Junction of a periodic family of elastic rods with a thin plate. Part II.

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Abstract

In this second paper, we consider again a set of elastic rods periodically distributed over an elastic plate whose thickness tends here to 0. This work is then devoted to describe the homogenization process for the junction of the rods and a thin plate. We use a technique based on two decompositions of the displacement field in each rod and in the plate. We obtain *a priori* estimates on each term of the two decompositions which permit to exhibit a few critical cases that distinguish the different possible limit behaviors. Then, we completely investigate one of these critical case which leads to a coupled bending-bending model for the rods and the 2d plate.

Résumé

Dans ce deuxième article, nous reprenons un ensemble de poutres élastiques périodiquement distribuées sur une plaque élastique dont l'épaisseur tend maintenant vers 0. Il s'agit donc de décrire des modèles d'homogénéisation pour la jonction de poutres et d'une plaque mince. Nous utilisons une technique de décomposition du champ de déplacement à la fois dans chaque poutre et dans la plaque. On obtient des estimations *a priori* sur chacun des termes de ces décompositions qui mettent en particulier en évidence les cas critiques qui séparent les différents modèles limites possibles. Ensuite, nous analysons en détail un de ces cas critiques pour lequel on obtient un modèle de couplage flexion-flexion entre les poutres et la plaque 2d.

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1 Introduction

This work follows paper [2] in which we assumed that a family of rods was placed over a $3d$ plate Ω^- of constant thickness. Here we investigate the case where the thickness of the plate

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Ω_δ^- below the rods vanishes with the periodicity of the rods (see Fig.1). Then, we have three small parameters: the radius of the rods r, the periodicity of the rods ε, and the thickness of the plate δ. Indeed, the first step in the derivation of limit problems consists in deriving accurate *a priori* estimates on the displacements, the strains and the stresses. As it was the case in [2], it appears that trying to quantify the dependence of the constant in Korn's inequality with respect to r, ε and δ is not relevant to provide such *a priori* estimates. Loosely speaking, this is due to the very different behaviors of each component of the displacement, strain and stress field. We use the same type of approach as in [2] but with two different decompositions of the 3d displacement in the rods and in the plate which take into account the fact that ε, r on the one hand and δ on the other hand are small (see [18] and [21]). Using the results proved in [21], each term of these decompositions are estimated in term of the total elastic energy. Since the two decompositions indeed match on the surface between the rods and the plate we deduce estimates of the type

$$\|u_i^{\varepsilon,r,\delta}\|^2_{L^2(\Omega^+_{\varepsilon,r,\delta})} \le c_i(\varepsilon,r,\delta)\mathcal{E}_{\Omega_{\varepsilon,r,\delta}}(u^{\varepsilon,r,\delta}), \quad i=1,2,3,$$

on the displacement $u^{\varepsilon,r,\delta} = (u_1^{\varepsilon,r,\delta}, u_2^{\varepsilon,r,\delta}, u_3^{\varepsilon,r,\delta})$, where $\mathcal{E}_{\Omega_{\varepsilon,r,\delta}}(u^{\varepsilon,r,\delta})$ is the elastic energy of the total domain $\Omega_{\varepsilon,r,\delta}$ (see Fig.1). This allows to scale the forces in order to derive *a priori* estimates on $u^{\varepsilon,r,\delta}$ and on the terms of the decompositions. Once these estimates are obtained, various limit models are possible. To shorten the length of the paper, we first assume that $r = k\varepsilon$ (the case $r/\varepsilon \to 0$ is analyzed in [2] for $\delta = 1$ and the reader is referred to this paper to adapt the analysis to the present setting). Loosely speaking, one can then distinguish two classes of limit models depending on the fact that the ratio ε/δ tends to zero or not.

• If $\varepsilon/\delta \to 0$, the plate is sufficiently thick to prevent any local effect of the rods on the 2d limit model for the plate (or the 3d plate if δ is a constant). As a consequence, the standard 2d membrane-bending model is obtained. Let us now describe the various models for the rods and the junction conditions depending of the relative order of δ with respect to ε. If δ is constant, the limit model for the rods is: standard bending equations uncoupled from the plate and standard compression-extension equation coupled with the plate. As soon as δ tends to 0, the compression-extension in the rods becomes uncoupled. Until $\varepsilon^{2/3}/\delta \to 0$, the bending in the rods remains uncoupled. For $\delta \sim \varepsilon^{2/3}$, which is a critical case, a coupling between the standard bending model for rods and the bending plate model appears. At least, if $\varepsilon^{2/3}/\delta \to +\infty$ each rod has a rigid displacement which is determined by the junction relations with the plate.

• If ε/δ does not tend to 0, then the plate is thin enough so that there is a microscopic geometrical effect on the surface between the rods and the plate. The 2d limit plate model becomes anisotropic with a modification of the elasticity coefficients which can be quantify through solving a few correctors problems.

In both cases (except for $\delta = 1$), we are prompt to consider oscillating test functions for the rods but also for the plate because the test displacements in order to obtain 1d and 2d models are very different. In some sense, this looks like an homogenization process was carried out also in the plate, even if it is homogeneous. Let us emphasize that it means that, even for a homogeneous material, one has to derive the correctors problems in the plate. In the present paper, we restrict the analysis to the case where $\varepsilon/\delta \to 0$ and more specifically we completely detail the critical case $\delta \sim \varepsilon^{2/3}$. An interested reader could easily adapt the analysis to the various cases mentioned above for $\varepsilon/\delta \to 0$. The situation where ε/δ does not tends to 0, for which the arguments must be modify, will be examined in a forthcoming paper.

The paper is organized as follows. Section 2 is devoted to specify the geometry and the equations of the problem. In Section 3 we give the two decompositions of the displacement field in the rods and in the plate. The *a priori* estimates on all the terms of the decompositions are established in Section 4 where the scaling of the forces is also specified. In Section 5, the plate is first rescaled and then we introduce the two unfolding operators in the rods and in the plate. We give the weak convergences of the unfold fields in Section 6. Section 7 is devoted to identify the weak limit of the unfold strain and to derive the junction conditions. In Section 8, we derive the equations for the homogenization correctors and we show that these correctors are equal to 0 because $\varepsilon/\delta \to 0$. Sections 9 and 10 are devoted to obtain the uncoupled "membrane" 2d model on the one hand, and the coupled bending model in the rods and the plate on the other hand. In Section 11, we show the strong concergence of the energy and we deduce a few strong convergences of the principal part of the displacement decomposition. All the results obtained in the paper are summarized in Section 12. At least, Sections 13 and 14 contains a few recalls and some complements on the periodic unfolding operator.

For the study of a scalar monotone problem in a multidomain as in this paper, we refer to [1] and [3]. For the study of the linearized elasticity system in the junction of a beam with a plate we refer to [15] and [16]. For the study of scalar second order and fourth order problems in the junction of a wire with a thin film, we refer to [14], and [17], respectively. For the study of plates, shells and thin films we refer to [4], [5], [6], [7], [10], [11], [13], [21], [22], [23], [24], [25] and [26]. About rods, multidomains and homogenization techniques see the references quoted in [2].

2 Notations and position of the problem

We recall first a few notations on the geometry of the problem.

Let us consider an open bounded domain ω with Lipschitz boundary contained in the (x_1, x_2) coordinate plane. For a real number $\varepsilon > 0$, $\mathcal{N}_\varepsilon$ denotes the following subset of $\mathbb{Z}^2$:

$$\mathcal{N}_\varepsilon = \Big\{(p,q) \in \mathbb{Z}^2 : \Big]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\Big[\subset \omega\Big\}. \tag{2.1}$$

Fix $L > 0$. For each $(p,q) \in \mathbb{Z}^2$, $\varepsilon > 0$ and $r > 0$, we consider a rod $\mathcal{P}^{\varepsilon,r}_{pq}$ whose cross section is the disk of center $(\varepsilon p, \varepsilon q)$ and radius r, and whose axis is x_3 and with a height equal to L:

$$\mathcal{D}^{\varepsilon,r}_{pq} = \big\{(x_1,x_2) \in \mathbb{R}^2 : (x_1 - \varepsilon p)^2 + (x_2 - \varepsilon q)^2 < r^2\big\}, \tag{2.2}$$

$$\mathcal{P}^{\varepsilon,r}_{pq} = \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1,x_2) \in \mathcal{D}^{\varepsilon,r}_{pq},\ 0 < x_3 < L\big\}. \tag{2.3}$$

Then, for $r \in]0, \frac{\varepsilon}{2}[$, we denote by $\Omega^+_{\varepsilon,r}$ the set of all the rods defined as above:

$$\Omega^+_{\varepsilon,r} = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{P}^{\varepsilon,r}_{pq}. \tag{2.4}$$

The lower cross sections of all the rods is denoted by $\omega_{\varepsilon,r}$:

$$\omega_{\varepsilon,r} = \bigcup_{(p,q)\in\mathcal{N}_\varepsilon} \mathcal{D}_{pq}^{\varepsilon,r} \times \{0\} \subset \omega. \tag{2.5}$$

We have assumed that $r \le \frac{\varepsilon}{2}$, in order to avoid the contact between two different rods.

The domain filled by the oscillating part $\Omega^+_{\varepsilon,r}$ (as ε tends to zero) is denoted by Ω^+:

$$\Omega^+ = \omega\times]0, L[. \tag{2.6}$$

Moreover, we set

$$\Omega^- = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega,\ -1 < x_3 < 0\right\}, \tag{2.7}$$

$$\Omega = \omega\times] - 1, L[. \tag{2.8}$$

The $3d$-plate Ω^-_δ is defined, for $\delta > 0$, by

$$\Omega^-_\delta = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega,\ -\delta < x_3 < 0\right\}, \tag{2.9}$$

and the elastic body under consideration is

$$\Omega_{\varepsilon,r,\delta} = \Omega^+_{\varepsilon,r} \cup \omega_{\varepsilon,r} \cup \Omega^-_\delta. \tag{2.10}$$

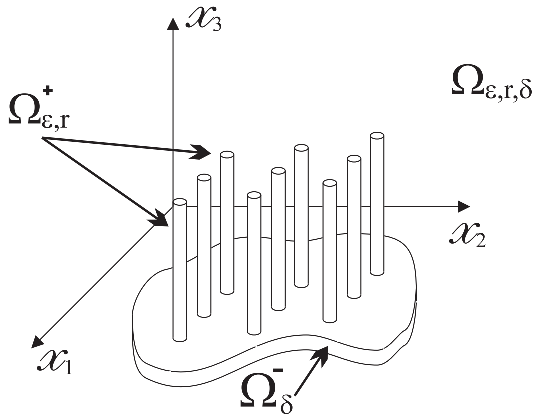

Figure 1: Elastic multistructure with highly oscillating boundary

We consider the standard linear equations of elasticity in $\Omega_{\varepsilon,r,\delta}$, and the displacement field in $\Omega_{\varepsilon,r,\delta}$ is denoted by

$$u^{\varepsilon,r,\delta} : \Omega_{\varepsilon,r,\delta} \to \mathbb{R}^3.$$

The linearized deformation field in $\Omega_{\varepsilon,r,\delta}$ is defined by

$$\gamma(u^{\varepsilon,r,\delta}) = \frac{1}{2}\left(Du^{\varepsilon,r,\delta} + (Du^{\varepsilon,r,\delta})^T\right), \tag{2.11}$$

or equivalently by its components:

$$\gamma_{ij}(u^{\varepsilon,r,\delta}) = \frac{1}{2}\left(\partial_i u_j^{\varepsilon,r,\delta} + \partial_j u_i^{\varepsilon,r,\delta}\right), \quad i,j = 1,2,3. \tag{2.12}$$

The Cauchy stress tensor in $\Omega_{\varepsilon,r,\delta}$ is linked to $\gamma(u^{\varepsilon,r,\delta})$ through the standard Hooke's law:

$$\sigma^{\varepsilon,r,\delta} = \lambda \left(\operatorname{Tr}\gamma(u^{\varepsilon,r,\delta})\right) I + 2\mu\gamma(u^{\varepsilon,r,\delta}), \tag{2.13}$$

where λ and μ denotes the Lamé coefficients of the elastic material, and I is the identity 3×3 matrix. Indeed (2.13) writes as

$$\sigma_{ij}^{\varepsilon,r,\delta} = \lambda\left(\sum_{k=1}^{3}\gamma_{kk}(u^{\varepsilon,r,\delta})\right)\delta_{ij} + 2\mu\gamma_{ij}(u^{\varepsilon,r,\delta}), \quad i,j=1,2,3, \tag{2.14}$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

The equations of equilibrium in $\Omega_{\varepsilon,r,\delta}$ write as

$$-\sum_{j=1}^{3}\partial_j\sigma_{ij}^{\varepsilon,r,\delta} = f_i^{\varepsilon,r,\delta} \text{ in } \Omega_{\varepsilon,r,\delta}, \quad i = 1,2,3, \tag{2.15}$$

where $f^{\varepsilon,r,\delta} : \Omega_{\varepsilon,r,\delta} \to \mathbb{R}^3$ denotes the volume applied force.

In order to specify the boundary conditions on $\partial\Omega_{\varepsilon,r,\delta}$, we will assume that:

- the $3D$ plate is clamped on its lateral boundary $\partial\omega\times]-\delta,0[=\Gamma_\delta$:

$$u^{\varepsilon,r,\delta} = 0 \text{ on } \Gamma_\delta, \tag{2.16}$$

- the boundary $\partial\Omega_{\varepsilon,r,\delta}\setminus\Gamma_\delta$ is free:

$$\sigma^{\varepsilon,r,\delta}\nu = 0 \text{ on } \partial\Omega_{\varepsilon,r,\delta}\setminus\Gamma_\delta, \tag{2.17}$$

where ν denotes the exterior unit normal to $\Omega_{\varepsilon,r,\delta}$.

Remark 2.1. *(2.17) means that the density of applied surface forces on the boundary $\partial\Omega_{\varepsilon,r}\setminus\Gamma_\delta$ is zero. This assumption is not necessary to carry on the analysis, but it is a bit natural as far as the fast oscillating boundary $\partial\Omega^+_{\varepsilon,r}$ is concerned.*

The variational formulation of (2.15)-(2.16)-(2.17) is very standard. If $V_{\varepsilon,r,\delta}$ denotes the space:

$$V_{\varepsilon,r,\delta} = \left\{v \in \left(H^1(\Omega_{\varepsilon,r,\delta})\right)^3 : v = 0 \text{ on } \Gamma_\delta\right\}, \tag{2.18}$$

it results that

$$\begin{cases} u^{\varepsilon,r,\delta} \in V_{\varepsilon,r,\delta}, \\ \displaystyle\int_{\Omega_{\varepsilon,r,\delta}}\sum_{i,j=1}^{3}\sigma_{ij}^{\varepsilon,r,\delta}\gamma_{ij}(v)dx = \int_{\Omega_{\varepsilon,r,\delta}}\sum_{i=1}^{3}f_i^{\varepsilon,r,\delta}v_i dx, \quad \forall v \in V_{\varepsilon,r,\delta}. \end{cases} \tag{2.19}$$

3 Decomposition of the displacement in $\Omega_{\varepsilon,r,\delta}$

As explained in [2], we will not seek for the dependance on ε, r and δ of the constant in a Korn's type inequality, but we will use the same decomposition of $u^{\varepsilon,r,\delta}$ in $\Omega^+_{\varepsilon,r}$ as in [2]. We drop for a while the indexes ε, r and δ in the notation of $u^{\varepsilon,r,\delta}$. Moreover, in order to shorten the notation, we set:

$$\widetilde{\omega}_\varepsilon = \bigcup_{(p,q)\in\mathcal{N}^\varepsilon} \Big(\Big]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\Big[\Big) \subset \omega.$$

Recall that the decomposition in $\Omega^+_{\varepsilon,r}$ is given by

- if $x = (x_1, x_2, x_3) \in \Big]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\Big[\times]0, L[$, $(p, q) \in \mathcal{N}_\varepsilon$

$$\mathcal{U}^+(x) = \frac{1}{\pi r^2} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} u(y_1, y_2, x_3) dy_1 dy_2, \tag{3.1}$$

$$\mathcal{R}^+_1(x) = \frac{1}{I_2 r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (y_2 - \varepsilon q) u_3(y_1, y_2, x_3) dy_1 dy_2, \tag{3.2}$$

$$\mathcal{R}^+_2(x) = -\frac{1}{I_1 r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (y_1 - \varepsilon p) u_3(y_1, y_2, x_3) dy_1 dy_2, \tag{3.3}$$

$$\mathcal{R}^+_3(x) = \frac{1}{(I_1 + I_2) r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} [(y_1 - \varepsilon p) u_2(y_1, y_2, x_3) - (y_2 - \varepsilon q) u_1(y_1, y_2, x_3)] dy_1 dy_2, \tag{3.4}$$

where $I_1 = \frac{1}{r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (x_1 - \varepsilon p)^2 dx_1 dx_2 = \frac{1}{r^4} \int_{\mathcal{D}^{\varepsilon,r}_{pq}} (x_2 - \varepsilon q)^2 dx_1 dx_2 = I_2 = \frac{\pi}{4}$.

- if $x = (x_1, x_2, x_3) \in \big(\omega \setminus \widetilde{\omega}_\varepsilon\big) \times]0, L[$

$$\mathcal{U}^+_i(x) = \mathcal{R}^+_i(x) = 0 \qquad \text{for } i = 1, 2, 3.$$

Let us denote by $\mathcal{R}^+$ the vectorial field $(\mathcal{R}^+_1, \mathcal{R}^+_2, \mathcal{R}^+_3)$ and define $\overline{u} \in (H^1(\Omega^+_{\varepsilon,r}))^3$ by

$$\overline{u}^+(x) = u(x) - \mathcal{U}^+(x) - \mathcal{R}^+(x) \wedge ((x_1 - \varepsilon p) e_1 + (x_2 - \varepsilon q) e_2) \qquad \text{for } x \in \mathcal{P}^{\varepsilon,r}_{pq}, \tag{3.5}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

We now introduce the decomposition of the displacement u in Ω^-_δ in order to take into account the specific geometry of Ω^-_δ. Let us define the following quantities:

$$\mathcal{U}^-_i(x_1, x_2) = \frac{1}{\delta} \int_{-\delta}^0 u_i(x_1, x_2, x_3) dx_3, \quad \text{for } i = 1, 2, 3, \tag{3.6}$$

$$\begin{cases} \mathcal{R}^-_1(x_1, x_2) = \dfrac{3}{2\delta^3} \displaystyle\int_{-\delta}^0 \Big(x_3 + \frac{\delta}{2}\Big) u_2(x_1, x_2, x_3) dx_3, \\[2ex] \mathcal{R}^-_2(x_1, x_2) = -\dfrac{3}{2\delta^3} \displaystyle\int_{-\delta}^0 \Big(x_3 + \frac{\delta}{2}\Big) u_1(x_1, x_2, x_3) dx_3, \end{cases} \tag{3.7}$$

$$\mathcal{R}_3^-(x_1,x_2)=0, \tag{3.8}$$

$$\overline{u}^-(x_1,x_2,x_3)=u(x_1,x_2,x_3)-\mathcal{U}^-(x_1,x_2)-\mathcal{R}^-(x_1,x_2)\wedge\left(x_3+\frac{\delta}{2}\right)e_3, \tag{3.9}$$

with $\mathcal{U}^-=\left(\mathcal{U}_1^-,\mathcal{U}_2^-,\mathcal{U}_3^-\right)$ and $\mathcal{R}^-=\left(\mathcal{R}_1^-,\mathcal{R}_2^-,0\right)$.

Indeed, due to the definitions of $\mathcal{U}^-$ and $\mathcal{R}^-$, we have

$$\int_{-\delta}^{0}\overline{u}_i^-(x_1,x_2,x_3)dx_3=0 \ \text{ a.e. in } \omega, \quad \text{for } i=1,2,3, \tag{3.10}$$

$$\int_{-\delta}^{0}\left(x_3+\frac{\delta}{2}\right)\overline{u}_\alpha^-(x_1,x_2,x_3)dx_3=0 \ \text{ a.e. in } \omega, \quad \text{for } \alpha=1,2. \tag{3.11}$$

Moreover since $u\in V_{\varepsilon,r,\delta}$, then $\mathcal{U}^-\in(H_0^1(\omega))^3$, $\mathcal{R}^-\in(H_0^1(\omega))^3$ and $\overline{u}\in(H^1(\Omega_\delta^-))^3$ with $\overline{u}^-=0$ on Γ_δ.

The elastic energy in Ω_δ^- is given by

$$\mathcal{E}^-(u)=\int_{\Omega_\delta^-}\left[\lambda\left(\sum_{k=1}^{3}\gamma_{kk}(u)\right)^2+2\mu\sum_{i,j=1}^{3}(\gamma_{ij}(u))^2\right]dx. \tag{3.12}$$

The following Lemma is established in [21].

Lemma 3.1. *There exists a constant c (independent of δ), such that*

$$\left\|\mathcal{U}_\alpha^-\right\|_{H^1(\omega)}^2\le\frac{c}{\delta}\mathcal{E}^-(u) \quad \textit{for } \alpha=1,2, \tag{3.13}$$

$$\left\|\mathcal{U}_3^-\right\|_{H^1(\omega)}^2\le\frac{c}{\delta^3}\mathcal{E}^-(u), \tag{3.14}$$

$$\left\|\mathcal{R}_\alpha^-\right\|_{H^1(\omega)}^2\le\frac{c}{\delta^3}\mathcal{E}^-(u) \quad \textit{for } \alpha=1,2, \tag{3.15}$$

$$\left\|\frac{\partial\mathcal{U}_3^-}{\partial x_1}+\mathcal{R}_2^-\right\|_{L^2(\omega)}^2\le\frac{c}{\delta}\mathcal{E}^-(u), \tag{3.16}$$

$$\left\|\frac{\partial\mathcal{U}_3^-}{\partial x_2}-\mathcal{R}_1^-\right\|_{L^2(\omega)}^2\le\frac{c}{\delta}\mathcal{E}^-(u), \tag{3.17}$$

$$\left\|\overline{u}_i^-\right\|_{L^2(\Omega_\delta^-)}^2\le c\delta^2\mathcal{E}^-(u) \quad \textit{for } i=1,2,3, \tag{3.18}$$

$$\left\|D\overline{u}_i^-\right\|_{(L^2(\Omega_\delta^-))^3}^2\le c\mathcal{E}^-(u) \quad \textit{for } i=1,2,3. \tag{3.19}$$

Let us remark that Korn's inequality and the L^2-estimates on u can be then deduced from (3.6)÷(3.9) and Lemma 3.1 (see [21]),

$$\begin{aligned}&\left\|\frac{\partial u_\beta}{\partial x_\alpha}\right\|_{L^2(\Omega_\delta^-)}^2+\delta^2\left\|\frac{\partial u_3}{\partial x_\alpha}\right\|_{L^2(\Omega_\delta^-)}^2+\delta^2\left\|\frac{\partial u_\alpha}{\partial x_3}\right\|_{L^2(\Omega_\delta^-)}^2+\left\|\frac{\partial u_3}{\partial x_3}\right\|_{L^2(\Omega_\delta^-)}^2\\&+\|Du_i\|_{(L^2(\Omega_\delta^-))^3}^2\le\frac{c}{\delta^2}\mathcal{E}^-(u) \quad \text{for } i=1,2,3, \text{ for } \alpha=1,2,\end{aligned} \tag{3.20}$$

$$\sum_{\alpha=1}^{2} \|u_\alpha\|^2_{L^2(\Omega_\delta^-)} + \delta^2 \|u_3\|^2_{L^2(\Omega_\delta^-)} \leq c\mathcal{E}^-(u); \tag{3.21}$$

but these last estimates are to loose to achieve the analysis. In the following section, we will use Lemma 4.2 (Section 4.4) of [2] and Lemma 3.1 of the present paper to derive a priori estimates on $u^{\varepsilon,r,\delta}$, and more precisely on its two decompositions in $\Omega^+_{\varepsilon,r}$ and Ω^-_δ.

4 *A priori* estimates

We follow the same strategy as in Section 3 of [2], and the displacement $u^{\varepsilon,r,\delta}$ is decomposed using (3.1)÷(3.5) in $\Omega^+_{\varepsilon,r}$ and in Ω^-_δ using (3.6)÷(3.9). In order to simplify the notations, we drop the indexes ε, r, δ, in all the considered fields and quantities. Recall that the elastic energy in $\Omega^+_{\varepsilon,r}$ is given by

$$\mathcal{E}^+(u) = \int_{\Omega^+_{\varepsilon,r}} \left[\lambda \left(\sum_{k=1}^{3} \gamma_{kk}(u) \right)^2 + 2\mu \sum_{i,j=1}^{3} (\gamma_{ij}(u))^2 \right] dx, \tag{4.1}$$

while the total elastic energy of u is

$$\mathcal{E}(u) = \mathcal{E}^+(u) + \mathcal{E}^-(u). \tag{4.2}$$

4.1 Uniform bound on $\mathcal{U}^+$ and $\mathcal{R}^+$ in terms of $\mathcal{E}(u)$

We use the same technique as in Section 4.1 of [2], and we first estimate $\mathcal{R}^+(0)$ and $\mathcal{U}^+(0)$ in Step 1, and then $\mathcal{U}^+$ and $\mathcal{R}^+$ in Step 2.

Step 1. To obtain sharp estimates on $\mathcal{U}^+(0)$ and $\mathcal{R}^+(0)$, we use the decomposition (3.9) of u in ω in the expressions of $\mathcal{R}^+(0)$ and $\mathcal{U}^+(0)$, and we first prove the following lemma on the behavior of the various terms entering the decomposition (3.9).

Lemma 4.1. *There exists a constant c (independent of ε, r and δ), such that*

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon} \left| \frac{1}{r^2} \int_{\mathcal{D}_{pq}} \mathcal{U}^-_\alpha(x_1,x_2)dx_1dx_2 \right|^2 \leq \frac{c}{r^2\delta}\mathcal{E}^-(u) \quad \textit{for } \alpha = 1,2, \tag{4.3}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon} \left| \frac{1}{r^2} \int_{\mathcal{D}_{pq}} \mathcal{U}^-_3(x_1,x_2)dx_1dx_2 \right|^2 \leq \frac{c}{r^2\delta^3}\mathcal{E}^-(u), \tag{4.4}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon} \left| \frac{1}{r^2} \int_{\mathcal{D}_{pq}} \mathcal{R}^-_\alpha(x_1,x_2)dx_1dx_2 \right|^2 \leq \frac{c}{r^2\delta^3}\mathcal{E}^-(u) \quad \textit{for } \alpha = 1,2, \tag{4.5}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon} \left| \frac{1}{r^4} \int_{\mathcal{D}_{pq}} (x_1 - p\varepsilon)\mathcal{U}^-_\alpha(x_1,x_2)dx_1dx_2 \right|^2 \leq \frac{c}{r^2\delta}\mathcal{E}^-(u), \quad \textit{for } \alpha = 1,2, \tag{4.6}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon}\left|\frac{1}{r^4}\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)\mathcal{U}_3^-(x_1,x_2)dx_1dx_2\right|^2\le\frac{c}{r^2\delta^3}\mathcal{E}^-(u),\tag{4.7}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon}\left|\frac{1}{r^4}\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)\mathcal{R}_\alpha^-(x_1,x_2)dx_1dx_2\right|^2\le\frac{c}{r^2\delta^3}\mathcal{E}^-(u)\quad \textit{for } \alpha=1,2,\tag{4.8}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon}\left|\frac{1}{r^2}\int_{\mathcal{D}_{pq}}\overline{u}_i^-(x_1,x_2,0)dx_1dx_2\right|^2\le\frac{c\delta}{r^2}\mathcal{E}^-(u)\quad \textit{for } i=1,2,3,\tag{4.9}$$

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon}\left|\frac{1}{r^4}\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)\overline{u}_i^-(x_1,x_2,0)dx_1dx_2\right|^2\le\frac{c\delta}{r^4}\mathcal{E}^-(u)\quad \textit{for } i=1,2,3.\tag{4.10}$$

Estimates (4.6), (4.7), (4.8) and (4.10) do not change, by replacing $(x_1-p\varepsilon)$ with $(x_2-q\varepsilon)$.

