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Josephson physics mediated by the Mott insulating phase

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We investigate the static and dynamic properties of bosonic lattice systems in which condensed and Mott insulating phases coexist due to the presence of a spatially varying potential. We formulate a description of these inhomogeneous systems and calculate the bulk energy at and near equilibrium. We derive the explicit form of the Josephson coupling between disjoint superfluid regions separated by Mott insulating regions. We obtain detailed estimates for the case of alternating superfluid and Mott insulating spherical shells in a radially symmetric parabolically confined cold atom system.

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I. INTRODUCTION

An important and generic situation which arises in inhomogeneous quantum many-body systems is that of competing states of matter coexisting in spatially separated regions. Diverse systems such as the quantum Hall system, metal-insulator compounds, high T_c superconductors, and more recently, cold atomic gases, display conducting regions embedded within insulating regions. Crucial to understanding thermodynamic and transport features of such systems is the manner in which conducting regions couple to one another through the insulating regions [1]. Classic examples of Josephson coupling in superconductors and cold atoms rely on an externally imposed potential barrier between condensed regions [2,3]. Here, on the other hand, germane to the instances mentioned above, we explore systems of bosons in which condensed (superfluid) regions exhibit Josephson physics mediated by Mott-insulating regions of the same bosons. This physics should be particularly applicable to granular superconductors and high T_c materials where Cooper pairs can be treated as the bosonic degrees of freedom [4], and trapped cold atoms in optical lattices where the atoms are bosons [5]. Through an explicit description of these phases in terms of microscopic parameters, we are able to go beyond phenomenological treatments for obtaining transport coefficients in these systems [1].

Toward understanding this physics of coexisting quantum phases, we study a system of interacting bosons on a lattice in the presence of a smooth potential $V(\mathbf{r})$ which varies on length scales much larger than the lattice spacing a. Within a local density approximation, the potential can be regarded as a shift in the local chemical potential $\tilde{\mu}(\mathbf{r})=\mu-V(\mathbf{r})$, where μ is the global chemical potential determined by the total number of bosons in the system, N. In the situations of interest, shown in Fig. 1, the potential $V(\mathbf{r})$ breaks the system into phase-separated domains of Mott insulator (wherein interactions pin the number of bosons per site) and of condensed bosons (having an associated order parameter and number fluctuation on each site). Potentially relevant to superconductors in the condensed matter setting, Fig. 1(a) represents a system having a long length-scale disorder potential with very slight variations resulting in Mott-insulator regions of a fixed number neighboring condensate regions which are in the immediate vicinity of the Mott insulator in the Mott-superfluid phase diagram. Figure 1(b) represents an optical lattice scenario [5] in which a spherically symmetric trapping potential breaks the system into Mott-insulating shells interspersed by condensates. In what follows, we derive the equilibrium properties of the domains, bulk energy costs for small deviations from equilibrium, dynamics of the condensed regions, the Josephson coupling between condensed regions mediated by a Mott-insulating interface, and detailed estimates for the situation illustrated in Fig. 1(b).

II. MODEL

The system at hand can be modeled by the Bose-Hubbard Hamiltonian, describing bosons whose tunneling between neighboring lattice sites has strength J and whose on-site repulsive interaction is U:

$$H_{BH}=-J\sum_{\langle ij\rangle} b_i^{\dagger}b_j+\sum_i\left[\frac{U}{2}\hat{n}_i(\hat{n}_i-1)-\tilde{\mu}_i\hat{n}_i\right]. \qquad (1)$$

Here, $\langle ij\rangle$ denotes a summation over nearest-neighbor sites, $b_i^{\dagger}$ and b_i denote bosonic creation and annhilation operators, respectively, on the site i and $\tilde{\mu}_i\equiv\tilde{\mu}(\mathbf{r}_i)$.

