Judgment Aggregation by Quota Rules: Majority Voting Generalized

Franz Dietrich

Department of Quantitative Economics

University of Maastricht

and

Christian List

Department of Government

London School of Economics

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Address for correspondence:

C. List

Department of Government

London School of Economics

London WC2A 2AE, UK

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Abstract: The widely discussed 'discursive dilemma' shows that majority voting in a group of individuals on logically connected propositions may produce irrational collective judgments. We generalize majority voting by considering quota rules, which accept each proposition if and only if the number of individuals accepting it exceeds a given threshold, where different thresholds may be used for different propositions. After characterizing quota rules, we prove necessary and sufficient conditions on the required thresholds for various collective rationality requirements. We also consider sequential quota rules, which ensure collective rationality by adjudicating propositions sequentially and letting earlier judgments constrain later ones. Sequential rules may be path-dependent and strategically manipulable. We characterize path-independence and prove its essential equivalence to strategy-proofness. Our results shed light on the rationality of simple-, super-, and sub-majoritarian decision-making.

Keywords: Judgment aggregation, quota rules, simple-, super- and sub-majority voting, collective rationality, path-dependence, strategy-proofness

1 Introduction

How can a group of individuals make collective judgments on some logically connected propositions based on the individuals' judgments on these propositions? This problem arises in many different collective decision-making bodies, such as legislative committees, multi-member courts, expert panels and monetary policy committees. A natural way to make collective judgments on a given set of propositions is to take a majority vote on each proposition. But a simple example illustrates that propositionwise majority voting does not guarantee 'rational' collective judgments. Suppose a three-member government has to make judgments on the following propositions:

a : Country X has weapons of mass destruction (hereafter WMD).

b : Action Y should be taken against country X.

$b \leftrightarrow a$: Action Y should be taken against country X if and only if country X has WMD.

Readers can make their preferred substitutions for X and Y. Suppose further that the judgments of the three government members are as shown in table 1, each individually consistent.

	a	$b \leftrightarrow a$	b
Individual 1	True	True	True
Individual 2	False	False	True
Individual 3	False	True	False
Majority	False	True	True

Table 1

Then a majority rejects a (i.e. holds that country X does not have WMD); a majority accepts $b \leftrightarrow a$ (i.e. holds that action Y should be taken if and only if country X has WMD); and yet a majority accepts b (i.e. holds that action Y should be taken), an inconsistent set of collective judgments. Problems of this kind are sometimes called 'discursive dilemmas' (Pettit 2001). Can we modify propositionwise majority voting so as to avoid such problems?

In this paper, we discuss a general class of judgment aggregation rules: *quota rules*. Here a proposition is collectively accepted if and only if the number of individuals accepting it is greater than or equal to some threshold, which may depend on the proposition in question. Propositionwise majority voting is a special quota rule with a simple majority threshold for every proposition. Generally, as propositions may differ in status and importance, the threshold may vary from proposition to proposition. In many real-world decision-making bodies, a higher acceptance threshold is required for more important propositions (e.g. constitutional amendments or taking action Y against country X) than for less important ones (e.g. ordinary legislation).

After characterizing the class of quota rules, we prove necessary and sufficient conditions under which a quota rule meets various requirements of 'collective rationality'. We discuss each of the following rationality conditions, defined formally below: weak and strong consistency, which require the collective judgments to be free from certain logical contradictions; deductive closure, which requires the group to accept the logical implications of its collective judgments; and completeness, which requires the group to form a determinate judgment on every proposition under consideration. We show that the agenda of propositions under consideration determines whether each of these conditions can be met. If the interconnections between the propositions are above a certain complexity, no quota rule guarantees full 'collective rationality'.

So how can rational collective judgments be achieved? In the real world, groups often consider different propositions not simultaneously, but sequentially, letting earlier judgments constrain later ones. Under a *sequential quota rule*, a group considers different propositions in a sequence and takes a vote (applying the relevant acceptance threshold) only on those propositions on which the judgments are not yet constrained by earlier judgments. Sequential quota rules guarantee collective consistency by design (and sometimes completeness and deductive closure), but may be *path-dependent*: the order in which the propositions are considered may affect the outcome. We show that a sequential quota rule is path-*in*dependent if and only if its corresponding ordinary quota rule is collectively rational in a relevant sense, which illustrates that path-independence is a demanding condition.

Path-dependence matters for two reasons. First, path-dependent sequential rules are obviously vulnerable to manipulation by changes of the decision-path. Second, and less obviously, path-dependent sequential rules are vulnerable to strategic voting. We show that strategy-proofness of a sequential quota rule is essentially equivalent to its path-independence. Our findings show that groups forming collective judgments on multiple propositions may face a trade-off between democratic responsiveness, collective rationality and strategy-proofness. At the end of the paper, we extend our main characterization result to a larger class of aggregation rules beyond quota rules.

The problem of judgment aggregation was formalized by List and Pettit (2002, 2004), who also proved a first impossibility theorem. Further impossibility results were proved by Pauly and van Hees (2006), van Hees (2007), Dietrich (2006, 2007), Gärdenfors (2006), Nehring and Puppe (2005) and Dietrich and List (2005). List (2003), Dietrich (2006) and Pigozzi (2006) proved possibility results. Nehring and Puppe (2002, 2005) investigated the related framework of 'property spaces' and proved a characterization of collective consistency similar to the one given here. We advance beyond their contribution by considering several other rationality conditions discussed in the literature over and above consistency – most importantly deductive closure – and by considering sequential aggregation rules. Path-dependence and strategy-proofness in judgment aggregation were discussed in List (2004) and Dietrich and List (2004), but not with respect to quota rules. We advance beyond the latter contributions by fully characterizing path-(in)dependence and strategy-proofness of sequential quota rules.[1]

Within political science, there has recently been a renewed interest in both supermajority voting (e.g. Goodin and List 2006) and submajority voting (e.g. Vermeule 2005). Earlier related contributions were papers by Craven (1971) and Ferejohn and Grether (1974) on conditions under which supermajority rules for preference aggreg-

[1] Other contributions related to judgment aggregation include Wilson's (1975) and Rubinstein and Fishburn's (1986) works on abstract aggregation theory and Konieczny and Pino-Perez's (2002) work on belief merging in computer science.

ation satisfy minimal requirements of collective rationality (specifically, acyclicity). Our results not only provide a broader theoretical background to the recent debates on super- and submajority voting, but they also generalize the earlier results by Craven as well as Ferejohn and Grether. All proofs are given in an appendix.

2 The model of judgment aggregation

We begin by introducing the key definitions of the judgment aggregation model. Let $N = \{1, 2, \ldots, n\}$ be a group of two or more individuals that seeks to make collective judgments on some logically connected propositions.

2.1 The propositions

Propositions are represented in formal logic.[2] Our examples use standard propositional logic, where the propositional language $\mathbf{L}$ contains

- a given set of *atomic propositions* a, b, c, ... without logical connectives, such as the proposition that country X has WMD and the proposition that action Y should be taken against country X, and
- *compound propositions* with the logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if-then), $\leftrightarrow$ (if and only if), such as the proposition that action Y should be taken against country X if and only if country X has WMD.[3]

The logical framework gives us the notions of *consistency* and *entailment*. In standard propositional logic, these are defined as follows. Let $S \subseteq \mathbf{L}$ be a set of propositions and let $p \in \mathbf{L}$ be a proposition.

- S is *consistent* if there exists a truth-value assignment for which all the propositions in S are true, and *inconsistent* otherwise;[4]

[2] We can use any logic $\mathbf{L}$ satisfying conditions L1-L4 in Dietrich (2007). Apart from propositional logic, this permits more expressive logics, including predicate, modal, conditional and deontic logics. Real-life judgment aggregation problems and disagreements in groups often involve propositions that contain not only classical logical connectives (not, and, or ...), but also non-classical ones such as subjunctive implications (if it were the case that p, it would be the case that q) or modal operators (it is necessary/possible that p), where the modality could be of a logical, physical, ethical or other kind. Our results apply to the logics of most realistic propositions.

[3] Formally, $\mathbf{L}$ is the smallest set such that a, b, c, ... $\in \mathbf{L}$ and if $p, q \in \mathbf{L}$ then $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$, $(p \leftrightarrow q) \in \mathbf{L}$. For notational simplicity, we drop external brackets around propositions.

[4] Formally, a *truth-value assignment* is a function assigning the value 'true' or 'false' to each

- S *entails* p if, for all truth-value assignments for which all the propositions in S are true, p is also true.

Examples of consistent sets are $\{b \leftrightarrow a, \neg a\}$ and $\{\neg(a \wedge b), \neg a, b\}$, examples of inconsistent ones $\{b \leftrightarrow a, \neg a, b\}$ and $\{a \wedge b, \neg a, \neg b\}$. Also, $\{b \leftrightarrow a, \neg a\}$ entails $\neg b$, whereas $\{a, \neg b\}$ does not entail $a \wedge b$.

Further, we say that S is *minimal inconsistent* if S is inconsistent, but every proper subset of S is consistent; we discuss the significance of this notion below. To illustrate, $\{b \leftrightarrow a, \neg a, b\}$ is minimal inconsistent (it becomes consistent as soon as one of the propositions is removed), whereas $\{a \wedge b, \neg a, \neg b\}$ is not (it remains inconsistent even if $\neg a$ or $\neg b$ are removed).

The *agenda* is the set of propositions on which judgments are to be made; it is a finite non-empty subset $X \subseteq \mathbf{L}$ consisting of proposition-negation pairs $p, \neg p$ (with p not a negated proposition) and containing no tautologies (propositions whose negations are inconsistent) or contradictions (propositions that are inconsistent by themselves).[5] We assume that double-negations cancel each other out, i.e. $\neg\neg p$ stands for p.[6] In the example above, the agenda is $X := \{a, b, b \leftrightarrow a, \neg a, \neg b, \neg(b \leftrightarrow a)\}$.[7]

2.2 Individual and collective judgment sets

Each individual i's *judgment set* is a subset $A_i \subseteq X$, where $p \in A_i$ means 'individual i accepts proposition p'. A *profile (of individual judgment sets)* is an n-tuple $(A_1, \ldots, A_n)$ of judgment sets across individuals.

A *(judgment) aggregation rule* is a function F that assigns to each profile $(A_1, \ldots, A_n)$ in a given domain a *collective judgment set* $F(A_1, \ldots, A_n) = A \subseteq X$, where $p \in A$ means 'the group accepts proposition p'.

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proposition in $\mathbf{L}$ such that, for any $p,q \in \mathbf{L}$, $\neg p$ is true if and only if p is false; $p \wedge q$ is true if and only if both p and q are true; $p \vee q$ is true if and only if at least one of p or q is true; $p \rightarrow q$ is true if and only if p is false or q is true; $p \leftrightarrow q$ is true if and only if p and q are both true or both false.

[5] Most of our results, including parts (c) and (d) of theorems 2 and 5, do not require the latter restrictions, i.e. they also hold for agendas containing tautologies and contradictions.

[6] Hereafter we use $\neg$ to represent a modified negation symbol $\sim$, where $\sim p := \neg p$ if p is not a negated proposition and $\sim p := q$ if $p = \neg q$ for some proposition q.

[7] Although for simplicity we use standard propositional logic for representing this agenda in this paper, one could argue that the biimplication $b \leftrightarrow a$ should be modelled as a *subjunctive* biimplication, not as a *material* one as done here. Then the negation $\neg(b \leftrightarrow a)$ becomes consistent with *any* judgments on a and b, also with $\{a, b\}$ and with $\{\neg a, \neg b\}$. This has consequences; for instance, the agenda then escapes the impossibility of corollary 3 below.

We introduce several rationality conditions on a judgment set A (individual or collective):

- A is *complete* if it contains at least one member of each pair $p, \neg p \in X$;
- A is *weakly consistent* if it contains at most one member of each pair $p, \neg p \in X$;
- A is *consistent* if it is a consistent set of propositions, as defined in the logic;
- A is *deductively closed* if any proposition $p \in X$ entailed by a consistent subset $B \subseteq A$ is also contained in A.

These rationality conditions are interrelated as follows.

Lemma 1 *For any judgment set A,*

(a) *consistency implies weak consistency;*

(b) *given deductive closure, consistency is equivalent to weak consistency;*

(c) *given completeness, consistency is equivalent to the conjunction of weak consistency and deductive closure.*

A judgment set is *fully rational* if it is complete and consistent (hence also weakly consistent and deductively closed, by lemma 1). We call the set of all possible profiles of fully rational judgment sets the *universal domain*. Finally, we call an aggregation rule *fully rational* (or *complete*, *weakly consistent*, *consistent*, *deductively closed*) if it generates, for every profile in its domain, a fully rational (or complete, weakly consistent, consistent, deductively closed) collective judgment set.

