕ between M_1 and M_2 is defined by

$$\cos \phi = \prod_{i=1}^{p_2} \cos \theta_i \tag{21}$$

where θ_i 's are the principal angles (see, e.g. References [2, 5, 25]). The product cosine and the similarly defined product sine are intensively studied by Miao and Ben-Israel [26, 27].

The following two angle concepts are defined for Hilbert spaces (see Reference [22]).

Definition 14 (Friedrichs)

The angle between the subspaces M_1 and M_2 of a Hilbert space H is the angle $\alpha(M_1, M_2)$ in $[0, \pi/2]$ whose cosine is given by

$$c(M_1, M_2) = \sup\{|\langle x, y \rangle| \mid x \in M_1 \cap (M_1 \cap M_2)^\perp, \|x\| \leq 1, \\ y \in M_2 \cap (M_1 \cap M_2)^\perp, \|y\| \leq 1\} \quad (22)$$

Definition 15 (Dixmier)

The minimal angle between the subspaces M_1 and M_2 is the angle $\alpha_0(M_1, M_2)$ in $[0, \pi/2]$ whose cosine is defined by

$$c_0(M_1, M_2) = \sup\{|\langle x, y \rangle| \mid x \in M_1, \|x\| \leq 1, y \in M_2, \|y\| \leq 1\} \quad (23)$$

The two definitions are clearly different if $M_1 \cap M_2 \neq \{0\}$ and agree, otherwise. If $H = \mathbb{C}^n$, then $\alpha_0 = \theta_1$ and $\alpha = \theta_{k+1}$ provided that $\dim(M_1 \cap M_2) = k$. Ipsen and Meyer [28] give interesting characterizations of the minimal angle for complementary subspaces of \mathbb{R}^n . An interesting application of the minimal angle is given in Reference [29] (see also Reference [22]).

The principal angles themselves can be defined with eigenvalues as well. The following theorem is an extension of Zassenhaus' classical result [8] (see also References [5, 30]).

Theorem 16 (Ben-Israel [12])

Let $M_1, M_2 \subset \mathbb{C}^n$ be subspaces with $p_i = \dim(M_i)$ ($i = 1, 2$) and let U and V be two matrices whose columns span M_1 and M_2 , respectively. Then the first $\min\{p_1, p_2\}$ eigenvalues of the matrix

$$U^+ V V^+ U \quad (24)$$

are the squares of the cosines of the principal angles between the subspaces M_1 and M_2 provided that the eigenvalues are given in descending order.

Here the non-zero eigenvalues of $U^+ V V^+ U$ and $V V^+ U U^+$ are the same and one can identify $P_2 = V V^+$ as the orthogonal projection onto M_2 and $P_1 = U U^+$ as the orthogonal projection onto M_1 according to the Penrose conditions on the pseudoinverse. Ben-Israel's theorem states

$$P_2 P_1 v_i = (\cos^2 \theta_i) v_i \quad \text{or} \quad P_1 P_2 u_i = (\cos^2 \theta_i) u_i \quad (25)$$

and these relations were already stated by Corollary 9 in a more general setting. As $v_i \in M_2$, another equivalent form of the first equation is $P_2 P_1 P_2 v_i = (\cos^2 \theta_i) v_i$ because $P_2 v_i = v_i$. The other equation can be handled similarly. These relations show the connection to the next definition (see, e.g. Reference [13, Problems 559, 560] or Bhatia [11, Exercise VII.1.10, p. 201]):

Definition 17

If P_1 and P_2 are finite dimensional orthogonal projections, then the principal angles between them (or, equivalently, between their ranges as subspaces) is defined as the arccos of the square root of the eigenvalues (counted according to multiplicity) of the positive (self-adjoint) finite rank operator $P_2 P_1 P_2$.

In general, formulas based on eigenvalues instead of singular values should be avoided in computations, since otherwise accuracy may be lost. An exception may exist however. It is shown

in Reference [31] that the principal angles can be recovered from the spectrum of $P_1 + P_2$, where P_1 and P_2 denote the orthogonal projection onto M_1 and M_2 , respectively.

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