As seen in the phase diagram of Fig. 1, coexisting Mott and superfluid regions can only be realized in the small-tunneling limit. For small J/U, each superfluid region is energetically near two Mott insulating phases, whose occupations we label as n_0 and n_0+1, as shown in Fig. 2. To describe each superfluid region in the coexisting system, we employ a pseudospin formulation of the Bose-Hubbard model [6–9] which truncates the Hilbert space on each site to these two occupation numbers. As we discuss later, this formulation can be generalized to include more number states if necessary, but here, for simplicity and because it can be easily realized in cold-atom systems, we assume that J/U is sufficiently small to justify the truncation. The resulting two-state Hilbert space can then be mapped to a spin-1/2 basis on each site, i, with the identifications $|n_0+1\rangle_i\leftrightarrow|\uparrow\rangle_i$ and $|n_0\rangle_i\leftrightarrow|\downarrow\rangle_i$, where $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ are eigenstates of the spin operator s_i^z with eigenvalues $\pm 1/2$, and $b_i^{\dagger}=\sqrt{n_0+1}s_i^{+}$ ($b_i=\sqrt{n_0+1}s_i^{-}$). The number operator $\hat{n}_i=b_i^{\dagger}b_i$ is related to the z component of the spin operator: $\hat{n}_i=n_0+1/2+s_i^z$. With this mapping, the Hamiltonian takes the form

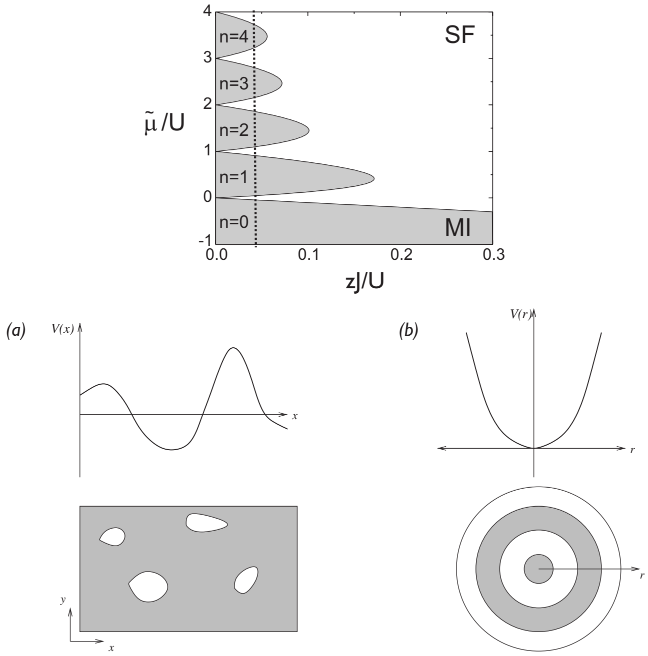

FIG. 1. Top: The perturbative zero temperature phases of the Bose-Hubbard model; the dotted line cuts through the phases that could coexist for a fixed small value of zJ/U. Below: Schematic (a) slowly varying random potential and (b) harmonic confining potential, and subsequent Mott-superfluid domains.

$$\mathcal{H} = -J(n_0+1)\sum_{\langle ij\rangle}(s_i^x s_j^x + s_i^y s_j^y) + \sum_i (Un_0 - \tilde{\mu}_i)s_i^z. \quad (2)$$

At the mean-field level, to which we confine ourselves in this work, pseudospins on each site can be thought of as experiencing an effective local "magnetic" field whose x-y component is determined by the surrounding spins and whose z component is determined by the local chemical potential. In the ground state, the spin on each site is aligned with this field,

$$\mathbf{B}_i^0 = zJ(n_0+1)[2f_i^x, 2f_i^y, \cos\theta_i], \quad (3)$$

$$\cos\theta_i \equiv \frac{\tilde{\mu}_i - Un_0}{zJ(n_0+1)} \quad (4)$$

(here z is the coordination number of the lattice), the fields $\mathbf{f}_i$ denote expectation values of spin operators (e.g., $f_i^z = \langle s_i^z\rangle$), and we have assumed $\mathbf{f}_i \approx \mathbf{f}_j$ for nearest neighbors. The equilibrium z component of the pseudospin has the value $f_{i0}^z = (1/2)\cos\theta_i$; the Mott phases correspond to complete polarization of the pseudospin along the z direction, i.e., $f_{i0}^z = \pm 1/2$, corresponding to $\langle\hat{n}_i\rangle = n_0+1$ or n_0, respectively. Within the mean-field approximation and to first order in zJ/U, we can thus identify the Mott-superfluid boundaries shown in Fig. 1 as occurring at the critical values of the external potential

$$\mu - V(\mathbf{r}_\pm^c) = Un_0 \pm zJ(n_0+1), \quad (5)$$

where $+$ and $-$ refer to the boundary at the Mott n_0+1 and n_0 phases, respectively.