2.3 Preference aggregation as a special case

To illustrate the generality of the judgment aggregation model, let us briefly explain how preference aggregation problems in the tradition of Condorcet, Arrow and Sen can be represented in it. Readers who wish to move on to our main results may skip this subsection.

To represent preference aggregation problems within the judgment aggregation model, take a simple predicate logic $\mathbf{L}$ with a set of two or more constants $K = \{x, y, ...\}$ representing options and a two-place predicate P representing (strict) preference, where, for any $x, y \in K$, xPy is interpreted as 'x is preferable to y'. To capture the structure of preference, define a set $S \subseteq \mathbf{L}$ to be *consistent* if $S \cup Z$ is consistent in the standard sense of predicate logic, where Z is the set of rationality conditions on preferences, i.e. asymmetry, transitivity and connectedness.[8] Now define the agenda to be $X = \{xPy, \neg xPy \in \mathbf{L} : x, y \in K \text{ with } x \neq y\}$. (For technical details, see Dietrich and List 2005; also List and Pettit 2004.)

Under this construction, each fully rational judgment set $A_i \subseteq X$ uniquely represents a fully rational (i.e. asymmetrical, transitive and connected) preference ordering $\succ_i$ on the set of options K, where, for any $x, y \in K$, $xPy \in A_i$ if and only if $x \succ_i y$. For example, if there are three options x, y and z, the preference ordering $x \succ_i y \succ_i z$ is represented by the judgment set $A_i = \{xPy, yPz, xPz, \neg yPx, \neg zPy, \neg zPx\}$. Now a judgment aggregation rule uniquely represents an Arrowian preference aggregation rule. In particular, a fully rational judgment aggregation rule represents a social welfare function as defined by Arrow.

Below we note that, by representing preference aggregation problems in the judgment aggregation model, we obtain some classic results by Craven (1971) and Ferejohn and Grether (1974) on supermajority rules for preference aggregation as corollaries of our results on quota rules for judgment aggregation.

3 Quota rules and collective rationality

We first define and characterize the class of quota rules for judgment aggregation. Then we prove necessary and sufficient conditions under which a quota rule satisfies various collective rationality conditions. Our results generalize the 'discursive dilemma'.

3.1 Quota rules

For each proposition $p \in X$, let an *acceptance threshold* $m_p \in \{1, 2, ..., n\}$ be given. Given a family $(m_p)_{p \in X}$ of such thresholds for the propositions in the agenda, the corresponding *quota rule* is the aggregation rule $F_{(m_p)_{p \in X}}$ with universal domain defined as follows. For each profile $(A_1, \ldots, A_n)$,

$$F_{(m_p)_{p \in X}}(A_1, ..., A_n) := \{p \in X : |\{i \in N : p \in A_i\}| \geq m_p\}.$$

Informally, each proposition $p \in X$ is collectively accepted if and only if it is accepted by at least m_p individuals. Since the total number of individuals n is fixed for our

[8] Formally, $Z := \{(\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1),\ (\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3),\ (\forall v_1)(\forall v_2)(v_1 \neq v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1))\}$.

analysis, this is equivalent to saying that p is collectively accepted if and only if it is accepted by a proportion of at least $\frac{m_p}{n}$ of the individuals.

We call a quota rule $F_{(m_p)_{p \in X}}$ $(= F_m)$ *uniform* if the acceptance threshold is the same for all propositions, i.e. $m_p = m$ for all $p \in X$. Examples of uniform quota rules are

- *propositionwise majority rule*, where $m = \lceil (n+1)/2 \rceil$, with $\lceil x \rceil$ defined as the smallest integer greater than or equal to x,
- *propositionwise special majority rule*, where $\lceil (n+1)/2 \rceil < m < n$, and
- *propositionwise unanimity rule*, where $m = n$.

To characterize the class of quota rules, let us introduce the following conditions.

Universal domain. The domain of F is the universal domain, i.e. the set of all possible profiles of fully rational individual judgment sets.

Anonymity. For every two profiles $(A_1, ..., A_n)$, $(A_{\pi(1)}, ..., A_{\pi(n)})$ in the domain of F, where $\pi : N \mapsto N$ is any permutation of the individuals, $F(A_1, ..., A_n) = F(A_{\pi(1)}, ..., A_{\pi(n)})$.

Responsiveness. For every proposition $p \in X$, there exist at least two profiles $(A_1, ..., A_n)$, $(A_1^*, ..., A_n^*)$ in the domain of F such that $p \in F(A_1, ..., A_n)$ and $p \notin F(A_1^*, ..., A_n^*)$.

Independence. For every proposition $p \in X$ and profiles $(A_1, ..., A_n)$, $(A_1^*, ..., A_n^*)$ in the domain of F, if [for all individuals i, $p \in A_i$ if and only if $p \in A_i^*$], then [$p \in F(A_1, ..., A_n)$ if and only if $p \in F(A_1^*, ..., A_n^*)$].

Monotonicity. For every proposition $p \in X$, individual i, and pair of i-variant profiles $(A_1, ..., A_n)$, $(A_1, ..., A_i^*, ..., A_n)$ in the domain of F with $p \notin A_i$ and $p \in A_i^*$, if $p \in F(A_1, ..., A_n)$ then $p \in F(A_1, ..., A_i^*, ..., A_n)$. (Two profiles are *$i$-variants* of each other if they coincide for all individuals except possibly i.)

Universal domain states that every possible profile of fully rational individual judgment sets is admissible. Anonymity requires giving equal consideration to all individuals' judgment sets. Responsiveness rules out that some proposition in the agenda is never accepted or never rejected. Independence requires propositionwise aggregation, i.e. the collective judgment on a proposition depends only on the individuals' judgments on that proposition and not on their judgments on other propositions. Monotonicity requires that an additional individual's support for an accepted proposition does not lead to the rejection of that proposition. Our first theorem shows that these five conditions uniquely characterize the class of quota rules.

Theorem 1 *An aggregation rule has universal domain, is anonymous, responsive, independent and monotonic if and only if it is a quota rule $F_{(m_p)_{p \in X}}$ for some family of thresholds $(m_p)_{p \in X}$.*[9]

3.2 Necessary and sufficient conditions for collective rationality

We began with the observation that propositionwise majority voting does not guarantee rational collective judgments: the 'discursive dilemma'. Quota rules generalize propositionwise majority voting by allowing any family of thresholds $(m_p)_{p \in X}$ instead of the same simple-majority threshold $m = \lceil (n+1)/2 \rceil$ for all propositions. Can we specify the thresholds such that the corresponding quota rule guarantees collective rationality?

Theorem 2 *A quota rule $F_{(m_p)_{p \in X}}$ is*

(a) *complete if and only if*

$$m_p + m_{\neg p} \leq n + 1 \text{ for every pair } p, \neg p \in X; \tag{1}$$

(b) *weakly consistent if and only if*

$$m_p + m_{\neg p} > n \text{ for every pair } p, \neg p \in X; \tag{2}$$

(c) *consistent if and only if*

$$\sum_{p \in Z} m_p > n(|Z| - 1) \text{ for every minimal inconsistent set } Z \subseteq X; \tag{3}$$

(d) *deductively closed if and only if*

$$\sum_{p \in Z \setminus \{q\}} m_p - m_{\neg q} \geq n(|Z| - 2) \quad \text{for every minimal inconsistent set } Z \subseteq X \text{ and every } q \in Z. \tag{4}$$

[9] The result remains true if responsiveness is dropped and each m_p is a member of $\{0, ..., n+1\}$ (rather than $\{1, ..., n\}$), which permits degenerate quota rules.

Nehring and Puppe (2002, 2005) have proved a result similar to part (c) of theorem 2 using an 'intersection property', which, in turn, generalizes an earlier 'intersection property' identified by Barberà et al. (1997). We discuss Nehring and Puppe's 'intersection property' for consistency in the penultimate section of this paper and identify a new 'intersection property' for deductive closure.

Theorem 2 shows that each of the four rationality conditions is guaranteed by a particular system of inequalities on the acceptance thresholds. The intuition behind parts (a) and (b) is obvious: here the inequalities guarantee that at least one proposition (in part (a)), or at most one proposition (in part (b)), from each proposition-negation pair is collectively accepted.

To illustrate part (c), notice that each minimal inconsistent subset of X represents one particular way in which a set of propositions from the agenda can be inconsistent. For instance, if the agenda includes only logically independent propositions and their negations, then the only inconsistencies that can ever arise are ones between a proposition and its negation; here the only minimal inconsistent subsets are proposition-negation pairs. By contrast, if there are richer logical connections between the propositions in the agenda, more subtle inconsistencies can arise, such as an inconsistency between $b \leftrightarrow a$, $\neg a$ and b in our initial example; here the agenda has larger minimal inconsistent subsets such as $\{b \leftrightarrow a, \neg a, b\}$. In light of this, the system of inequalities in part (c) can be interpreted as follows. For each minimal inconsistent subset of the agenda (i.e. for each particular way in which an inconsistency can arise), the system of inequalities contains a corresponding inequality that rules out that particular inconsistency. Moreover, this inequality states that the sum of the acceptance thresholds across the propositions in the given minimal inconsistent set must be sufficiently large. Obviously, the larger the required sum of the thresholds, the harder it is for these propositions to be simultaneously accepted; and if a sufficiently large sum is required, they can never be simultaneously accepted, which means that the inconsistency in question cannot arise. Part (d), finally, can be interpreted analogously.

How difficult is it to satisfy the systems of inequalities? For example, to see how strongly the condition of consistency restricts the thresholds $(m_p)_{p \in X}$, note that the inequality in (3) is equivalent to $\frac{1}{|Z|}\sum_{p \in Z} m_p > n(1-1/|Z|)$. So the average threshold m_p for the acceptance of p (averaging over $p \in Z$) must exceed $n(1-1/|Z|)$. This value approaches n as the size of a minimal inconsistent set Z increases, which illustrates that, for non-trivial agendas, supermajoritarian decision-making is usually needed to guarantee collective consistency.

By combining the inequalities in theorem 2, we obtain conditions under which a quota rule satisfies two or more of the rationality conditions simultaneously.

Corollary 1 *A quota rule $F_{(m_p)_{p\in X}}$ is*

(a) *complete and weakly consistent if and only if*

$$m_p + m_{\neg p} = n + 1 \text{ for every pair } p, \neg p \in X;$$

(b) *consistent and deductively closed if and only if*

$$\sum_{p\in Z\backslash\{q\}} m_p + \min\{m_q, n+1-m_{\neg q}\} > n(|Z|-1) \quad \begin{array}{l}\text{for every minimal inconsistent set } Z \subseteq X \\ \text{and every } q \in Z;\end{array} \tag{5}$$

(c) *fully rational if and only if*

$$\begin{array}{ll} m_p + m_{\neg p} = n+1 & \text{for every pair } p, \neg p \in X, \text{ and} \\ \sum_{p\in Z} m_p > n(|Z|-1) & \text{for every minimal inconsistent set } Z \subseteq X. \end{array} \tag{6}$$

3.3 The special case of uniform quota rules

As noted above, an important special class of quota rules are the uniform ones, where the acceptance threshold is the same for all propositions. Propositionwise majority rule is the most prominent example. Here the inequalities characterizing consistency and deductive closure reduce to some simple conditions.

Corollary 2 *Let z be the size of the largest minimal inconsistent set $Z \subseteq X$.*

(a) *A uniform quota rule F_m is consistent if and only if $m > n - n/z$.*[10] *In particular, for $n \neq 2, 4$, propositionwise majority rule (where $m = \lceil (n+1)/2 \rceil$) is consistent if and only if $z \leq 2$; if $n = 2$, it is always consistent; if $n = 4$, it is consistent if and only if $z \leq 3$.*

(b) *A uniform quota rule F_m is deductively closed if and only if $m = n$ (i.e. F_m is propositionwise unanimity rule) or $z \leq 2$. In particular, if $n \geq 3$, propositionwise majority rule is deductively closed if and only if $z \leq 2$; if $n = 2$, it is always deductively closed.*

(c) *A uniform quota rule F_m is consistent and deductively closed if and only if $m = n$ (i.e. it is propositionwise unanimity rule) or [$z \leq 2$ and $m > n/2$]. In*

[10] This generalizes a result on the consistency of supermajority rules in List (2001, ch. 9); see also List and Pettit (2002).

particular, if $n \geq 3$, propositionwise majority rule is consistent and deductively closed if and only if $z \leq 2$; if $n = 2$, it is always consistent and deductively closed.