Proof. Estimates (4.3), (4.4) and (4.5) are direct consequences of (3.13), (3.14) and (3.15). In order to prove (4.6) we use the fact that $\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)dx_1dx_2=0$ so that denoting by $\mathcal{M}_{\mathcal{D}_{pq}}(v)$ the mean over the disc $\mathcal{D}_{pq}$ of a function $v\in L^2(\Omega^+_{\varepsilon,r})$, it results that

$$\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)\left[\mathcal{U}_\alpha^-(x_1,x_2)-\mathcal{M}_{\mathcal{D}_{pq}}(\mathcal{U}_\alpha^-)\right]dx_1,dx_2=\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)\mathcal{U}_\alpha^-(x_1,x_2)dx_1,dx_2.$$

Then Poincaré-Wirtinger inequality in $\mathcal{D}_{pq}$ (see e.g. (4.18) in [2]) and Cauchy-Schwarz inequality permit to obtain

$$\sum_{(p,q)\in\mathcal{N}_\varepsilon}\left|\frac{1}{r^4}\int_{\mathcal{D}_{pq}}(x_1-p\varepsilon)\mathcal{U}_\alpha^-(x_1,x_2)dx_1,dx_2\right|^2\le\frac{c}{r^2}\|\mathcal{U}_\alpha^-\|^2_{H^1(\omega)},$$

which in turn gives (4.6) in view of (3.13).

The estimates (4.7) and (4.8) are obtained with the same technique using also (3.14) and (3.15).

At last, remark that (3.18) and (3.19) imply the following bound on the value of $\overline{u}^-$ on ω

$$\left\|\overline{u}_i^-\right\|^2_{L^2(\omega)}\le c\delta\mathcal{E}^-(u),$$

which then yields (4.9) and (4.10). Then the proof of the lemma is complete. □

We turn back to the derivation of bounds on $\mathcal{R}^+(0)$ and $\mathcal{U}^+(0)$. We only detail the arguments for $\mathcal{R}_1^+(0)$. Recalling the definition (3.2) of $\mathcal{R}_1^+$, we have

$$\mathcal{R}_1^+(x_1,x_2,0)=\frac{1}{I_2r^4}\int_{\mathcal{D}_{pq}}(y_2-\varepsilon q)u_3(y_1,y_2,0)dy_1dy_2.$$

Inserting the decomposition (3.9) for $u_3(y_1,y_2,0)$ in the above expression leads to two terms

$$\frac{1}{I_2r^4}\int_{\mathcal{D}_{pq}}(y_2-\varepsilon q)\mathcal{U}_3^-(y_1,y_2,0)dy_1dy_2.$$

and

$$\frac{1}{I_2 r^4}\int_{\mathcal{D}_{pq}}(y_2-\varepsilon q)\overline{u_3^-}(y_1,y_2,0)dy_1dy_2.$$

which are estimated in Lemma 4.1. Consequently,

$$\left\|\mathcal{R}_1^+(\cdot,\cdot,0)\right\|^2_{L^2(\omega)}=\varepsilon^2\sum_{(p,q)\in\mathcal{N}^\varepsilon}\left|\mathcal{R}_1^+(p\varepsilon,q\varepsilon,0)\right|^2\le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{\delta^3}+\frac{\delta}{r^2}\right)\mathcal{E}^-(u).$$

The analysis is identical for the other components of $\mathcal{R}^+(0)$ and $\mathcal{U}^+(0)$, and we obtain, by using Lemma 4.1,

$$\left\|\mathcal{U}_\alpha^+(\cdot,\cdot,0)\right\|^2_{L^2(\omega)}\le c\frac{\varepsilon^2}{r^2\delta}\mathcal{E}^-(u),\qquad \text{for } \alpha=1,2, \tag{4.11}$$

$$\left\|\mathcal{U}_3^+(\cdot,\cdot,0)\right\|^2_{L^2(\omega)}\le c\frac{\varepsilon^2}{r^2\delta^3}\mathcal{E}^-(u), \tag{4.12}$$

$$\left\|\mathcal{R}_\alpha^+(\cdot,\cdot,0)\right\|^2_{L^2(\omega)}\le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{\delta^3}+\frac{\delta}{r^2}\right)\mathcal{E}^-(u),\qquad \text{for } \alpha=1,2, \tag{4.13}$$

$$\left\|\mathcal{R}_3^+(\cdot,\cdot,0)\right\|^2_{L^2(\omega)}\le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{\delta}+\frac{\delta}{r^2}\right)\mathcal{E}^-(u). \tag{4.14}$$

Step 2. Since the estimates on $\dfrac{\partial\mathcal{R}^+}{\partial x_3}$ and $\dfrac{\partial\mathcal{U}^+}{\partial x_3}$ are identically to those of Step 2 in Section 4.1 of [2] (they only depend on the decomposition of $\Omega^+_{\varepsilon,r}$) namely

$$\left\|\frac{\partial\mathcal{R}_i^+}{\partial x_3}\right\|^2_{(L^2(\Omega^+))^3}\le c\frac{\varepsilon^2}{r^4}\mathcal{E}^+(u),\qquad \text{for } i=1,2,3, \tag{4.15}$$

$$\left\|\frac{\partial\mathcal{U}_\alpha^+}{\partial x_3}\right\|^2_{L^2(\Omega^+)}\le c\left[\left\|\mathcal{R}_\beta^+\right\|^2_{L^2(\Omega^+)}+\frac{\varepsilon^2}{r^2}\mathcal{E}^+(u)\right],\qquad \text{for } \alpha,\beta=1,2, \text{ and } \alpha\neq\beta,$$

$$\left\|\frac{\partial\mathcal{U}_3^+}{\partial x_3}\right\|^2_{L^2(\Omega^+)}\le c\frac{\varepsilon^2}{r^2}\mathcal{E}^+(u), \tag{4.16}$$

we finally obtain

$$\left\|\mathcal{U}_\alpha^+\right\|^2_{L^2(\Omega^+)}\le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)\mathcal{E}(u),\qquad \text{for } \alpha=1,2, \tag{4.17}$$

$$\left\|\mathcal{U}_3^+\right\|^2_{L^2(\Omega^+)}\le c\frac{\varepsilon^2}{r^2\delta^3}\mathcal{E}(u), \tag{4.18}$$

$$\left\|\mathcal{R}_\alpha^+\right\|^2_{L^2(\Omega^+)}\le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)\mathcal{E}(u),\qquad \text{for } \alpha=1,2, \tag{4.19}$$

$$\left\|\mathcal{R}_3^+\right\|^2_{L^2(\Omega^+)}\le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{r^2}+\frac{1}{\delta}\right)\mathcal{E}(u). \tag{4.20}$$

Remark 4.2. *Actually, the estimate (4.19) on $\mathcal{R}^+_\alpha$ can be obtained directly, i.e. without using the decomposition (3.9) of the displacement in Ω^-_δ, through using the same technique as in [2] (see step 1 and 2 of Section 4.1) and the estimates (3.20) on u. This is not the case for the bound in (4.20) for $\mathcal{R}^+_3$. If one uses directly the method of [2] with the actual estimates (3.20) on u, one obtains*

$$\left\|\mathcal{R}^+_3\right\|^2_{L^2(\Omega^+)} \le c\frac{\varepsilon^2}{r^2}\left(\frac{1}{r^2\delta}+\frac{1}{\delta}\right)\mathcal{E}(u),$$

which is worse than (4.20). This means that the estimates of Lemma 3.1 are sharper than (3.20) (which comes directly from Korn's inequality in Ω^-_δ).

4.2 Estimates on $u^{\varepsilon,r,\delta}$ in term of $\mathcal{E}(u^{\varepsilon,r,\delta})$

Recall that we have, upon still dropping the index ε, r and δ in $\mathcal{U}^+$, $\mathcal{R}^+$ and $\overline{u}^+$, (see Section 4.3 of [2])

$$\|u^{\varepsilon,r,\delta}_\alpha\|^2_{L^2(\Omega^+_{\varepsilon,r})} \le c\left[\frac{r^2}{\varepsilon^2}\left\|\mathcal{U}^+_\alpha\right\|^2_{L^2(\Omega^+)}+\frac{r^4}{\varepsilon^2}\left\|\mathcal{R}^+_3\right\|^2_{L^2(\Omega^+)}+\|\overline{u}^+_\alpha\|^2_{L^2(\Omega^+_{\varepsilon,r})}\right],\qquad \text{for } \alpha=1,2,$$

$$\|u^{\varepsilon,r,\delta}_3\|^2_{L^2(\Omega^+_{\varepsilon,r})} \le c\left[\frac{r^2}{\varepsilon^2}\left\|\mathcal{U}^+_3\right\|^2_{L^2(\Omega^+)}+\frac{r^4}{\varepsilon^2}\left(\left\|\mathcal{R}^+_1\right\|^2_{L^2(\Omega^+)}+\left\|\mathcal{R}^+_2\right\|^2_{L^2(\Omega^+)}\right)+\|\overline{u}^+_3\|^2_{L^2(\Omega^+_{\varepsilon,r})}\right].$$

The field $\overline{u}^+$ still satisfies the following estimates (see (4.36) and (4.37) of [2]

$$\left\|\overline{u}^+\right\|^2_{\left(L^2(\Omega^+_{\varepsilon,r})\right)^3} \le cr^2\mathcal{E}^+(u^{\varepsilon,r,\delta}), \tag{4.21}$$

and

$$\left\|D\overline{u}^+\right\|^2_{\left(L^2(\Omega^+_{\varepsilon,r})\right)^9} \le c\mathcal{E}^+(u^{\varepsilon,r,\delta}). \tag{4.22}$$

Then using (4.17)÷(4.20) yields

$$\|u^{\varepsilon,r,\delta}_\alpha\|^2_{L^2(\Omega^+_{\varepsilon,r})} \le c\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)\mathcal{E}(u^{\varepsilon,r,\delta}),\qquad \text{for } \alpha=1,2. \tag{4.23}$$

$$\|u^{\varepsilon,r,\delta}_3\|^2_{L^2(\Omega^+_{\varepsilon,r})} \le \frac{c}{\delta^3}\mathcal{E}(u^{\varepsilon,r,\delta}). \tag{4.24}$$

4.3 ***A priori*** estimates on $u^{\varepsilon,r,\delta}$

We have

$$\mathcal{E}(u^{\varepsilon,r,\delta}) \le \sum_{\alpha=1}^{2}\|f^+_\alpha\|_{L^2(\Omega^+_{\varepsilon,r})}\|u^{\varepsilon,r,\delta}_\alpha\|_{L^2(\Omega^+_{\varepsilon,r})}+\|f^+_3\|_{L^2(\Omega^+_{\varepsilon,r})}\|u^{\varepsilon,r,\delta}_3\|_{L^2(\Omega^+_{\varepsilon,r})}+$$

$$\sum_{\alpha=1}^{2}\|f^-_\alpha\|_{L^2(\Omega^-_\delta)}\|u^{\varepsilon,r,\delta}_\alpha\|_{L^2(\Omega^-_\delta)}+\|f^-_3\|_{L^2(\Omega^-_\delta)}\|u^{\varepsilon,r,\delta}_3\|_{L^2(\Omega^-_\delta)}.$$

Inserting estimates (3.21), (4.23) and (4.24) in the above inequality gives

$$\begin{aligned}\mathcal{E}(u^{\varepsilon,r,\delta}) \le c\Bigg[\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)^{\frac12} \sum_{\alpha=1}^{2} \|f_\alpha^+\|_{L^2(\Omega^+_{\varepsilon,r})} + \frac{1}{\delta^{\frac32}}\|f_3^+\|_{L^2(\Omega^+_{\varepsilon,r})}+ \\ \sum_{\alpha=1}^{2} \|f_\alpha^-\|_{L^2(\Omega^-_{\delta})} + \frac{1}{\delta}\|f_3^-\|_{L^2(\Omega^-_{\delta})}\Bigg] \left(\mathcal{E}(u^{\varepsilon,r,\delta})\right)^{\frac12}.\end{aligned} \tag{4.25}$$

Now the choice of the order of $\mathcal{E}(u^{\varepsilon,r,\delta})$ has to be specified in order to fit the orders of the forces f_i^- and f_i^+. For a single plate of thickness δ, the energy is usually assumed to be of order δ. We keep the same choice here and then to obtain $\mathcal{E}(u^{\varepsilon,r,\delta}) \le c\delta$, the inequality (4.25) shows that the forces are chosen such that

$$\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)\|f_\alpha^+\|^2_{L^2(\Omega^+_{\varepsilon,r})} \le c\delta, \quad \text{ for } \alpha=1,2, \tag{4.26}$$

$$\|f_3^+\|^2_{L^2(\Omega^+_{\varepsilon,r})} \le c\delta^4, \tag{4.27}$$

$$\|f_\alpha^-\|^2_{L^2(\Omega^-_{\delta})} \le c\delta, \quad \text{ for } \alpha=1,2, \tag{4.28}$$

$$\|f_3^-\|^2_{L^2(\Omega^-_{\delta})} \le c\delta^3. \tag{4.29}$$

We are now in a position to state the following lemma which is valid under conditions (4.26)÷(4.29) (see also Lemma 4.2 in [2]).

Lemma 4.3. *If the forces satisfy conditions (4.26)÷(4.29), then there exists a constant c (independent of ε, r and δ), such that*

$$\|u_\alpha^{\varepsilon,r,\delta}\|_{L^2(\Omega^+_{\varepsilon,r})} \le c\left(\frac{\delta}{r^2}+\frac{1}{\delta^2}\right)^{\frac12} \quad \textit{for } \alpha=1,2, \tag{4.30}$$

$$\|u_3^{\varepsilon,r,\delta}\|_{L^2(\Omega^+_{\varepsilon,r})} \le \frac{c}{\delta}, \tag{4.31}$$

$$\|\gamma_{ij}(u^{\varepsilon,r,\delta})\|_{L^2(\Omega^+_{\varepsilon,r})} \le c\delta^{\frac12} \quad \textit{for } i,j=1,2,3, \tag{4.32}$$

$$\|u_\alpha^{\varepsilon,r,\delta}\|_{L^2(\Omega^-_{\delta})} \le c\delta^{\frac12} \quad \textit{for } \alpha=1,2, \tag{4.33}$$

$$\|u_3^{\varepsilon,r,\delta}\|_{L^2(\Omega^-_{\delta})} \le c\delta^{-\frac12}, \tag{4.34}$$

$$\|\gamma_{ij}(u^{\varepsilon,r,\delta})\|_{L^2(\Omega^-_{\delta})} \le c\delta^{\frac12} \quad \textit{for } i,j=1,2,3, \tag{4.35}$$

$$\|\mathcal{U}_\alpha^{\varepsilon,r,\delta+}\|_{L^2(\Omega^+)} \le c\frac{\varepsilon\delta^{\frac12}}{r}\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)^{\frac12}, \quad \textit{for } \alpha=1,2, \tag{4.36}$$

$$\left\|\mathcal{U}_3^{\varepsilon,r,\delta+}\right\|_{L^2(\Omega^+)} \le c\frac{\varepsilon}{r\delta}, \tag{4.37}$$

$$\|\mathcal{R}_\alpha^{\varepsilon,r,\delta+}\|_{L^2(\Omega^+)} \le c\frac{\varepsilon\delta^{\frac12}}{r}\left(\frac{1}{r^2}+\frac{1}{\delta^3}\right)^{\frac12}, \quad \textit{for } \alpha=1,2, \tag{4.38}$$

$$\left\|\mathcal{R}_3^{\varepsilon,r,\delta+}\right\|_{L^2(\Omega^+)} \leq c\frac{\varepsilon\delta^{\frac{1}{2}}}{r}\left(\frac{1}{r^2}+\frac{1}{\delta}\right)^{\frac{1}{2}}, \tag{4.39}$$

$$\left\|\frac{\partial \mathcal{U}^{\varepsilon,r,\delta+}}{\partial x_3} - \mathcal{R}^{\varepsilon,r,\delta+}\wedge e_3\right\|_{(L^2(\Omega^+))^3} \leq c\frac{\varepsilon\delta^{\frac{1}{2}}}{r}, \tag{4.40}$$

$$\|\overline{u}_i^{\varepsilon,r,\delta+}\|_{L^2(\Omega^+_{\varepsilon,r})} \leq cr\delta^{\frac{1}{2}} \quad \textit{for } i=1,2,3, \tag{4.41}$$

$$\|D\overline{u}_i^{\varepsilon,r,\delta+}\|_{(L^2(\Omega^+_{\varepsilon,r}))^3} \leq c\delta^{\frac{1}{2}} \quad \textit{for } i=1,2,3, \tag{4.42}$$

$$\|\overline{u}_i^{\varepsilon,r,\delta-}\|_{L^2(\Omega^-_{\delta})} \leq c\delta^{\frac{3}{2}} \quad \textit{for } i=1,2,3, \tag{4.43}$$

$$\|D\overline{u}_i^{\varepsilon,r,\delta-}\|_{(L^2(\Omega^-_{\delta}))^3} \leq c\delta^{\frac{1}{2}} \quad \textit{for } i=1,2,3, \tag{4.44}$$

$$\|\sigma_{ij}^{\varepsilon,r,\delta}\|_{L^2(\Omega^+_{\varepsilon,r})} \leq c\delta^{\frac{1}{2}} \quad \textit{for } i,j=1,2,3, \tag{4.45}$$

$$\|\sigma_{ij}^{\varepsilon,r,\delta}\|_{L^2(\Omega^-_{\delta})} \leq c\delta^{\frac{1}{2}} \quad \textit{for } i,j=1,2,3. \tag{4.46}$$

Until now we have derived the *a priori* estimates on all the fields in terms of arbitrary parameters ε, r and δ, so that an interested reader may investigate various limit models depending on the respective asymptotic behavior of these parameters.

From now on, we will first restrict the analysis to the case where $r = k\varepsilon$ ($0 < k < \frac{1}{2}$); the case $\dfrac{r_\varepsilon}{\varepsilon} \to 0$ can be studied as in [2].

Secondly, and as explained in the introduction, we will develop the complete analysis, in what follows, for a critical case where one obtains a standard $2d$-model for the plate. Assuming e.g. that $r \sim \varepsilon \sim \delta^\alpha$, this is the case if $\alpha > 1$ and for such values of α, what differs from a model to another is the limit junction conditions between the rods and the plate (see the concluding remarks in Section 11). In view of estimates (4.30), (4.36) and (4.39), we have decided to completely develop the case where $r^2 = k^2\varepsilon^2 = a^2\delta^3$. The techniques developed below can be easily reproduced for others asymptotic behaviors of ε, r and δ, to obtain various limit models.

In order to satisfy (4.26)÷(4.29) with $r^2 = a^2\delta^3$, we assume that

$$f_i^+ = \delta^2 F^+_{i\,|_{\Omega^+_\varepsilon}}, \quad \text{for } i=1,2,3, \tag{4.47}$$

$$f_\alpha^-(x_1,x_2,x_3) = F_\alpha^-\left(x_1,x_2,\frac{x_3}{\delta}\right), \quad \text{a.e. in } \Omega_\delta^-, \quad \text{for } \alpha=1,2, \tag{4.48}$$

$$f_3^-(x_1,x_2,x_3) = \delta F_3^-\left(x_1,x_2,\frac{x_3}{\delta}\right), \quad \text{a.e. in } \Omega_\delta^-, \tag{4.49}$$

where $F_i^+ \in L^2(\Omega^+)$ and $F_i^- \in L^2(\Omega^-)$, for $i=1,2,3$.

Let us explicitly give the a priori estimates which follow from Lemma 4.3 in the case where $r^2 = k^2\varepsilon^2 = a^2\delta^3$.

Lemma 4.4. *If the forces satisfy conditions (4.26)÷(4.29), then there exists a constant c (independent δ), such that*

$$\|u_i^\delta\|_{L^2(\Omega_\varepsilon^+)} \le \frac{c}{\delta} \quad \textit{for } i = 1,2,3, \tag{4.50}$$

$$\|\gamma_{ij}(u^\delta)\|_{L^2(\Omega_\varepsilon^+)} \le c\delta^{\frac{1}{2}} \quad \textit{for } i,j = 1,2,3, \tag{4.51}$$

$$\|u_\alpha^\delta\|_{L^2(\Omega_\delta^-)} \le c\delta^{\frac{1}{2}} \quad \textit{for } \alpha = 1,2, \tag{4.52}$$

$$\|u_3^\delta\|_{L^2(\Omega_\delta^-)} \le c\delta^{-\frac{1}{2}}, \tag{4.53}$$

$$\|\gamma_{ij}(u^\delta)\|_{L^2(\Omega_\delta^-)} \le c\delta^{\frac{1}{2}} \quad \textit{for } i,j = 1,2,3, \tag{4.54}$$

$$\big\|\mathcal{U}_\alpha^{\delta+}\big\|_{L^2(\omega;H^1((0,L)))} \le \frac{c}{\delta} \quad \textit{for } \alpha = 1,2, \tag{4.55}$$

$$\big\|\mathcal{U}_3^{\delta+}\big\|_{L^2(\Omega^+)} \le \frac{c}{\delta}, \tag{4.56}$$

$$\left\|\frac{\partial \mathcal{U}_3^{\delta+}}{\partial x_3}\right\|_{L^2(\Omega^+)} \le c\delta^{\frac{1}{2}}, \tag{4.57}$$

$$\big\|\mathcal{R}_i^{\delta+}\big\|_{L^2(\omega;H^1((0,L)))} \le \frac{c}{\delta}, \quad \textit{for } i = 1,2,3, \tag{4.58}$$

$$\left\|\frac{\partial \mathcal{U}^{\delta+}}{\partial x_3} - \big(\mathcal{R}^{\delta+} \wedge e_3\big)\right\|_{(L^2(\Omega^+))^3} \le c\delta^{\frac{1}{2}}, \tag{4.59}$$

$$\|\overline{u}_i^{\delta+}\|_{L^2(\Omega_{\varepsilon,r}^+)} \le c\delta^2 \quad \textit{for } i = 1,2,3, \tag{4.60}$$

$$\|D\overline{u}_i^{\delta+}\|_{(L^2(\Omega_{\varepsilon,r}^+))^3} \le c\delta^{\frac{1}{2}} \quad \textit{for } i = 1,2,3, \tag{4.61}$$

$$\|\overline{u}_i^{\delta-}\|_{L^2(\Omega_\delta^-)} \le c\delta^{\frac{3}{2}} \quad \textit{for } i = 1,2,3, \tag{4.62}$$

$$\|D\overline{u}_i^{\delta-}\|_{(L^2(\Omega_\delta^-))^3} \le c\delta^{\frac{1}{2}} \quad \textit{for } i = 1,2,3, \tag{4.63}$$

$$\|\sigma_{ij}^\delta\|_{L^2(\Omega_{\varepsilon,r}^+)} \le c\delta^{\frac{1}{2}} \quad \textit{for } i,j = 1,2,3, \tag{4.64}$$

$$\|\sigma_{ij}^\delta\|_{L^2(\Omega_\delta^-)} \le c\delta^{\frac{1}{2}} \quad \textit{for } i,j = 1,2,3. \tag{4.65}$$

5 Rescaling of Ω_δ^- and unfolding operators in Ω_ε^+ and Ω^-

We denote by D the unit disk of $\mathbb{R}^2$. We first recall the definition of the unfolding operator $\mathcal{T}^\varepsilon$ given in Section 5 of [2] which is defined for any $v \in L^2(\Omega_\varepsilon^+)$ by, for almost $(x_1, x_2, x_3) \in \Omega^+$

and $(X_1, X_2) \in D$,

$$\mathcal{T}^\varepsilon(v)(x_1, x_2, x_3, X_1, X_2) =$$

$$\begin{cases} v(p\varepsilon + r_\varepsilon X_1, q\varepsilon + r_\varepsilon X_2, x_3), \\ \qquad \text{if } (x_1, x_2) \in \left]\varepsilon p - \dfrac{\varepsilon}{2}, \varepsilon p + \dfrac{\varepsilon}{2}\right[\times \left]\varepsilon q - \dfrac{\varepsilon}{2}, \varepsilon q + \dfrac{\varepsilon}{2}\right[, \quad (p, q) \in \mathcal{N}_\varepsilon, \\ 0, \quad \text{if } (x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon. \end{cases} \tag{5.1}$$

We refer to Lemma 5.1 of [2] for the properties of this operator. Then, in order to take into account the necessary rescaling on Ω_δ^-, we introduce the following new operator Π_δ defined for any function $v \in L^2(\Omega_\delta^-)$:

$$\Pi_\delta(v)(x_1, x_2, X_3) = v(x_1, x_2, \delta X_3) \quad \text{for } (x_1, x_2, X_3) \in \Omega^- = \omega \times]-1, 0[. \tag{5.2}$$

Remark that $\Pi_\delta(v) \in L^2(\Omega^-)$. Indeed we have for any $v \in L^2(\Omega_\delta^-)$ and any $w \in L^2(\Omega_\delta^-)$

$$\int_{\Omega^-} \Pi_\delta(v)\Pi_\delta(w) dx_1 dx_2 dX_3 = \frac{1}{\delta} \int_{\Omega_\delta^-} vw dx_1 dx_2 dx_3, \tag{5.3}$$

$$\frac{\partial \Pi_\delta(v)}{\partial x_\alpha} = \Pi_\delta\left(\frac{\partial v}{\partial x_\alpha}\right), \quad \text{for } \alpha = 1, 2, \tag{5.4}$$

$$\frac{\partial \Pi_\delta(v)}{\partial X_3} = \delta \Pi_\delta\left(\frac{\partial v}{\partial x_3}\right). \tag{5.5}$$

Thirdly and since we will use a few oscillating test functions in Ω^- in Section 6.2, we also introduce the usual unfolding operator in homogenization theory (see [8] and [9]). The operator $\mathcal{T}_-^\varepsilon$ is defined on $\omega \times]-\frac{1}{2}, \frac{1}{2}[^2 \times]-1, 0[$, for almost $(x_1, x_2) \in \omega$ and $(X_1, X_2, X_3) \in]-\frac{1}{2}, \frac{1}{2}[^2 \times]-1, 0[$, by

$$\mathcal{T}_-^\varepsilon(v)(x_1, x_2, X_1, X_2, X_3) =$$

$$\begin{cases} v(p\varepsilon + \varepsilon X_1, q\varepsilon + \varepsilon X_2, X_3), \ \text{if } (x_1, x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[\times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \\ \hfill \text{and } (p, q) \in \mathcal{N}_\varepsilon, \\ 0 \text{ if } (x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon. \end{cases} \tag{5.6}$$

The main properties of $\mathcal{T}_-^\varepsilon$ that we will use in this paper are recalled in Annex 1 and Annex 2, and we refer to [8] and [9] for the proofs and various applications in homogenization.