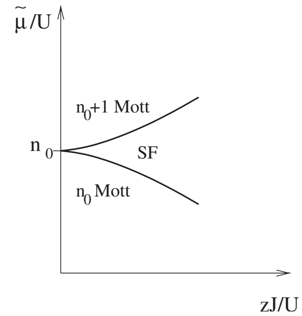

FIG. 2. Schematic Bose-Hubbard phase diagram showing the superfluid region between the n_0 and n_0+1 Mott regions, which is well-described at small J/U by truncating the Hilbert space on each site to occupations n_0 and n_0+1.

In the condensed phase, a local order parameter can be defined as $\psi = \langle b^\dagger\rangle = \sqrt{n_0+1}f^+$ for $0 < f^z \leqslant 1/2$ and $\psi = \langle b\rangle = \sqrt{n_0+1}f^-$ for $-1/2 < f^z \leqslant 0$, corresponding to condensates of particles and holes, respectively. To first order in J/U and in the continuum limit, the equilibrium order parameter profile [as a function of $\tilde{\mu} = \mu - V(\mathbf{r})$] follows from the normalization of the spins: $f_0^\pm = \sqrt{1-f_0^{z2}}/2$. Ignoring the energy cost of variations of $f_0^\pm$ from site to site (the Thomas-Fermi approximation), the order parameter is found to be

$$\psi(\mathbf{r}) = \sqrt{\frac{z^2J^2(n_0+1)^2 - (\tilde{\mu} - n_0U)^2}{4z^2J^2(n_0+1)}}. \quad (6)$$

This is of the same form as the Thomas-Fermi order parameter for a free (no lattice) condensate in an external potential V_{ext} with interaction strength g: $\psi_{TF} = \sqrt{(\mu - V_{ext})/g}$ [3]. One can thus identify the "effective" confining potential for the superfluid between two Mott regions in the optical lattice system: $(\mu - V_{ext})_{eff} = (\tilde{\mu} - n_0U)^2/(zJ[n_0+1])$.

The boson density in the condensed phase is found from $\langle\hat{n}\rangle = (n_0+1/2) + f^z$ and in equilibrium in the Thomas-Fermi approximation is given by

$$\langle\hat{n}\rangle = (n_0+1/2) + \frac{\tilde{\mu} - n_0U}{2zJ(n_0+1)}, \quad (7)$$

which smoothly interpolates between densities of n_0+1 at $\mathbf{r}_c^+$ and n_0 at $\mathbf{r}_c^-$.

The model above gives a description of each superfluid region in the inhomogeneous system, along with its two nearby Mott phases, valid for small J/U. Equipped with this description, we turn to the coupling of two spatially separated superfluid regions in the system and the resulting Josephson physics. In what follows, we find that the transfer of δN particles between two superfluid regions having a relative phase $\Delta\varphi$ is described by the usual Josephson Hamiltonian

$$H(\delta N, \Delta\phi) \approx E_B(\delta N) + E_J[1 - \cos(\Delta\phi)]. \quad (8)$$

We derive explicit formulas for the relevant energy scales for the exchange of particles, namely, the bulk energy change E_B due to out-of-equilbrium transfer of bosons and the Josephson energy E_J, which measures the strength of tunneling between the two superfluid regions. We pinpoint the differences and similarities between Josephson physics in the "self-organized" Mott-superfluid system and in conventional systems.

III. BULK ENERGY

As derived here, for mesoscopic superfluid regions, the energy cost for deviations from equilibrium is found to be non-negligible. Within the Thomas-Fermi approximation, which suffices to derive the form of E_B, the Hamiltonian, Eq. (2), can be expressed in terms of $f^z(\mathbf{r})$:

$$E_B(N) = \int \frac{1}{a^3}[zJ(n_0+1)(f^{z2}-1) + (n_0 U - \tilde{\mu})f^z]d\mathbf{r},$$

$$N = \int \langle \hat{n} \rangle d\mathbf{r} = \int \left[n_0 + \frac{1}{2} + f^z \right] d\mathbf{r}, \tag{9}$$

where, assuming that variations in the density are over length scales greater than the lattice spacing, a continuum approximation has been made. We see that, in the Thomas-Fermi description, the Mott and superfluid regions decouple from each other and have separate contributions to the bulk energy, $E_B(N)=E_B^{\mathrm{Mott}}(N_M)+E_B^{sf}(N_S)$, where N_M and N_S are the total number of particles in the Mott and superfluid phases, respectively. These are obtained by integrating the boson density, $\langle \hat{n} \rangle$, over the appropriate region and thus it is clear that $N=N_S+N_M$.