We have already explained why we must consider minimal inconsistent subsets of the agenda to obtain necessary and sufficient conditions for the rationality of a quota rule. Further, we can interpret the size of the largest minimal inconsistent subset of the agenda as a simple indicator of how complex the logical interconnections between the propositions in the agenda are. For example, $z \leq 2$ corresponds to an agenda without any non-trivial interconnections, i.e. without any minimal inconsistent sets of more than two propositions, whereas $z > 2$ corresponds to an agenda with richer interconnections.

Propositionwise unanimity rule is always consistent and deductively closed, at the expense of significant incompleteness. By contrast, propositionwise special majority rule is consistent if and only if the acceptance threshold for every proposition exceeds $n(1 - 1/z)$, which approaches 1 as z increases, and it is deductively closed only in the special case $z \leq 2$. Propositionwise majority rule (when $n \geq 3$) is consistent and deductively closed only in the special case $z \leq 2$. These results generalize the 'discursive dilemma' with which we began.

As announced above, we can obtain a classic result on preference aggregation as a corollary of our present results. Craven (1971) and Ferejohn and Grether (1974) have shown that pairwise supermajority rules for preference aggregation guarantee acyclic collective preferences if the supermajority threshold is greater than $n - n/k$, where k is the number of options. Under the representation of preference aggregation problems in the judgment aggregation model, this result follows from part (a) of corollary 2. To see this, notice that if the agenda is $X = \{xPy, \neg xPy \in \mathbf{L} : x, y \in K \text{ with } x \neq y\}$ as defined above, then the largest minimal inconsistent subset of X is a set of k binary preference propositions representing a preference cycle of length k, where k is the number of options in K.[11] Thus the size of the largest minimal inconsistent subset $Z \subseteq X$ is simply $z = k$. Also, notice that, for the given agenda, a consistent judgment set $A \subseteq X$ precisely represents an acyclic preference ordering $\succ$ on the set of options K. So Craven's and Ferejohn and Grether's result follows immediately.

[11] For $k > 2$, this largest minimal inconsistent subset of X is not unique, as there can be different cycles of length k.

3.4 A general (im)possibility result

By combining theorems 1 and 2, we can characterize the types of agendas X for which there exist fully rational aggregation rules that satisfy the conditions introduced in the previous section.

Corollary 3 *An aggregation rule with universal domain is anonymous, responsive, independent, monotonic and fully rational if and only if it is a quota rule* $F_{(m_p)_{p\in X}}$ *satisfying* (6) *above. In particular, there exists an aggregation rule with these properties if and only if the system* (6) *admits a solution* $(m_p)_{p\in X}$ *in* $\{1,...,n\}^X$.

This corollary can be seen as an impossibility result: the (in)equalities in (6) have solutions only for special agendas with few logical connections between propositions.[12]

3.5 An example

For the agenda $X := \{a, b, b \leftrightarrow a, \neg a, \neg b, \neg(b \leftrightarrow a)\}$ from our initial example, the minimal inconsistent subsets $Z \subsetneq X$ are $\{a, \neg a\}$, $\{b, \neg b\}$, $\{b \leftrightarrow a, \neg(b \leftrightarrow a)\}$, $\{a, \neg b, b \leftrightarrow a\}$, $\{\neg a, b, b \leftrightarrow a\}$, $\{a, b, \neg(b \leftrightarrow a)\}$ and $\{\neg a, \neg b, \neg(b \leftrightarrow a)\}$. We show that there exists no fully rational quota rule for this agenda. Assume, for a contradiction, that $F_{(m_p)_{p\in X}}$ is fully rational. Then, by part (c) of corollary 1,

$$m_a + m_{\neg a} = m_b + m_{\neg b} = m_{b\leftrightarrow a} + m_{\neg(b\leftrightarrow a)} = n + 1, \tag{7}$$

$$m_a + m_{\neg b} + m_{b\leftrightarrow a} > 2n \text{ and } m_{\neg a} + m_b + m_{b\leftrightarrow a} > 2n, \tag{8}$$

$$m_a + m_b + m_{\neg(b\leftrightarrow a)} > 2n \text{ and } m_{\neg a} + m_{\neg b} + m_{\neg(b\leftrightarrow a)} > 2n. \tag{9}$$

By adding the two inequalities in (8), we obtain $m_a + m_{\neg a} + m_b + m_{\neg b} + 2m_{b\leftrightarrow a} > 4n$. By (7), $n+1+n+1+2m_{b\leftrightarrow a} > 4n$, hence $2m_{b\leftrightarrow a} > 2n-2$, i.e. $m_{b\leftrightarrow a} = n$. An analogous argument for the two inequalities in (9) yields $m_{\neg(b\leftrightarrow a)} = n$. So $m_{b\leftrightarrow a} + m_{\neg(b\leftrightarrow a)} = 2n > n + 1$, which violates (7).

But, for a slightly modified agenda, there is a fully rational quota rule. Replace the biconditional $b \leftrightarrow a$ (action should be taken if and only if country X has WMD) by the simple conditional $a \rightarrow b$ (if country X has WMD, then action Y should be taken). The new agenda is thus $X := \{a, b, a \rightarrow b, \neg a, \neg b, \neg(a \rightarrow b)\}$. The minimal inconsistent sets $Z \subseteq X$ are now $\{a, \neg a\}$, $\{b, \neg b\}$, $\{a \rightarrow b, \neg(a \rightarrow b)\}$, $\{\neg a, \neg(a \rightarrow b)\}$,

[12] As shown by Nehring and Puppe (2002) in the framework of 'property spaces', the existence of fully rational quota rules can also be elegantly characterized in terms of certain conditional entailment relations within the agenda.

$\{b, \neg(a \to b)\}$ and $\{a, \neg b, a \to b\}$. By part (c) of corollary 1, a quota rule $F_{(m_p)_{p\in X}}$ is fully rational if and only if

$$m_a + m_{\neg a} = m_b + m_{\neg b} = m_{a\to b} + m_{\neg(a\to b)} = n + 1,$$
$$m_{\neg a} + m_{\neg(a\to b)} > n \text{ and } m_b + m_{\neg(a\to b)} > n \text{ and } m_a + m_{\neg b} + m_{a\to b} > 2n.$$

By expressing each $m_{\neg p}$ as $n + 1 - m_p$, the three inequalities become

$$-m_a + m_{\neg(a\to b)} > -1 \text{ and } -m_{\neg b} + m_{\neg(a\to b)} > -1 \text{ and } m_a + m_{\neg b} - m_{\neg(a\to b)} > n - 1;$$

equivalently,

$$m_{\neg(a\to b)} \geq m_a \text{ and } m_{\neg(a\to b)} \geq m_{\neg b} \text{ and } m_{\neg(a\to b)} \leq m_a + m_{\neg b} + 1.$$

The only solution to these inequalities in $\{1, ..., n\}^X$ is $m_a = m_{\neg b} = m_{\neg(a\to b)} = n$, i.e. a unanimity threshold for each of a, $\neg b$ and $\neg(a \to b)$ and a threshold of 1 for each of $\neg a$, b and $a \to b$. So, in our example, the proposition that country X has WMD is accepted only if all individuals accept that proposition, whereas the proposition that action Y should be taken and the proposition that WMD require action are each accepted as soon as they are accepted by just one individual, a questionable aggregation rule.

Further, for the original agenda and also the modified one, the size of the largest minimal inconsistent set is $z = 3$, so by corollary 2 a uniform quota rule F_m is consistent if and only if $m > \frac{2}{3}n$ and deductively closed if and only if $m = n$. By implication, for both agendas, there exists no fully rational uniform quota rule.

3.6 The computational usefulness of the inequalities

Apart from giving us theoretical insights into when quota rules satisfy various rationality conditions, the inequalities in theorem 2 and its corollaries are also computationally useful. First, suppose we wish to verify whether a *given* quota rule $F_{(m_p)_{p\in X}}$ satisfies some rationality condition. Without theoretical results, we would have to consider every profile in the universal domain and determine whether the collective judgment set for that profile satisfies the required condition. The number of such profiles grows exponentially in the group size n (of course, it also depends on the structure of the agenda). By contrast, the number of inequalities in each part of theorem 2 does not depend on n; it is determined only by the structure of the agenda. So, by using our inequalities, verifying the rationality of a given quota rule is computationally feasible even for large group sizes.

Second, suppose we wish to verify, for a given agenda and a given number of individuals, whether there *exists* a fully rational quota rule. Even for a small n, this task is computationally hard. There are n^k possible quota rules for n individuals and k propositions, and, for each of these n^k rules, we would have to consider every possible profile and check the rationality of the outcome under that profile, where the number of such profiles grows exponentially in n. But, if we use corollary 3, the problem reduces to verifying whether the system of linear (in)equalities (6) admits a solution $(m_p)_{p \in X}$ in $\{1, ..., n\}^X$. This is a computationally feasible task; it is a problem of linear programming and can be solved using the well-known simplex algorithm (for an overview, see Dantzig 2002).

4 Sequential quota rules and path-dependence

We have seen that, for agendas above a certain complexity, there exists no fully rational quota rule. A group can solve this problem by making judgments on multiple propositions sequentially, letting earlier judgments constrain later ones. We now consider the class of sequential quota rules, which are always consistent, but may be path-dependent. After formally defining sequential quota rules and giving an example of path-dependence, we prove necessary and sufficient conditions for the avoidance of path-dependence. In the subsequent section, we address the relation between path-dependence of a sequential quota rule and its manipulability by strategic voting.

4.1 Sequential quota rules

Let us begin by introducing the concept of a *sequential quota rule* informally, generalizing the approach in List (2004). Under such a rule, the group considers the propositions in the agenda not simultaneously, but one by one in a given sequence. This sequence may reflect either the temporal order in which the propositions come up for consideration or some order of priority among the propositions. For each new proposition considered in the sequence, if the proposition is logically unconstrained by propositions accepted earlier, then the group takes a vote on the new proposition, applying the appropriate acceptance threshold. But if the new proposition is logically constrained by propositions accepted earlier, then the group derives its judgment on it from its earlier judgments.

On a temporal interpretation, a sequential quota rule captures two important characteristics of many real-world decision processes. The first characteristic is that real-world collective decision-making bodies often consider different propositions at different points in the time. The second one is that, in such decision processes, earlier decisions often constrain later ones: for example, earlier decisions may create legal or other commitments that cannot subsequently be overruled, for instance when earlier decisions are legally binding, serve as precedents, or are simply too costly to reverse.

On a priority interpretation, a sequential quota rule captures the fact that collective decision processes are often highly structured and implement a collective reasoning process. Some propositions may serve as premises for others such that judgments on premises are made before judgments on conclusions are derived.

Formally, a *decision-path* is a one-to-one function $\Omega : \{1, 2, ..., k\} \to X$, with $k = |X|$, where $p_1 := \Omega(1)$, $p_2 := \Omega(2)$, ..., $p_k := \Omega(k)$ are the propositions considered first, second, ..., last in the sequence. Given a decision-path Ω and a family of thresholds $(m_p)_{p \in X}$, a *sequential quota rule* $F_{\Omega,(m_p)_{p \in X}}$ is the aggregation rule with universal domain defined as follows. For each profile $(A_1, \ldots, A_n)$,

$$F_{\Omega,(m_p)_{p \in X}}(A_1, ..., A_n) := \Phi_k,$$

where the set Φ_k is obtained recursively in k steps: for $t = 1, ..., k$,

$$\Phi_t := \begin{cases} \Phi_{t-1} \cup \{p_t\} & \text{if } \left[\begin{array}{l} \Phi_{t-1} \cup \{p_t\} \text{ is consistent } \textit{and} \\ |\{i \in N : p_t \in A_i\}| \geq m_{p_t} \end{array} \right] \textit{ or } \Phi_{t-1} \text{ entails } p_t, \\ \Phi_{t-1} & \text{otherwise,} \end{cases}$$

with $\Phi_0 := \varnothing$.