6 Estimates and weak convergence of the unfold fields in Ω^+ and of the rescaled and unfold fields in Ω^-

As far as the fields defined in Ω^+ are concerned, recalling Lemma 5.1 of [2] and the fact that $r^2 = k^2\varepsilon^2 = a^2\delta^3$, we obtain from Lemma 4.4 above

Lemma 6.1. *If the forces satisfy conditions (4.47)÷(4.49), then there exists a constant c (independent of δ), such that (recall that $\varepsilon = \frac{a}{k}\delta^{\frac{3}{2}}$)*

$$\delta \| \mathcal{T}^{\varepsilon}(u_i^{\delta}) \|_{L^2(\Omega^+ \times D)} \leq c \quad \text{for } i = 1, 2, 3, \tag{6.1}$$

$$\| \mathcal{T}^{\varepsilon}(\gamma_{ij}(u^{\delta})) \|_{L^2(\Omega^+ \times D)} \leq c\delta^{\frac{1}{2}} \quad \text{for } i, j = 1, 2, 3, \tag{6.2}$$

$$\delta \| \mathcal{T}^{\varepsilon}(\mathcal{U}_i^{\delta +}) \|_{L^2(\Omega^+ \times D)} \leq c \quad \text{for } i = 1, 2, 3, \tag{6.3}$$

$$\delta \| \mathcal{T}^{\varepsilon}(\mathcal{R}_i^{\delta +}) \|_{L^2(\Omega^+ \times D)} \leq c, \quad \text{for } i = 1, 2, 3, \tag{6.4}$$

$$\| \mathcal{T}^{\varepsilon}(\overline{u}_i^{\delta +}) \|_{L^2(\Omega^+ \times D)} \leq c\delta^2, \quad \text{for } i = 1, 2, 3, \tag{6.5}$$

$$\begin{cases} \left\| \dfrac{\partial}{\partial X_\alpha} \mathcal{T}^{\varepsilon}(\overline{u}^{\delta +}) \right\|_{(L^2(\Omega^+ \times D))^3} \leq c\delta^2, \quad \text{for } \alpha = 1, 2, \\ \left\| \dfrac{\partial}{\partial x_3} \mathcal{T}^{\varepsilon}(\overline{u}^{\delta +}) \right\|_{(L^2(\Omega^+ \times D))^3} \leq c\delta^{\frac{1}{2}}, \end{cases} \tag{6.6}$$

$$\| \mathcal{T}^{\varepsilon}(\sigma_{ij}^{\delta}) \|_{L^2(\Omega^+ \times D)} \leq c\delta^{\frac{1}{2}}. \quad \text{for } i = 1, 2, 3. \tag{6.7}$$

As far as the fields defined in Ω^- are concerned, Lemma 4.3, Lemma 4.4, the properties (5.3), (5.4) and (5.5) of Π^{δ} and those of $\mathcal{T}_-^{\varepsilon}$ recalled in Annex 1 permit to obtain the following lemma:

Lemma 6.2. *If the forces satisfy conditions (4.47)÷(4.49), then there exists a constant c (independent δ), such that (recall that $\varepsilon = \frac{a}{k}\delta^{\frac{3}{2}}$)*

$$\| \Pi^{\delta}(u_\alpha^{\delta}) \|_{H^1(\Omega^-)} \leq c \quad \text{for } \alpha = 1, 2, \tag{6.8}$$

$$\delta \| \Pi^{\delta}(u_3^{\delta}) \|_{H^1(\Omega^-)} \leq c, \tag{6.9}$$

$$\| \Pi^{\delta}(\gamma_{ij}(u^{\delta})) \|_{L^2(\Omega^-)} \leq c \quad \text{for } i, j = 1, 2, 3, \tag{6.10}$$

$$\| \Pi^{\delta}(\sigma_{ij}^{\delta -}) \|_{L^2(\Omega^-)} \leq c, \quad \text{for } i, j = 1, 2, 3, \tag{6.11}$$

$$\| \Pi^{\delta}(\overline{u}_i^{\delta -}) \|_{L^2(\Omega^-)} \leq c\delta, \quad \text{for } i = 1, 2, 3, \tag{6.12}$$

$$\text{for } i = 1, 2, 3, \qquad \begin{cases} \| \Pi^{\delta}(D\overline{u}_i^{\delta -}) \|_{(L^2(\Omega^-))^3} \leq c \\ \left\| \dfrac{\partial}{\partial x_\alpha} \Pi^{\delta}(\overline{u}_i^{\delta -}) \right\|_{L^2(\Omega^-)} \leq c, \quad \text{for } \alpha = 1, 2, \\ \left\| \dfrac{\partial}{\partial X_3} \Pi^{\delta}(\overline{u}_i^{\delta -}) \right\|_{L^2(\Omega^-)} \leq c\delta, \end{cases} \tag{6.13}$$

$$\| \mathcal{T}_-^{\varepsilon}(\Pi^{\delta}(\gamma_{ij}(u^{\delta}))) \|_{L^2(\Omega^- \times Y)} \leq c, \quad \text{for } i, j = 1, 2, 3, \tag{6.14}$$

$$\| \mathcal{T}_-^{\varepsilon}(\Pi^{\delta}(\sigma_{ij}^{\delta})) \|_{L^2(\Omega^- \times Y)} \leq c, \quad \text{for } i, j = 1, 2, 3. \tag{6.15}$$

Indeed, the two last lemmas together with the properties of $\mathcal{T}_-^{\varepsilon}$ recalled in Annex 1 lead to the following weak convergence results:

Lemma 6.3. *Assume that the forces satisfy conditions (4.47)÷(4.49).*

For a subsequence still indexed by δ:

• *there exist $u_i^{0+} \in L^2(\Omega^+ \times D)$ and $\overline{u}_i^{0+} \in L^2(\Omega^+, H^1(D))$, for $i = 1, 2, 3$, such that, as δ tends to zero, (recall that $\varepsilon = \dfrac{a}{k}\delta^{\frac{3}{2}}$)*

$$\delta \mathcal{T}^\varepsilon(u_i^\delta) \rightharpoonup u_i^{0+} \text{ weakly in } L^2(\Omega^+ \times D), \quad \text{for } i = 1, 2, 3, \tag{6.16}$$

$$\frac{1}{\delta^2}\mathcal{T}^\varepsilon(\overline{u}_i^{\delta+}) \rightharpoonup \overline{u}_i^{0+} \text{ weakly in } L^2(\Omega^+, H^1(D)), \quad \text{for } i = 1, 2, 3, \tag{6.17}$$

• *there exist $\mathcal{U}_i^{0+} \in L^2(\omega, H^1((0, L)))$, $\mathcal{R}_i^{0+} \in L^2(\omega, H^1((0, L)))$, for $i = 1, 2, 3$, and $Z^+ \in (L^2(\Omega^+))^3$ such that, as δ tends to zero,*

$$\delta \mathcal{U}_i^{\delta+} \rightharpoonup \mathcal{U}_i^{0+}, \text{ weakly in } L^2\left(\omega, H^1((0, L))\right), \text{ for } i = 1, 2, 3 \tag{6.18}$$

$$\delta \mathcal{R}_i^{\delta+} \rightharpoonup \mathcal{R}_i^{0+}, \text{ weakly in } L^2\left(\omega, H^1((0, L))\right), \text{ for } i = 1, 2, 3, \tag{6.19}$$

$$\delta^{-\frac{1}{2}}\left(\frac{\partial \mathcal{U}^{\delta+}}{\partial x_3} - \mathcal{R}^{\delta+} \wedge e_3\right) \rightharpoonup Z^+, \text{ weakly in } \left(L^2(\Omega^+)\right)^3, \tag{6.20}$$

• *there exist $X_{ij}^+ \in L^2(\Omega^+ \times D)$ and $\Sigma_{ij}^+ \in L^2(\Omega^+ \times D)$, for $i, j = 1, 2, 3$, such that, as δ tends to zero,*

$$\delta^{-\frac{1}{2}}\mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta)) \rightharpoonup X_{ij}^+, \text{ weakly in } L^2(\Omega^+ \times D), \text{ for } i, j = 1, 2, 3, \tag{6.21}$$

$$\delta^{-\frac{1}{2}}\mathcal{T}^\varepsilon(\sigma_{ij}^\delta) \rightharpoonup \Sigma_{ij}^+, \text{ weakly in } L^2(\Omega^+ \times D), \text{ for } i, j = 1, 2, 3, \tag{6.22}$$

Lemma 6.4. *Assume that the forces satisfy conditions (4.47)÷(4.49).*

For a subsequence still indexed by δ:

• *there exist $u_i^{0-} \in L^2(\Omega^-)$ and $\overline{u}_i^{0-} \in L^2(\omega, H^1((-1, 0)))$, for $i = 1, 2, 3$, such that, as δ tends to zero,*

$$\Pi^\delta(u_\alpha^\delta) \rightharpoonup u_\alpha^{0-} \text{ weakly in } H^1(\Omega^-), \quad \text{for } \alpha = 1, 2, \tag{6.23}$$

$$\delta \Pi^\delta(u_3^\delta) \rightharpoonup u_3^{0-} \text{ weakly in } H^1(\Omega^-), \tag{6.24}$$

$$\frac{1}{\delta}\Pi^\delta(\overline{u}_i^{\delta-}) \rightharpoonup \overline{u}_i^{0-} \text{ weakly in } L^2(\omega, H^1((-1, 0))), \quad \text{for } i = 1, 2, 3, \tag{6.25}$$

• *there exist $\mathcal{U}_i^{0-} \in H_0^1(\omega)$, $\mathcal{R}_i^{0-} \in H_0^1(\omega)$, for $i = 1, 2, 3$, and $Z_\alpha^- \in L^2(\omega)$, for $\alpha = 1, 2$, such that, as δ tends to zero,*

$$\mathcal{U}_\alpha^{\delta-} \rightharpoonup \mathcal{U}_\alpha^{0-} \text{ weakly in } H^1(\omega), \text{ for } \alpha = 1, 2, \tag{6.26}$$

$$\delta \mathcal{U}_3^{\delta-} \rightharpoonup \mathcal{U}_3^{0-} \text{ weakly in } H^1(\omega), \tag{6.27}$$

$$\delta \mathcal{R}_\alpha^{\delta-} \rightharpoonup \mathcal{R}_\alpha^{0-} \text{ weakly in } H^1(\omega), \text{ for } \alpha = 1, 2, \tag{6.28}$$

$$\frac{\partial \mathcal{U}_3^{\delta-}}{\partial x_1} + \mathcal{R}_2^{\delta-} \rightharpoonup Z_1^- \text{ weakly in } L^2(\omega), \tag{6.29}$$

$$\frac{\partial \mathcal{U}_3^{\delta-}}{\partial x_2} - \mathcal{R}_1^{\delta-} \rightharpoonup Z_2^- \textit{ weakly in } L^2(\omega), \tag{6.30}$$

• *there exist* $X_{ij}^- \in L^2(\Omega^-)$ *and* $\Sigma_{ij}^- \in L^2(\Omega^-)$*, for* $i, j = 1, 2, 3$*, such that, as* δ *tends to zero,*

$$\Pi^\delta(\gamma_{ij}(u^\delta)) \rightharpoonup X_{ij}^- \textit{ weakly in } L^2(\Omega^-), \textit{ for } i, j = 1, 2, 3, \tag{6.31}$$

$$\Pi^\delta(\sigma_{ij}^\delta) \rightharpoonup \Sigma_{ij}^- \textit{ weakly in } L^2(\Omega^-), \textit{ for } i, j = 1, 2, 3. \tag{6.32}$$

• *there exist* $\overline{X}_{ij}^- \in L^2(\Omega^- \times Y)$ *and* $\overline{\Sigma}_{ij}^- \in L^2(\Omega^- \times Y)$*, for* $i, j = 1, 2, 3$*, such that, as* δ *tends to zero,*

$$\mathcal{T}_-^\varepsilon(\Pi^\delta(\gamma_{ij}(u^\delta))) \rightharpoonup \overline{X}_{ij}^- \textit{ weakly in } L^2(\Omega^- \times Y), \quad \textit{for } i, j = 1, 2, 3, \tag{6.33}$$

$$\mathcal{T}_-^\varepsilon(\Pi^\delta(\sigma_{ij}^\delta)) \rightharpoonup \overline{\Sigma}_{ij}^- \textit{ weakly in } L^2(\Omega^- \times Y), \quad \textit{for } i, j = 1, 2, 3. \tag{6.34}$$

7 Relations between the limit fields

We begin with the limit fields defined in Lemma 6.1 for which the derivations are similar to the ones performed in Section 5.4 of [2].

7.1 Relations between the limit fields in Ω^+

Proceeding exactly as in Section 5.4 of [2], we deduce from (6.18), (6.19) and (6.20) that

$$\frac{\partial \mathcal{U}_1^{0+}}{\partial x_3} = \mathcal{R}_2^{0+} \text{ in } \Omega^+, \tag{7.1}$$

$$\frac{\partial \mathcal{U}_2^{0+}}{\partial x_3} = -\mathcal{R}_1^{0+} \text{ in } \Omega^+. \tag{7.2}$$

Then $\mathcal{U}_\alpha^{0+} \in L^2(\omega, H^2((0, L)))$, for $\alpha = 1, 2$. We also have, still following Section 5.4 of [2], that

$$\begin{aligned} &u_\alpha^{0+}(x_1, x_2, x_3, X_1, X_2) = \mathcal{U}_\alpha^{0+}(x_1, x_2, x_3), \\ &\qquad \text{for almost any } (x_1, x_2, x_3) \in \Omega^+, \quad (X_1, X_2) \in D, \quad \text{for } \alpha = 1, 2. \end{aligned} \tag{7.3}$$

Now using (3.5), (6.16), (6.17), (6.18) and (6.19) first yields

$$\begin{aligned} &u_3^{0+}(x_1, x_2, x_3, X_1, X_2) = \mathcal{U}_3^{0+}(x_1, x_2, x_3), \\ &\qquad \text{for almost any } (x_1, x_2, x_3) \in \Omega^+, \quad (X_1, X_2) \in D, \end{aligned}$$

while estimate (4.16) then gives that $\mathcal{U}_3^{0+}$ does not depend on x_3, so that

$$\begin{aligned} &u_3^{0+}(x_1, x_2, x_3, X_1, X_2) = \mathcal{U}_3^{0+}(x_1, x_2), \\ &\qquad \text{for almost any } (x_1, x_2, x_3) \in \Omega^+, \quad (X_1, X_2) \in D. \end{aligned} \tag{7.4}$$

To identify $X^+_{\alpha\beta}$, we proceed as in [2] and write (note that $r = r_\varepsilon = k\varepsilon$)

$$r_\varepsilon \mathcal{T}^\varepsilon(\gamma_{\alpha\beta}(u^\delta)) = r_\varepsilon \mathcal{T}^\varepsilon(\gamma_{\alpha\beta}(\overline{u}^{\delta+})) = \Gamma_{\alpha\beta}\big(\mathcal{T}^\varepsilon(\overline{u}^{\delta+})\big) \text{ a.e. in } \Omega^+ \times D, \text{ for } \alpha,\beta = 1,2,$$

and we recall the definition of $\Gamma_{\alpha\beta}(v)$:

$$\Gamma_{\alpha\beta}(v) = \frac{1}{2}\left(\partial_{X_\beta} v_\alpha + \partial_{X_\alpha} v_\beta\right), \text{ a.e. in } \Omega^+ \times D, \text{ for } \alpha,\beta = 1,2.$$

Multiplying the above equality by $\dfrac{1}{\delta^2}$ and using (6.17) and (6.21) yields

$$aX^+_{\alpha\beta} = \Gamma_{\alpha\beta}(\overline{u}^{0+}), \text{ a.e. in } \Omega^+ \times D, \qquad \text{for } \alpha,\beta = 1,2. \tag{7.5}$$

As far as $X^+_{\alpha 3}$, for $\alpha = 1,2$, is concerned, we use (6.17), (6.19), (6.20) and (6.21) and proceed as in [2]

$$X^+_{13} = \frac{1}{2}\left[Z^+_1 - X_2 a\frac{\partial \mathcal{R}^{0+}_3}{\partial x_3} + \frac{1}{a}\frac{\partial \overline{u}^{0+}_3}{\partial X_1}\right], \text{ a.e. in } \Omega^+ \times D,$$

or equivalently

$$X^+_{13} = \frac{1}{2}\left[\frac{\partial}{\partial X_1}\left(X_1 Z^+_1 + \frac{1}{a}\overline{u}^{0+}_3\right) - X_2 a\frac{\partial \mathcal{R}^{0+}_3}{\partial x_3}\right], \text{ a.e. in } \Omega^+ \times D. \tag{7.6}$$

Similarly,

$$X^+_{23} = \frac{1}{2}\left[\frac{\partial}{\partial X_2}\left(X_2 Z^+_2 + \frac{1}{a}\overline{u}^{0+}_3\right) + X_1 a\frac{\partial \mathcal{R}^{0+}_3}{\partial x_3}\right], \text{ a.e. in } \Omega^+ \times D. \tag{7.7}$$

To obtain the expression of X^+_{33}, we first introduce the sequence $W^{\delta+}_3$ of $L^2(\omega; H^1((0,L))$ through

$$W^{\delta+}_3(x_1,x_2,x_3) = \frac{1}{\delta^{\frac{1}{2}}}\int_0^{x_3} \frac{\partial \mathcal{U}^{\delta+}_3}{\partial x_3}(x_1,x_2,\zeta)d\zeta = \frac{1}{\delta^{\frac{1}{2}}}(\mathcal{U}^{\delta+}_3(x_1,x_2,x_3) - \mathcal{U}^{\delta+}_3(x_1,x_2,0)), \tag{7.8}$$

and estimate (4.16) shows that, up to a subsequence

$$W^{\delta+}_3 \rightharpoonup \mathcal{W}^{0+}_3 \text{ weakly in } L^2(\omega; H^1((0,L))). \tag{7.9}$$

Since $\delta^{-\frac{1}{2}}\dfrac{\partial \mathcal{U}^{\delta+}_3}{\partial x_3} = \dfrac{\partial \mathcal{W}^{\delta+}_3}{\partial x_3}$, proceeding as in [2] leads to

$$X^+_{33} = \frac{\partial \mathcal{W}^{0+}_3}{\partial x_3} - aX_1\frac{\partial^2 \mathcal{U}^{0+}_1}{\partial x_3^2} - aX_2\frac{\partial^2 \mathcal{U}^{0+}_2}{\partial x_3^2}, \text{ a.e. in } \Omega^+ \times D. \tag{7.10}$$

Once the above expressions of X^+_{ij} are obtained, using the constitutive law (0.13) and (6.21), (6.22) lead to

$$\Sigma_{11}^+ = \frac{1}{a}\left[(\lambda+2\mu)\Gamma_{11}(\overline{u}^{0+}) + \lambda\Gamma_{22}(\overline{u}^{0+})\right] + \lambda\left[\frac{\partial \mathcal{W}_3^{0+}}{\partial x_3} - aX_1\frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} - aX_2\frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2}\right], \quad (7.11)$$

$$\Sigma_{22}^+ = \frac{1}{a}\left[(\lambda+2\mu)\Gamma_{22}(\overline{u}^{0+}) + \lambda\Gamma_{11}(\overline{u}^{0+})\right] + \lambda\left[\frac{\partial \mathcal{W}_3^{0+}}{\partial x_3} - aX_1\frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} - aX_2\frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2}\right], \quad (7.12)$$

$$\Sigma_{12}^+ = 2\frac{\mu}{a}\Gamma_{12}(\overline{u}^{0+}), \quad (7.13)$$

$$\Sigma_{13}^+ = \mu\left[\frac{\partial}{\partial X_1}\left(X_1 Z_1^+ + \frac{1}{a}\overline{u}_3^{0+}\right) - aX_2\frac{\partial \mathcal{R}_3^{0+}}{\partial x_3}\right], \quad (7.14)$$

$$\Sigma_{23}^+ = \mu\left[\frac{\partial}{\partial X_2}\left(X_2 Z_2^+ + \frac{1}{a}\overline{u}_3^{0+}\right) + aX_1\frac{\partial \mathcal{R}_3^{0+}}{\partial x_3}\right], \quad (7.15)$$

$$\Sigma_{33}^+ = (\lambda+2\mu)\left(\frac{\partial \mathcal{W}_3^{0+}}{\partial x_3} - aX_1\frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} - aX_2\frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2}\right) + \frac{\lambda}{a}\left(\Gamma_{11}(\overline{u}^{0+}) + \Gamma_{22}(\overline{u}^{0+})\right), \quad (7.16)$$

almost everywhere in $\Omega^+ \times D$.

7.2 Relations between the limit fields in Ω^-

7.2.1 Limit displacement

In Ω_δ^-, we have by (3.9)

$$u_1^\delta = \mathcal{U}_1^{\delta-} + \left(x_3 + \frac{\delta}{2}\right)\mathcal{R}_2^{\delta-} + \overline{u}_1^{\delta-},$$

so that in Ω^-

$$\Pi^\delta(u_1^\delta) = \mathcal{U}_1^{\delta-} + \left(X_3 + \frac{1}{2}\right)\delta\mathcal{R}_2^{\delta-} + \Pi^\delta(\overline{u}_1^{\delta-}).$$

Passing to the limit as δ tends to 0 in the above equality, using (6.23), (6.26) and (6.28), gives

$$u_1^{0-} = \mathcal{U}_1^{0-} + \left(X_3 + \frac{1}{2}\right)\mathcal{R}_2^{0-}. \quad (7.17)$$

Proceeding as above for u_2^δ leads to

$$u_2^{0-} = \mathcal{U}_2^{0-} - \left(X_3 + \frac{1}{2}\right)\mathcal{R}_1^{0-}. \quad (7.18)$$

As far as u_3^δ is concerned, we have in ω_δ^- (still by (3.9))

$$u_3^\delta = \mathcal{U}_3^{\delta-} + \overline{u}_3^{\delta-},$$

so that in Ω^-

$$\Pi^\delta(u_3^\delta) = \mathcal{U}_3^{\delta-} + \Pi^\delta(\overline{u}_3^{\delta-}).$$

Passing to the limit as δ tends to 0 with the help of (6.12), (6.24) and (6.27) leads to

$$u_3^{0-} = \mathcal{U}_3^{0-}, \tag{7.19}$$

and in particular u_3^{0-} is independent of X_3.

Let us now derive a standard relation between $\mathcal{R}_\alpha^{0-}$ and $\mathcal{U}_3^{0-}$ in ω which leads to a Kirchhoff-Love displacement for u^{0-} in Ω^-. Indeed (6.27), (6.28) and (6.29) show that

$$\mathcal{R}_1^{0-} = \frac{\partial \mathcal{U}_3^{0-}}{\partial x_2}, \tag{7.20}$$

$$\mathcal{R}_2^{0-} = -\frac{\partial \mathcal{U}_3^{0-}}{\partial x_1}, \tag{7.21}$$

and we first deduce that $\mathcal{U}_3^{0-} \in H_0^2(\omega)$. Secondly, inserting (7.20) and (7.21) into (7.17) and (7.18) yields in Ω^-

$$u_\alpha^{0-} = \mathcal{U}_\alpha^{0-} - \left(X_3 + \frac{1}{2}\right)\frac{\partial \mathcal{U}_3^{0-}}{\partial x_\alpha}, \quad \text{for } \alpha = 1, 2, \tag{7.22}$$

and this means that $(u_1^{0-}, u_2^{0-}, u_3^{0-})$ is a displacement field of Kirchhoff-Love's type.

7.2.2 Limit of the unfold deformation

In this subsection we derive the relations between the weak limit of the unfold deformation $\mathcal{T}_-^\varepsilon(\Pi^\delta(\gamma_{ij}(u^\delta)))$ in $\Omega^- \times Y$ and those of the unfold derivatives of $\mathcal{U}^{\delta-}$, $\mathcal{R}^{\delta-}$ and $\overline{u}^{\delta-}$. We begin with a lemma which describes the behaviors of $\mathcal{U}^{\delta-}$, $\mathcal{R}^{\delta-}$ and $\overline{u}^{\delta-}$.