We seek the change in the bulk energy when the number of particles in the superfluid region changes. Therefore we consider a situation in which the superfluid region slightly shrinks from its equilibrium configuration by transferring a small number of particles δN to the Mott region: $N_S \to N_S - \delta N$ and $N_M \to N_M + \delta N$. We find that the bulk energy takes the form

$$E_B \approx E_B^{\mathrm{Mott}}(N_{M0}) + E_B^{sf}(N_{S0}) + \frac{1}{2}\left(\left.\frac{\partial^2 E_B^{\mathrm{Mott}}}{\partial N_M^2}\right|_0 + \left.\frac{\partial^2 E_B^{sf}}{\partial N_S^2}\right|_0 \right) \times (\delta N)^2, \tag{10}$$

where the subscript "0" denotes equilibrium. The energy scale associated with the transfer of particles to the superfluid region, E_C (often called the "capacitive energy" in reference to Josephson physics in mesoscopic superconductors), is defined by $E_B(\delta N)=E_C(\delta N)^2/2$ and from Eq. (10) is found to be

$$E_C = \left.\frac{\partial^2 E_B^{\mathrm{Mott}}}{\partial N_M^2}\right|_0 + \left.\frac{\partial^2 E_B^{sf}}{\partial N_S^2}\right|_0 . \tag{11}$$

The formulas in the previous section provide the equilibrium configuration $f_0^z(\mathbf{r})$, including the locations of the boundaries of the superfluid and insulating regions. For a given external potential $V(\mathbf{r})$, Eqs. (9)–(11) therefore allow one to calculate the energy E_C. In Sec. V, below, we give an explicit formula for E_C in the case of spherical shells.

We remark that the deviation from equilibrium described here differs both from that of mesoscopic superconductors and that of two externally trapped superfluids that couple when brought near each other. Specifically, in the latter cases, transfer of particles is between two condensed regions and thus the contribution to the system's bulk energy that is linear in δN only vanishes when the energy change of the two coupled condensed regions is combined [10]. In the superfluid-Mott coexisting system, the transfer of particles is a local one between one superfluid region and the surrounding Mott phase, whose boundary is determined by the external potential and the ratio J/U and is in this sense self-organized. The coexisting system thus obeys an equilibrium condition between the Mott and superfluid regions: $\partial E_B^{\mathrm{Mott}}/\partial N_M|_0 = \partial E_B^{sf}/\partial N_S|_0$ that is not present in the more familiar examples.

IV. DYNAMICS AND JOSEPHSON ENERGY

To describe the tunneling of particles from one superfluid region to another, we must understand the dynamics of particle transfer in the Mott-superfluid system. We proceed as follows: first, we consider the equations of motion for a superfluid region and its neighboring Mott regions in the pseudospin approximation, which allows us to identify the continuity equation in the inhomogeneous system and the corresponding current density of bosons. Next, we derive expressions for the equilibrium order parameter and its decay in the neighboring insulating regions. Finally, we consider two superfluid regions separated by a single Mott region and derive the Josephson equation governing the transfer of particles between the two.

Concentrating on a single superfluid region in the pseudospin approximation, the "spins" obey Heisenberg equations of motion. Furthermore, in the mean-field approximation, they obey Bloch equations, $\partial_t \mathbf{f}_i = \mathbf{f}_i \times \mathbf{B}_i$. To properly capture the Josephson coupling between superfluid regions, we must go beyond the Thomas-Fermi approximation and account for the energy of spatial variations between neighboring spin operators: $\Sigma_j \mathbf{f}_j \approx z\mathbf{f} + a^2 \nabla^2 \mathbf{f}$. The resulting equations of motion for the local order parameter can be used to derive, for instance, the collective modes within each superfluid region [9]. The equation of motion for the z component of the pseudospin, on the other hand, can be written in the form of a continuity equation, $\partial_t \langle n \rangle + \nabla \cdot \boldsymbol{\mathcal{J}} = 0$. This allows us to identify

$$\boldsymbol{\mathcal{J}} = iJa^2(\psi \nabla \psi^* - \psi^* \nabla \psi) \tag{12}$$

as the supercurrent density of bosons.