Here, for each t, Φ_t is the set of propositions accepted up to step t in the group's sequential decision process. Moreover, proposition p_t is accepted at step t if *either* [past judgments are consistent with p_t *and* the group votes the accept p_t] *or* past judgments require the acceptance of p_t. Our definition generalizes the one in List (2004) by allowing different acceptance thresholds for different propositions.[13]

In analogy to our earlier definition, a sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ $(= F_{\Omega,m})$ is *uniform* if the acceptance threshold takes the same value $m_p = m$ for all $p \in X$.

[13] The acceptance threshold m_p for each proposition $p \in X$ does not depend on the decision-path Ω or the collective judgments made prior to the occurrence of p along that path. However, as pointed out by a referee, there may be scenarios in which one may plausibly wish to introduce such dependencies. For example, if military interventions are costly and several interventions have already been approved, then one may require a higher acceptance threshold for a new intervention than if this were the first one under consideration. We acknowledge this point and leave its treatment as a challenge for future research.

A sequential quota rule is always consistent by design (hence also weakly consistent). Whether it is also complete and deductively closed depends on the decision-path Ω and the family of thresholds $(m_p)_{p \in X}$.[14]

4.2 An example

To illustrate that the outcome of a sequential quota rule may depend on the decision-path, consider our first example, where the agenda is $X := \{a,\ b,\ b \leftrightarrow a,\ \neg a,\ \neg b,\ \neg(b \leftrightarrow a)\}$ and there are three individuals with judgment sets $A_1 = \{a,\ b \leftrightarrow a,\ b\}$, $A_2 = \{\neg a,\ \neg(b \leftrightarrow a),\ b\}$ and $A_3 = \{\neg a,\ b \leftrightarrow a,\ \neg b\}$, as shown in table 1. Suppose the group uses a sequential quota rule $F_{\Omega,m}$, with a simple majority threshold $m = 2$ for every proposition $p \in X$. Consider two different decision-paths, Ω_1 and Ω_2, as shown in table 2.

t	1	2	3	4	5	6
$\Omega_1(t)$	a	$\neg a$	$b \leftrightarrow a$	$\neg(b \leftrightarrow a)$	b	$\neg b$
$\Omega_2(t)$	b	$\neg b$	$b \leftrightarrow a$	$\neg(b \leftrightarrow a)$	a	$\neg a$

Table 2

It is easy to see that the decision-paths Ω_1 and Ω_2 lead to different outcomes. Under Ω_1, $\neg a$ and $b \leftrightarrow a$ are each accepted by a vote and $\neg b$ is accepted by inference, resulting in the judgment set $\{\neg a, b \leftrightarrow a, \neg b\}$. In the example, the government first makes the judgment that country X has no WMD and that WMD are the required justification for action before deriving the judgment that no action should be taken. So the government's relevant judgment of fact and its judgment on the appropriate normative principle determine its judgment on how to act. Under Ω_2, b and $b \leftrightarrow a$ are each accepted by a vote and a is accepted by inference, resulting in the judgment set $\{a, b \leftrightarrow a, b\}$. Here the government first makes the judgment that action should be taken and that WMD are the required justification for action before deriving the judgment that country X has WMD. This means, more disturbingly, that the

[14] Consider a (natural) decision-path in which each proposition $p \in X$ and its negation $\neg p$ are adjacent, i.e. $\neg p$ comes immediately before or after p, and suppose that the thresholds m_p and $m_{\neg p}$ satisfy $m_p + m_{\neg p} \leq n + 1$, meaning that the corresponding quota rule $F_{(m_p)_{p \in X}}$ is complete. Then the sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ is complete (and hence deductively closed by consistency). Informally, the reason is that, when the sequential decision process reaches a pair of adjacent propositions $p, \neg p$, either the past judgments entail p or $\neg p$, in which case p or $\neg p$ is accepted, or the past judgments entail neither p nor $\neg p$, in which case again p or $\neg p$ is accepted since the relation $m_p + m_{\neg p} \leq n + 1$ ensures that the support for p or $\neg p$ exceeds the appropriate threshold.

government's judgment on how to act and its judgment on the appropriate normative principle determine its judgment of fact.

4.3 Necessary and sufficient conditions for path-independence

A sequential quota rule is *path-dependent* if the order in which the propositions are considered can affect its outcome. Formally, a sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$ is *path-dependent* if there exist two decision-paths Ω_1 and Ω_2, a profile $(A_1, \ldots, A_n)$ and a proposition $p \in X$ such that

$$p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n) \text{ and } p \notin F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n),$$

and *path-independent* otherwise; $F_{\Omega,(m_p)_{p\in X}}$ is *strongly path-dependent* if there exist two decision-paths Ω_1 and Ω_2, a profile $(A_1, \ldots, A_n)$ and a proposition $p \in X$ such that

$$p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n) \text{ and } \neg p \in F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n),$$

and *weakly path-independent* otherwise. Strong path-dependence implies path-dependence; path-independence implies weak path-independence.

When is a sequential quota rule path-dependent, when not? By combining a result in List (2004) with theorem 2 above, we can answer this question.

Theorem 3 *A sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is*

(a) *weakly path-independent if and only if the corresponding ordinary quota rule* $F_{(m_p)_{p\in X}}$ *is consistent, i.e. if and only if* (3) *above holds;*

(b) *path-independent if and only if the corresponding ordinary quota rule* $F_{(m_p)_{p\in X}}$ *is consistent and deductively closed, i.e. if and only if* (5) *above holds.*

We can also address the special case of a uniform sequential quota rule, combining theorem 3 and corollary 2.

Corollary 4 *Let* z *be the size of the largest minimal inconsistent set* $Z \subseteq X$*.*

(a) *A uniform sequential quota rule* $F_{\Omega,m}$ *is weakly path-independent if and only if* $m > n - n/z$*. In particular, for* $n \neq 2, 4$*, a sequential majority rule (where* $m = \lceil (n+1)/2 \rceil$*) is weakly path-independent if and only if* $z \leq 2$*; if* $n = 2$*, it is always weakly path-independent; if* $n = 4$*, it is weakly path-independent if and only if* $z \leq 3$*.*

(b) *A uniform sequential quota rule* $F_{\Omega,m}$ *is path-independent if and only if* $m = n$ *(i.e. it is a sequential unanimity rule) or [*$z \leq 2$ *and* $m > n/2$*].*

Our example above illustrates this result: sequential majority voting is path-dependent because we have $z = 3$ (with $n = 3$), which violates $z \leq 2$.

5 Path-independence and strategy-proofness

Path-dependent sequential quota rules are obviously vulnerable to manipulation by agenda setters who can influence the order in which the propositions are considered. In our example, an agenda setter who cares about taking action Y will set the decision-path Ω_2, whereas one who cares about avoiding action Y will set the decision-path Ω_1. But path-dependent rules are also vulnerable to strategic voting, i.e. to the misrepresentation of judgments by the individuals. We show that, under mild conditions, strategy-proofness is equivalent to path-independence. We also note that ordinary quota rules are always strategy-proof, although their use is limited given their rationality violations.

5.1 Strategy-proofness[15]

We now assume that each individual has not only a judgment set, but also an underlying preference relation – possibly only partial – over all possible judgment sets. This assumption captures the idea that, in comparing different collective judgment sets as potential outcomes, individuals will prefer some judgment sets to others.

Formally, each individual i has a preference relation $\succsim_i$ over all possible judgment sets of the form $A \subseteq X$. We assume that preference relations are reflexive and transitive.[16] We also require that $\succsim_i$ is *compatible* with individual i's judgment set A_i as follows. We say that one judgment set, A, *agrees* with another, A^*, on a proposition $p \in X$ if either both or neither of A and A^* contains p. Now $\succsim_i$ is *compatible* with A_i if the following holds: whenever two judgment sets A and A^* are such that [for all propositions $p \in X$, if A^* agrees with A_i on p, then so does A], then $A \succsim_i A^*$. Informally, compatibility of $\succsim_i$ with A_i requires that, if one judgment set is at least as close as another to an individual's own judgments on the propositions, then the individual weakly prefers the first judgment set to the second. In particular, an individual most prefers his or her own judgment set.

Now we can define strategy-proofness of an aggregation rule F.

Strategy-proofness. For every profile $(A_1, ..., A_n)$ in the domain of F, every individual i and any preference relation $\succsim_i$ compatible with A_i, $F(A_1, ..., A_n) \succsim_i F(A_1, ..., A_i^*, ..., A_n)$ for every i-variant $(A_1, ..., A_i^*, ..., A_n)$ in the domain of F.

[15] The present approach to strategy-proofness is based on List (2004) and Dietrich and List (2004). For related analyses of strategy-proofness, see Barberà et al. (1997) and Nehring and Puppe (2002).

[16] The assumption of completeness is not required.

Informally, strategy-proofness requires that, for every profile, each individual weakly prefers the collective judgment set that is obtained from expressing his or her own judgment set truthfully to any collective judgment set that would be obtained from misrepresenting his or her judgment set (where other individuals' expressed judgment sets are held fixed). Game-theoretically, this requires that, for each individual, the expression of his or her true judgment set is a weakly dominant strategy. Strategy-proofness is also equivalent to the simpler (and preference-free) condition that F is *non-manipulable* (Dietrich and List 2004): For every individual i, proposition p in X, and profile $(A_1, \ldots, A_n)$ in the domain, if A_i does not agree with $F(A_1, \ldots, A_n)$ on p, then A_i still does not agree with $F(A_1, \ldots, A_i^*, \ldots, A_n)$ on p for every i-variant $(A_1, \ldots, A_i^*, \ldots, A_n)$ in the domain. This means that no individual i can make it the case, by expressing an untruthful judgment set A_i^*, that the collective judgment on some proposition p switches to his or her true judgment on p.

Proposition 1 *(Dietrich and List 2004) An aggregation rule with universal domain is strategy-proof if and only if it is independent and monotonic.*

This proposition, a judgment-aggregation version of a classic result by Barberà et al. (1997), immediately implies that ordinary quota rules are strategy-proof, as they are independent and monotonic by theorem 1. But, as we have seen, such rules often generate rationality violations. Are sequential quota rules ever strategy-proof?

5.2 An example

Consider again our example of the three-member government with judgments as shown in table 1. Suppose the government uses a sequential majority rule with decision-path Ω_1 as shown in table 2. Assuming that all three government members express their judgments truthfully, the decision-path Ω_1 leads to the collective judgment set $\{\neg a, b \leftrightarrow a, \neg b\}$, i.e. a decision not to take action Y against country X, as shown above. But suppose individual 2 cares strongly about taking action Y, i.e. the acceptance of proposition b. Specifically, the following preference relation is compatible with individual 2's judgment set A_2:

$$\{\neg a, \neg(b \leftrightarrow a), b\} \succ_2 \{a, (b \leftrightarrow a), b\} \succ_2 \{a, \neg(b \leftrightarrow a), \neg b\} \succ_2 \{\neg a, b \leftrightarrow a, \neg b\},$$

where $\succ_2$ is the strong component of $\succsim_2$.

If individual 2 strategically expresses the judgment set $A_2^* = \{a,\ b \leftrightarrow a, b\}$ instead of his or her truthful judgment set $A_2 = \{\neg a,\ \neg(b \leftrightarrow a), b\}$, then sequential majority voting leads to the collective judgment set $\{a, b \leftrightarrow a, b\}$ instead of $\{\neg a, b \leftrightarrow a, \neg b\}$, where $\{a, b \leftrightarrow a, b\} \succ_2 \{\neg a, b \leftrightarrow a, \neg b\}$. So, by pretending to believe that country X has WMD and that WMD justify action, individual 2 can bring about the preferred decision to take action against the country. Hence sequential majority rule on the given agenda with decision-path Ω_1 is not strategy-proof.

5.3 Necessary and sufficient conditions for strategy-proofness

To state necessary and sufficient conditions for strategy-proofness of a sequential quota rule, we first introduce a simple condition on the representation of such a rule.

Note that, for a fixed decision-path Ω, two different families of thresholds $(m_p)_{p \in X}$ and $(m_p^*)_{p \in X}$ may yield the same aggregation rule, i.e. the same mapping from profiles to collective judgments. For example, let $X = \{a, \neg a\}$, $\Omega(1) = a$, $\Omega(2) = \neg a$, $m_a = m_{\neg a} = m_a^* = (n+1)/2$ (with n odd) and $m_{\neg a}^* = 1$. The rules $F_{\Omega,(m_p)_{p \in X}}$ and $F_{\Omega,(m_p^*)_{p \in X}}$ both accept a whenever a majority supports a, and $\neg a$ whenever a majority supports $\neg a$. This is obvious for $F_{\Omega,(m_p)_{p \in X}}$ and holds for $F_{\Omega,(m_p^*)_{p \in X}}$ because any submajority acceptance of $\neg a$ at step 2 in the recursive decision process is overruled by the majority acceptance of a at step 1. So $F_{\Omega,(m_p)_{p \in X}}$ and $F_{\Omega,(m_p^*)_{p \in X}}$ represent the same aggregation rule, though $F_{\Omega,(m_p)_{p \in X}}$ does so more transparently.