Lemma 7.1. *Assume the forces satisfy conditions (4.47)÷(4.49). Then there exist $\widehat{\mathcal{U}}_\alpha^0$, $\widehat{\mathcal{R}}_\alpha^0$, $\breve{u} \in L^2(\omega, H^1_{per}(Y))$, for $\alpha = 1, 2$, $\widehat{\overline{u}}_i^0 \in L^2(\Omega^-, H^1_{per}(Y))$, for $i = 1, 2, 3$, such that for a subsequence still indexed by δ*

$$\mathcal{T}_-^\varepsilon(\mathcal{U}_\alpha^{\delta-}) \to \mathcal{U}_\alpha^{0-} \text{ strongly in } L^2(\omega \times Y), \text{ for } \alpha = 1, 2, \tag{7.23}$$

$$\mathcal{T}_-^\varepsilon\left(\frac{\partial \mathcal{U}_\alpha^{\delta-}}{\partial x_\beta}\right) \rightharpoonup \frac{\partial \mathcal{U}_\alpha^{0-}}{\partial x_\beta} + \frac{\partial \widehat{\mathcal{U}}_\alpha^0}{\partial X_\beta} \text{ weakly in } L^2(\omega \times Y), \text{ for } \alpha, \beta = 1, 2, \tag{7.24}$$

$$\delta\mathcal{T}_-^\varepsilon(\mathcal{U}_3^{\delta-}) \to \mathcal{U}_3^{0-} \text{ strongly in } L^2(\omega \times Y), \tag{7.25}$$

$$\delta\mathcal{T}_-^\varepsilon\left(\frac{\partial \mathcal{U}_3^{\delta-}}{\partial x_\alpha}\right) \rightharpoonup \frac{\partial \mathcal{U}_3^{0-}}{\partial x_\alpha} \text{ weakly in } L^2(\omega \times Y), \text{ for } \alpha = 1, 2, \tag{7.26}$$

$$\delta\mathcal{T}_-^\varepsilon(\mathcal{R}_\alpha^{\delta-}) \to \mathcal{R}_\alpha^{0-} \text{ strongly in } L^2(\omega \times Y), \text{ for } \alpha = 1, 2, \tag{7.27}$$

$$\delta\mathcal{T}_-^\varepsilon\left(\frac{\partial \mathcal{R}_\alpha^{\delta-}}{\partial x_\beta}\right) \rightharpoonup \frac{\partial \mathcal{R}_\alpha^{0-}}{\partial x_\beta} + \frac{\partial \widehat{\mathcal{R}}_\alpha^0}{\partial X_\beta} \text{ weakly in } L^2(\omega \times Y), \text{ for } \alpha, \beta = 1, 2, \tag{7.28}$$

$$\mathcal{T}_{-}^{\varepsilon}\left(\frac{\partial \mathcal{U}_3^{\delta -}}{\partial x_1}+\mathcal{R}_2^{\delta -}\right) \rightharpoonup Z_1^- + \frac{\partial \check{u}}{\partial X_1} \text{ weakly in } L^2(\omega\times Y), \tag{7.29}$$

$$\mathcal{T}_{-}^{\varepsilon}\left(\frac{\partial \mathcal{U}_3^{\delta -}}{\partial x_2}+\mathcal{R}_1^{\delta -}\right) \rightharpoonup Z_2^- + \frac{\partial \check{u}}{\partial X_2} \text{ weakly in } L^2(\omega\times Y), \tag{7.30}$$

$$\mathcal{T}_{-}^{\varepsilon}(\Pi^\delta(\overline{u}_i^{\delta -})) \to 0 \text{ strongly in } L^2(\Omega^-\times Y), \text{ for } i=1,2,3, \tag{7.31}$$

$$\frac{1}{\delta}\left(\mathcal{T}_{-}^{\varepsilon}(\Pi^\delta(\overline{u}_i^{\delta -}))\right) \rightharpoonup \overline{u}_i^{0-} \text{ weakly in } L^2(\omega\times Y, H^1((-1,0))), \text{ for } i=1,2,3, \tag{7.32}$$

$$\mathcal{T}_{-}^{\varepsilon}\left(\Pi^\delta\left(\frac{\partial \overline{u}_i^{\delta -}}{\partial x_\alpha}\right)\right) \rightharpoonup \frac{\partial \widehat{\overline{u}}_i^0}{\partial X_\alpha} \text{ weakly in } L^2(\Omega^-\times Y), \text{ for } i=1,2,3, \text{ and } \alpha=1,2, \tag{7.33}$$

$$\mathcal{T}_{-}^{\varepsilon}\left(\Pi^\delta\left(\frac{\partial \overline{u}_i^{\delta -}}{\partial x_3}\right)\right) \rightharpoonup \frac{\partial \overline{u}_i^{0-}}{\partial X_3} \text{ weakly in } L^2(\Omega^-\times Y), \text{ for } i=1,2,3, \tag{7.34}$$

as δ tends to 0. Moreover the functions $\widehat{\overline{u}}_i^0$ satisfy

$$\int_{-1}^0 \widehat{\overline{u}}_i^0 dX_3 = \int_{-1}^0 \left(X_3+\frac{1}{2}\right)\widehat{\overline{u}}_\alpha^0 dX_3 = 0 \text{ a.e. in } \omega. \tag{7.35}$$

Proof. Convergences (7.23)÷(7.28), (7.31)÷(7.34) are mainly direct consequences of Lemma A1 of Appendix A. Indeed by (6.26)÷(6.28) one may assume that $\mathcal{U}_\alpha^{\delta -}$, $\delta\mathcal{U}_3^{\delta -}$ and $\delta\mathcal{R}_\alpha^{\delta -}$ strongly converge in $L^2(\omega)$ so that Lemma A1 of Appendix A shows that (7.23), (7.25) and (7.27) hold true. The same lemma also leads to the existence of $\widehat{\mathcal{U}}_i^0$, $\widehat{\mathcal{R}}_\alpha^0 \in L^2(\omega, H^1_{per}(Y))$ such that (7.24) and (7.28) are valid and a priori

$$\delta\mathcal{T}_{-}^{\varepsilon}\left(\frac{\partial \mathcal{U}_3^{\delta -}}{\partial x_\alpha}\right) \rightharpoonup \frac{\partial \mathcal{U}_3^{0-}}{\partial x_\alpha}+\frac{\partial \widehat{\mathcal{U}}_3^0}{\partial X_\alpha} \text{ weakly in } L^2(\Omega^-\times Y), \text{ for } \alpha=1,2. \tag{7.36}$$

Actually $\widehat{\mathcal{U}}_3^0=0$. Indeed we have by (6.29)

$$\delta\left\|\mathcal{T}_{-}^{\varepsilon}\left(\frac{\partial \mathcal{U}_3^{\delta -}}{\partial x_1}\right)+\mathcal{T}_{-}^{\varepsilon}\left(\mathcal{R}_2^{\delta -}\right)\right\|_{L^2(\Omega^-\times Y)} \le c\delta$$

and by (7.27) and (7.36)

$$\delta\left(\mathcal{T}_{-}^{\varepsilon}\left(\frac{\partial \mathcal{U}_3^{\delta -}}{\partial x_1}\right)+\mathcal{T}_{-}^{\varepsilon}\left(\mathcal{R}_2^{\delta -}\right)\right) \rightharpoonup \frac{\partial \mathcal{U}_3^{0-}}{\partial x_1}+\frac{\partial \widehat{\mathcal{U}}_3^0}{\partial X_1}+\mathcal{R}_2^{0-} \text{ weakly in } L^2(\Omega^-\times Y).$$

In view of (7.21), we obtain $\dfrac{\partial \widehat{\mathcal{U}}_3^0}{\partial X_1}=0$. Similarly, (7.20) leads to $\dfrac{\partial \widehat{\mathcal{U}}_3^0}{\partial X_2}=0$. Since $\widehat{\mathcal{U}}_3^0$ is defined up to a constant (it is only defined through its gradient with respect to (X_1,X_2); see Lemma A1 of Appendix A), one obtains $\widehat{\mathcal{U}}_3^0=0$ and (7.26) is established.

As far as $\overline{u}_i^{\delta-}$ is concerned, we have using (6.25)

$$\frac{1}{\delta}\|\mathcal{T}_-^\varepsilon(\Pi^\delta\overline{u}_i^{\delta-})\|_{L^2(\Omega^-\times Y)} \leq \frac{c}{\delta}\|\Pi^\delta\overline{u}_i^{\delta-}\|_{L^2(\Omega^-)} \leq c \tag{7.37}$$

so which leads to (7.31).

Moreover (see Appendix A)

$$\frac{1}{\delta}\left\|\frac{\partial}{\partial X_\alpha}\left(\mathcal{T}_-^\varepsilon(\Pi^\delta\overline{u}_i^{\delta-})\right)\right\|_{L^2(\Omega^-\times Y)} \leq \frac{\varepsilon}{\delta}\left\|\frac{\partial}{\partial x_\alpha}\left(\Pi^\delta\overline{u}_i^{\delta-}\right)\right\|_{L^2(\Omega^-)} \leq c\delta^{\frac{1}{2}},$$

$$\text{for } i=1,2,3 \text{ and } \alpha=1,2,$$

by (6.13). Then (7.37) shows that the weak limit in $L^2(\Omega^-\times Y)$ of $\frac{1}{\delta}\mathcal{T}_-^\varepsilon(\Pi^\delta\overline{u}_i^{\delta-})$ is actually independent of (X_1,X_2). Let us emphasize that this last result strongly uses $\frac{\varepsilon}{\delta}\to 0$. As a consequence of Lemma A1 *iv)*, we deduce that the weak limits of $\frac{1}{\delta}\mathcal{T}_-^\varepsilon(\Pi^\delta\overline{u}_i^{\delta-})$ and $\frac{1}{\delta}\Pi^\delta(\overline{u}_i^{\delta-})$ are the same and (7.32) is proved.

The existence of $\widehat{\overline{u}}^0$ such that (7.33) holds true is a direct consequence of (6.13), (7.31) and Lemma A1 of Appendix A. Since

$$\mathcal{T}_-^\varepsilon\left(\Pi^\delta\left(\frac{\partial\overline{u}_i^{\delta-}}{\partial x_3}\right)\right) = \frac{1}{\delta}\mathcal{T}_-^\varepsilon\left(\frac{\partial}{\partial X_3}\Pi^\delta\left(\overline{u}_i^{\delta-}\right)\right) = \frac{1}{\delta}\frac{\partial}{\partial X_3}\left(\mathcal{T}_-^\varepsilon\left(\Pi^\delta\left(\overline{u}_i^{\delta-}\right)\right)\right),$$

the convergence (7.32) implies (7.34).

To show that $\widehat{\overline{u}}_i^0$ satisfy (7.35), we recall the kinematic conditions (3.10) and (3.11) on $\overline{u}^{\delta-}$. Then we have e.g. by (3.10), $\mathcal{T}_-^\varepsilon\left(\int_{-1}^0\frac{\partial\Pi^\delta\left(\overline{u}_i^{\delta-}\right)}{\partial x_\alpha}dX_3\right)=0$ so that (7.33) shows that $\int_{-1}^0\frac{\partial\widehat{\overline{u}}_i^0}{\partial X_\alpha}dX_3=0$. Then the function $\int_{-1}^0\widehat{\overline{u}}_i^0 dX_3$ is independent of the local variable (X_1,X_2). Proceeding identically, starting form (3.11) yields that the function $\int_{-1}^0\left(X_3+\frac{1}{2}\right)\widehat{\overline{u}}_\alpha^0 dX_3$ is also independent of (X_1,X_2). Since $\widehat{\overline{u}}_i^0$ is defined up to a function of X_3, we obtain (7.35).

It remains to show (7.29) and (7.30) which are not direct consequences of Lemma A1 of Appendix A and of the a priori estimates on $\mathcal{U}_3^{\delta-}$ and $\mathcal{R}_\alpha^{\delta-}$. Loosely speaking (7.29) and (7.30) show that the oscillations of the fields $\frac{\partial\mathcal{U}_3^{\delta-}}{\partial x_\alpha}\pm\mathcal{R}_\beta^{\delta-}$ $(\alpha\neq\beta)$ can be asymptotically described by the gradient with respect to the local variable (X_1,X_2) of a function $\breve{u}\in L^2(\omega,H^1_{per}(Y))$, as if $(\mathcal{R}_2^{\delta-},\mathcal{R}_1^{\delta-})$ was a gradient with respect to the variable (x_1,x_2). Actually, this is a consequence of the $H^1(\omega)$-estimate on $\mathcal{R}_\alpha^{\delta-}$ and to shorten the proof of the actual lemma, (7.29) and (7.30) is established in Lemma A3 of Appendix B. $\square$

We are now in a position to identify the $\overline{X}^{-}_{ij}$'s which are defined in (6.33). Due to the decomposition (3.9) of u^δ

$$\gamma_{\alpha\beta}(u^\delta) = \gamma_{\alpha\beta}(\mathcal{U}^{\delta-}) + \gamma_{\alpha\beta}(\mathcal{R}^{\delta-} \wedge e_3)\left(x_3 + \frac{\delta}{2}\right) + \gamma_{\alpha\beta}(\overline{u}^{\delta-}), \tag{7.38}$$

$$\gamma_{13}(u^\delta) = \frac{1}{2}\left(\mathcal{R}_2^{\delta-} + \frac{\partial \mathcal{U}_3^{\delta-}}{\partial x_1} + \frac{\partial \overline{u}_1^{\delta-}}{\partial x_3} + \frac{\partial \overline{u}_3^{\delta-}}{\partial x_1}\right), \tag{7.39}$$

$$\gamma_{23}(u^\delta) = \frac{1}{2}\left(-\mathcal{R}_1^{\delta-} + \frac{\partial \mathcal{U}_3^{\delta-}}{\partial x_2} + \frac{\partial \overline{u}_2^{\delta-}}{\partial x_3} + \frac{\partial \overline{u}_3^{\delta-}}{\partial x_2}\right), \tag{7.40}$$

$$\gamma_{33}(u^\delta) = \frac{\partial \overline{u}_3^{\delta-}}{\partial x_3}, \tag{7.41}$$

Remark that $\Pi^\delta(w) = w$ for any function w which is independent of x_3.

Applying $\mathcal{T}_-^\varepsilon \circ \Pi^\delta$ to (7.38)÷(7.41) and passing to the limit as δ tends to 0 with the help of Lemma 7.1 give

$$\begin{aligned} \overline{X}^{-}_{\alpha\beta} &= \gamma_{\alpha\beta}(\mathcal{U}^{0-}) + \left(X_3 + \tfrac{1}{2}\right)\gamma_{\alpha\beta}(\mathcal{R}^{0-} \wedge e_3) + \\ &\Gamma_{\alpha\beta}(\widehat{\mathcal{U}}^0) + \left(X_3 + \tfrac{1}{2}\right)\Gamma_{\alpha\beta}(\widehat{\mathcal{R}}^0 \wedge e_3) + \Gamma_{\alpha\beta}(\widehat{\overline{u}}^0), \end{aligned} \tag{7.42}$$

$$\overline{X}^{-}_{13} = \frac{1}{2}\left(Z_1^- + \frac{\partial \overline{u}_1^{0-}}{\partial X_3} + \frac{\partial \check{u}}{\partial X_1} + \frac{\partial \widehat{\overline{u}}_3^0}{\partial X_1}\right), \tag{7.43}$$

$$\overline{X}^{-}_{23} = \frac{1}{2}\left(Z_2^- + \frac{\partial \overline{u}_2^{0-}}{\partial X_3} + \frac{\partial \check{u}}{\partial X_2} + \frac{\partial \widehat{\overline{u}}_3^0}{\partial X_2}\right), \tag{7.44}$$

$$\overline{X}^{-}_{33} = \frac{\partial \overline{u}_3^{0-}}{\partial X_3}. \tag{7.45}$$

7.3 Limit kinematic conditions

Proceeding again as in Section 5.5 of [2], we first obtain from (4.11), (6.18) on one hand, and (4.14), (6.19) on the other hand

$$\mathcal{U}_\alpha^{0+}(x_1, x_2, 0) = 0, \text{ a.e. in } \omega, \text{ for } \alpha = 1, 2, \tag{7.46}$$

and

$$\mathcal{R}_3^{0+}(x_1, x_2, 0) = 0, \text{ a.e. in } \omega. \tag{7.47}$$

By contrast with [2], here estimates (4.13) and (6.19) does not permit to conclude that the components $\mathcal{R}_\alpha^{0+}$ vanish on ω (and we will see later that this is not the case). We turn now to the continuity condition between $\mathcal{U}_3^{0+}$ and $\mathcal{U}_3^{0-}$ on ω and the argument is identical of that used in [2]. We consider the function defined by $w_\delta = \delta u_3^\delta$ in Ω_ε^+ and $w_\delta = \delta \Pi^\delta(u_3^\delta)$ in Ω^-. Since $\mathcal{T}^\varepsilon(w_\delta)$ is bounded in $L^2(\omega \times D, H^1((-1, L))$ by (6.21) and (6.31), we can repeat the argument used in [2] (see again Section 5.5) for u_3^ε, with w_δ in place of u_3^ε, in order to obtain

$$\mathcal{U}_3^{0-}(x_1,x_2) = \mathcal{U}_3^{0+}(x_1,x_2) \quad \text{ in } \omega. \tag{7.48}$$

Now we investigate the more intricate question of the transmission condition on $\mathcal{R}_\alpha^{0+}(\cdot,\cdot,0)$ on ω. To this end we go back to the definition of $\mathcal{R}_\alpha^{\delta+}$ (see (3.2) and (3.3)) and use the continuity of the trace of u^δ on ω to write

$$\begin{aligned}
&\mathcal{R}_1^{\delta+}(x_1,x_2,0) = \frac{1}{I_2 r^4}\int_{\mathcal{D}_{pq}} (x_2-\varepsilon q)u_3^\delta(x_1,x_2,0)dx_1dx_2,\\
&\qquad\qquad \text{if } (x_1,x_2) \in \Big]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\Big[, \quad (p,q)\in \mathcal{N}^\varepsilon,\\
&\mathcal{R}_1^{\delta+}(x_1,x_2,0) = 0, \quad \text{ if } (x_1,x_2)\in \omega\setminus\widetilde{\omega}_\varepsilon.
\end{aligned} \tag{7.49}$$

Then we use the decomposition of u_3^δ given in (3.9) which leads to

$$\delta\mathcal{R}_1^{\delta+}(x_1,x_2,0) = T_1^\delta + T_2^\delta, \tag{7.50}$$

where T_1^δ and T_2^δ are the functions which are constant and equal to

$$T_1^\delta = \frac{\delta}{I_2 r^4}\int_{\mathcal{D}_{pq}} (x_2-\varepsilon q)\mathcal{U}_3^{\delta-}(x_1,x_2)dx_1dx_2,$$

$$T_2^\delta = \frac{\delta}{I_2 r^4}\int_{\mathcal{D}_{pq}} (x_2-\varepsilon q)\overline{u}_3^{\delta-}(x_1,x_2,0)dx_1dx_2,$$

on each cell $\Big]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\Big[\times \Big]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\Big[$, $(p,q)\in\mathcal{N}^\varepsilon$, and $T_1^\delta = T_2^\delta = 0$ if $(x_1,x_2)\in$ $\omega\setminus\widetilde{\omega}_\varepsilon$. In view of (4.62) and (4.63) and since $r^2 = k^2\varepsilon^2 = a^2\delta^3$, we have

$$\|T_2^\delta\|^2_{L^2(\omega)} \le c\delta,$$

so that $T_2^\delta \to 0$ strongly in $L^2(\omega)$.

As far as T_1^δ is concerned, we write

$$T_1^\delta = \frac{\delta}{I_2 r}\int_D X_2\mathcal{U}_3^{\delta-}(p\varepsilon + rX_1, q\varepsilon + rX_2)dX_1dX_2.$$

According to (6.27)÷(6.30) the sequence $\{\delta\mathcal{U}_3^{\delta-}\}$ is actually compact in $H^1(\omega)$ and converges to $\mathcal{U}_3^{0-}$. We appeal now to Lemma A2 (ii) of Appendix A to claim that

$$T_1^\delta \to \frac{\partial\mathcal{U}_3^{0-}}{\partial x_2} \text{ strongly in } L^2(\omega), \tag{7.51}$$

as $\delta\to 0$.

Then passing to the limit in (7.50), and with an identical proof for $\mathcal{R}_2^{\delta+}$, we obtain in view of (6.19), (7.1), (7.2), (7.51),

$$\mathcal{R}_1^{0+} = -\frac{\partial \mathcal{U}_2^{0+}}{\partial x_3} = \frac{\partial \mathcal{U}_3^{0-}}{\partial x_2} \text{ in } \omega, \tag{7.52}$$

$$\mathcal{R}_2^{0+} = \frac{\partial \mathcal{U}_1^{0+}}{\partial x_3} = -\frac{\partial \mathcal{U}_3^{0-}}{\partial x_1} \text{ in } \omega, \tag{7.53}$$

which are the kinematic transmission between the flexion in the rods and in the plate.

To end this subsection, let us notice that the kinematic conditions (3.10) and (3.11) on $\overline{u}_i^{\delta-}$ together with the definition (6.25) of $\overline{u}_i^{0-}$ give

$$\int_{-1}^{0} \overline{u}_i^{0-} dX_3 = 0 \text{ a.e. in } \omega, \text{ for } i = 1,2,3, \tag{7.54}$$

$$\int_{-1}^{0} \left(X_3 + \frac{1}{2}\right) \overline{u}_\alpha^{0-} dX_3 = 0 \text{ a.e. in } \omega, \text{ for } \alpha = 1,2. \tag{7.55}$$

8 Determination of the fields $\overline{u}^{0+}$, $\mathcal{R}_3^{0+}$, $\widehat{u}^0$, $\widehat{\mathcal{R}}^0$, $\check{u}$ and $\widehat{\overline{u}}^0$

8.1 Determination of $\overline{u}^{0+}$ and $\mathcal{R}_3^{0+}$

Let us remark that, since $\varepsilon \sim \delta^{\frac{3}{2}}$, the ratios between the order of the estimates on $\mathcal{T}^\varepsilon(\sigma^\delta)$, $\mathcal{T}^\varepsilon(\overline{u}^{\delta+})$ and $\mathcal{T}^\varepsilon(\mathcal{R}_3^{\delta+})$ are exactly the same as in [2]. Using the expressions (7.11)÷(7.16) of Σ_{ij}^+ and repeating exactly the argument developed in Subsections 6.1 and 6.2 of [2] permit to obtain $\overline{u}_3^{0+} = \mathcal{R}_3^{0+} = 0$ and

$$\overline{u}_1^{0+} = \nu \left\{ -aX_1 \frac{\partial \mathcal{W}_3^{0+}}{\partial x_3} + a^2 \frac{X_1^2 - X_2^2}{2} \frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} + a^2 X_1 X_2 \frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2} \right\},$$

$$\overline{u}_2^{0+} = \nu \left\{ -aX_2 \frac{\partial \mathcal{W}_3^{0+}}{\partial x_3} + a^2 X_1 X_2 \frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} + a^2 \frac{X_2^2 - X_1^2}{2} \frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2} \right\},$$

where $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is the Poisson coefficient of the material.

As a consequence, we obtain

$$\Sigma_{11}^+ = \Sigma_{22}^+ = \Sigma_{12}^+ = \Sigma_{13}^+ = \Sigma_{23}^+ = 0, \text{ a.e. in } \Omega^+ \times D, \tag{8.1}$$

$$\Sigma_{33}^+ = E \left(\frac{\partial \mathcal{W}_3^{0+}}{\partial x_3} - aX_1 \frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} - aX_2 \frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2} \right), \text{ a.e. in } \Omega^+ \times D, \tag{8.2}$$

where $E = \dfrac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$ is the Young modulus of the elastic material.

8.2 Determination of $\mathcal{W}_3^{0+}$

We recall the definition (7.8) of $W_3^{\delta+}$ so that $W_3^{\delta+}(x_1,x_2,0)=0$ and then by (7.9)

$$\mathcal{W}_3^{0+}(x_1,x_2,0)=0 \text{ a.e. in } \omega. \tag{8.3}$$

In order to show that $\mathcal{W}_3^{0+}=0$ in Ω^+, we repeat the analysis of Section 6.3 of [2] with $\mathcal{W}_3^{0+}$ in place of $\mathcal{U}_3^0$ and for a test function $v=(0,0,v_3)$ with $v_3\in C_0^\infty(\overline{\omega}\times]0,L])$. We obtain

$$2\int_{\Omega^+\times D}\Sigma_{13}^+\gamma_{13}(v)dx_1dx_2dx_3dX_1dX_2+2\int_{\Omega^+\times D}\Sigma_{23}^+\gamma_{23}(v)dx_1dx_2dx_3dX_1dX_2+$$

$$\int_{\Omega^+\times D}\Sigma_{33}^+\gamma_{33}(v)dx_1dx_2dx_3dX_1dX_2=\lim_{\varepsilon\to 0}\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(f_3^+)\mathcal{T}^\varepsilon(v_3)dx_1dx_2dx_3dX_1dX_2.$$

In view of the assumption (4.47), we have $\mathcal{T}^\varepsilon(f_3^+)=\delta^2\mathcal{T}^\varepsilon(F_3^+)$ and then, using (8.1)

$$\int_{\Omega^+\times D}\Sigma_{33}^+\gamma_{33}(v)dx_1dx_2dx_3dX_1dX_2=0.$$

Appealing now the expression (8.2) of Σ_{33}^+ shows that $\mathcal{W}_3^{0+}\in L^2(\omega,H^1(0,L))$ is a solution of the problem (see again Section 6.3 of [2])

$$\frac{\partial^2\mathcal{W}_3^{0+}}{\partial x_3^2}=0 \text{ in } \Omega^+,$$

$$\frac{\partial\mathcal{W}_3^{0+}}{\partial x_3}=0 \text{ on } \omega\times\{L\},$$

which together with the boundary condition (8.3) yields

$$\mathcal{W}_3^{0+}=0 \text{ in } \Omega^+. \tag{8.4}$$

8.3 Determination of $\widehat{\mathcal{U}}^0$, $\widehat{\mathcal{R}}^0$, $\breve{u}$ and $\widehat{\overline{u}}^0$

In this subsection, we prove that $\widehat{u}^0=\widehat{\mathcal{R}}^0=\breve{u}=\widehat{\overline{u}}^0=0$. Let us emphasize that this result is not only the consequence of the homogeneous character of the plate Ω^-, it is also strongly linked to the fact that $\frac{\varepsilon}{\delta}$ tends to 0. This means that even for a homogeneous material plate Ω^-, if $\frac{\varepsilon}{\delta}$ does not tends to 0 (for example if $\delta=\varepsilon$) then the periodic character of the rods above the plate can induce microscopic effects on the limit model in Ω^-. This phenomenon has already investigated in [3] for a conduction problem. We will examine the case $\delta=\varepsilon$ for the elastic problem in a forthcoming paper.

In order to focus on the microscopic behavior of σ_{ij}^δ in Ω_δ^-, we choose in (2.19) the following test function

$$v_i(x_1,x_2,x_3)=\varepsilon\varphi(x_1,x_2)\psi\left(\frac{x_3}{\delta}\right)\chi_i\left(\frac{x_1-p\varepsilon}{\varepsilon},\frac{x_2-q\varepsilon}{\varepsilon}\right), \tag{8.5}$$

where $\varphi \in C_0^\infty(\omega)$, $\psi \in C_0^\infty((-1,0))$ and $\chi_i \in H^1_{per}(Y)$. Remark that, since φ and ψ are smooth and χ_i is periodic, we have $v_i \in V^\varepsilon$, (V^ε is defined in (2.18) with $r^2 = k^2\varepsilon^2 = a^2\delta^3$), and also $v_i = 0$ in Ω_ε^+.