To calculate the Josephson coupling between spatially separated superfluid regions, we first need an accurate description of the order parameter in each region. Equation (6) gives such a description deep in the superfluid region, where the Thomas-Fermi approximation is valid. Near the Mott regions, however, this approximation clearly breaks down, and the abrupt transition between Mott and superfluid is replaced with a smooth decay of the order parameter as we move into the Mott region. From the Bloch equations for the pseudospin, we find that close to a Mott-superfluid interface, the order parameter respects the equation (for $f^z < 0$, i.e., the boundary with the n_0-Mott region)

$$i\partial_t\psi \approx -J(n_0+1)a^2\nabla^2\psi + [Un_0 - \tilde{\mu} - zJ(n_0+1)]\psi + 2Jz|\psi|^2\psi. \quad (13)$$

(A similar equation is respected close to the n_0+1-Mott boundary where $f^z>0$.) This equation governs the decay of the order parameter beyond the Thomas-Fermi boundary of the Mott region. We remark that while Eq. (13) is identical to a Gross-Pitaevskii equation for a superfluid order parameter [3], this analogy breaks down well within the condensed region. In particular, as seen above in Eqs. (6) and (7), the Mott-superfluid system does not have a density of bosons directly proportional to the square of the order parameter.

While the two-component pseudospin description suffices at the Mott-superfluid boundary, and is in fact ideally suited to connect the magnitude of the order parameter at the boundary to its value in the bulk of the superfluid, it does not capture the physics deep in the Mott region between superfluid regions. Specifically, deep in any n_0-Mott region, it is clear that the states nearby in energy have occupation numbers n_0+1 and n_0-1, thus requiring at least a spin-1 pseudospin description [8]. The relevant equations of motion for this case are easily calculated by going beyond the two-component truncation and employing a perturbative mean-field analysis [11]. As detailed in Ref. [11], one finds to lowest nonvanishing order in ψ:

$$i\kappa_\tau\partial_t\psi \approx -\kappa_r\nabla^2\psi + \alpha\psi,$$

$$\alpha = \frac{1}{a^3}\left[\frac{1}{zJ} - \frac{n_0+1}{Un_0-\tilde{\mu}} - \frac{n_0}{\tilde{\mu}-U(n_0-1)}\right], \quad (14)$$

where $\kappa_\tau = a^{-3}\frac{\partial\alpha}{\partial\mu}$ and $\kappa_r = \frac{a^{-1}}{z^2J}$. At the mean-field level, the Mott-superfluid boundary is captured by the relationship $\alpha=0$, which can be used to generate the Mott lobes of the Bose-Hubbard phase diagram shown in Fig. 1. We remark that this result, being perturbative in the tunneling, breaks down well within the superfluid where the difference in energies for occupation numbers n_0 and n_0+1, for instance, becomes much smaller than J. In this region, we return to the original spin-1/2 pseudospin model, which provides the correct description of the system. We note that the equations of motion obtained by the perturbative approach deep in the Mott region, as required, coincide with Eq. (13) close to the superfluid boundary (where terms of order $|\psi|^3$ can be ignored).

Thus we have a complete mean-field description of the order parameter for the superfluid region between n_0 and n_0+1 regions and have shown how to extend this description deep into the neighboring Mott regions. We are now prepared to consider the coupling of two superfluid regions through an intervening Mott region. Each superfluid region can be described by a pseudospin model with an appropriate choice of n_0, giving the order parameters in the two regions, $\psi_A e^{i\varphi_A}$ and $\psi_B e^{i\varphi_B}$, where $\psi_{A/B}$ are real. The total order parameter of the two-superfluid region is thus $\psi=\psi_A e^{i\varphi_A}+\psi_B e^{i\varphi_B}$.