We say that m_p is the *effective threshold* for proposition $p \in X$ under the aggregation rule F if, for all profiles $(A_1, ..., A_n)$ in the domain of F, $p \in F(A_1, ..., A_n)$ if and only if $|\{i \in N : p \in A_i\}| \geq m_p$. A sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ is *transparent* if, for any proposition $p \in X$ for which there exists an effective threshold, m_p is this threshold. Transparency is a weak requirement: every sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ can – if not yet transparent – be made transparent by adjusting some of the thresholds, as shown by the next proposition.

Proposition 2 *For every sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$, there exists a transparent sequential quota rule $F_{\Omega,(m_p^*)_{p \in X}}$ with the same decision-path Ω such that, for every profile $(A_1, ..., A_n)$ in the universal domain, $F_{\Omega,(m_p)_{p \in X}}(A_1, ..., A_n) = F_{\Omega,(m_p^*)_{p \in X}}(A_1, ..., A_n)$.*

To obtain $(m_p^*)_{p \in X}$, simply define, for each $p \in X$, m_p^* to be the effective threshold for p if there exists such an effective threshold and $m_p^* = m_p$ otherwise.

Now we can state the logical relation between strategy-proofness and path-independence.

Theorem 4 *A complete or deductively closed transparent sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is strategy-proof if and only if it is path-independent.*[17]

By combining theorem 4 with theorem 3 above, we can characterize strategy-proofness in terms of our inequalities on the family of thresholds.

Corollary 5 *A complete or deductively closed transparent sequential quota rule* $F_{\Omega,(m_p)_{p\in X}}$ *is strategy-proof if and only if the corresponding ordinary quota rule is consistent, i.e.* (3) *above holds.*

Together with corollary 4 above, theorem 4 finally implies a result on sequential majority and unanimity rules.

Corollary 6 *Let* z *be the size of the largest minimal inconsistent set* $Z \subseteq X$*.*

(a) *If* n *is odd, a sequential majority rule* $F_{\Omega,m}$ *(where* $m = \lceil (n+1)/2 \rceil$*) is strategy-proof if and only if* $z \leq 2$*.*

(b) *A sequential unanimity rule* $F_{\Omega,m}$ *(where* $m = n$*) is always strategy-proof.*

Our results show that strategy-proofness of a sequential quota rule is a demanding condition. Moreover, among the class of uniform sequential quota rules, a sequential majority rule is strategy-proof only in the special case $z \leq 2$; a sequential unanimity rule is always strategy-proof, but again only at the expense of significant incompleteness.

6 Beyond quota rules: generalizing the necessary and sufficient conditions for collective rationality

Before concluding, let us explain an avenue for generalizing our results. Quota rules are by definition anonymous. The non-anonymous generalization of a quota rule is a *committee rule*. Here each proposition $p \in X$ is endowed not with a threshold m_p but with a set $\mathcal{C}_p$ of winning coalitions $C \subseteq N$ satisfying the following conditions: (i) $N \in \mathcal{C}_p$, (ii) $\emptyset \notin \mathcal{C}_p$, and (iii) if $C \in \mathcal{C}_p$ and $C \subseteq C^* \subseteq N$, then $C^* \in \mathcal{C}_p$. For each family $(\mathcal{C}_p)_{p\in X}$ of sets of winning coalitions, a *committee rule* $F_{(\mathcal{C}_p)_{p\in X}}$ is the aggregation rule with universal domain given by

$$F_{(\mathcal{C}_p)_{p\in X}}(A_1, ..., A_n) = \{p \in X : \{i \in N : p \in A_i\} \in \mathcal{C}_p\} \text{ for each profile } (A_1, ..., A_n).$$

[17] This result and corollary 5 also holds if, instead of requiring the sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$ to be complete or deductively closed, we require the corresponding ordinary quota rule $F_{(m_p)_{p\in X}}$ to be complete or deductively closed.

Nehring and Puppe's 'voting by committees' (2002, 2005) is a committee rule with the additional property that $F_{(\mathcal{C}_p)_{p\in X}}$ is complete and weakly consistent.[18]

Can our results on collective rationality under quota rules be generalized to committee rules? Nehring and Puppe (2002, 2005) have proved that 'voting by committees' is consistent if and only if the family $(\mathcal{C}_p)_{p\in X}$ satisfies the 'intersection property'. Generally, the following theorem can be shown in analogy to theorem 2 above, where part (c) corresponds to Nehring and Puppe's result (without assuming completeness and weak consistency). Part (d) is the first 'intersection property' for deductive closure rather than consistency.

Theorem 5 *A committee rule* $F_{(\mathcal{C}_p)_{p\in X}}$ *is*

(a) *complete if and only if*

$$C \in \mathcal{C}_p \text{ or } N\backslash C \in \mathcal{C}_{\neg p} \text{ for every pair } p, \neg p \in X \text{ and every coalition } C;$$

(b) *weakly consistent if and only if*

$$C \notin \mathcal{C}_p \text{ or } N\backslash C \notin \mathcal{C}_{\neg p} \text{ for every pair } p, \neg p \in X \text{ and every coalition } C;$$

(c) *consistent if and only if*

$$\bigcap_{p\in Z} C_p \neq \emptyset \quad \begin{array}{l}\text{for every minimal inconsistent set } Z \subseteq X \\ \text{and all coalitions } C_p \in \mathcal{C}_p \text{ with } p \in Z;\end{array}$$

(d) *deductively closed if and only if*

$$\bigcap_{p\in Z\backslash\{q\}} C_p \in \mathcal{C}_{\neg q} \quad \begin{array}{l}\text{for every minimal inconsistent set } Z \subseteq X, \\ \text{every proposition } q \in Z \text{ and all coalitions } C_p \in \mathcal{C}_p \text{ with } p \in Z\backslash\{q\}.\end{array}$$

Using this generalization of theorem 2, it is clear that the other results of this paper can be generalized to the non-anonymous case too.

7 Concluding remarks

Our findings have clarified the scope for rational judgment aggregation under ordinary and sequential quota rules, which generalize propositionwise majority voting. For each of the four rationality conditions of completeness, weak and full consistency and deductive closure, we have shown that a quota rule satisfies the given rationality

[18] Formally, [$C \in \mathcal{C}_p$ if and only if $N\backslash C \notin \mathcal{C}_{\neg p}$] for each $p \in X$ and each $C \subseteq N$.

condition if and only if its family of acceptance thresholds satisfies an appropriate system of inequalities. Furthermore, that system of inequalities depends crucially on the structure of the agenda of propositions under consideration. The richer the logical connections between these propositions, the more demanding the inequalities. As a corollary of our results, we can derive a classic result by Craven (1971) and Ferejohn and Grether (1974) on conditions for the acyclicity of supermajority voting for preference aggregation.

As full rationality is often impossible to achieve under an ordinary quota rule, we have also considered sequential quota rules, which adjudicate propositions in a sequence, letting earlier judgments constrain later ones. Such rules capture a large class of real-world decision processes. Sequential quota rules guarantee consistency – and sometimes also completeness and deductive closure – but they are path-dependent whenever the corresponding ordinary quota rule exhibits certain rationality violations. Path-dependence, in turn, matters because path-dependent rules are vulnerable to various forms of strategic manipulation. In particular, we have shown that strategy-proofness of a sequential quota rule is essentially equivalent to its path-independence.

So we can conclude that a group making judgments on interconnected propositions, such as whether to take a certain action, what counts as a justification for that action and whether the justification holds, may have a hard time doing so in a way that is simultaneously democratic, rational and strategy-proof.

8 References

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A Appendix

In the following we write $S \vDash p$ as an abbreviation for 'S entails p'. For each $S \subseteq \mathbf{L}$, we define $\overline{S} := \{p \in \mathbf{L} : S \vDash p\}$. For each judgment set A and each proposition $p \in X$, we define $A(p) := \begin{cases} 1 & \text{if } p \in A, \\ 0 & \text{if } p \notin A. \end{cases}$

The proof of lemma 1 uses the following result.

Lemma 2 *(List 2004) A set $S \subseteq X$ is inconsistent if and only if there exist two consistent subsets $S_1, S_2 \subseteq S$ and a proposition $p \in X$ such that S_1 entails p and S_2 entails $\neg p$.*

Proof of lemma 1. Part (a) is trivial.

(b) Assume $A \subseteq X$ is deductively closed. By part (a), consistency implies weak consistency. Now assume A is not consistent. By lemma 2, there exist consistent sets $S_1, S_2 \subseteq A$ and a proposition $p \in X$ such that $S_1 \vDash p$ and $S_2 \vDash \neg p$. By deductive closure, $p, \neg p \in A$. Hence A is not weakly consistent.

(c) Assume $A \subseteq X$ is complete. First, let A be consistent. By (a), A is weakly consistent. To prove deductive closure, consider any $p \in X$ such that $A \vDash p$. Then $A \cup \{\neg p\}$ is inconsistent. So, since A is consistent, $A \neq A \cup \{\neg p\}$. Hence $\neg p \notin A$. By completeness, $p \in A$. Conversely, suppose A is not consistent. We must show that A is not weakly consistent or not deductively closed. By lemma 2, there exist consistent sets $S_1, S_2 \subseteq A$ and a proposition $p \in X$ such that $S_1 \vDash p$ and $S_2 \vDash \neg p$. If A is deductively closed, then $p, \neg p \in X$, hence X violates deductive closure. ■

Proof of theorem 1. It is easy to see that a quota rule $F_{(m_p)_{p \in X}}$ satisfies the specified conditions. Conversely, assume that F satisfies the conditions. We show that, for any $p \in X$, there exists a threshold $m_p \in \{1, ..., n\}$ such that p is accepted if and only if at least m_p individuals accept p. Consider any $p \in X$. By responsiveness, there exists at least one profile $(A_1, ..., A_n)$ such that p is accepted; among all such profiles, choose one for which the number of individuals accepting p is minimal, and call this number m_p. By independence and anonymity, p is accepted for *every* profile with exactly m_p individuals accepting p. Using monotonicity, it follows that p is accepted in every profile with at least m_p individuals accepting p. On the other hand, p is rejected in every profile with less than m_p individuals accepting p, by definition of m_p. Since by responsiveness p is not always accepted, $m_p \neq 0$. Hence $m_p \in \{1, ..., n\}$. ■

Proof of theorem 2. We denote $F_{(m_p)_{p \in X}}$ simply by F. Also, for each $p \in X$, let n_p be the number of individuals i such that $p \in A_i$ for a given profile $(A_1, ..., A_n)$. Note that, as the profile ranges over the universal domain, for each pair $p, \neg p \in X$, the pair of numbers $(n_p, n_{\neg p})$ ranges over the set $\{(k, n-k) : k = 0, 1, ..., n\}$.