Transforming $\displaystyle\int_{\Omega_\delta^-}\sum_{i,j=1}^3 \sigma_{ij}^\delta \gamma_{ij}(v)$ by application of $\mathcal{T}_-^\varepsilon \circ \Pi_\delta$ (see (5.3) and Appendix A), we obtain (for ε small enough such that supp$\varphi \subset \widetilde{\omega}_\varepsilon$)

$$\begin{aligned}
&\delta \int_{\Omega^-\times Y} \mathcal{T}_-^\varepsilon\left(\Pi_\delta(\sigma_{ij}^\delta)\right) \mathcal{T}_-^\varepsilon\left(\Pi_\delta(\gamma_{ij}(v))\right) dx_1dx_2dX_3dX_1dX_2 = \\
&\delta \sum_{\alpha=1}^2 \int_{\Omega^-\times Y} \mathcal{T}_-^\varepsilon\left(F_\alpha^-\right) \mathcal{T}_-^\varepsilon\left(\Pi_\delta v_\alpha\right) dx_1dx_2dX_3dX_1dX_2 + \\
&\delta^2 \int_{\Omega^-\times Y} \mathcal{T}_-^\varepsilon\left(F_3^-\right) \mathcal{T}_-^\varepsilon\left(\Pi_\delta v_3\right) dx_1dx_2dX_3dX_1dX_2.
\end{aligned} \tag{8.6}$$

To pass to the limit as ε tends to 0 and δ tends to 0 in (8.6), we first notice that

$$\mathcal{T}_-^\varepsilon\left(\Pi_\delta v_i\right) = \varepsilon\left(\mathcal{T}_-^\varepsilon \varphi\right)(x_1,x_2,X_1,X_2)\psi(X_3)\chi_i(X_1,X_2), \qquad \text{for i=1,2,3}$$

so that

$$\mathcal{T}_-^\varepsilon\left(\Pi_\delta v_i\right) \to 0 \text{ strongly in } L^2(\Omega^-\times Y), \qquad \text{for i=1,2,3,} \tag{8.7}$$

as δ tends to zero. Moreover,

$$\mathcal{T}_-^\varepsilon\left(F_i^-\right) \to F_i^- \text{ strongly in } L^2(\Omega^-\times Y), \qquad \text{for i=1,2,3.} \tag{8.8}$$

Then, using the rules (5.4), (5.5) and Appendix A to commute the spatial derivatives with the operators $\mathcal{T}_-^\varepsilon$ and Π_δ,

$$\begin{aligned}
&\mathcal{T}_-^\varepsilon\left(\Pi_\delta \gamma_{\alpha\beta}(v)\right) = \\
&\varepsilon\psi\left[\mathcal{T}_-^\varepsilon(\varphi)\frac{1}{\varepsilon}\Gamma_{\alpha\beta}(\chi) - \frac{1}{2}\left(\mathcal{T}_-^\varepsilon\left(\frac{\partial\varphi}{\partial x_\alpha}\right)\chi_\beta + \mathcal{T}_-^\varepsilon\left(\frac{\partial\varphi}{\partial x_\beta}\right)\chi_\alpha\right)(\delta_{\alpha\beta}-1) + \right. \\
&\left.\mathcal{T}_-^\varepsilon\left(\frac{\partial\varphi}{\partial x_\alpha}\right)\chi_\beta\delta_{\alpha\beta}\right],
\end{aligned} \tag{8.9}$$

$$\mathcal{T}_-^\varepsilon\left(\Pi_\delta \gamma_{\alpha 3}(v)\right) = \frac{\varepsilon}{2}\left[\frac{1}{\delta}\psi'\mathcal{T}_-^\varepsilon(\varphi)\chi_\alpha + \mathcal{T}_-^\varepsilon\left(\frac{\partial\varphi}{\partial x_\alpha}\right)\chi_3\psi + \psi\varphi\frac{1}{\varepsilon}\frac{\partial\chi_3}{\partial X_\alpha}\right], \tag{8.10}$$

$$\mathcal{T}_-^\varepsilon\left(\Pi_\delta \gamma_{33}(v)\right) = \frac{\varepsilon}{\delta}\psi'\mathcal{T}_-^\varepsilon(\varphi)\chi_3. \tag{8.11}$$

Appealing now to the strong convergence of $\mathcal{T}_-^\varepsilon(\varphi)$ and of $\mathcal{T}_-^\varepsilon\left(\dfrac{\partial\varphi}{\partial x_\alpha}\right)$ for $\alpha = 1,2$, in $L^2(\omega\times Y)$, and using the fact that $\dfrac{\varepsilon}{\delta} \to 0$, we obtain

$$\mathcal{T}_-^\varepsilon\left(\Pi_\delta \gamma_{\alpha\beta}(v)\right) \to \psi\varphi\Gamma_{\alpha\beta}(\chi) \text{ strongly in } L^2(\Omega^-\times Y), \tag{8.12}$$

$$\mathcal{T}^{\varepsilon}_{-}\left(\Pi_\delta\gamma_{\alpha 3}(v)\right) \to \frac{1}{2}\psi\varphi\frac{\partial\chi_3}{\partial X_\alpha} \text{ strongly in } L^2(\Omega^-\times Y), \tag{8.13}$$

$$\mathcal{T}^{\varepsilon}_{-}\left(\Pi_\delta\gamma_{33}(v)\right) \to 0 \text{ strongly in } L^2(\Omega^-\times Y), \tag{8.14}$$

as δ tends to 0.

At last, the constitutive law (2.13) together with the convergence (6.33) and (6.34) lead to

$$\overline{\Sigma}^-_{ij} = \lambda\left(\sum_{k=1}^{3}\overline{X}^-_{kk}\right)\delta_{ij} + 2\mu\overline{X}^-_{ij} \text{ for } i,j=1,2,3. \tag{8.15}$$

Passing to the limit in (8.6) is now easy in view of (8.7), (8.8), (8.12)÷(8.15) and it yields

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{2}\int_{\Omega^-\times Y}\psi\varphi\overline{\Sigma}^-_{\alpha\beta}\Gamma_{\alpha\beta}(\chi)dx_1dx_2dX_3dX_1dX_2\\&+\sum_{\alpha=1}^{2}\int_{\Omega^-\times Y}\psi\varphi\overline{\Sigma}^-_{\alpha 3}\frac{\partial\chi_3}{\partial X_\alpha}dx_1dx_2dX_3dX_1dX_2=0.\end{aligned} \tag{8.16}$$

The equation (8.16) only concerns the local variables dependent fields $\widehat{\mathcal{U}}^0$, $\widehat{\mathcal{R}}^0$ and $\widehat{\overline{u}}^0$. Indeed the expression (8.15) of $\overline{\Sigma}^-_{ij}$ together with the values (7.42)÷(7.45) of $\overline{X}^-_{ij}$ show that the contribution of the macroscopic fields $\mathcal{U}^{0-}$, $\mathcal{R}^{0-}$ and $\overline{u}^{0-}$ vanish in (8.16) because χ is periodic with respect to the variables (X_1,X_2) (recall that $\varphi(x_1,x_2),\psi(X_3)$) and the plate is homogeneous.

Recall that equation (8.16) is valid for any function $\varphi\in C_0^\infty(\omega)$ and any function $\psi\in C_0^\infty((-1,0))$ (and any $\chi\in(H^1_{per}(Y))^3$). Then it is true for any, say, $\varphi\in C(\overline{\omega})$ (which is standard for a local problem) but also for any $\psi\in C([-1,0])$ which is a consequence of (8.14) that itself comes from the fact that $\dfrac{\varepsilon}{\delta}\to 0$ (see (8.11)). This means that the local problem (8.16) is independent of the periodic asymptotic behavior of the rods on the surface $x_3=0$ as soon as $\dfrac{\varepsilon}{\delta}\to 0$. This will lead to the nullity of the local fields $\widehat{\mathcal{U}}^0$, $\widehat{\mathcal{R}}^0$ and $\widehat{\overline{u}}^0$ as shown below.

We localize (8.16) with respect to $(x_1,x_2)\in\omega$. Then we first choose $\chi_3=0$ and define the displacement

$$\begin{aligned}&\widetilde{u}_1(x_1,x_2,X_3,X_1,X_2)=\\&\widehat{\mathcal{U}}^0_1(x_1,x_2,X_1,X_2)+\left(X_3+\frac{1}{2}\right)\widehat{\mathcal{R}}^0_2(x_1,x_2,X_1,X_2)+\widehat{\overline{u}}^0_1(x_1,x_2,X_3,X_1,X_2),\\&\widetilde{u}_2(x_1,x_2,X_3,X_1,X_2)=\\&\widehat{\mathcal{U}}^0_2(x_1,x_2,X_1,X_2)-\left(X_3+\frac{1}{2}\right)\widehat{\mathcal{R}}^0_1(x_1,x_2,X_1,X_2)+\widehat{\overline{u}}^0_1(x_1,x_2,X_3,X_1,X_2)\end{aligned}$$

and we obtain

$$\sum_{\alpha,\beta=1}^{2}\int_{-1}^{0}\int_{Y}\left[\lambda\left(\sum_{k=1}^{2}\Gamma_{kk}(\widetilde{u})\right)I+2\mu\Gamma_{\alpha\beta}(\widetilde{u})\right]\Gamma_{\alpha\beta}(\chi)\psi(X_3)dX_1dX_2dX_3=0 \text{ a.e. in } \omega, \tag{8.17}$$

for any $(\chi_1,\chi_2)\in(H^1_{per}(Y))^2$. Remark that $\widetilde{u}_\alpha\in L^2(\Omega^-,H^1_{per}(Y))$ and that one can always assume that $\displaystyle\int_Y \widetilde{u}_\alpha dX_1dX_2=0$ for $\alpha=1,2$.

For almost $X_3\in]-1,0[$, Problem (8.17) is then an elastic $2d$-problem with periodic boundary conditions on ∂Y and with no applied forces. As a consequence of a standard result, we obtain $\widetilde{u}_\alpha=0$, $\alpha=1,2$. Now because of (7.35),

$$\begin{aligned}\widehat{\mathcal{U}}^0_\alpha&=\int_{-1}^{0}\widetilde{u}_\alpha dX_3=0,\\ \widehat{\mathcal{R}}^0_1&=-\frac{1}{12}\int_{-1}^{0}\left(X_3+\frac{1}{2}\right)\widetilde{u}_2dX_3=0,\\ \widehat{\mathcal{R}}^0_2&=\frac{1}{12}\int_{-1}^{0}\left(X_3+\frac{1}{2}\right)\widetilde{u}_1dX_3=0,\end{aligned}\tag{8.18}$$

and then, $\widehat{\overline{u}}^0_\alpha=0$.

It remains to show that $\widehat{\overline{u}}^0_3=0$. To this end, we choose $\chi_\alpha=0$, $\alpha=1,2$, in (8.16) and this gives

$$\int_{-1}^{0}\int_{Y}\mu\Big[\frac{\partial}{\partial X_1}(\breve{u}+\widehat{\overline{u}}^0_3)\frac{\partial\chi_3}{\partial X_1}+\frac{\partial}{\partial X_2}(\breve{u}+\widehat{\overline{u}}^0_3)\frac{\partial\chi_3}{\partial X_2}\Big]\psi(X_3)dX_1dX_2dX_3=0 \tag{8.19}$$

for any $\chi_3\in H^1_{per}(Y)$ and for almost any $(x_1,x_2)\in\omega$.

Since $\breve{u}+\widehat{\overline{u}}^0_3\in L^2(\Omega^-,H^1_{per}(Y))$ and $\displaystyle\int_Y(\breve{u}+\widehat{\overline{u}}^0_3)dX_1dX_2=0$, we obtain $\breve{u}+\widehat{\overline{u}}^0_3=0$ for a.e. $(x_1,x_2)\in\omega$ and a.e. $X_3\in]-1,0[$ which in turn implies that $\breve{u}=0$ and $\widehat{\overline{u}}^0_3=0$ because of (7.35).

As a conclusion of this subsection, since $\widehat{\mathcal{U}}^0=\widehat{\mathcal{R}}^0=\widehat{\overline{u}}^0=\breve{u}=0$, the weak limits X^-_{ij} and Σ^-_{ij} on one hand and $\overline{X}^-_{ij}$ and $\overline{\Sigma}^-_{ij}$ on the other hand are the same and in particular

$$\begin{aligned}\overline{\Sigma}^-_{\alpha\beta}=\Sigma^-_{\alpha\beta}=\lambda&\left(\sum_{z=1}^{2}\Big[\gamma_{zz}(\mathcal{U}^{0-})+\left(X_3+\frac{1}{2}\right)\gamma_{zz}(\mathcal{R}^{0-}\wedge e_3)\Big]+\frac{\partial\overline{u}^{0-}_3}{\partial X_3}\right)\delta_{\alpha\beta}+\\ 2\mu&\left(\gamma_{\alpha\beta}(\mathcal{U}^{0-})+\left(X_3+\frac{1}{2}\right)\gamma_{\alpha\beta}(\mathcal{R}^{0-}\wedge e_3)+\Gamma_{\alpha\beta}(\overline{u}^0)\right),\quad \text{for } \alpha=1,2,\end{aligned}\tag{8.20}$$

$$\overline{\Sigma}^-_{\alpha3}=\Sigma^-_{\alpha3}=\mu\left(Z^-_\alpha+\frac{\partial\overline{u}^{0-}_\alpha}{\partial X_3}\right),\quad \text{for } \alpha=1,2. \tag{8.21}$$

$$\overline{\Sigma}^-_{33}=\Sigma^-_{33}=\lambda\left(\sum_{z=1}^{2}\Big[\gamma_{zz}(\mathcal{U}^{0-})+\left(X_3+\frac{1}{2}\right)\gamma_{zz}(\mathcal{R}^{0-}\wedge e_3)\Big]+\frac{\partial\overline{u}^{0-}_3}{\partial X_3}\right)+2\mu\frac{\partial\overline{u}^{0-}_3}{\partial X_3}. \tag{8.22}$$

Remark 8.1. *Although this is not the goal of the present paper, let us briefly explain how the analysis developed above can permit to handle the case where the elastic coefficients* λ_1, μ_1 *in the rods are different from the plate ones* λ_2, μ_2 *and for the physical case where the rods* Ω_ε^+ *are clamped into the plate. This means that each rod would have a length equal to* $\delta + L$ *so that the elastic coefficients in the plate would be* $\lambda(x_1, x_2) = \lambda_1 \chi_{\mathcal{D}_{pq}^\varepsilon}(x_1, x_2) + \lambda_2(1 - \chi_{\mathcal{D}_{pq}^\varepsilon}(x_1, x_2))$, $\mu(x_1, x_2) = \mu_1 \chi_{\mathcal{D}_{pq}^\varepsilon}(x_1, x_2) + \mu_2(1 - \chi_{\mathcal{D}_{pq}^\varepsilon}(x_1, x_2))$ *if* $(x_1, x_2) \in \widetilde{\omega}_\varepsilon$ *and* $\lambda(x_1, x_2) = \lambda_2$, $\mu(x_1, x_2) = \mu_2$ *if* $(x_1, x_2) \in \omega \setminus \widetilde{\omega}_\varepsilon$. *Then the right hand side of (8.16) is not zero and it involves the macroscopic fields* $\mathcal{U}^{0-}$, $\mathcal{R}^{0-}$ *and* $\overline{u}^{0-}$. *It results two right hand sides with the same dependence in the two uncoupled problems (8.17) and (8.19). Using the relations (8.18) and the properties of* $\breve{u}$ *and* $\widehat{\overline{u}}_3^0$, *each field* $\widehat{\mathcal{U}}_\alpha^0$, $\widehat{\mathcal{R}}_\alpha^0$, $\widehat{\overline{u}}_\alpha^0$, $\breve{u}$ *and* $\widehat{\overline{u}}_3^0$ *can be expressed in terms of* $\mathcal{U}^{0-}$, $\mathcal{R}^{0-}$ *and* $\overline{u}^{0-}$. *Inserting this dependence into (7.42)÷(7.45), one obtains a constitutive law between* $\mathcal{U}^{0-}$, $\mathcal{R}^{0-}$, $\overline{u}^{0-}$ *and* $\overline{\Sigma}$ *which takes into account the homogenization process in the plate* Ω^-. *With this new constitutive law, the analysis that follows can be achieved with the same tools (as far as homogenization of plate models are concerned the reader is referred to [10]).*

8.4 Determination of $\overline{u}^{0-}$ and Z_1^-, Z_2^-.

We prove that $Z_\alpha^- = \overline{u}_\alpha^{0-} = 0$ for $\alpha = 1, 2$ and we give the expression of $\overline{u}_3^{0-}$ as a function of $\mathcal{U}^{0-}$.

We start with (2.19) and plug the test function

$$v_\alpha(x_1, x_2, x_3) = \delta \varphi_\alpha(x_1, x_2) \psi\left(\frac{x_3}{\delta}\right) \qquad \text{for } \alpha = 1, 2,$$
$$v_3(x_1, x_2, x_3) = 0,$$

where $\varphi_\alpha \in C_0^\infty(\omega)$ for $\alpha = 1, 2$ and $\psi \in C^\infty([-1, 0])$ and $\psi(0) = 0$. Then we transform the integrals on Ω_δ^- through application of Π_δ (the last function is identically 0 on Ω_ε^+), it gives after passing to the limit as $\delta \to 0$ (using (8.21))

$$\sum_{\alpha=1}^2 \int_{\Omega^-} \varphi_\alpha \left(Z_\alpha^- + \frac{\partial \overline{u}_\alpha^{0-}}{\partial X_3} \right) \psi' dx_1 dx_2 dX_3 = 0.$$

Since the φ_α's are arbitrary in $C_0^\infty(\omega)$ and $Z_\alpha^- \in L^2(\omega)$, we obtain

$$Z_\alpha^- X_3 + \overline{u}_\alpha^{0-} = 0 \quad \text{ for a.e. } (x_1, x_2, X_3) \in \Omega^- \text{ and } \alpha = 1, 2.$$

Using now the kinematic conditions (7.54) and (7.55) on $\overline{u}_\alpha^{0-}$, it is easy to deduce that $\overline{u}_\alpha^{0-} = Z_\alpha^- = 0$. In order to derive $\overline{u}_3^{0-}$, we use, as test function in (2.19)

$$v_\alpha(x_1, x_2, x_3) = 0,$$
$$v_3(x_1, x_2, x_3) = \delta \varphi(x_1, x_2) \psi\left(\frac{x_3}{\delta}\right),$$

where $\varphi \in C_0^\infty(\omega)$, $\psi \in C^\infty([-1, 0])$ and $\psi(0) = 0$. Proceeding as above now leads to (using (8.22))

$$\int_{\Omega^-} \varphi \left[\lambda \left(\sum_{\alpha=1}^2 \frac{\partial \mathcal{U}_\alpha^{0-}}{\partial x_\alpha} + \left(X_3 + \frac{1}{2} \right) \frac{\partial \mathcal{R}_2^{0-}}{\partial x_1} - \left(X_3 + \frac{1}{2} \right) \frac{\partial \mathcal{R}_1^{0-}}{\partial x_2} \right) \psi' + (\lambda + 2\mu) \frac{\partial \overline{u}_3^{0-}}{\partial X_3} \psi' \right] = 0$$

According to (7.20) and (7.21) and to the kinematic conditions (7.54) and (7.55) on $\overline{u}_3^{0-}$, the solution of the above problem is given by (see [21])

$$\overline{u}_3^{0-} = \frac{\lambda}{\lambda+2\mu}\left[-\left(X_3+\frac{1}{2}\right)\left(\frac{\partial \mathcal{U}_1^{0-}}{\partial x_1}+\frac{\partial \mathcal{U}_2^{0-}}{\partial x_2}\right)+\left(\frac{(X_3+\frac{1}{2})^2}{2}-\frac{1}{24}\right)\Delta\mathcal{U}_3^{0-}\right]. \tag{8.23}$$

As a conclusion of this subsection we have, through inserting (8.23) into (8.20)÷(8.22) and in (7.42)÷(7.45)

$$\overline{\Sigma}_{11}^{-} = \Sigma_{11}^{-} =$$

$$\frac{E}{1-\nu^2}\left[\left(\frac{\partial \mathcal{U}_1^{0-}}{\partial x_1}-\left(X_3+\frac{1}{2}\right)\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_1^2}\right)+\nu\left(\frac{\partial \mathcal{U}_2^{0-}}{\partial x_2}-\left(X_3+\frac{1}{2}\right)\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_2^2}\right)\right], \tag{8.24}$$

$$\overline{\Sigma}_{22}^{-} = \Sigma_{22}^{-} =$$

$$\frac{E}{1-\nu^2}\left[\left(\frac{\partial \mathcal{U}_2^{0-}}{\partial x_2}-\left(X_3+\frac{1}{2}\right)\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_2^2}\right)+\nu\left(\frac{\partial \mathcal{U}_1^{0-}}{\partial x_1}-\left(X_3+\frac{1}{2}\right)\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_1^2}\right)\right], \tag{8.25}$$

$$\overline{\Sigma}_{12}^{-} = \Sigma_{12}^{-} = \mu\left(\frac{\partial \mathcal{U}_1^{0-}}{\partial x_2}+\frac{\partial \mathcal{U}_2^{0-}}{\partial x_1}-2\left(X_3+\frac{1}{2}\right)\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_1\partial x_2}\right), \tag{8.26}$$

$$\overline{\Sigma}_{13}^{-} = \overline{\Sigma}_{23}^{-} = \overline{\Sigma}_{33}^{-} = \Sigma_{13}^{-} = \Sigma_{23}^{-} = \Sigma_{33}^{-} = 0. \tag{8.27}$$

$$\overline{X}_{\alpha\beta}^{-} = X_{\alpha\beta}^{-} = \gamma_{\alpha\beta}(\mathcal{U}^{0-}) - \left(X_3+\frac{1}{2}\right)\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_\alpha\partial x_\beta}, \tag{8.28}$$

$$\overline{X}_{\alpha 3}^{-} = 0 \text{ for } \alpha = 1,2 \tag{8.29}$$

$$\overline{X}_{33}^{-} = X_{33}^{-} = \frac{\lambda}{\lambda+2\mu}\left[-\left(\frac{\partial \mathcal{U}_1^{0-}}{\partial x_1}+\frac{\partial \mathcal{U}_2^{0-}}{\partial x_2}\right)+\left(X_3+\frac{1}{2}\right)\Delta\mathcal{U}_3^{0-}\right]. \tag{8.30}$$

The two next sections are devoted to derive the PDE's of the limit problem.

9 The uncoupled "membrane" model in Ω^-.

For a single plate $\Omega^- = \omega\times]-1,0[$, the very standard method to obtain the "membrane" equation in ω consists in choosing in the variational formulation a test function of the type $v_\alpha(x_1,x_2)$, $\alpha = 1,2$ and $v_3 = 0$ with v_i smooth enough and $v_\alpha = 0$ on $\partial\omega$. It results a deformation field such that $\gamma_{i3}(v) = 0$ in Ω_δ^-, for $i = 1,2,3$. For the present problem under investigation, this simple choice of the test function in (2.19) indeed involves a contribution of the part Ω_ε^+ of the domain. But the requested type of test functions in Ω_ε^+ in order to obtain rods models is not compatible with the structure of the field $v_\alpha(x_1,x_2)$, $v_3 = 0$ (see [2]). This means that we have to choose a test function in (2.19) which does not depend on x_3 and which vanishes in Ω_ε^+. Then it has to depend on the "microscopic" variables $\dfrac{x_1-p\varepsilon}{\varepsilon}$,

$\dfrac{x_2 - q\varepsilon}{\varepsilon}$, i.e. it exhibits oscillations with respect to x_1, x_2. As a consequence we will have to deal with an oscillating test function of (x_1, x_2) in ω. This is the very reason why we introduce the unfold periodic fields of $\gamma_{ij}(u^\delta)$ and σ_{ij}^δ in Ω_δ^-, even if Ω_δ^- is homogeneous.

We are now in a position to construct an adequate test function in (2.19). Let us consider a function $\Phi \in H_0^1(Y)$ such that $\Phi = 1$ in D_k (here $D_k = D(0,k)$). Let V_α be in $C_0^\infty(\omega)$ for $\alpha = 1, 2$. We define then the functions in Ω^ε

$$v_\alpha^\varepsilon(x_1, x_2) = \Phi\left(\frac{x_1 - p\varepsilon}{\varepsilon}, \frac{x_2 - q\varepsilon}{\varepsilon}\right) V_\alpha(p\varepsilon, q\varepsilon) +$$

$$\left(1 - \Phi\left(\frac{x_1 - p\varepsilon}{\varepsilon}, \frac{x_2 - q\varepsilon}{\varepsilon}\right)\right) V_\alpha(x_1, x_2), \tag{9.1}$$

$$\text{for } \alpha = 1, 2, \text{ for } (x_1, x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[\times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \quad (p,q) \in \mathcal{N}^\varepsilon.$$

Remark that v_α^ε is well defined for ε small enough such that $\mathrm{supp}(v_\alpha) \subset \omega^\varepsilon$. Indeed $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, 0) \in V_\varepsilon$ (see (2.18)) and it is then an admissible test function in (2.19). Moreover, since $\Phi = 1$ in D_k and $v_3^\varepsilon = 0$, we have firstly

$$\gamma_{ij}(v^\varepsilon) = 0 \text{ a.e. in } \Omega_\varepsilon^+, \text{ for } i, j = 1, 2, 3, \tag{9.2}$$

and secondly, in Ω_δ^-

$$\begin{aligned}\gamma_{\alpha\beta}(v^\varepsilon)(x_1, x_2) &= \left(1 - \Phi\left(\frac{x_1 - p\varepsilon}{\varepsilon}, \frac{x_2 - q\varepsilon}{\varepsilon}\right)\right) \gamma_{\alpha\beta}(V)(x_1, x_2) + \\ &\frac{1}{2\varepsilon}\left[(V_\alpha(p\varepsilon, q\varepsilon) - V_\alpha(x_1, x_2)) \frac{\partial \Phi}{\partial X_\beta}\left(\frac{x_1 - p\varepsilon}{\varepsilon}, \frac{x_2 - q\varepsilon}{\varepsilon}\right) + \right. \\ &\left.(V_\beta(p\varepsilon, q\varepsilon) - V_\beta(x_1, x_2)) \frac{\partial \Phi}{\partial X_\alpha}\left(\frac{x_1 - p\varepsilon}{\varepsilon}, \frac{x_2 - q\varepsilon}{\varepsilon}\right)\right]\end{aligned} \tag{9.3}$$

$$\text{for } \alpha, \beta = 1, 2, \text{ for } (x_1, x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[\times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \quad (p,q) \in \mathcal{N}^\varepsilon.$$

$$\gamma_{i3}(v^\varepsilon) = 0 \text{ a.e. in } \Omega_\delta^- \text{ for } i = 1, 2, 3. \tag{9.4}$$

Defining the piecewise constant function $\tilde{V}_\alpha^\varepsilon$ by (see Section 5.5 of [2])

$$\begin{cases} \tilde{V}_\alpha^\varepsilon(x_1, x_2) = V_\alpha(p\varepsilon, q\varepsilon) \text{ if } (x_1, x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[\times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[, \\ \tilde{V}_\alpha^\varepsilon(x_1, x_2) = 0 \text{ otherwise}, \end{cases} \tag{9.5}$$

and applying $\mathcal{T}_-^\varepsilon \circ \Pi^\delta$ to (9.3) give

$$\begin{aligned}\mathcal{T}_-^\varepsilon\left(\Pi^\delta(\gamma_{\alpha\beta}(v^\varepsilon))\right) &= (1 - \Phi(X_1, X_2))\, \mathcal{T}_-^\varepsilon\left(\Pi^\delta \gamma_{\alpha\beta}(V)\right) + \\ &\frac{1}{2\varepsilon}\left[\left(\tilde{V}_\alpha^\varepsilon - \mathcal{T}_-^\varepsilon(\Pi^\delta V_\alpha)\right)\frac{\partial \Phi}{\partial X_\beta} + \left(\tilde{V}_\beta^\varepsilon - \mathcal{T}_-^\varepsilon(\Pi^\delta V_\beta)\right)\frac{\partial \Phi}{\partial X_\alpha}\right] \text{ a.e. in } \Omega^- \times Y,\end{aligned} \tag{9.6}$$

while (9.4) gives

$$\mathcal{T}_-^\varepsilon\left(\Pi^\delta(\gamma_{i3}(v^\varepsilon))\right)=0 \text{ a.e. in } \Omega^-\times Y. \tag{9.7}$$

Now since $\gamma_{ij}(v^\varepsilon)=0$ in Ω_ε^+, starting with (2.19) with $v=v^\varepsilon$ gives (as soon as $\mathrm{supp}(v_\alpha^\varepsilon)\subset \widetilde{\omega}_\varepsilon$, for $\alpha=1,2$), i.e.

$$\begin{aligned}&\sum_{i,j=1}^{3}\int_{\Omega^-\times Y}\mathcal{T}_-^\varepsilon\left(\Pi^\delta(\sigma_{ij}^\delta)\right)\mathcal{T}_-^\varepsilon\left(\Pi^\delta(\gamma_{ij}(v^\varepsilon))\right)dx_1dx_2dx_3dX_1dX_2=\\&\sum_{\alpha=1}^{2}\int_{\Omega^-\times Y}\mathcal{T}_-^\varepsilon\left(F_\alpha^-\right)\mathcal{T}_-^\varepsilon\left(\Pi^\delta v_\alpha^\varepsilon\right)dx_1dx_2dx_3dX_1dX_2.\end{aligned}\tag{9.8}$$

In order to pass to the limit as ε tends to zero (and δ tends to zero) in (9.8), first remark that (see again Section 5.5 of [2])

$$\mathcal{T}_-^\varepsilon\left(\Pi^\delta(v_\alpha^\varepsilon)\right)=\Phi\tilde{V}_\alpha^\varepsilon+(1-\Phi)\mathcal{T}_-^\varepsilon(V_\alpha)\qquad\text{a.e. in } \Omega^-\times Y,$$

so that applying Lemma A1 of Appendix A (and since e.g. $\tilde{V}_\alpha^\varepsilon\to V_\alpha$ strongly in $L^\infty(\Omega^-\times Y)$),

$$\mathcal{T}_-^\varepsilon\left(\Pi^\delta(v_\alpha^\varepsilon)\right)\to V_\alpha \text{ strongly in } L^2(\Omega^-\times Y),\ \text{for } \alpha=1,2,$$

as ε tends to zero. We obtain from (9.8)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sum_{\alpha,\beta=1}^{2}\int_{\Omega^-\times Y}\mathcal{T}_-^\varepsilon\left(\Pi_\delta(\sigma_{\alpha\beta}^\delta)\right)\mathcal{T}_-^\varepsilon\left(\Pi_\delta(\gamma_{\alpha\beta}(v^\varepsilon))\right)dx_1dx_2dx_3dX_1dX_2=\\&\sum_{\alpha=1}^{2}\int_{\Omega^-\times Y}F_\alpha^-V_\alpha dx_1dx_2dx_3dX_1dX_2.\end{aligned}\tag{9.9}$$

To compute the limit in (9.9), we first use the smooth character of $\gamma_{\alpha\beta}(V)$ to obtain

$$\mathcal{T}_-^\varepsilon\left(\Pi^\delta(\gamma_{\alpha\beta}(V))\right)\to\gamma_{\alpha\beta}(V) \text{ strongly in } L^2(\Omega^-\times Y),\ \text{for } \alpha=1,2,\tag{9.10}$$

as ε tends to zero.