The continuity equations in each region can be combined to give the continuity equation across the two superfluid regions:

$$\partial_t(\langle n\rangle_A - \langle n\rangle_B) + \nabla\cdot\mathcal{J} = 0, \quad (15)$$

where $\mathcal{J}$ is given by Eq. (12) with the ψ above. Plugging ψ in, we find that $\mathbf{J}$ has the Josephson form

$$\mathcal{J} = 2Ja^2(\psi_B\nabla\psi_A - \psi_A\nabla\psi_B)\sin(\Delta\phi), \quad (16)$$

where the relative phase between the superfluids is given by $\Delta\varphi=\varphi_A-\varphi_B$. The Josephson energy is defined by $\partial_t(\delta N_{A\to B}) = -E_J\sin(\varphi_A-\varphi_B)$, where when particles are transferred from the A region to the B region, $\delta N_A=-\delta N_B=\delta N_{A\to B}$. E_J can be explicitly calculated by integrating Eq. (15) over an appropriate surface enclosing one of the superfluid regions and using Eq. (16). One finds, as expected, that E_J is proportional to the overlap of the order parameters ψ_A and ψ_B in the region separating the two superfluids.

In the two situations depicted in Fig. 1, this Josephson coupling (a) behaves as a weak link bridging the two superfluid domains along the line of closest approach or (b) has a radially symmetric form connecting two concentric superfluid shells, and its evaluation can be reduced to a one-dimensional problem along the appropriate direction. In fact, the equilibrium configuration given by Eq. (14) has a direct correspondence with the Ginzburg-Landau form for superconductors [2] and with the Gross-Pitaevksii (GP) form for a superfluid [3] trapped in a potential, given in this case by $\alpha(\mathbf{r})$. Hence we can use standard techniques for calculating the Josephson coupling for a one-dimensional system [10,12] and by employing the WKB approximation for the superfluid order parameters in the Mott region, we find

$$E_J \approx A_J \exp\left[-\int_C \sqrt{Q(\mathbf{r}')}d\mathbf{r}'\right], \quad (17)$$

where $Q(\mathbf{r}')=z^2Ja\alpha(\mathbf{r}')$. The contour C can be evaluated using the method of steepest descent and is the least-action path linking the two superfluids through the Mott-insulating barrier. Its end points correspond to the two turning points at the Mott-superfluid interface for A and B at which the function α vanishes. The constant A_J depends on the precise forms of ψ_A and ψ_B. As in the case of condensates in free space [12], A_J can be obtained by using a linearized potential approximation and matching the boundary condition imposed at the Mott-superfluid interface by Eq. (13).

From Eq. (17), a lower bound can be placed on the exponential dependence of the Josephson coupling by setting α to its maximum value of $1/(zJa^3)$ along the entire path C to obtain a value of $\exp(-\sqrt{z}\ell_{AB}/a)$, where ℓ_{AB} is the path length. Strikingly, to first order, the Josephson coupling is dominated in an exponential manner only by the path length between superfluid regions which in turn is determined by the potential landscape. We remark that for the Bose-Hubbard system, Eq. (17) represents an explicit derivation of the transport coefficient postulated in Ref. [1] on phenomenological grounds.

V. MOTT-CONDENSATE SHELLS

To demonstrate the above formalism and to obtain estimates of the bulk and Josephson energies for an alreadyrealized experimental system [5], we now consider $N=10^6$ ultracold ^{87}Rb atoms in a three-dimensional optical lattice of spacing $a=0.43$ μm (corresponding to a laser wavelength $\lambda=2a$), hopping parameter $J=h\times 120$ Hz, and on-site repulsion $U=h\times 10^4$ Hz confined by a harmonic trap $V(r)=br^2$ with $b=h\times 24$ Hz/μm^2. This system has an inner Mott core with two atoms per site surrounded by a superfluid shell (SFA), a Mott shell with $n=1$ atom per site (1-Mott), and finally an outer superfluid shell (SFB); the Josephson coupling between the SFA and SFB shells is mediated through the 1-Mott shell. Equation (5) can be solved to yield the boundaries of all the shells in the system. We label the boundary of SFA (SFB) with the n-Mott region by r_A (r_B).

To calculate the capacitive energy, E_C, one considers a transfer of a small number of particles from SFA to SFB, which leads to a change in the location of the regions' boundaries. Then, Eqs. (9)–(11) can be used to find the capacitive energy. Linearizing the external potential in each superfluid region (which is valid at small J/U since the shells are thin) we obtain the following expression for the capacitive energy for thin shell systems where the coupling is through the n-Mott region:

$$E_C=\frac{(ba^2)^{3/2}}{6\pi\sqrt{U}}\times\left[\frac{1}{(2n+1)^2\sqrt{\mu/U-n}}+\frac{1}{(2n-1)^2\sqrt{\mu/U-(n-1)}}\right]. \tag{18}$$

For the parameters detailed above, we find that $E_C\approx h\times 5\times 10^{-3}$ Hz.