(a) F is complete if and only if, for each pair $p, \neg p \in X$, we have

$$\begin{array}{ll} & \text{for each profile, if } p \text{ is rejected then } \neg p \text{ is accepted,} \\ \text{equivalently,} & \text{for each profile, if } n_p < m_p \text{ then } n_{\neg p} \geq m_{\neg p}, \\ \text{equivalently,} & \text{for each } 0 \leq k \leq n, \text{ if } k < m_p \text{ then } n - k \geq m_{\neg p}, \\ \text{equivalently,} & \text{if } k = m_p - 1 \text{ then } n - k \geq m_{\neg p}, \\ \text{equivalently,} & m_p + m_{\neg p} \leq n + 1. \end{array}$$

(b) F is weakly consistent if and only if, for each pair $p, \neg p \in X$, we have

$$\begin{array}{ll} & \text{for each profile, if } p \text{ is accepted then } \neg p \text{ is rejected,} \\ \text{equivalently,} & \text{for each profile, if } n_p \geq m_p \text{ then } n_{\neg p} < m_{\neg p}, \\ \text{equivalently,} & \text{for each } 0 \leq k \leq n, \text{ if } k \geq m_p \text{ then } n - k < m_{\neg p}, \\ \text{equivalently,} & \text{if } k = m_p \text{ then } n - k < m_{\neg p}, \\ \text{equivalently,} & m_p + m_{\neg p} > n. \end{array}$$

(c) First, assume that F is not consistent. We show that at least one of the inequalities is violated. By assumption, there exists a profile $(A_1, ..., A_n)$ for which $F(A_1, ..., A_n)$ is inconsistent. Let $Z \subseteq F(A_1, ..., A_n)$ be a minimal inconsistent set. Since in the profile $(A_1, ..., A_n)$ exactly $n - n_p$ individuals reject each given $p \in Z$, a rejection of some proposition in Z by some individual i occurs exactly $\sum_{p \in Z}(n - n_p)$ times in $(A_1, ..., A_n)$. On the other hand, since Z is inconsistent, each of the n individuals rejects at least one proposition in Z. So, a rejection of some proposition in Z by some individual i occurs at least n times in $(A_1, ..., A_n)$. Hence $\sum_{p \in Z}(n - n_p) \geq n$. So, since for all $p \in Z$ we have $n_p \geq m_p$ (by $p \in F(A_1, ..., A_n)$), it follows that

$$\begin{array}{ll} & \sum_{p \in Z}(n - m_p) \geq n, \\ \text{equivalently,} & n|Z| - \sum_{p \in Z} m_p \geq n \\ \text{equivalently,} & \sum_{p \in Z} m_p \leq n(|Z| - 1). \end{array} \tag{10}$$

This violates the inequality for Z.

Conversely, assume that there is some minimal inconsistent set $Z \subseteq X$ with $\sum_{p\in Z} m_p \leq n(|Z|-1)$, hence by (10) $\sum_{p\in Z}(n-m_p) \geq n$. We construct a profile $(A_1,...,A_n)$ for which the group accepts each $p \in Z$, and hence generates an inconsistent judgment set. Since Z is minimal inconsistent, for each $p \in Z$ the set $Z\backslash\{p\}$ is consistent, and so $Z\backslash\{p\}$ may be extended to a (complete and consistent) judgment set, denoted $A_{\neg p}$. By $\sum_{p\in Z}(n-m_p) \geq n$, it is possible to assign to every individual i exactly one proposition $p_i \in Z$ in such a way that each $p \in Z$ is assigned to at most $n - m_p$ individuals. Define A_i as $A_{\neg p_i}$. For each $p \in Z$, at most $n - m_p$ individuals do not accept p, hence at least m_p individuals accept p. So $p \in F(A_1,...,A_n)$ for each $p \in Z$.

(d) First, assume that F is not deductively closed. We show that at least one of the inequalities is violated. By assumption, there exists a profile $(A_1,...,A_n)$, a consistent subset $R \subseteq F(A_1,...,A_n)$ and a $p^* \in X$ such that $R \vDash p^*$ but $p^* \notin F(A_1,...,A_n)$. Let $S \subseteq R$ be minimal such that $S \vDash p^*$. Writing q for $\neg p^*$, the set $Z := S \cup \{q\}$ is minimal inconsistent. Since in the profile $(A_1,...,A_n)$ exactly $n - n_p$ individuals reject each given $p \in Z$, a rejection of some proposition in Z by some individual i occurs exactly $\sum_{p\in Z}(n-n_p)$ times in $(A_1,...,A_n)$. On the other hand, since Z is inconsistent, each of the n individuals rejects at least one proposition in Z, so that a rejection of some proposition in Z by some individual i occurs at least n times in $(A_1,...,A_n)$. Hence $\sum_{p\in Z}(n-n_p) \geq n$, or $\sum_{p\in Z\backslash\{q\}}(n-n_p) + (n-n_q) \geq n$, or $\sum_{p\in Z\backslash\{q\}}(n-n_p) + n_{\neg q} \geq n$. Using that $n_{\neg q} < m_{\neg q}$ (by $\neg q = p^* \notin F(A_1,...,A_n)$) and that, for all $p \in Z\backslash\{q\}$, $n_p \geq m_p$ (by $p \in F(A_1,...,A_n)$), it follows that

$$\begin{aligned}
& \textstyle\sum_{p\in Z\backslash\{q\}}(n-m_p) + m_{\neg q} > n, \\
\text{equivalently,}\quad & n|Z\backslash\{q\}| - \textstyle\sum_{p\in Z\backslash\{q\}} m_p + m_{\neg q} > n, \\
\text{equivalently,}\quad & n(|Z|-1) - \textstyle\sum_{p\in Z\backslash\{q\}} m_p + m_{\neg q} > n \\
\text{equivalently,}\quad & \textstyle\sum_{p\in Z\backslash\{q\}} m_p - m_{\neg q} < n(|Z|-2).
\end{aligned} \tag{11}$$

This violates the inequality for Z.

Conversely, assume there is a minimal inconsistent set $Z \subseteq X$ and an element $q \in Z$ such that $\sum_{p\in Z\backslash\{q\}} m_p - m_{\neg q} < n(|Z|-2)$, i.e. by (11) $\sum_{p\in Z\backslash\{q\}}(n-m_p) + m_{\neg q} > n$. We construct a profile $(A_1,...,A_n)$ for which each $p \in Z\backslash\{q\}$ but not $\neg q$ is accepted. This is a violation of deductive closure because $Z\backslash\{q\}$ is consistent and entails $\neg q$. For each $p \in Z$, let $A_{\neg p}$ be some extension of $Z\backslash\{p\}$ to a (complete and consistent) judgment set. By $\sum_{p\in Z\backslash\{q\}}(n-m_p) + m_{\neg q} > n$ we have $\sum_{p\in Z\backslash\{q\}}(n-m_p) + (m_{\neg q} - 1) \geq n$. So it is possible to assign to every individual i exactly one proposition $p_i \in Z$

in such a way that each $p \in Z\backslash\{q\}$ is assigned to at most $n - m_p$ individuals and q is assigned to at most $m_{\neg q} - 1$ individuals. Let A_i be $A_{\neg p_i}$. Then, for each $p \in Z\backslash\{q\}$, at most $n - m_p$ individuals do not accept p, hence at least m_p individuals accept p. So $p \in F(A_1, ..., A_n)$ for each $p \in Z\backslash\{p\}$. Moreover, at most $m_{\neg q} - 1$ individuals do not accept q, i.e. accept $\neg q$. So $\neg q \notin F(A_1, ..., A_n)$. ■

Proof of corollary 1. Part (a) is trivial.

(b) Let $\mathcal{Z}$ be the set of minimal inconsistent sets $Z \subseteq X$. $F_{(m_p)_{p\in X}}$ is consistent if and only if $\sum_{p\in Z} m_p > n(|Z| - 1)$ for every $Z \in \mathcal{Z}$, or equivalently

$$\sum_{p\in Z\backslash\{q\}} m_p + m_q > n(|Z| - 1) \quad \text{for every member } q \text{ of every } Z \in \mathcal{Z}. \tag{12}$$

Further, $F_{(m_p)_{p\in X}}$ is deductively closed if and only if $\sum_{p\in Z\backslash\{q\}} m_p - m_{\neg q} \geq n(|Z| - 2)$ for every member q of every $Z \in \mathcal{Z}$, or equivalently

$$\sum_{p\in Z\backslash\{q\}} m_p + n + 1 - m_{\neg q} > n(|Z| - 1) \quad \text{for every member } q \text{ of every } Z \in \mathcal{Z}. \tag{13}$$

The claim follows from the fact that the conjunction of (12) and (13) is equivalent to (5).

(c) $F_{(m_p)_{p\in X}}$ is fully rational if and only if it is (i) complete and weakly consistent, and (ii) consistent; by part (a), (i) is equivalent to the equations in (6), and (ii) is equivalent to the inequalities in (6). ■

Proof of corollary 2. (a) By theorem 4, F_m is consistent if and only if, for all minimal inconsistent $Z \subseteq X$, $n(|Z| - 1) < \sum_{p\in Z} m$, i.e. $n|Z| - n < |Z|m$, i.e. $m > n - n/|Z|$. The latter inequality holds for all minimal inconsistent $Z \subseteq X$ just in case $m > n - n/z$. Let $m = \lceil (n + 1)/2 \rceil$. First, assume n is odd, hence $m = (n+1)/2$. Then F_m is consistent if and only if $(n + 1)/2 > n - n/z$, which is easily seen to be equivalent to $z \leq 2$. Now let n be even, hence $m = n/2 + 1$. Then F_m is consistent if and only if $n/2 + 1 > n - n/z$, i.e. $n/z > n/2 - 1$. This inequality always holds if $n = 2$; if $n = 4$, it holds just in case $z \leq 3$; if $n \geq 6$, it holds just in case $z \leq 2$.

(b) By theorem 4, F_m is deductively closed if and only if, for all minimal inconsistent $Z \subseteq X$ and any $q \in Z$, $\sum_{p\in Z\backslash\{q\}} m - m \geq n(|Z| - 2)$, i.e. $m(|Z| - 2) \geq n(|Z| - 2)$, i.e. $m = n$ or $|Z| \leq 2$. The latter inequality holds for all minimal inconsistent $Z \subseteq X$ just in case $z \leq 2$. Now let $m = \lceil (n + 1)/2 \rceil$. First, let $n \geq 3$. Hence $m \neq n$. So F_m is deductively closed if and only if $z \leq 3$. Second, let $n = 2$. Then $m = n$. So F_m is deductively closed.

(c) By parts (a) and (b), F_m is consistent and deductively closed if and only if $m > n - n/z$ and $[z \leq 2$ or $m = n]$, i.e. (i) $[m > n - n/z$ and $m = n]$ or (ii) $[m > n - n/z$ and $z \leq 2]$. Note that (i) is equivalent to $m = n$. Further, (ii) is equivalent to $[m > n/2$ and $z \leq 2]$: if $z \leq 2$, then we have $z = 2$ (because $z \neq 1$, as X contains no contradictions), and hence $n - n/z = n/2$. ■

Proof of corollary 3. The result follows immediately from theorems 1 and 2. ■

Given theorem 2 and corollary 2, only the following equivalence remains to be shown in order to prove theorem 3.

Proposition 3 *A sequential quota rule $F_{\Omega,(m_p)_{p \in X}}$ is*

(a) *weakly path-independent if and only if the corresponding quota rule $F_{(m_p)_{p \in X}}$ is consistent;*

(b) *path-independent if and only if the corresponding quota rule $F_{(m_p)_{p \in X}}$ is consistent and deductively closed.*

The proof of proposition 3 relies on two lemmas.

Lemma 3 *For every sequential rule $F_{\Omega,(m_p)_{p \in X}}$, profile $(A_1, ..., A_n)$, and step $t \in \{0, ..., k\}$, we have $\Phi_t \subseteq \overline{\Phi_t \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}$ (where Φ_t is as in the definition of $F_{\Omega,(m_p)_{p \in X}}(A_1, ..., A_n)$).*

Proof of lemma 3. Consider any family $(m_p)_{p \in X}$ and profile $(A_1, ..., A_n)$. We prove $\Phi_t \subseteq \overline{\Phi_t \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}$ by induction on $t \in \{0, ..., k\}$.

If $t = 0$, the claim follows from $\Phi_0 = \emptyset$.

Now let $t > 0$ and assume $\Phi_{t-1} \subseteq \overline{\Phi_{t-1} \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}$. If $p_t \notin \Phi_t$, then $\Phi_t = \Phi_{t-1}$; hence the claim holds by induction hypothesis.

Now suppose $p_t \in \Phi_t$. Then $\Phi_t = \Phi_{t-1} \cup \{p_t\}$. So $\Phi_t \subseteq \overline{\Phi_{t-1} \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)} \cup \{p_t\}$ by induction hypothesis. Hence it is sufficient to prove that

$$\overline{\Phi_{t-1} \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)} \cup \{p_t\} \subseteq \overline{\Phi_t \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}.$$

Since $\overline{\Phi_{t-1} \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)} \subseteq \overline{\Phi_t \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}$ (by $\Phi_{t-1} \subseteq \Phi_t$), it is sufficient to show that $p_t \in \overline{\Phi_t \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}$. By $p_t \in \Phi_t$, there are two cases: (i) $\Phi_{t-1} \vDash p_t$, or (ii) $p_t \in F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)$. Under case (ii), the claim is trivial. Under case (i), the induction hypothesis implies $\overline{\Phi_{t-1} \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)} \vDash p_t$; hence $p_t \in \overline{\Phi_{t-1} \cap F_{(m_p)_{p \in X}}(A_1, \ldots, A_n)}$, as required. ■

Lemma 4 *Let $(m_p)_{p\in X}$ be given. For any profile $(A_1,\ldots,A_n)$ and any proposition $p\in X$, [some consistent subset $S\subseteq F_{(m_p)_{p\in X}}(A_1,\ldots,A_n)$ entails p] if and only if [$p\in F_{\Omega,(m_p)_{p\in X}}(A_1,\ldots,A_n)$ for some decision-path Ω].*

Proof of lemma 4. Consider any profile $(A_1, \ldots, A_n)$ and proposition $p\in X$.