Then we appeal to a result established in Lemma A1 *(iii)* of Appendix A, namely

$$\frac{1}{\varepsilon}[\tilde{V}_\alpha^\varepsilon-\mathcal{T}_-^\varepsilon\left(\Pi^\delta V_\alpha)\right)]\to-\frac{\partial V_\alpha}{\partial x_1}X_1-\frac{\partial V_\alpha}{\partial x_2}X_2 \text{ strongly in } L^2(\Omega^-\times Y),\ \text{for } \alpha=1,2,\tag{9.11}$$

as $\varepsilon\to 0$. In view of (6.34), (8.15), (9.4)÷(9.7), (9.10) and (9.11), the equality (9.9) implies that

$$\begin{aligned}&\sum_{\alpha,\beta=1}^{2}\int_{\Omega^-\times Y}\overline{\Sigma}_{\alpha\beta}^-\left[(1-\Phi)\gamma_{\alpha\beta}(V)-\frac{1}{2}\left(\frac{\partial V_\alpha}{\partial x_1}X_1+\frac{\partial V_\alpha}{\partial x_2}X_2\right)\frac{\partial\Phi}{\partial X_\beta}-\right.\\&\left.\frac{1}{2}\left(\frac{\partial V_\beta}{\partial x_1}X_1+\frac{\partial V_\beta}{\partial x_2}X_2\right)\frac{\partial\Phi}{\partial X_\alpha}\right]dx_1dx_2dX_1dX_2=\sum_{\alpha=1}^{2}\int_{\Omega^-}F_\alpha V_\alpha dx_1dx_2,\end{aligned}\tag{9.12}$$

for any $V_\alpha \in C_0^\infty(\omega)$, $\alpha = 1,2$, and any $\Phi \in H_0^1(Y)$ such that $\Phi = 1$ in D_k.

Now, remark that for $\Phi \in H_0^1(Y)$

$$\int_Y \frac{\partial \Phi}{\partial X_\alpha} X_\gamma dX_1 dX_2 = -\delta_{\alpha\gamma} \int_Y \Phi dX_1 dX_2 \quad \text{for } \alpha, \gamma = 1, 2.$$

Then, we have for $\alpha, \beta = 1, 2$,

$$\sum_{\gamma=1}^{2} \frac{\partial V_\alpha}{\partial x_\gamma} \int_Y X_\gamma \frac{\partial \Phi}{\partial X_\beta} dX_1 dX_2 = -\frac{\partial V_\alpha}{\partial x_\beta} \int_Y \Phi dX_1 dX_2$$

and

$$\sum_{\gamma=1}^{2} \frac{\partial V_\beta}{\partial x_\gamma} \int_Y X_\gamma \frac{\partial \Phi}{\partial X_\alpha} dX_1 dX_2 = -\frac{\partial V_\beta}{\partial x_\alpha} \int_Y \Phi dX_1 dX_2.$$

It follows that, from (9.12) and because $\overline{\Sigma}_{\alpha\beta}^-$ does not depend on (X_1, X_2) (see (8.24)÷(8.26))

$$\sum_{\alpha,\beta=1}^{2} \int_{\Omega^-} \overline{\Sigma}_{\alpha\beta}^- \left[\gamma_{\alpha\beta}(V) \int_Y (1-\Phi) dX_1 dX_2 + \gamma_{\alpha\beta}(V) \int_Y \Phi dX_1 dX_2 \right] dx_1 dx_2 = \sum_{\alpha=1}^{2} \int_{\Omega^-} F_\alpha V_\alpha dx_1 dx_2,$$

or equivalently

$$\sum_{\alpha,\beta=1}^{2} \int_{\Omega^-} \overline{\Sigma}_{\alpha\beta}^- \gamma_{\alpha\beta}(V) dx_1 dx_2 = \sum_{\alpha=1}^{2} \int_{\Omega^-} F_\alpha V_\alpha dx_1 dx_2, \tag{9.13}$$

for any $V_\alpha \in C_0^\infty(\omega)$, $\alpha = 1, 2$.

In view of the expressions (8.24)÷(8.26) of $\overline{\Sigma}_{\alpha\beta}^- = \Sigma_{\alpha\beta}^-$, the variational problem (9.13) (indeed by density, one can take $V_\alpha \in H_0^1(\omega)$, for $\alpha = 1, 2$) is the standard "membrane" problem for $(\mathcal{U}_1^{0-}, \mathcal{U}_2^{0-}) \in H_0^1(\omega)$, which reads as

$$\begin{gathered} \frac{E}{1-\nu^2} \int_\omega \left[(1-\nu) \sum_{\alpha,\beta=1}^{2} \gamma_{\alpha\beta}(\mathcal{U}^{0-}) \gamma_{\alpha\beta}(V) + \nu \sum_{\delta=1}^{2} \gamma_{\delta\delta}(\mathcal{U}^{0-}) \sum_{\delta=1}^{2} \gamma_{\delta\delta}(V) \right] dx_1 dx_2 = \\ \int_\omega \left(\int_{-1}^{0} F_\alpha^- dX_3 \right) V_\alpha dx_1 dx_2, \end{gathered} \tag{9.14}$$

for any $V = (V_1, V_2) \in \left(H_0^1(\omega)\right)^2$.

10 The coupled model for the bending in the rods and in the plate.

In view of the transmission conditions (7.52) and (7.53) on ω between $(\mathcal{U}_1^{0+}, \mathcal{U}_2^{0+})$ (which describes the bending in the rods) and $\mathcal{U}_3^{0-}$ (which describes the bending in the plate), we have to built a test function in Ω_ε in such a way that these two behaviors are coupled after passing to the limit as ε tends to 0. Then, loosely speaking, this test function must be a displacement of Bernoulli-Navier's type in Ω^+_ε (see e.g. Section 6.2 of [2]) and a displacement of Kirchhoff-love's type in Ω^-_δ. As a consequence and as in the previous section, this leads to deal with oscillating functions in Ω^-_δ. Recall that we denote by Φ a function in $H^1_0(Y)$ such that $\Phi = 1$ in D_k.

Let us consider ϕ and V_3 in $C^\infty_0(\omega)$ and $z_1,\ z_2 \in C^\infty_0((0,L])$. We construct a test field $v^\varepsilon = (v^\varepsilon_1, v^\varepsilon_2, v^\varepsilon_3) \in V_\varepsilon$ as follows. In Ω^+_ε, we set for $(x_1,x_2) \in \mathcal{D}^\varepsilon_{pq}$ $((p,q)\in\mathcal{N}_\varepsilon)$ and $x_3 \in [0,L]$

$$v^\varepsilon_\alpha(x_1,x_2,x_3) = \frac{1}{\delta}\left[\varphi(\varepsilon p,\varepsilon q)z_\alpha(x_3) - \frac{\partial V_3}{\partial x_\alpha}(\varepsilon p,\varepsilon q)\left(x_3+\frac{\delta}{2}\right)\right], \tag{10.1}$$

$$\begin{aligned}v^\varepsilon_3(x_1,x_2,x_3) = \frac{1}{\delta}\Bigg[&V_3(\varepsilon p,\varepsilon q) - (x_1-\varepsilon p)\left(\varphi(\varepsilon p,\varepsilon q)z'_1(x_3) - \frac{\partial V_3}{\partial x_1}(\varepsilon p,\varepsilon q)\right) - \\ &(x_2-\varepsilon q)\left(\varphi(\varepsilon p,\varepsilon q)z'_2(x_3) - \frac{\partial V_3}{\partial x_2}(\varepsilon p,\varepsilon q)\right)\Bigg],\end{aligned} \tag{10.2}$$

Remark again that v^ε_i is well defined in Ω^+_ε for ε small enough, since φ and V_3 have compact support in ω.

In Ω^-_δ, we set

$$\begin{aligned}v^\varepsilon_\alpha(x_1,x_2,x_3) = \frac{1}{\delta}\left(x_3+\frac{\delta}{2}\right)\Bigg[&\Phi\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\left(-\frac{\partial V_3}{\partial x_\alpha}(\varepsilon p,\varepsilon q)\right) + \\ &\left(1-\Phi\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\right)\left(-\frac{\partial V_3}{\partial x_\alpha}(x_1,x_2)\right)\Bigg] \quad \text{for } \alpha=1,2,\end{aligned} \tag{10.3}$$

$$\begin{aligned}v^\varepsilon_3(x_1,x_2,x_3) = \frac{1}{\delta}\Bigg[&\Phi\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\Bigg(V_3(\varepsilon p,\varepsilon q) + (x_1-\varepsilon p)\frac{\partial V_3}{\partial x_1}(\varepsilon p,\varepsilon q) + \\ &+(x_2-\varepsilon q)\frac{\partial V_3}{\partial x_2}(\varepsilon p,\varepsilon q)\Bigg) + \left(1-\Phi\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\right)V_3(x_1,x_2)\Bigg],\end{aligned} \tag{10.4}$$

for $(x_1,x_2) \in \left]\varepsilon p - \frac{\varepsilon}{2}, \varepsilon p + \frac{\varepsilon}{2}\right[\times \left]\varepsilon q - \frac{\varepsilon}{2}, \varepsilon q + \frac{\varepsilon}{2}\right[$ $((p,q)\in\mathcal{N}_\varepsilon)$ and $x_3 \in]-\delta, 0[$ (remark again that v^ε_i is well defined in Ω^-_δ since V_3 has a compact support).

Let us first note that the two expressions of v^ε_i given by (10.1), (10.2) (for $x_3 \geq 0$) and (10.3), (10.4) (for $x_3 \leq 0$) match at $x_3 = 0$ because $\Phi = 1$ in D_k and $z_\alpha \in C^\infty_0((0,L])$ $(z_1(0) = z_2(0) = z'_1(0) = z'_2(0) = 0)$

Proceeding as in Section 6.2 of [2], we have since φ, z_1, z_2 and V_3 are smooth

$$\delta\mathcal{T}^\varepsilon(v^\varepsilon_\alpha) \to \varphi z_\alpha - x_3\frac{\partial V_3}{\partial x_\alpha} \text{ strongly in } L^2(\Omega^+\times D), \text{ for } \alpha=1,2, \tag{10.5}$$

$$\delta\mathcal{T}^\varepsilon(v^\varepsilon_3) \to V_3 \text{ strongly in } L^2(\Omega^+\times D), \tag{10.6}$$

and

$$\mathcal{T}_{-}^{\varepsilon}(\Pi^{\delta}(v_{\alpha}^{\varepsilon})) \to -\left(X_3+\frac{1}{2}\right)\frac{\partial V_3}{\partial x_\alpha} \text{ strongly in } L^2(\Omega^-\times Y), \text{ for } \alpha=1,2, \tag{10.7}$$

$$\delta\mathcal{T}_{-}^{\varepsilon}(\Pi^{\delta}(v_{3}^{\varepsilon})) \to V_3 \text{ strongly in } L^2(\Omega^-\times Y), \tag{10.8}$$

as ε tends to 0 (or as δ tends to 0).

We now derive the deformations $\gamma_{ij}(v^\varepsilon)$ separately in Ω_ε^+ and Ω_δ^-. Firstly, an easy calculation shows that

$$\gamma_{ij}(v^\varepsilon)=0 \text{ in } \Omega_\varepsilon^+ \text{ for} (i,j)\neq(3,3), \tag{10.9}$$

and

$$\gamma_{33}(v^\varepsilon)=-\frac{1}{\delta}\left[(x_1-\varepsilon p)\varphi(\varepsilon p,\varepsilon q)z_1''(x_3)+(x_2-\varepsilon q)\varphi(\varepsilon p,\varepsilon q)z_2''(x_3)\right] \text{ in } \Omega_\varepsilon^+. \tag{10.10}$$

Secondly in Ω_δ^-, we have

$$\begin{aligned}\gamma_{\alpha\alpha}(v^\varepsilon)=\frac{1}{\delta}\left(x_3+\frac{\delta}{2}\right)&\left[\frac{1}{\varepsilon}\frac{\partial\Phi}{\partial X_\alpha}\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\left(\frac{\partial V_3}{\partial x_\alpha}(x_1,x_2)-\frac{\partial V_3}{\partial x_\alpha}(\varepsilon p,\varepsilon q)\right)\right.\\ &\left.-\left(1-\Phi\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\right)\frac{\partial^2 V_3}{\partial x_\alpha^2}(x_1,x_2)\right] \quad \text{for } \alpha=1,2,\end{aligned} \tag{10.11}$$

$$\begin{aligned}\gamma_{12}(v^\varepsilon)=\frac{1}{\delta}\left(x_3+\frac{\delta}{2}\right)&\left[\frac{1}{2\varepsilon}\frac{\partial\Phi}{\partial X_1}\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\left(\frac{\partial V_3}{\partial x_2}(x_1,x_2)-\frac{\partial V_3}{\partial x_2}(\varepsilon p,\varepsilon q)\right)\right.\\ &+\frac{1}{2\varepsilon}\frac{\partial\Phi}{\partial X_2}\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\left(\frac{\partial V_3}{\partial x_1}(x_1,x_2)-\frac{\partial V_3}{\partial x_1}(\varepsilon p,\varepsilon q)\right)-\\ &\left.\left(1-\Phi\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)\right)\frac{\partial^2 V_3}{\partial x_1\partial x_2}(x_1,x_2)\right],\end{aligned} \tag{10.12}$$

$$\begin{aligned}\gamma_{\alpha3}(v^\varepsilon)=\frac{1}{2\varepsilon\delta}\frac{\partial\Phi}{\partial X_\alpha}\left(\frac{x_1-\varepsilon p}{\varepsilon},\frac{x_2-\varepsilon q}{\varepsilon}\right)&\left[V_3(\varepsilon p,\varepsilon q)-V_3(x_1,x_2)\right.\\ &\left.+(x_1-\varepsilon p)\frac{\partial V_3}{\partial x_1}(\varepsilon p,\varepsilon q)+(x_2-\varepsilon q)\frac{\partial V_3}{\partial x_2}(\varepsilon p,\varepsilon q)\right] \quad \text{for } \alpha=1,2,\end{aligned} \tag{10.13}$$

$$\gamma_{33}(v^\varepsilon)=0. \tag{10.14}$$

Remark that, since $V_3\in C_0^\infty(\omega)$, the relation (10.13) shows that

$$\|\gamma_{\alpha3}(v^\varepsilon)\|_{L^\infty(\omega)}\leq c\frac{\varepsilon}{\delta} \quad \text{for } \alpha=1,2. \tag{10.15}$$

where c is a constant independent of ε.

Then we apply $\mathcal{T}^\varepsilon$ to the relations (10.9) and (10.10) (as in Section 6.2 of [2]) and using the notation $\tilde{\varphi}^\varepsilon$ for the analog of (9.5) (with φ in place of V_α), it gives

$$\mathcal{T}^\varepsilon(\gamma_{ij}(v^\varepsilon)) = 0 \quad \text{for } (i,j) \neq (3,3), \tag{10.16}$$

$$\mathcal{T}^\varepsilon(\gamma_{33}(v^\varepsilon)) = -\frac{r}{\delta}\Bigg[X_1\widetilde{\varphi}^\varepsilon z_1''(x_3) + X_2\widetilde{\varphi}^\varepsilon z_2''(x_3)\Bigg]. \tag{10.17}$$

Due to $r^2 = a^2\delta^3$ and to the convergence of the function $\tilde{\varphi}^\varepsilon$ to φ e.g. in $L^\infty(\omega)$, we deduce that

$$\delta^{-\frac{1}{2}}\mathcal{T}^\varepsilon(\gamma_{33}(v^\varepsilon)) \to -a\Bigg[X_1\varphi z_1'' + X_2\varphi z_2''\Bigg] \text{ strongly in } L^2(\Omega^+ \times D), \tag{10.18}$$

as ε tends to zero.

Now applying $\mathcal{T}_-^\varepsilon \circ \diamond^\delta$ to (10.11)÷(10.13), as in the previous section, leads to

$$\begin{aligned}\mathcal{T}_-^\varepsilon\left(\Pi^\delta\left(\gamma_{\alpha\beta}(v^\varepsilon)\right)\right) &= \left(X_3 + \frac{1}{2}\right)\Bigg[\frac{1}{2\varepsilon}\frac{\partial\Phi}{\partial X_\alpha}\left(\mathcal{T}_-^\varepsilon\left(\frac{\partial V_3}{\partial x_\beta}\right) - \left(\widetilde{\frac{\partial V_3}{\partial x_\beta}}\right)^\varepsilon\right) + \\ &\frac{1}{2\varepsilon}\frac{\partial\Phi}{\partial X_\beta}\left(\mathcal{T}_-^\varepsilon\left(\frac{\partial V_3}{\partial x_\alpha}\right) - \left(\widetilde{\frac{\partial V_3}{\partial x_\alpha}}\right)^\varepsilon\right) - (1-\Phi)\,\mathcal{T}_-^\varepsilon\left(\frac{\partial^2 V_3}{\partial x_\alpha \partial x_\beta}\right)\Bigg] \text{ for } \alpha,\beta = 1,2.\end{aligned} \tag{10.19}$$

Using the smooth character of V_3 and the results of Lemma A1 *(iii)* of Appendix A (as in the previous section) permit to obtain

$$\begin{aligned}\mathcal{T}_-^\varepsilon\left(\Pi^\delta\left(\gamma_{\alpha\beta}(v^\varepsilon)\right)\right) \to \left(X_3 + \frac{1}{2}\right)\Bigg[&\frac{1}{2}\frac{\partial\Phi}{\partial X_\alpha}\left(\sum_{\gamma=1}^{2}\frac{\partial^2 V_3}{\partial x_\gamma \partial x_\beta}X_\gamma\right) + \frac{1}{2}\frac{\partial\Phi}{\partial X_\beta}\left(\sum_{\gamma=1}^{2}\frac{\partial^2 V_3}{\partial x_\gamma \partial x_\alpha}X_\gamma\right) \\ &- (1-\Phi)\frac{\partial^2 V_3}{\partial x_\alpha \partial x_\beta}\Bigg] \qquad \text{strongly in } L^2(\Omega^- \times Y), \text{ for } \alpha,\beta = 1,2.\end{aligned} \tag{10.20}$$

Note also that the estimates (10.15) indeed imply that

$$\mathcal{T}_-^\varepsilon\left(\Pi^\delta\left(\gamma_{\alpha 3}(v^\varepsilon)\right)\right) \to 0 \text{ strongly in } L^2(\Omega^- \times Y), \text{ for } \alpha = 1,2. \tag{10.21}$$

as ε tends to 0.

In order to obtain the limit problem as δ tends to 0, we choose $v = v^\varepsilon$ in (2.19) and we transform the integral over Ω_ε^+ through application of $\mathcal{T}_+^\varepsilon$ and the integral over Ω_δ^- through application of $\mathcal{T}_{-}^{\varepsilon} \circ \Pi^{\delta}$. We obtain, using the assumptions (4.47)÷(4.49) on the forces,

$$
\begin{aligned}
& k^2 \sum_{i,j=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^{\varepsilon}(\sigma_{ij}^{\delta}) \mathcal{T}^{\varepsilon}(\gamma_{ij}(v^{\varepsilon})) dx_1 dx_2 dx_3 dX_1 dX_2 \\
& + \delta \sum_{i,j=1}^{3} \int_{\Omega^- \times Y} \left(\mathcal{T}_{-}^{\varepsilon} \circ \Pi^{\delta}\right)(\sigma_{ij}^{\delta}) \left(\mathcal{T}_{-}^{\varepsilon} \circ \Pi^{\delta}\right)(\gamma_{ij}(v^{\varepsilon})) dx_1 dx_2 dX_3 dX_1 dX_2 \\
& = \delta^2 k^2 \sum_{i=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^{\varepsilon}(F_i^+) \mathcal{T}^{\varepsilon}(v^{\varepsilon}) dx_1 dx_2 dx_3 dX_1 dX_2 \qquad (10.22) \\
& + \delta \sum_{\alpha=1}^{2} \int_{\Omega^- \times Y} \mathcal{T}_{-}^{\varepsilon}(F_{\alpha}^-) \mathcal{T}_{-}^{\varepsilon}(\Pi^{\delta}(v_{\alpha}^{\varepsilon})) dx_1 dx_2 dX_3 dX_1 dX_2 \\
& + \delta^2 \int_{\Omega^- \times Y} \mathcal{T}_{-}^{\varepsilon}(F_3^-) \mathcal{T}_{-}^{\varepsilon}(\Pi^{\delta}(v_3^{\varepsilon})) dx_1 dx_2 dX_3 dX_1 dX_2 .
\end{aligned}
$$

In what follows, we pass to the limit in the relation (10.22) divided by δ. We first have, in view of (10.4)÷(10.8)

$$
\begin{aligned}
& \lim_{\delta \to 0} \Bigg[\delta k^2 \sum_{i=1}^{3} \int_{\Omega^+ \times D} \mathcal{T}^{\varepsilon}(F_i^+) \mathcal{T}^{\varepsilon}(v^{\varepsilon}) dx_1 dx_2 dx_3 dX_1 dX_2 \\
& + \sum_{\alpha=1}^{2} \int_{\Omega^- \times Y} \mathcal{T}_{-}^{\varepsilon}(F_{\alpha}^-) \mathcal{T}_{-}^{\varepsilon}(\Pi^{\delta}(v_{\alpha}^{\varepsilon})) dx_1 dx_2 dX_3 dX_1 dX_2 \\
& + \delta \int_{\Omega^- \times Y} \mathcal{T}_{-}^{\varepsilon}(F_3^-) \mathcal{T}_{-}^{\varepsilon}(\Pi^{\delta}(v_3^{\varepsilon})) dx_1 dx_2 dX_3 dX_1 dX_2 \Bigg] \qquad (10.23) \\
& = k^2 \int_{\Omega^+ \times D} \left[\sum_{\alpha=1}^{2} F_{\alpha}^+ \left(\varphi z_{\alpha} - x_3 \frac{\partial V_3}{\partial x_{\alpha}} \right) + F_3^+ V_3 \right] dx_1 dx_2 dx_3 dX_1 dX_2 \\
& + \int_{\Omega^- \times Y} \left[- \sum_{\alpha=1}^{2} F_{\alpha}^- \left(X_3 + \frac{1}{2} \right) \frac{\partial V_3}{\partial x_{\alpha}} + F_3^- V_3 \right] dx_1 dx_2 dX_3 dX_1 dX_2
\end{aligned}
$$

Secondly, to pass to the limit as δ tends to zero in the left-hand side of (10.22) (divided by δ), we use (6.22), (10.16) and (10.18) for the integral over $\Omega^+ \times D$ and (8.15), (10.14), (10.20)÷(10.21) for the integral over $\Omega^- \times Y$, it gives

$$\lim_{\delta\to 0}\left[\frac{k^2}{\delta}\sum_{i,j=1}^{3}\int_{\Omega^+\times D}\mathcal{T}^\varepsilon(\sigma_{ij}^\delta)\mathcal{T}^\varepsilon(\gamma_{ij}(v^\varepsilon))dx_1dx_2dx_3dX_1dX_2\right.$$

$$\left.+\sum_{i,j=1}^{3}\int_{\Omega^-\times Y}\left(\mathcal{T}_-^\varepsilon\circ\Pi^\delta\right)(\sigma_{ij}^\delta)\left(\mathcal{T}_-^\varepsilon\circ\Pi^\delta\right)(\gamma_{ij}(v^\varepsilon))dx_1dx_2dX_3dX_1dX_2\right]$$

$$= -k^2a\int_{\Omega^+\times D}\Sigma_{33}^+[X_1\varphi z_1'' + X_2\varphi z_2'']dx_1dx_2dx_3dX_1dX_2 \tag{10.24}$$

$$+\int_{\Omega^-\times Y}\sum_{\alpha,\beta=1}^{2}\left(X_3+\frac{1}{2}\right)\overline{\Sigma}_{\alpha\beta}^{\,-}\left[\frac{1}{2}\frac{\partial\Phi}{\partial X_\alpha}\left(\sum_{\gamma=1}^{2}\frac{\partial^2V_3}{\partial x_\gamma\partial x_\beta}X_\gamma\right)\right.$$

$$\left.+\frac{1}{2}\frac{\partial\Phi}{\partial X_\beta}\left(\sum_{\gamma=1}^{2}\frac{\partial^2V_3}{\partial x_\gamma\partial x_\alpha}X_\gamma\right)-(1-\Phi)\frac{\partial^2V_3}{\partial x_\alpha\partial x_\beta}\right]dx_1dx_2dX_3dX_1dX_2.$$