The Josephson energy can be calculated using Eq. (16) after solving for the order parameter solutions near their respective boundaries using Eq. (13). Because Eq. (13) is identical to the GP equation for a Bose-Einstein condensate in an external trap, the solutions are as described in Ref. [12]. One finds that each order parameter decays near its Mott boundary with a characteristic decay length,

$$d_A=\sqrt[3]{J(n+1)a^2/q_A}, \tag{19}$$

$$d_B=\sqrt[3]{Jna^2/q_B}, \tag{20}$$

respectively, where $q_{A/B}=dV/dr|_{r_{A/B}}$ is the slope of the external potential at the boundary of each superfluid shell ($r_{A/B}$). In terms of these quantities, the constant in Eq. (17) is found to be

$$A_J=(\pi JA^2/z)\sqrt{n(n+1)}(r_Ar_Ba)/(d_Ad_B)^{3/2}, \tag{21}$$

where $A\approx 0.397$ [12]. After a numerical integration of the exponent in Eq. (16), for the system parameters detailed above we find $E_J\approx A_Je^{-28}\approx h\times 2\times 10^{-8}$ Hz.

From the energies above, we can predict that the thin shell system has Josephson oscillations in the strongly quantum regime ($E_J\ll E_C$ [10]) and that the Josephson plasmon frequency $\omega_{JP}=\sqrt{E_JE_C}\sim 10^{-4}$ Hz is quite small. This suggests that the system will be very slow (on the order of hours) to transfer particles between the two shells and that a phase difference initially present between the superfluids will remain for the duration of most current experiments. Whether the two shells have established a common phase can be ascertained via interference experiments [13].

VI. CONCLUDING REMARKS

In the Josephson effect, a difference in the complex phase of the macroscopic quantum wave function of two nearby superfluid regions leads to the exchange of particles between the regions. Previous examples of this effect have involved superfluids that are separated by vacuum or by a foreign material. Here, we have considered the physics of a phase-separated system where regions of condensed bosons are coupled through Mott-insulating regions of the same bosons. Such systems can arise whenever bosons described by the Bose-Hubbard model are subject to an external confining potential and the tunneling strength is sufficiently small. In cold atom systems, such inhomogeneities are generic, owing to the presence of the confining trap. Our analysis is likely to also be relevant to granular and high-T_c systems where superconductivity exists in the presence of mesoscopic inhomogeneities.

We have provided a description of the spatial profile of the coexistent Mott-superfluid system, including the decay of the condensate order parameter into the Mott phase. We have derived explicit equations for the two energy scales, E_J and E_C, governing Josephson oscillations in these systems and have arrived at estimates for their values in the case of trapped ultracold bosons in an optical lattice. A complete description of such coexistent Mott insulating and superfluid phases in inhomogeneous systems will need to go beyond mean-field treatments and will likely need to include effects of finite temperature and of dissipation, for instance, due to quasiparticle excitations. We hope that our initial discussion here generates interest in pursuing these avenues.

Our analyses of the spherically symmetric optical lattice geometry suggest that the Josephson coupling between superfluid shells in typical three-dimensional traps will be vanishingly small when J/U is small. Our results confirm that the shells will be essentially independent of each other and this should have consequences for the dynamics of the system as J/U is changed [14]. These results are particularly important to proposals where bosons in the Mott state act as units of quantum information and where having minimal number fluctuation is essential; our analyses can be used to show the extent to which the proximity to a superfluid phase can influence the fidelity of the Mott state.

With regards to actually observing Josephson physics mediated via the Mott insulator, our analyses provide expressions from which the conditions for significant Josephson coupling can be deduced. Because the Josephson energy is exponentially dependent on the distance between the coupled superfluid regions, it should be possible to obtain a significantly larger Josephson coupling in the case of a random (or pseudorandom) external potential yielding the geometry illustrated in Fig. 1(a), where the distance between superfluid regions could be more easily tuned. Furthermore, a knowledge of the spatial profile in disordered coexistent condensed matter systems where Josephson physics is known to be significant, when combined with our analyses, ought to provide realistic estimates of the Josephson coupling which has thus far been treated as a phenomenological parameter.

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