First, let there exist a decision-path Ω with $p\in F_{\Omega,(m_p)_{p\in X}}(A_1,\ldots,A_n)$. Let $\Phi_0,...,\Phi_k$ ($k=|X|$) be as in the definition of $F_{\Omega,(m_p)_{p\in X}}(A_1,\ldots,A_n)$, and $t\in\{1,2,...,k\}$ be such that $p=p_t$. By $p\in F_{\Omega,(m_p)_{p\in X}}(A_1,\ldots,A_n)$ we have $p\in\Phi_t$. So there are only two possible cases: (i) $\Phi_{t-1}\vDash p$ *or* (ii) $\left[\begin{array}{l}\Phi_{t-1}\cup\{p\}\text{ is consistent } \textit{and}\\ |\{i\in 1,2,\ldots,n:p\in A_i\}|\geq m_p\end{array}\right]$.
In case (i), we put $S:=\Phi_{t-1}\cap F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$, which is consistent; since $\Phi_{t-1}\vDash p$ and since $\Phi_{t-1}\subseteq\overline{S}$ by lemma 3, we have $\overline{S}\vDash p$, hence $S\vDash p$. In case (ii), we put $S:=\{p\}$, which is again consistent and entails p.

Conversely, assume that $S\subseteq F_{(m_p)_{p\in X}}(A_1,\ldots,A_n)$ is consistent and entails p. Let Ω be a decision-path that begins with the propositions in S, followed by proposition p, followed by all other propositions; specifically, $\Omega(m)=p_m\in S$ for all $m=1,...,s=|S|$, and $\Omega(s+1)=p$. Let the sets $\Phi_0,...,\Phi_k$ ($k=|X|$) be as in the definition of $F_{\Omega,(m_p)_{p\in X}}(A_1,\ldots,A_n)$. We show by induction that $\Phi_m=\{p_1,...,p_m\}$ for each $m=1,...,s$.

If $m=1$, then $\Phi_1=\{p_1\}$ since $p_1\in F_{\Omega,(m_p)_{p\in X}}(A_1,...,A_n)$.

Now let $1<m\leq s$ and assume that $\Phi_{m-1}=\{p_1,...,p_{m-1}\}$. Since $\Phi_{m-1}\cup\{p_m\}=\{p_1,...,p_m\}\subseteq S$, $\Phi_{m-1}\cup\{p_m\}$ is consistent. So, as $p_m\in F_{\Omega,(m_p)_{p\in X}}(A_1,...,A_n)$, we have $\Phi_m=\Phi_{m-1}\cup\{p_m\}=\{p_1,...,p_m\}$, as desired.

In particular, $\Phi_s=\{p_1,...,p_s\}=S$. By $S\vDash p$, we have $p\in\Phi_{s+1}$, so $p\in F_{\Omega,(m_p)_{p\in X}}(A_1,\ldots,A_n)$. ■

Proof of proposition 3. (a) First, suppose $F_{\Omega,(m_p)_{p\in X}}$ is not weakly path-independent, i.e. strongly path-dependent. Then there exist a profile $(A_1,\ldots,A_n)$, a proposition $p\in X$ and two decision-paths Ω_1 and Ω_2 such that

$$p\in F_{\Omega_1,(m_p)_{p\in X}}(A_1,\ldots,A_n)\text{ and }\neg p\in F_{\Omega_2,(m_p)_{p\in X}}(A_1,\ldots,A_n).$$

So, by lemma 4, there exists a consistent set $S_1\subseteq F_{(m_p)_{p\in X}}(A_1,\ldots,A_n)$ that entails p, and a consistent set $S_2\subseteq F_{(m_p)_{p\in X}}(A_1,...,A_n)$ that entails $\neg p$. Hence, by lemma 2, $F_{(m_p)_{p\in X}}(A_1,\ldots,A_n)$ is inconsistent, i.e. $F_{(m_p)_{p\in X}}$ is not consistent.

Conversely, suppose $F_{(m_p)_{p\in X}}$ is not consistent. Consider any profile $(A_1,\ldots,A_n)$ for which $F_{(m_p)_{p\in X}}(A_1,\ldots,A_n)$ is inconsistent. By lemma 2, there exist two consistent subsets $S_1, S_2 \subseteq F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ such that $S_1 \vDash p$ and $S_2 \vDash \neg p$. Hence, by lemma 4, there exist decision-path Ω_1 and Ω_2 such that $p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n)$ and $\neg p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n)$. So $F_{\Omega,(m_p)_{p\in X}}$ is not weakly path-independent.

(b) First, suppose $F_{(m_p)_{p\in X}}$ is consistent and deductively closed. We show that, for every decision-path Ω_1 and profile $(A_1, ..., A_n)$, $F_{\Omega_1,(m_p)_{p\in X}}(A_1, ..., A_n) = F_{(m_p)_{p\in X}}(A_1, ..., A_n)$, which implies path-independence. Consider any Ω_1 and $(A_1, ..., A_n)$. Let the sets Φ_0, ..., Φ_k and propositions p_1, ..., p_k ($k = |X|$) be as in the definition of $F_{\Omega_1,(m_p)_{p\in X}}(A_1, ..., A_n)$. We show by induction that $\Phi_t = F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$; the case $t = k$ then yields our claim.

For $t = 0$, the claim is trivial by $\Phi_0 = \emptyset$.

Now let $0 < t \leq k$ and assume $\Phi_{t-1} = F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_{t-1}\}$. We have to show that $p_t \in \Phi_t$ is equivalent to $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. First, assume $p_t \in \Phi_t$. Then, by definition of Φ_t, either (i) $\Phi_{t-1} \vDash p_t$ or (ii) [$\Phi_{t-1} \cup \{p_t\}$ is consistent and $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$]. In case (ii), we obviously have $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. In case (i), we have $F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_{t-1}\} \vDash p_t$ by induction hypothesis; since $F_{(m_p)_{p\in X}}(A_1, ..., A_n) \cap \{p_1, ..., p_{t-1}\}$ is consistent by the consistency of $F_{(m_p)_{p\in X}}$, we have $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$ by the deductive closure of $F_{(m_p)_{p\in X}}$. Now assume that $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. By induction hypothesis, $\Phi_{t-1} \subseteq F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. Hence, $\Phi_{t-1} \cup \{p_t\} \subseteq F_{(m_p)_{p\in X}}(A_1, ..., A_n)$. So, by the consistency of $F_{(m_p)_{p\in X}}$, $\Phi_{t-1} \cup \{p_t\}$ is consistent. Hence, as $p_t \in F_{(m_p)_{p\in X}}(A_1, ..., A_n)$, we have $p_t \in \Phi_t$ by definition of Φ_t.

Conversely, suppose $F_{(m_p)_{p\in X}}$ is not consistent or not deductively closed. If $F_{(m_p)_{p\in X}}$ is not consistent, then the result follows from part (a). Suppose now $F_{(m_p)_{p\in X}}$ is consistent, but not deductively closed. Then there is a profile $(A_1, \ldots, A_n)$, a consistent set $S \subseteq F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$ and a proposition $p \in X$ such that $S \vDash p$ and $p \notin F_{(m_p)_{p\in X}}(A_1, \ldots, A_n)$. So, on the one hand, by lemma 4, there exists a decision-path Ω_1 such that $p \in F_{\Omega_1,(m_p)_{p\in X}}(A_1, \ldots, A_n)$, and, on the other hand, $p \notin F_{\Omega_2,(m_p)_{p\in X}}(A_1, \ldots, A_n)$ for any decision-path Ω_2 with $\Omega_2(1) = p$. This implies path-dependence. ■

Proof of theorem 3. Given theorem 2 and corollary 2, the result follows from proposition *3*. ■

Proof of proposition 2. Consider any sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$. For each $p \in X$, define the new threshold m_p^* as the effective threshold for p if p has an effective threshold, and as m_p otherwise.

Claim 1: $F_{\Omega,(m_p)_{p\in X}}$ and $F_{\Omega,(m_p^*)_{p\in X}}$ generate the same judgment sets. Consider any profile $(A_1,...,A_n) \in \mathbf{A}^n$. Let the sets Φ_t $(t = 0,...,k,\ k = |X|)$ be as given in the definition of $F_{\Omega,(m_p)_{p\in X}}(A_1,...,A_n)$, and let Φ_t^* $(t = 0,...,k)$ be the corresponding sets for $F_{\Omega,(m_p^*)_{p\in X}}(A_1,...,A_n)$. By a straightforward induction on t, we have $\Phi_t = \Phi_t^*$ for all t. In particular, $\Phi_k = \Phi_k^*$, i.e. $F_{\Omega,(m_p)_{p\in X}}(A_1,...,A_n) = F_{\Omega,(m_p^*)_{p\in X}}(A_1,...,A_n)$.

Claim 2: $F_{\Omega,(m_p^*)_{p\in X}}$ is transparent. Consider any proposition $p \in X$, and assume p has an effective threshold under $F_{\Omega,(m_p^*)_{p\in X}}$. By claim 1, p has the same effective threshold under $F_{\Omega,(m_p)_{p\in X}}$. So, by definition of m_p^*, p has effective threshold m_p^* under $F_{\Omega,(m_p)_{p\in X}}$. So, by claim 1, p has effective threshold m_p^* under $F_{\Omega,(m_p^*)_{p\in X}}$. ■

The proof of theorem 4 relies on the following lemma.

Lemma 5 *For every sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$,*

(a) path-independence implies $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$;

(b) the converse also holds in case $F_{\Omega,(m_p)_{p\in X}}$ or $F_{(m_p)_{p\in X}}$ is complete or deductively closed.

Proof of lemma 5. (a) Let $F_{\Omega,(m_p)_{p\in X}}$ be path-independent. Consider any profile $(A_1,...,A_n)$ and proposition $p \in X$. We have to show that $F_{\Omega,(m_p)_{p\in X}}(A_1,...,A_n)(p) = F_{(m_p)_{p\in X}}(A_1,...,A_n)(p)$. Let Ω_1 be some decision-path with $\Omega_1(1) = p$. Then, by path-independence, $F_{\Omega,(m_p)_{p\in X}}(A_1,...,A_n)(p) = F_{\Omega_1,(m_p)_{p\in X}}(A_1,...,A_n)(p)$, which equals $F_{(m_p)_{p\in X}}(A_1,...,A_n)(p)$ by definition of Ω_1.

(b) Now let $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$, and assume this aggregation rule is complete or deductively closed. If it is deductively closed, then, as it is also consistent (by definition of sequential rules), it is path-independent by proposition 3 (b). If it is complete, then, as it is also consistent, it is deductively closed by lemma 1; hence it is again path-independent by proposition 3 (b). ■

Proof of theorem 4. Consider any complete or deductively closed transparent sequential quota rule $F_{\Omega,(m_p)_{p\in X}}$.

1. First, assume $F_{\Omega,(m_p)_{p\in X}}$ is path-independent. By lemma 5, $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$. So $F_{\Omega,(m_p)_{p\in X}}$ is independent and monotonic, hence strategy-proof by proposition 1.

2. Now assume $F_{\Omega,(m_p)_{p\in X}}$ is strategy-proof. Then $F_{\Omega,(m_p)_{p\in X}}$ is independent and monotonic by proposition 1. So, since $F_{\Omega,(m_p)_{p\in X}}$ is also anonymous, there exists a family $(m_p^*)_{p\in X} \in \{0,...,n+1\}^X$ such that $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p^*)_{p\in X}}$, where $F_{(m_p^*)_{p\in X}}$

denotes the obvious generalisation of our definition of quota rules to the case where each m_p^* can also be 0 or $n+1$ (in which case p is always or never accepted).

Claim 1: For each $p \in X$, $m_p^* \leq n$.