Repeating exactly the argument which allowed to pass from (9.11) to (9.13) in the previous section (i.e. integrating by parts the contribution of $\dfrac{\partial\Phi}{\partial X_\alpha}$ in the above equation) and using (10.23) and (10.24) lead to

$$-k^2a\int_{\Omega^+\times D}\Sigma_{33}^+[X_1\varphi z_1'' + X_2\varphi z_2'']dx_1dx_2dx_3dX_1dX_2$$

$$-\int_{\Omega^-\times Y}\left(X_3+\frac{1}{2}\right)\overline{\Sigma}_{\alpha\beta}^{\,-}\frac{\partial^2V_3}{\partial x_\alpha\partial x_\beta}dx_1dx_2dX_3dX_1dX_2$$

$$= k^2\int_{\Omega^+\times D}\left[\sum_{\alpha=1}^{2}F_\alpha^+\left(\varphi z_\alpha - x_3\frac{\partial V_3}{\partial x_\alpha}\right)+F_3^+V_3\right]dx_1dx_2dx_3dX_1dX_2 \tag{10.25}$$

$$+\int_{\Omega^-\times Y}\left[-\sum_{\alpha=1}^{2}F_\alpha^-\left(X_3+\frac{1}{2}\right)\frac{\partial V_3}{\partial x_\alpha}+F_3^-V_3\right]dx_1dx_2dX_3dX_1dX_2$$

We first choose $z_1 = z_2 = 0$ in (10.25). Using the expression (8.24)÷(8.26) of $\overline{\Sigma}_{\alpha\beta}^{\,-}$ this gives the usual weak formulation of the plate problem for the displacement $\mathcal{U}_3^{0-}$

$$
\begin{aligned}
&\frac{E}{12(1-\nu)}\int_\omega \left[(1-\nu^2)\sum_{\alpha,\beta=1}^{2}\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_\alpha\partial x_\beta}\frac{\partial^2 V_3}{\partial x_\alpha\partial x_\beta}+\nu\Delta\mathcal{U}_3^{0-}\Delta V_3\right]dx_1dx_2\\
&=k^2\pi\int_\omega\left[-\sum_{\alpha=1}^{2}\int_0^L x_3F_\alpha^+dx_3\frac{\partial V_3}{\partial x_\alpha}+\int_0^L F_3^+dx_3V_3\right]dx_1dx_2\\
&+\int_\omega\left[-\sum_{\alpha=1}^{2}\int_{-1}^{0}\left(X_3+\frac{1}{2}\right)F_\alpha^-dX_3\frac{\partial V_3}{\partial x_\alpha}+\int_{-1}^{0}F_3^-dX_3V_3\right]dx_1dx_2,
\end{aligned}
\tag{10.26}
$$

for any V_3 in $C_0^\infty(\omega)$ and then by density for any V_3 in $H_0^2(\omega)$. Indeed (10.26) leads to he usual operator Δ^2 in the PDE for $\mathcal{U}_3^{0-}$ (but remark that the forces F_i^+ in the rods induce a bending in the plate):

$$
\begin{aligned}
&\frac{E}{12(1-\nu^2)}\Delta^2\mathcal{U}_3^{0-}=k^2\pi\left(\int_0^L F_3^+dx_3+\sum_{\alpha=1}^{2}\int_0^L x_3\frac{\partial F_\alpha^+}{\partial x_\alpha}dx_3\right)\\
&+\int_{-1}^{0}F_3^-dX_3+\sum_{\alpha=1}^{2}\int_{-1}^{0}\left(X_3+\frac{1}{2}\right)\frac{\partial F_\alpha^-}{\partial x_\alpha}dX_3,
\end{aligned}
\tag{10.27}
$$

which has a unique solution $\mathcal{U}_3^{0-}\in H_0^2(\omega)$.

In order to obtain the rods equations in Ω^+, we choose now $V_3=0$ in (10.25) and the expression (8.2) of Σ_{33}^+ leads to

$$
\begin{aligned}
&a^2E\int_{\Omega^+\times D}\varphi\left(X_1\frac{\partial^2\mathcal{U}_1^{0+}}{\partial x_3^2}+X_2\frac{\partial^2\mathcal{U}_2^{0+}}{\partial x_3^2}\right)(X_1z_1''+X_2z_2'')dx_1dx_2dx_3dX_1dX_2\\
&=\sum_{\alpha=1}^{2}\int_{\Omega^+\times D}\varphi F_\alpha^+z_\alpha dx_1dx_2dx_3dX_1dX_2.
\end{aligned}
\tag{10.28}
$$

Since z_1, z_2 are arbitrary in $C_0^\infty(]0,L])$, (10.28) gives the same equations for $(\mathcal{U}_1^{0+},\mathcal{U}_2^{0+})$ as in [2] :

$$
\begin{cases}
a^2EI_\alpha\dfrac{\partial^4\mathcal{U}_\alpha^{0+}}{\partial x_3^4}=\pi F_\alpha^+ \text{ in } \Omega^+,\ \text{ for } \alpha=1,2,\\[2ex]
\dfrac{\partial^2\mathcal{U}_\alpha^{0+}}{\partial x_3^2}(x_1,x_2,L)=\dfrac{\partial^3\mathcal{U}_\alpha^{0+}}{\partial x_3^3}(x_1,x_2,L)=0 \text{ a.e. in } \omega,\ \text{ for } \alpha=1,2.
\end{cases}
\tag{10.29}
$$

The bending problem (10.29) in the rods is coupled with the bending $\mathcal{U}_3^{0-}$ in the plate through the transmission conditions (7.52) and (7.53) for $x_3=0$. Since $\mathcal{U}_\alpha^{0+}(x_1,x_2,0)=0$ for $\alpha=1,2$ (due to (7.46)), the functions $\mathcal{U}_\alpha^{0+}$ are uniquely determined in $L^2(\omega,H^2((0,L)))$.

11 Convergence of the energies.

We take $v = u^\delta$ in (2.19) (recall that u^δ denotes $u^{\varepsilon,r,\delta}$ for $r^2 = k^2\varepsilon^2 = a^2\delta^3$) to obtain the energy identity:

$$\mathcal{E}(u^\delta) = \int_{\Omega_\varepsilon^+ \cup \Omega_\delta^-} \sum_{i,j=1}^{3} \sigma_{ij}^\delta \gamma_{ij}(u^\delta) dx_1 dx_2 dx_3 = \int_{\Omega_\varepsilon^+ \cup \Omega_\delta^-} \sum_{i=1}^{3} f_i^\delta u_i^\delta dx_1 dx_2 dx_3. \tag{11.1}$$

Due to the properties of $\mathcal{T}^\varepsilon$ and Π_δ and the assumption on the forces (4.47)÷(4.49), we have

$$\begin{aligned} & k^2 \int_{\Omega^+ \times D} \sum_{i,j=1}^{3} \mathcal{T}^\varepsilon(\sigma_{ij}^\delta) \mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta)) dx_1 dx_2 dx_3 dX_1 dX_2 \\ & + \delta \int_{\Omega^-} \sum_{i,j=1}^{3} \Pi_\delta(\sigma_{ij}^\delta) \Pi_\delta(\gamma_{ij}(u^\delta)) dx_1 dx_2 dX_3 dX_1 dX_2 \\ & = \delta^2 k^2 \int_{\Omega^+ \times D} \sum_{i=1}^{3} F_i^+ \mathcal{T}^\varepsilon(u_i^\delta) dx_1 dx_2 dx_3 dX_1 dX_2 + \delta \int_{\Omega^-} \sum_{\alpha=1}^{2} F_\alpha^- \Pi_\delta(u_\alpha^\delta) dx_1 dx_2 dX_3 \\ & + \delta^2 \int_{\Omega^-} F_3^- \Pi_\delta(u_3^\delta) dx_1 dx_2 dX_3. \end{aligned} \tag{11.2}$$

Dividing (11.2) by δ and using the weak convergences (6.16), (6.23) and (6.25), we obtain

$$\begin{aligned} & \lim_{\delta \to 0} \left[\frac{k^2}{\delta} \int_{\Omega^+ \times D} \sum_{i,j=1}^{3} \mathcal{T}^\varepsilon(\sigma_{ij}^\delta) \mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta)) dx_1 dx_2 dx_3 dX_1 dX_2 + \int_{\Omega^-} \sum_{i,j=1}^{3} \Pi_\delta(\sigma_{ij}^\delta) \Pi_\delta(\gamma_{ij}(u^\delta)) dx_1 dx_2 dX_3 \right] \\ & = k^2 \int_{\Omega^+ \times D} \sum_{i=1}^{3} F_i^+ u_i^{0+} dx_1 dx_2 dx_3 dX_1 dX_2 + \int_{\Omega^-} \sum_{i=1}^{3} F_i^- u_i^{0-} dx_1 dx_2 dX_3 = A. \end{aligned} \tag{11.3}$$

With the help of (7.3), (7.4), (7.19), (7.22) and (7.48) we have

$$\begin{aligned} A = & k^2 \pi \int_{\Omega^+} \sum_{\alpha=1}^{2} F_\alpha^+ \mathcal{U}_\alpha^{0+} dx_1 dx_2 dx_3 + k^2 \pi \int_\omega \left(\int_0^L F_3^+ dx_3 \right) \mathcal{U}_3^{0-} dx_1 dx_2 \\ & + \int_\omega \sum_{i=1}^{3} \int_{-1}^{0} F_i^- dX_3 \mathcal{U}_i^{0-} dx_1 dx_2 - \int_\omega \left(\int_{-1}^{0} \left(X_3 + \frac{1}{2} \right) F_\alpha^- dX_3 \right) \frac{\partial \mathcal{U}_3^{0-}}{\partial x_\alpha}. \end{aligned} \tag{11.4}$$

Now, using $(w_1, w_2) = \left(\mathcal{U}_1^{0+} + x_3 \frac{\partial \mathcal{U}_3^{0-}}{\partial x_1}, \mathcal{U}_2^{0+} + x_3 \frac{\partial \mathcal{U}_3^{0-}}{\partial x_2} \right)$ as a test function in problem (10.29), we obtain since (w_1, w_2) satisfies the boundary conditions $w_1 = w_2 = 0$ and $\frac{\partial w_1}{\partial x_3} = \frac{\partial w_2}{\partial x_3} = 0$ due to (7.52)÷(7.53)

$$a^2E\int_{\Omega^+}\sum_{\alpha=1}^{2} I_\alpha\left(\frac{\partial^2\mathcal{U}_\alpha^{0+}}{\partial x_3^2}\right)^2 dx_1dx_2dx_3 = \int_{\Omega^+}\sum_{\alpha=1}^{2} F_\alpha^+\mathcal{U}_\alpha^{0+}dx_1dx_2dx_3$$
$$+\int_\omega\sum_{\alpha=1}^{2}\int_0^L x_3F_\alpha^+dx_3\frac{\partial\mathcal{U}_3^{0-}}{\partial x_\alpha}dx_1dx_2. \tag{11.5}$$

Using $\mathcal{U}_3^{0-}\in H_0^2(\omega)$ as test function in (10.26) and $(\mathcal{U}_1^{0-},\mathcal{U}_2^{0-})$ as test function in (9.14) leads to

$$\frac{E}{12(1-\nu^2)}\int_\omega\left[(1-\nu)\sum_{\alpha,\beta=1}^{2}\left(\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_\alpha\partial x_\beta}\right)^2+\nu\left(\Delta\mathcal{U}_3^{0-}\right)^2\right]dx_1dx_2$$
$$=-k^2\pi\sum_{\alpha=1}^{2}\int_\omega\left(\int_0^L x_3F_\alpha^+dx_3\right)\frac{\partial\mathcal{U}_3^{0-}}{\partial x_\alpha}dx_1dx_2+k^2\pi\int_\omega\left(\int_0^L F_3^+dx_3\right)\mathcal{U}_3^{0-}dx_1dx_2 \tag{11.6}$$
$$-\sum_{\alpha=1}^{2}\int_\omega\left(\int_{-1}^{0}\left(X_3+\frac{1}{2}\right)F_\alpha^-dX_3\right)\frac{\partial\mathcal{U}_3^{0-}}{\partial x_\alpha}dx_1dx_2+\int_\omega\left(\int_{-1}^{0}F_3^-dX_3\right)\mathcal{U}_3^{0-}dx_1dx_2$$

and

$$\frac{E}{1-\nu^2}\int_\omega\left[(1-\nu)\sum_{\alpha,\beta=1}^{2}\gamma_{\alpha\beta}(\mathcal{U}^{0-})\gamma_{\alpha\beta}(\mathcal{U}^{0-})+\nu\left(\sum_{\delta=1}^{2}\gamma_{\delta\delta}(\mathcal{U}^{0-})\right)^2\right]dx_1dx_2$$
$$=\int_\omega\sum_{\alpha=1}^{2}\left(\int_{-1}^{0}F_\alpha^-dX_3\right)\mathcal{U}_\alpha^{0-}dx_1dx_2, \tag{11.7}$$

Inserting (11.5)÷(11.7) into (11.4) yields

$$A=\frac{E}{1-\nu^2}\int_\omega\left[(1-\nu)\sum_{\alpha,\beta=1}^{2}\gamma_{\alpha\beta}(\mathcal{U}^{0-})\gamma_{\alpha\beta}(\mathcal{U}^{0-})+\nu\left(\sum_{\delta=1}^{2}\gamma_{\delta\delta}(\mathcal{U}^{0-})\right)^2\right]dx_1dx_2$$
$$+k^2a^2E\int_{\Omega^+}\sum_{\alpha=1}^{2}I_\alpha\left(\frac{\partial^2\mathcal{U}_\alpha^{0+}}{\partial x_3^2}\right)dx_1dx_2dx_3 \tag{11.8}$$
$$+\frac{E}{12(1-\nu^2)}\int_\omega\left[(1-\nu)\sum_{\alpha,\beta=1}^{2}\left(\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_\alpha\partial x_\beta}\right)^2+\nu\left(\Delta\mathcal{U}_3^{0-}\right)^2\right]dx_1dx_2.$$

Proceeding exactly as in [2] (Section 8), we first have

$$\int_{\Omega^+\times D}\sum_{i,j=1}^{3}\Sigma_{ij}^+X_{ij}^+dx_1dx_2dx_3dX_1dX_2=a^2E\int_{\Omega^+}\sum_{\alpha=1}^{2}I_\alpha\left(\frac{\partial^2\mathcal{U}_\alpha^{0+}}{\partial x_3^2}\right)^2dx_1dx_2dx_3. \tag{11.9}$$

As far as the plate contribution is concerned in (11.8), an easy calculation shows that (using the expressions (8.28), (8.29), (8.30) of the X^-'s and those (8.24)÷(8.27) of the Σ^-'s)

$$\frac{E}{1-\nu^2}\int_\omega \left[(1-\nu)\sum_{\alpha,\beta=1}^{2}\gamma_{\alpha\beta}(\mathcal{U}^{0-})\gamma_{\alpha\beta}(\mathcal{U}^{0-})+\nu\left(\sum_{\delta=1}^{2}\gamma_{\delta\delta}(\mathcal{U}^{0-})\right)^2\right]dx_1dx_2$$

$$+\frac{E}{12(1-\nu^2)}\int_\omega \left[(1-\nu)\sum_{\alpha,\beta=1}^{2}\left(\frac{\partial^2\mathcal{U}_3^{0-}}{\partial x_\alpha\partial x_\beta}\right)^2+\nu\left(\Delta\mathcal{U}_3^{0-}\right)^2\right]dx_1dx_2 \tag{11.10}$$

$$=\int_{\Omega^-}\sum_{i,j=1}^{3}\Sigma_{ij}^{-}X_{ij}^{-}dx_1dx_2dX_3.$$

Finally, (11.3), (11.8), (11.9) and (11.10) permit to conclude that

$$\lim_{\delta\to 0}\left[\frac{k^2}{\delta}\int_{\Omega^+\times D}\sum_{i,j=1}^{3}\mathcal{T}^\varepsilon(\sigma_{ij}^\delta)\mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta))dx_1dx_2dx_3dX_1dX_2\right.$$

$$\left.+\int_{\Omega^-}\sum_{i,j=1}^{3}\Pi_\delta(\sigma_{ij}^\delta)\Pi_\delta(\gamma_{ij}(u^\delta))dx_1dx_2dX_3\right] \tag{11.11}$$

$$=k^2\int_{\Omega^+\times D}\sum_{i,j=1}^{3}\Sigma_{ij}^{+}X_{ij}^{+}dx_1dx_2dx_3dX_1dX_2+\int_{\Omega^-}\sum_{i,j=1}^{3}\Sigma_{ij}^{-}X_{ij}^{-}dx_1dx_2dX_3.$$

In view of the weak convergences (6.21), (6.22), (6.33), (6.34), the strict convexity of the elastic energy implies that the weak convergences mentioned above are strongly in L^2. From this fact, we deduce exactly as in Section 8 of [2] that

$$\delta\mathcal{U}_i^{\delta+}\to\mathcal{U}_i^{0+} \text{ strongly in } L^2(\omega;H^1((0,L))) \text{ for } i=1,2,3 \tag{11.12}$$

and also (using the result in [21] for the displacement field in the plate)

$$\begin{cases}\mathcal{U}_\alpha^{\delta-}\to\mathcal{U}_\alpha^{0-} \text{ strongly in } H_0^1(\omega) \text{ for } \alpha=1,2,\\ \delta\mathcal{U}_3^{\delta-}\to\mathcal{U}_3^{0-} \text{ strongly in } H_0^1(\omega).\end{cases} \tag{11.13}$$

The strong convergence of the stress fields $\Pi^\delta(\sigma_{ij}^\delta)$ in $(L^2(\Omega^-))^{3\times 3}$ imply that

$$\Pi^\delta({u^\delta}_\alpha)\to {u^0}_\alpha^- \text{ strongly in } H^1(\Omega^-) \text{ for } \alpha=1,2, \tag{11.14}$$

$$\delta\Pi^\delta({u^\delta}_3)\to {u^0}_3^- \text{ strongly in } H^1(\Omega^-), \tag{11.15}$$

(see e.g. [7] and [21]).

12 Summarize

Let ε be a sequence of positive real numbers which tends to 0 and set $r = k\varepsilon$ and $\delta = \left(\frac{r}{a}\right)^{\frac{2}{3}}$ $(0 < k < \frac{1}{2},\ 0 < a)$. Denote by $(u^\delta, \sigma^\delta)$ the solution of Problem (2.19) on $\Omega_\varepsilon^+ \cup \Omega_\delta^- = \Omega_{\varepsilon,\delta}$. The field u^δ is decomposed as follows:

- in Ω_ε^+, we use the decomposition given in (3.5) (see also Section 3 of [2])

$$\overline{u}^{\delta +}(x_1, x_2, x_3) = u^\delta(x_1, x_2, x_3) - \mathcal{U}^{\delta +}(x_1, x_2, x_3) - \mathcal{R}^{\delta +}(x_1, x_2, x_3) \wedge ((x_1 - \varepsilon p)e_1 + (x_2 - \varepsilon q)e_2).$$

- in Ω_δ^-, we use the decomposition given in Section 1 of the present paper

$$\overline{u}^{\delta -}(x_1, x_2, x_3) = u^\delta(x_1, x_2, x_3) - \mathcal{U}^{\delta -}(x_1, x_2) - \mathcal{R}^{\delta -}(x_1, x_2) \wedge \left(x_3 + \frac{\delta}{2}\right) e_3.$$

In order to state the convergence theorem below on $(u^\delta, \sigma^\delta)$ as δ tends to 0 (or ε tends to 0), we introduce the limit problem for any $(F_1^+, F_2^+) \in (L^2(\Omega^+))^2$ and any $(F_1^-, F_2^-, F_3^-) \in (L^2(\Omega^-))^3$:

• "membrane" problem in the plate: let $\mathcal{U}^{0-} = (\mathcal{U}_1^{0-}, \mathcal{U}_2^{0-}) \in (H_0^1(\omega))^2$ be the unique solution of

$$-\frac{E}{1-\nu^2} \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left[(1-\nu)\gamma_{\alpha\beta}(\mathcal{U}^{0-}) + \nu \gamma_{\alpha\alpha}(\mathcal{U}^{0-})\delta_{\alpha\beta} \right] = \int_{-1}^{0} F_\beta^- dX_3 \text{ in } \omega.$$

• coupled bending problems in the rods and in the plate: let $(\mathcal{U}_1^{0+}, \mathcal{U}_2^{0+}) \in L^2(\omega, H^2((0,L)))^2$ and $\mathcal{U}_3^{0-} \in H_0^2(\omega)$ be the unique solution of the problem:

$$a^2 E I_\alpha \frac{\partial^4 \mathcal{U}_\alpha^{0+}}{\partial x_3^4} = \pi F_\alpha^+ \text{ in } \Omega^+, \text{ for } \alpha = 1, 2.$$

$$\frac{E}{12(1-\nu^2)} \Delta^2 \mathcal{U}_3^{0-} = k^2 \pi \left(\int_0^L F_3^+ dx_3 + \sum_{\alpha=1}^{2} \int_0^L x_3 \frac{\partial F_\alpha^+}{\partial x_\alpha} dx_3 \right)$$
$$+ \int_{-1}^{0} F_3^- dX_3 + \sum_{\alpha=1}^{2} \int_{-1}^{0} \left(X_3 + \frac{1}{2} \right) \frac{\partial F_\alpha^-}{\partial x_\alpha} dX_3$$

together with the boundary and transmission conditions:

$$\mathcal{U}_\alpha^{0+}(x_1, x_2, 0) = 0 \text{ a.e. in } \omega, \text{ for } \alpha = 1, 2,$$

$$\frac{\partial^2 \mathcal{U}_\alpha^{0+}}{\partial x_3^2}(x_1, x_2, L) = \frac{\partial^3 \mathcal{U}_\alpha^{0+}}{\partial x_3^3}(x_1, x_2, L) = 0 \text{ a.e. in } \omega, \text{ for } \alpha = 1, 2,$$

$$\frac{\partial \mathcal{U}_\alpha^{0+}}{\partial x_3}(x_1, x_2, 0) + \frac{\partial \mathcal{U}_3^{0-}}{\partial x_\alpha}(x_1, x_2) = 0 \text{ a.e. in } \omega, \text{ for } \alpha = 1, 2.$$

According to the previous sections, we have proved the following theorem:

Theorem 12.1. *Under the assumptions (4.47)÷(4.49) on the applied forces f_i^ε, the sequence $(u^\delta, \sigma^\delta)$ satisfies the following convergences:*

- $$\delta \mathcal{T}^\varepsilon(u_i^\delta) \to \mathcal{U}_i^{0+} \textit{strongly in } L^2(\Omega^+ \times D), \textit{ for } i = 1,2,3,$$
$$\delta \mathcal{U}_i^{\delta+} \to \mathcal{U}_i^{0+} \textit{ strongly in } L^2(\omega, H^1((0,L))), \textit{ for } i = 1,2,3,$$
$$\delta^{-\frac{1}{2}} \mathcal{T}^\varepsilon(\gamma_{ij}(u^\delta)) \to X_{ij}^+ \textit{ strongly in } L^2(\Omega^+ \times D), \textit{ for } i,j = 1,2,3,$$

where

$$X_{11}^+ = X_{22}^+ = \nu \left\{ aX_1 \frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} + aX_2 \frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2} \right\}, \qquad X_{12}^+ = X_{13}^+ = X_{23}^+ = 0,$$

$$X_{33}^+ = -aX_1 \frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} - aX_2 \frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2},$$

- $$\delta^{-\frac{1}{2}} \mathcal{T}^\varepsilon(\sigma_{ij}^\delta) \to \Sigma_{ij}^+ \textit{ strongly in } L^2(\Omega^+ \times D), \textit{ for } i,j = 1,2,3,$$

where

$$\Sigma_{11}^+ = \Sigma_{22}^+ = \Sigma_{12}^+ = \Sigma_{13}^+ = \Sigma_{23}^+ = 0,$$
$$\Sigma_{33}^+ = -E \left\{ aX_1 \frac{\partial^2 \mathcal{U}_1^{0+}}{\partial x_3^2} + aX_2 \frac{\partial^2 \mathcal{U}_2^{0+}}{\partial x_3^2} \right\}.$$

We also have

- $$\Pi^\delta({u^\delta}_\alpha) \to {u^0}^-_\alpha \textit{ strongly in } H^1(\Omega^-) \textit{ for } \alpha = 1,2,$$
$$\delta \Pi^\delta({u^\delta}_3) \to {u^0}^-_3 \textit{ strongly in } H^1(\Omega^-),$$

where u^{0-} is the Kirchoff-love displacement

$$u_\alpha^{0-}(x_1, x_2, X_3) = \mathcal{U}_\alpha^{0-}(x_1, x_2) - \left(X_3 + \frac{1}{2} \right) \frac{\partial \mathcal{U}_3^{0-}}{\partial x_\alpha}(x_1, x_2)$$

$$u_3^{0-}(x_1, x_2, X_3) = \mathcal{U}_3^{0-}(x_1, x_2) = \mathcal{U}_3^{0+}(x_1, x_2),$$

- $$\mathcal{U}_\alpha^{\delta-} \to \mathcal{U}_\alpha^{0-} \textit{ strongly in } H_0^1(\omega), \textit{ for } \alpha = 1,2,$$
$$\delta \mathcal{U}_3^{\delta-} \to \mathcal{U}_3^{0-} \textit{ strongly in } H_0^1(\omega),$$
$$\Pi^\delta(\gamma_{ij}(u^\delta)) \to X_{ij}^- \textit{ strongly in } L^2(\Omega^-), \textit{ for } i,j = 1,2,3,$$

where

$$X_{\alpha\beta}^- = \gamma_{\alpha\beta}(\mathcal{U}^{0-}) - \left(X_3 + \frac{1}{2} \right) \frac{\partial^2 \mathcal{U}_3^{0-}}{\partial x_\alpha \partial x_\beta}, \qquad X_{\alpha 3}^- = 0 \textit{ for } \alpha = 1,2,$$

$$X_{33}^- = \frac{\lambda}{\lambda + 2\mu} \left\{ -\left(\frac{\partial \mathcal{U}_1^{0-}}{\partial x_1} + \frac{\partial \mathcal{U}_2^{0-}}{\partial x_2} \right) + \left(X_3 + \frac{1}{2} \right) \Delta \mathcal{U}_3^{0-} \right\},$$

$$\Pi^\delta(\sigma_{ij}^\delta) \to \Sigma_{ij}^- \textit{ strongly in } L^2(\Omega^-), \textit{ for } i,j = 1,2,3,$$

where

$$\Sigma^-_{11} = \frac{E}{1-\nu^2}\Bigg[\Bigg(\frac{\partial \mathcal{U}^{0-}_1}{\partial x_1} - \Big(X_3 + \frac{1}{2}\Big)\frac{\partial^2 \mathcal{U}^{0-}_3}{\partial x_1^2}\Bigg) + \nu\Bigg(\frac{\partial \mathcal{U}^{0-}_2}{\partial x_2} - \Big(X_3 + \frac{1}{2}\Big)\frac{\partial^2 \mathcal{U}^{0-}_3}{\partial x_2^2}\Bigg)\Bigg],$$

$$\Sigma^-_{22} = \frac{E}{1-\nu^2}\Bigg[\Bigg(\frac{\partial \mathcal{U}^{0-}_2}{\partial x_2} - \Big(X_3 + \frac{1}{2}\Big)\frac{\partial^2 \mathcal{U}^{0-}_3}{\partial x_2^2}\Bigg) + \nu\Bigg(\frac{\partial \mathcal{U}^{0-}_1}{\partial x_1} - \Big(X_3 + \frac{1}{2}\Big)\frac{\partial^2 \mathcal{U}^{0-}_3}{\partial x_1^2}\Bigg)\Bigg],$$

$$\Sigma^-_{12} = \mu\Bigg(\frac{\partial \mathcal{U}^{0-}_1}{\partial x_2} + \frac{\partial \mathcal{U}^{0-}_2}{\partial x_1} - 2\Big(X_3 + \frac{1}{2}\Big)\frac{\partial^2 \mathcal{U}^{0-}_3}{\partial x_1 \partial x_2}\Bigg),$$

$$\Sigma^-_{13} = \Sigma^-_{23} = \Sigma^-_{33} = 0.$$

13 Appendix A

In this section we recall some properties of the periodic unfolding operator $\mathcal{T}_\varepsilon$. Let ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. We denote $Y =]-1/2, 1/2[^N$ the unit cell in $\mathbb{R}^N$. For almost every z in $\mathbb{R}^N$ there exists a unique element $[z]$ in $\mathbb{Z}^N$ such that $z - [z] = \{z\}$ belongs to Y.