Consider any $p \in X$. Let A be a (complete and consistent) judgment set such that $p \in A$. Let the propositions p_t and the sets Φ_t ($t = 0, ..., k$, $k = |X|$) be as in the definition of $F_{\Omega,(m_p)_{p\in X}}(A, ..., A)$. Note that $F_{(m_p)_{p\in X}}(A, ..., A) = A$ (since $1 \leq m_p \leq n$ for all $p \in X$). By a straightforward induction, it follows that $\Phi_t = A \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$. In particular, $\Phi_k = A \cap \{p_1, ..., p_k\} = A$, i.e. $F_{\Omega,(m_p)_{p\in X}}(A, ..., A) = A$. So, by $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p^*)_{p\in X}}$, $F_{(m_p^*)_{p\in X}}(A, ..., A) = A$. In particular, $p \in F_{(m_p^*)_{p\in X}}(A, ..., A)$, hence $m_p^* \leq n$.

Claim 2: For each $p \in X$, $m_p^* \geq 1$.

Consider any $p \in X$. Let A be a (complete and consistent) judgment set such that $p \notin A$. By the argument used to prove claim 1, $F_{(m_p^*)_{p\in X}}(A, ..., A) = A$. In particular, $p \notin F_{(m_p^*)_{p\in X}}(A, ..., A)$, hence $m_p^* \geq 1$.

By the claims 1 and 2, each m_p^* belongs to $\{1, ..., n\}$, i.e. is a threshold in our standard sense, which will allow us to use a transparency argument. By $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p^*)_{p\in X}}$, each $p \in X$ has effective threshold m_p^*. So, by transparency, $m_p = m_p^*$ for each $p \in X$. Hence $F_{\Omega,(m_p)_{p\in X}} = F_{(m_p)_{p\in X}}$. So, by lemma 5 (b), $F_{\Omega,(m_p)_{p\in X}}$ is path-independent. ■

Proof of corollary 6. (a) Let n be odd and consider a sequential majority rule $F_{\Omega,m}$ ($m = (n+1)/2$). To apply theorem 4 to $F_{\Omega,m}$, it is sufficient to show that $F_{\Omega,m}$ is transparent and that F_m is complete (see footnote 17). As n is odd, F_m is complete. To prove that $F_{\Omega,m}$ is transparent, consider any $p \in X$ and let there be an effective threshold m_p for p. We show that $m_p = m$ by proving first that $m_p \leq m$ and then that $m_p > m - 1$. Let A_p and $A_{\neg p}$ be (complete and consistent) judgment sets with $p \in A_p$ and $p \notin A_{\neg p}$.

$m_p \leq m$: Let $(A_1, ..., A_n)$ be a profile in which exactly m individuals i have $A_i = A_p$ and the other $n - m = m - 1$ individuals i have $A_i = A_{\neg p}$. Let the propositions $p_1, ..., p_k$ and the sets $\Phi_0, ..., \Phi_k$ ($k = |X|$) be as in the recursive definition of $F_{\Omega,m}(A_1, ..., A_n)$. By a straightforward induction that uses the fact that a majority submits the judgment set A_p, we have $\Phi_t = A_p \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$. In particular, $\Phi_k = A_p \cap \{p_1, ..., p_k\} = A_p$, i.e. $F_{\Omega,m}(A_1, ..., A_n) = A_p$. Hence $p \in F_{\Omega,m}(A_1, ..., A_n)$. This implies $m_p \leq m$, as m_p is the effective threshold for p and m individuals accept p in $(A_1, ..., A_n)$.

$m_p > m - 1$: Now let $(A_1, ..., A_n)$ be a profile in which exactly m individuals

i have $A_i = A_{\neg p}$ and the other $n - m = m - 1$ individuals i have $A_i = A_p$. By an argument analogous to the above one, we have $F_{\Omega,m}(A_1, ..., A_n) = A_{\neg p}$. Hence $p \notin F_{\Omega,m}(A_1, ..., A_n)$. This implies $m_p > m - 1$, as m_p is the effective threshold for p and $m - 1$ individuals accept p in $(A_1, ..., A_n)$.

Having shown transparency, by theorem 4 strategy-proofness is equivalent to path-independence, which is equivalent to $z \leq 2$ by corollary 4.

(b) Now consider a sequential unanimity rule $F_{\Omega,m}$ $(m = n)$. By corollary 2, $F_{\Omega,m}$ is deductively closed. To apply theorem 4, we need to show that $F_{\Omega,m}$ is transparent. Consider any $p \in X$ and let there be an effective threshold m_p for p. We show that $m_p = m$, i.e. that $m_p = n$. As in part (a), let A_p and $A_{\neg p}$ be (complete and consistent) judgment sets with $p \in A_p$ and $p \notin A_{\neg p}$. Let $(A_1, ..., A_n)$ be a profile in which one individual i has $A_i = A_{\neg p}$ and $n - 1$ individuals i have $A_i = A_p$. Let the propositions $p_1, ..., p_k$ and the sets $\Phi_0, ..., \Phi_k$ $(k = |X|)$ be as in the definition of $F_{\Omega,m}(A_1, ..., A_n)$. By a straightforward induction, we have $\Phi_t = A_p \cap A_{\neg p} \cap \{p_1, ..., p_t\}$ for all $t \in \{0, ..., k\}$. In particular, $\Phi_k = A_p \cap A_{\neg p} \cap \{p_1, ..., p_k\} = A_p \cap A_{\neg p}$, i.e. $F_{\Omega,m}(A_1, ..., A_n) = A_p \cap A_{\neg p}$. Hence $p \notin F_{\Omega,m}(A_1, ..., A_n)$. This implies $m_p > n - 1$, as m_p is the effective threshold for p and $n - 1$ individuals accept p in $(A_1, ..., A_n)$. So $m_p = n$. This proves transparency.

Now by theorem 4 strategy-proofness is equivalent to path-independence, which is satisfied by corollary 4. ■

Proof of theorem 5. We denote $F_{(\mathcal{C}_p)_{p \in X}}$ simply by F. Also, for each $p \in X$ we denote by N_p the set of persons i such that $p \in A_i$ (for the relevant profile $(A_1, ..., A_n)$). Note that, as the profile ranges over the universal domain, for each pair $p \in X$ the pair coalitions $(N_p, N_{\neg p})$ ranges over the set of pairs $\{(C, N \backslash C) : C \subseteq N\}$.

(a) F is complete if and only if, for each $p \in X$, we have

	for each profile, p or $\neg p$ is accepted,
equivalently,	for each profile, $N_p \in \mathcal{C}_p$ or $N_{\neg p} \in \mathcal{C}_{\neg p}$,
equivalently,	for each coalition C, $C \in \mathcal{C}_p$ or $N \backslash C \in \mathcal{C}_{\neg p}$.

(b) F is weakly consistent if and only if, for each $p \in X$, we have

	for each profile, p or $\neg p$ is rejected,
equivalently,	for each profile, $N_p \notin \mathcal{C}_p$ or $N_{\neg p} \notin \mathcal{C}_{\neg p}$,
equivalently,	for each coalition C, $C \notin \mathcal{C}_p$ or $N \backslash C \notin \mathcal{C}_{\neg p}$.

(c) First, assume that F is *not* consistent, and let us show that at least one of the intersections is empty. By assumption, there exists a profile $(A_1, ..., A_n)$ for which

$F(A_1, ..., A_n)$ is inconsistent. Let $Z \subseteq F(A_1, ..., A_n)$ be a minimal inconsistent set. Since no person accepts each $p \in Z$ (by the inconsistency of Z), we have $\bigcap_{p \in Z} N_p = \emptyset$. But, for all $p \in Z$ we have $N_p \in \mathcal{C}_p$ by $p \in F(A_1, ..., A_n)$

Conversely, assume now that there is some minimal inconsistent set $Z \subseteq X$ and coalitions $C_p \in \mathcal{C}_p$, $p \in Z$, such that $\emptyset = \bigcap_{p \in Z} C_p$, and let us construct a profile $(A_1, ..., A_n)$ with $Z \subseteq F(A_1, ..., A_n)$ (which implies that F is not consistent). We have

$$N = N \backslash \left[\bigcap_{p \in Z} C_p \right] = \bigcup_{p \in Z} (N \backslash C_p) = \bigcup_{p \in Z} G_p, \text{ where } G_p := N \backslash C_p.$$

As one can easily see, it is possible to choose subsets $H_p \subseteq G_p$ such that we still have $\bigcup_{p \in Z} H_p = N$, but now the sets H_p, $p \in Z$ are pairwise disjoint, i.e. $H_p \cap H_q = \emptyset$ for any distinct $p, q \in Z$. In other words, the sets $H_{p,}$, $p \in Z$, form a partition of N. Let $(A_1, ..., A_n)$ be a profile such that, for each $p \in Z$, each person in H_p accepts $\neg p$ and all member of $Z \backslash \{p\}$. For each $p \in Z$, we have $\{i : p \in A_i\} = N \backslash H_p \supseteq N \backslash G_p = C_p \in \mathcal{C}_p$, hence $\{i : p \in A_i\} \in \mathcal{C}_p$, hence $p \in F(A_1, ..., A_n)$. So $Z \subseteq F(A_1, ..., A_n)$.

(d) First, assume that $\bigcap_{p \in Z \backslash \{q\}} C_p \in \mathcal{C}_{\neg q}$ holds for every minimal inconsistent set $Z \subseteq X$, $p \in Z$ and $C_p \in \mathcal{C}_p$, $p \in Z \backslash \{q\}$. Consider any profile $(A_1, ..., A_n)$, consistent set $R \subseteq F(A_1, ..., A_n)$ and $p \in X$ such that $R \models p$, and let us show that $p \in F(A_1, ..., A_n)$. Let $S \subseteq R$ be minimal such that $S \models p$. Then $Z := S \cup \{q\}$ is minimal inconsistent, where $q := \neg p$. For each $p^* \in Z \backslash \{q\} = S$, we have $N_{p^*} \in \mathcal{C}_{p^*}$ by $p^* \in F(A_1, ..., A_n)$. So, by assumption, $\bigcap_{p^* \in Z \backslash \{q\}} N_{p^*} \in \mathcal{C}_{\neg q} = \mathcal{C}_p$. But $\bigcap_{p^* \in Z \backslash \{q\}} N_{p^*} \subseteq N_p$, since $Z \backslash \{q\} = S \models p$. So $N_p \in \mathcal{C}_p$. Hence $p \in F(A_1, ..., A_n)$.

Conversely, assume there is a minimal inconsistent set $Z \subseteq X$, a proposition $q \in Z$ and coalitions $C_p \in \mathcal{C}_p$, $p \in Z \backslash \{q\}$, such that $C := \bigcap_{p \in Z \backslash \{q\}} C_p \notin \mathcal{C}_{\neg q}$. We construct a profile $(A_1, ..., A_n)$ for which each $p \in Z \backslash \{q\}$ but not $\neg q$ is collectively accepted. This is a violation of deductive closure because $Z \backslash \{q\}$ is consistent and entails $\neg q$. We have

$$N \backslash C = N \backslash \left[\bigcap_{p \in Z \backslash \{q\}} C_p \right] = \bigcup_{p \in Z \backslash \{q\}} (N \backslash C_p) = \bigcup_{p \in Z \backslash \{q\}} G_p, \text{ where } G_p := N \backslash C_p.$$

As in the proof of part (c), we choose subsets $H_p \subseteq G_p$ such that we still have $\bigcup_{p \in Z} H_p = N \backslash C$, but now the sets H_p, $p \in Z$ are pairwise disjoint, i.e. $H_p \cap H_{p'} = \emptyset$ for any distinct $p, p' \in Z \backslash \{q\}$. Note that $\{H_p : p \in Z \backslash \{q\}\} \cup \{C\}$ is a partition of N. Let $(A_1, ..., A_n)$ be a profile such that, for each $p \in Z \backslash \{q\}$, each person in H_p accepts $\neg p$ and all member of $Z \backslash \{p\}$, and each person in C accepts $\neg q$ and all members of

$Z\backslash\{q\}$. For each $p \in Z\backslash\{q\}$, we have $\{i : p \in A_i\} = N\backslash H_p \supseteq N\backslash G_p = C_p \in \mathcal{C}_p$, hence $\{i : p \in A_i\} \in \mathcal{C}_p$, hence $p \in F(A_1, ..., A_n)$. Moreover, $\{i : \neg q \in A_i\} = C = \bigcap_{p \in Z\backslash\{q\}} C_p \notin \mathcal{C}_{\neg q}$, hence $\neg q \notin F(A_1, ..., A_n)$. ■