Let us now recall the definition of the periodic unfolding operator $\mathcal{T}_\varepsilon$. For any function ϕ in $L^1(\omega)$ we define $\mathcal{T}_\varepsilon(\phi)$ by

$$\mathcal{T}_\varepsilon(\phi)(x,y) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon y\Big), & \text{if} \quad \varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon Y \subset \omega, \\ 0, & \text{if} \quad \varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon Y \not\subset \omega, \end{cases} \qquad \text{for a.e. } (x,y) \in \omega \times Y.$$

The function $\mathcal{T}_\varepsilon(\phi)$ belongs to $L^1(\omega \times Y)$ and verifies

$$||\mathcal{T}_\varepsilon(\phi)||_{L^1(\omega\times Y)} \le ||\phi||_{L^1(\omega)}.$$

Usually we do not have the integration formula $\displaystyle\int_\omega \phi = \frac{1}{|Y|}\int_{\omega\times Y} \mathcal{T}_\varepsilon(\phi)$. We have the following estimate of the difference between the left hand side and the right hand side:

$$\Big|\int_\omega \phi - \frac{1}{|Y|}\int_{\omega\times Y} \mathcal{T}_\varepsilon(\phi)\Big| \le ||\phi||_{L^1(\omega\setminus\omega_\varepsilon)},$$

where

$$\omega_\varepsilon = int\big(\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\big), \quad \Xi_\varepsilon = \big\{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \omega\big\}.$$

Notice that the distance between ω_ε and the boundary of ω is less than $\sqrt{N}\varepsilon$. If ω' is an open set strongly included in ω and if ϕ vanish over $\omega \setminus \omega'$ then the integration formula is exact with ε sufficiently small. Obviously, for any $\phi,\ \psi \in L^2(\omega)$, we have

$$\mathcal{T}_\varepsilon(\phi\psi) = \mathcal{T}_\varepsilon(\phi)\ \mathcal{T}_\varepsilon(\psi).$$

Let $\mathcal{O} \subset \mathbb{R}^q$ be an open set of parameters. In the same way, for any $\phi \in L^1(\omega \times \mathcal{O})$ we define the unfold function $\mathcal{T}_\varepsilon(\phi)$ by

$$\mathcal{T}_\varepsilon(\phi)(x,y,z) = \begin{cases} \phi\Big(\varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon y, z\Big), & \text{if} \quad \varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon Y \subset \omega, \\ \\ 0, & \text{if} \quad \varepsilon\Big[\dfrac{x}{\varepsilon}\Big] + \varepsilon Y \not\subset \omega, \end{cases} \qquad \text{for a.e. } (x,y,z) \in \omega \times Y \times \mathcal{O}.$$

This function belongs to $L^1(\omega \times Y \times \mathcal{O})$. Of course if $\phi \in L^2(\omega; H^1(\mathcal{O}))$, we have $\mathcal{T}_\varepsilon(\phi) \in L^2(\omega \times Y; H^1(\mathcal{O}))$ and moreover

$$\nabla_z \mathcal{T}_\varepsilon(\phi) = \mathcal{T}_\varepsilon(\nabla_z \phi), \qquad\qquad ||\nabla_z \mathcal{T}_\varepsilon(\phi)||_{[L^2(\omega\times Y\times\mathcal{O})]^q} \le ||\nabla_z\phi||_{[L^2(\omega\times\mathcal{O})]^q}.$$

For any function $\phi \in L^2(\omega)$, we define the local average $M^\varepsilon_Y \ : \ L^2(\omega) \longrightarrow L^2(\omega)$, by

$$M^\varepsilon_Y(\phi)(x) = \frac{1}{|Y|}\int_Y \mathcal{T}_\varepsilon(\phi)(x,y)dy, \quad x \in \omega.$$

For any function $\phi \in \mathcal{C}(\omega)$, we define $\widetilde{\phi}_\varepsilon \in L^\infty(\omega)$, by

$$\widetilde{\phi}_\varepsilon(x) = \begin{cases} \phi\Big(\Big[\dfrac{x}{\varepsilon}\Big]\Big), & \text{for a.e. } x \in \omega_\varepsilon, \\ \\ 0, & \text{for a.e. } x \in \omega \setminus \omega_\varepsilon. \end{cases}$$

The result mentioned in the following lemma can found in [8] and [9].

Lemma A1 : *(i) For any ϕ belonging to $L^2(\omega)$, we have*

$$\begin{cases} \mathcal{T}_\varepsilon(\phi) \longrightarrow \phi \quad \textit{strongly in} \quad L^2(\omega \times Y), \\ \\ M^\varepsilon_Y(\phi) \longrightarrow \phi \quad \textit{strongly in} \quad L^2(\omega). \end{cases}$$

(ii) For any ϕ belonging to $H^1(\omega)$, we have

$$\frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon(\phi) - M^\varepsilon_Y(\phi)\big) \longrightarrow y\cdot\nabla\phi \quad \textit{strongly in} \quad L^2(\omega\times Y).$$

(iii) For any ϕ belonging to $\mathcal{C}^1_0(\omega)$, we have

$$\frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon(\phi) - \widetilde{\phi}_\varepsilon\big) \longrightarrow y\cdot\nabla\phi \quad \textit{strongly in} \quad L^\infty(\omega\times Y).$$

(iv) Let $\big(\phi_\varepsilon)_{\varepsilon>0}$ be a sequence of functions uniformly bounded in $L^2(\omega)$. There exists $\widehat{\phi} \in L^2(\omega\times Y)$ such that, up to a subsequence we have

$$\begin{cases} \phi_\varepsilon \rightharpoonup \phi \quad \textit{weakly in} \quad L^2(\omega), \quad \phi = \dfrac{1}{|Y|}\displaystyle\int_Y \widehat{\phi}(.,y)dy, \\ \\ \mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \quad \textit{weakly in} \quad L^2(\omega\times Y). \end{cases}$$

(v) Let $\big(\phi_\varepsilon\big)_{\varepsilon>0}$ be a sequence of functions uniformly bounded in $H^1(\omega)$. There exists $\phi \in H^1(\omega)$ and $\widehat{\phi} \in L^2(\omega; H^1_{per}(Y))$ such that, up to a subsequence we have

$$\begin{cases} \phi_\varepsilon \rightharpoonup \phi \quad \textit{weakly in} \quad H^1(\omega), \\ \\ \mathcal{T}_\varepsilon(\phi_\varepsilon) \longrightarrow \phi \quad \textit{strongly in} \quad L^2(\omega \times Y), \\ \\ \mathcal{T}_\varepsilon(\nabla \phi_\varepsilon) \rightharpoonup \nabla_x \phi + \nabla_y \widehat{\phi} \quad \textit{weakly in} \quad [L^2(\omega \times Y)]^N. \end{cases}$$

If we choose $\widehat{\phi}$ such that $\displaystyle\int_Y \widehat{\phi}(.,y)dy = 0$ then we have

$$\frac{1}{\varepsilon}\big(\mathcal{T}_\varepsilon(\phi_\varepsilon) - M^\varepsilon_Y(\phi_\varepsilon)\big) \rightharpoonup y \cdot \nabla_x \phi + \widehat{\phi}, \quad \textit{weakly in} \quad L^2(\omega \times Y).$$

(vi) Let $\big(\phi_\varepsilon\big)_{\varepsilon>0}$ be a sequence of functions in $H^1(\omega)$ such that

$$||\phi_\varepsilon||_{L^2(\omega)} + \varepsilon||\nabla \phi_\varepsilon||_{(L^2(\omega))^N} \leq C.$$

There exists $\widehat{\phi} \in L^2(\omega; H^1_{per}(Y))$ such that, up to a subsequence we have

$$\begin{cases} \phi_\varepsilon \rightharpoonup \phi \quad \textit{weakly in} \quad L^2(\omega), \quad \phi = \dfrac{1}{|Y|}\displaystyle\int_Y \widehat{\phi}(.,y)dy, \\ \\ \mathcal{T}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \quad \textit{weakly in} \quad L^2(\omega \times Y), \\ \\ \varepsilon\mathcal{T}_\varepsilon(\nabla \phi_\varepsilon) \rightharpoonup \nabla_y \widehat{\phi} \quad \textit{weakly in} \quad [L^2(\omega \times Y)]^N. \end{cases}$$

Let i be in $\{1, \dots, N\}$ and let B be the ball included in Y with center O and radius $r < 1/2$. For any function $\phi \in L^2(\omega)$, we define the local momentum $\mathcal{M}^\varepsilon_{i,B} : L^2(\omega) \longrightarrow L^2(\omega)$, by

$$\mathcal{M}^\varepsilon_{i,B}(\phi)(x) = \frac{1}{\varepsilon I_i}\int_B y_i \mathcal{T}_\varepsilon(\phi)(x,y)dy, \quad x \in \omega, \quad \text{where} \quad I_i = \int_B y_i^2 dy.$$

Lemma A2 : *(i) Let $\big(\phi_\varepsilon\big)_{\varepsilon>0}$ be a sequence of functions in $H^1(\omega)$ such that*

$$\phi_\varepsilon \rightharpoonup \phi \quad \textit{weakly in} \quad H^1(\omega),$$

$$\mathcal{T}_\varepsilon(\nabla \phi_\varepsilon) \rightharpoonup \nabla_x \phi + \nabla_y \widehat{\phi} \quad \textit{weakly in} \quad [L^2(\omega \times Y)]^N,$$

where $\widehat{\phi} \in L^2(\omega; H^1_{per}(Y))$ and verifies $\displaystyle\int_Y \widehat{\phi}(.,y)dy = 0$. Then we have

$$\mathcal{M}^\varepsilon_{i,B}(\phi_\varepsilon) \rightharpoonup \frac{\partial \phi}{\partial x_i} + \frac{1}{I_i}\int_B y_i \widehat{\phi}(\cdot,y)dy \quad \textit{weakly in} \quad L^2(\omega \times Y).$$

(ii) Let $\big(\phi_\varepsilon\big)_{\varepsilon>0}$ be a sequence of functions in $H^1(\omega)$ such that

$$\phi_\varepsilon \longrightarrow \phi \quad \textit{strongly in} \quad H^1(\omega).$$

We have

$$\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon) \longrightarrow \frac{\partial \phi}{\partial x_i} \quad \textit{strongly in} \quad L^2(\omega).$$

Proof : First we prove (i). For any $\phi \in H^1(\omega)$ we have

$$\mathcal{M}^{\varepsilon}_{i,B}(\phi) = \frac{1}{I_i}\int_B y_i \frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\phi - M^\varepsilon_Y(\phi)\big)(.,y)dy.$$

We apply the Poincar-Wirtinger inequality and we deduce that

$$\big\|\mathcal{M}^{\varepsilon}_{i,B}(\phi)\big\|_{L^2(\omega)} \le C||\nabla\phi||_{[L^2(\omega)]^N}.$$

The constant is independent of ε. Hence the sequence $\big(\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon)\big)_{\varepsilon>0}$ is uniformly bounded in $L^2(\omega)$. Let ψ be in $\mathcal{C}^\infty_0(\omega)$. If ε is sufficiently small we have

$$\int_\omega \mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon)M^\varepsilon_Y(\psi) = \frac{1}{I_i}\int_{\omega\times B} y_i\frac{1}{\varepsilon}\mathcal{T}_\varepsilon\big(\phi_\varepsilon - M^\varepsilon_Y(\phi_\varepsilon)\big)(x,y)M^\varepsilon_Y\big(\psi\big)(x)dxdy.$$

We pass to the limit and due to Lemma A1 (i) and (iv) we obtain

$$\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon) \rightharpoonup \frac{1}{I_i}\int_B y_i\big(y\cdot\nabla_x\phi + \widehat{\phi}\big)dy = \frac{\partial\phi}{\partial x_i} + \frac{1}{I_i}\int_B y_i\widehat{\phi}(\cdot,y)dy, \quad \text{weakly in} \quad L^2(\omega).$$

Now we prove (ii). We have $\widehat{\phi} = 0$ and then

$$\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon) \rightharpoonup \frac{\partial\phi}{\partial x_i} \quad \text{weakly in} \quad L^2(\omega).$$

For any function $\psi \in \mathcal{C}^1(\overline{\omega})$ the sequence $\big(\mathcal{M}^{\varepsilon}_{i,B}(\psi)\big)_{\varepsilon>0}$ converges strongly in $L^2(\omega)$ to $\dfrac{\partial\psi}{\partial x_i}$. Let $\big(\phi_n\big)_{n\in\mathbb{N}}$ be a sequence of functions belonging to $\mathcal{C}^1(\overline{\omega})$ such that

$$\phi_n \longrightarrow \phi \quad \text{strongly in} \quad H^1(\omega).$$

We have

$$\begin{aligned}
&\big\|\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon) - \frac{\partial\phi}{\partial x_i}\big\|_{L^2(\omega)}\\
&\le \big\|\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon) - \mathcal{M}^{\varepsilon}_{i,B}(\phi_n)\big\|_{L^2(\omega)} + \big\|\mathcal{M}^{\varepsilon}_{i,B}(\phi_n) - \frac{\partial\phi_n}{\partial x_i}\big\|_{L^2(\omega)} + \big\|\frac{\partial\phi_n}{\partial x_i} - \frac{\partial\phi}{\partial x_i}\big\|_{L^2(\omega)}\\
&\le C\big\{||\nabla(\phi_\varepsilon - \phi)||_{(L^2(\omega))^N} + ||\nabla(\phi_n - \phi)||_{(L^2(\omega))^N}\big\} + \big\|\mathcal{M}^{\varepsilon}_{i,B}(\phi_n) - \frac{\partial\phi_n}{\partial x_i}\big\|_{L^2(\omega)}.
\end{aligned}$$

With these inequalities and the strong convergences of the sequences $\big(\phi_\varepsilon\big)_{\varepsilon>0}$ and $\big(\phi_n\big)_{\varepsilon>0}$ we immediately deduce the strong convergence in $L^2(\omega)$ of the sequence $\big(\mathcal{M}^{\varepsilon}_{i,B}(\phi_\varepsilon)\big)_{\varepsilon>0}$.

14 Appendix B

In this section we prove Lemma A3. We use the notation of the previous section. Throughout this appendix the constants appearing in the estimates are independent from δ. Let ω be a bounded domain in $\mathbb{R}^2$ with lipschitz boundary and let γ be a part of $\partial\omega$ with positive measure. We set

$$H^1_\gamma(\omega) = \Big\{\Phi \in H^1(\omega) \ \mid \ \Phi = 0 \ \text{ on } \ \gamma\Big\}.$$

Let $\big(U^\delta\big)_{\delta>0}$, $\big(R^\delta_1\big)_{\delta>0}$ and $\big(R^\delta_2\big)_{\delta>0}$ be sequences of functions in $H^1_\gamma(\omega)$ such that

$$\delta||\nabla R^\delta_\alpha||_{[L^2(\omega)]^2} + \Big\|\frac{\partial U^\delta}{\partial x_1} + R^\delta_2\Big\|_{L^2(\omega)} + \Big\|\frac{\partial U^\delta}{\partial x_2} - R^\delta_1\Big\|_{L^2(\omega)} \le C, \tag{1}$$

which implies that

$$||R^\delta_\alpha||_{H^1(\omega)} + ||U^\delta||_{H^1(\omega)} \le \frac{C}{\delta}.$$

Let $\mathbf{U}^\delta$ be the solution of the variational problem

$$\begin{cases} \mathbf{U}^\delta \in H^1_\gamma(\omega), \\ \displaystyle\int_\omega \nabla\mathbf{U}^\delta\nabla\Phi = \int_\omega \Big[R^\delta_2\frac{\partial\Phi}{\partial x_1} - R^\delta_1\frac{\partial\Phi}{\partial x_2}\Big], \\ \forall\Phi \in H^1_\gamma(\omega). \end{cases} \tag{2}$$

The function $\mathbf{U}^\delta$ belong to $H^1_\gamma(\omega)\cap H^2_{loc}(\omega)$ and it verifies the following estimates :

$$||\mathbf{U}^\delta||_{H^1(\omega)} \le \frac{C}{\delta}, \quad \Big\|\rho\frac{\partial^2\mathbf{U}^\delta}{\partial x_\alpha\partial x_\beta}\Big\|_{L^2(\omega)} \le \frac{C}{\delta}. \tag{3}$$

where ρ is defined by

$$\rho(x) = dist(x, \partial\omega), \quad x \in \omega, \quad \rho \in W^{1,\infty}(\omega).$$

We put

$$\mathbf{u}^\delta = U^\delta - \mathbf{U}^\delta, \quad r^\delta_1 = \frac{\partial\mathbf{U}^\delta}{\partial x_1} + R^\delta_1, \quad r^\delta_2 = -\frac{\partial\mathbf{U}^\delta}{\partial x_2} + R^\delta_2.$$

The function $\mathbf{u}^\delta$ belongs to $H^1_\gamma(\omega)$ and thanks to (1) and (2) we have

$$||\mathbf{u}^\delta||_{H^1(\omega)} \le C. \tag{4}$$

The functions r^δ_1 and r^δ_2 belong to $L^2(\omega)\cap H^1_{loc}(\omega)$ and due to (1), (2), (3) and (4) they verify the estimates

$$||r^\delta_\alpha||_{L^2(\omega)} \le C, \quad ||\rho\nabla r^\delta_\alpha||_{L^2(\omega)} \le \frac{C}{\delta}. \tag{5}$$

There exists $\mathbf{u} \in H^1_\gamma(\omega)$ and $r_1,\ r_2,\ Z_1,\ Z_2 \in L^2(\omega)$ such that, up to subsequences we have the following weak convergences

$$\begin{cases} \mathbf{u}^\delta \rightharpoonup \mathbf{u}, \quad \text{weakly in } H^1_\gamma(\omega), \\ \\ r^\delta_\alpha \rightharpoonup r_\alpha, \quad \text{weakly in } L^2(\omega), \\ \\ \dfrac{\partial U^\delta}{\partial x_1} + R^\delta_2 \rightharpoonup Z_1, \quad \text{weakly in } L^2(\omega), \\ \\ \dfrac{\partial U^\delta}{\partial x_2} - R^\delta_1 \rightharpoonup Z_2, \quad \text{weakly in } L^2(\omega). \end{cases}$$

Due to the definition of $\mathbf{u}^\delta$, r^δ_1 and r^δ_2 we have

$$\frac{\partial U^\delta}{\partial x_1} + R^\delta_2 = \frac{\partial \mathbf{u}^\delta}{\partial x_1} + r^\delta_2, \quad \frac{\partial U^\delta}{\partial x_2} - R^\delta_1 = \frac{\partial \mathbf{u}^\delta}{\partial x_2} - r^\delta_1. \tag{6}$$

Then we obtain

$$Z_1 = \frac{\partial \mathbf{u}}{\partial x_1} + r_2, \quad Z_2 = \frac{\partial \mathbf{u}}{\partial x_2} - r_1. \tag{7}$$

Let us consider two sequences ε and δ of positive real numbers which converge to 0 with

$$\frac{\varepsilon}{\delta} \longrightarrow 0. \tag{8}$$

Lemma A3 : *There exists $\widehat{\mathbf{u}} \in L^2(\omega; H^1_{per}(Y))$ such that, up to subsequences we have the following weak convergences :*

$$\begin{cases} \mathcal{T}_\varepsilon\Big(\dfrac{\partial U^\delta}{\partial x_1} + R^\delta_2\Big) \rightharpoonup Z_1 + \dfrac{\partial \widehat{\mathbf{u}}}{\partial X_1} \quad \textit{weakly in } L^2(\omega \times Y), \\ \\ \mathcal{T}_\varepsilon\Big(\dfrac{\partial U^\delta}{\partial x_2} - R^\delta_1\Big) \rightharpoonup Z_2 + \dfrac{\partial \widehat{\mathbf{u}}}{\partial X_2} \quad \textit{weakly in } L^2(\omega \times Y). \end{cases} \tag{9}$$

Proof : There exists $\widehat{r}_1,\ \widehat{r}_2,\ \widehat{Z}_1,\ \widehat{Z}_2 \in L^2(\omega \times Y)$, $\widehat{\mathbf{u}} \in L^2(\omega; H^1_{per}(Y))$ such that, up to subsequences, we have the following weak convergences ($\alpha \in \{1,2\}$):

$$\begin{cases} \mathcal{T}_\varepsilon\Big(\dfrac{\partial \mathbf{u}^\delta}{\partial x_\alpha}\Big) \rightharpoonup \dfrac{\partial \mathbf{u}}{\partial x_\alpha} + \dfrac{\partial \widehat{\mathbf{u}}}{\partial X_\alpha} \quad \text{weakly in } L^2(\omega \times Y), \\ \\ \mathcal{T}_\varepsilon(r^\delta_\alpha) \rightharpoonup \widehat{r}_\alpha \quad \text{weakly in } L^2(\omega \times Y), \\ \\ \mathcal{T}_\varepsilon\Big(\dfrac{\partial U^\delta}{\partial x_1} + R^\delta_2\Big) \rightharpoonup \widehat{Z}_1 \quad \text{weakly in } L^2(\omega \times Y). \\ \\ \mathcal{T}_\varepsilon\Big(\dfrac{\partial U^\delta}{\partial x_2} - R^\delta_1\Big) \rightharpoonup \widehat{Z}_2 \quad \text{weakly in } L^2(\omega \times Y). \end{cases}$$

Let $\mathcal{O}$ be an open set such that $\overline{\mathcal{O}} \subset \omega$. Thanks to estimates (5) of r_α, to Lemma A1 of Appendix A and (8) we obtain

$$||\mathcal{T}_\varepsilon(r_\alpha)||_{L^2(\omega\times Y)} \leq C, \quad \Big\|\frac{\partial \mathcal{T}_\varepsilon(r_\alpha)}{\partial X_\beta}\Big\|_{L^2(\mathcal{O}\times Y)} \leq C\frac{\varepsilon}{\delta} \leq C.$$

Then we have the weak convergences

$$\mathcal{T}_\varepsilon(r^\delta_\alpha) \rightharpoonup \widehat{r}_\alpha, \quad \text{weakly in } L^2(\omega\times Y),$$

$$\mathcal{T}_\varepsilon(r^\delta_\alpha) \rightharpoonup \widehat{r}_\alpha, \quad \text{weakly in } L^2(\mathcal{O}; H^1(Y)).$$

and due to (8)

$$\nabla_X \widehat{r}_\alpha = 0, \quad \text{in} \quad \mathcal{O}\times Y.$$

So $\widehat{r}_\alpha = r_\alpha$ in $L^2(\mathcal{O}\times Y)$ where r_α is the weak limit in $L^2(\omega)$ of the sequence $(r^\delta_\alpha)_{\delta>0}$. There results that $\widehat{r}_\alpha = r_\alpha$ in $L^2(\omega\times Y)$. We transform the equalities (6) by unfolding and we pass to the limit. Due to (7) the convergences (9) are proved.

Remark : If $\dfrac{\varepsilon}{\delta} \longrightarrow \kappa \in \mathbf{R}^*_+$ we can prove that

$$\begin{cases} \mathcal{T}_\varepsilon(r^\delta_\alpha) \rightharpoonup \widehat{r}_\alpha, \quad \text{weakly in } L^2(\omega\times Y), \\ \mathcal{T}_\varepsilon\Big(\dfrac{\partial U^\delta}{\partial x_1} + R^\delta_2\Big) \rightharpoonup \dfrac{\partial \mathbf{u}}{\partial x_1} + \dfrac{\partial \widehat{\mathbf{u}}}{\partial X_1} + \widehat{r}_2, \quad \text{weakly in } L^2(\omega\times Y), \\ \mathcal{T}_\varepsilon\Big(\dfrac{\partial U^\delta}{\partial x_2} - R^\delta_1\Big) \rightharpoonup \dfrac{\partial \mathbf{u}}{\partial x_2} + \dfrac{\partial \widehat{\mathbf{u}}}{\partial X_2} - \widehat{r}_1, \quad \text{weakly in } L^2(\omega\times Y). \end{cases}$$

where

$$\widehat{r}_\alpha \in L^2_\rho(\omega; H^1_{per}(Y)) = \Big\{\widehat{\phi} \in L^2(\omega\times Y) \mid \rho(.)\widehat{\phi}(.,..) \in L^2(\omega; H^1_{per}(Y))\Big\}.$$

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