Just What *Did* Archimedes Say About Buoyancy?

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"A body immersed in a fluid is buoyed up with a force equal to the weight of the displaced fluid." So goes a venerable textbook[1] statement of the hydrostatic principle that bears Archimedes' name. Archimedes' principle is often proved for the special case of a right-circular cylinder or rectangular solid by considering the difference in hydrostatic forces between the (flat, horizontal) upper and lower surfaces, and then generalized by the even more venerable "it can be shown…" that the principle is in fact true for bodies of arbitrary shape.

From time to time in this journal,[2,3,4] the question has arisen as to how to deal with an immersed object if that object happens to be resting at the bottom of the fluid, in such a way that there is no fluid between the object and the bottom. This immediately raises two thought-provoking questions:

1. If there is no fluid beneath the object, is there a buoyant force? The standard proof might reasonably lead one to believe that there would be no buoyant force.
2. If there is no buoyant force, would this case not be an "exception" to Archimedes' principle that the great man somehow overlooked?

What Archimedes Really Said

Going back to the source,[5] Archimedes' *On Floating Bodies, Book I* contains a number of statements and proofs that are relevant to the present discussion. Archimedes considers three cases separately, namely, objects with a density equal to that of the fluid, objects "lighter" (less dense) than the fluid, and objects "heavier" (more dense) than the fluid.

He proves the following propositions:

"**Proposition 3.** Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.

"**Proposition 4.** A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the surface.

"**Proposition 5.** Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.

"**Proposition 6.** If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced."

So far, so good. The modern reader will be familiar with these results, although the term "buoyant force" does not explicitly appear (at least in this translation). In Proposition 6, Archimedes mentions an upward force that today we would say is the difference be-

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tween the buoyant force and the gravitational force on the object. Although in the present translation the word "driven" (perhaps implying motion) is used, it is clear from his proof of this proposition that Archimedes is considering a purely *static* case: His "upward" force is countered by an equal and opposite downward force to hold the object submerged and motionless. It is also clear from the proofs, (which make fascinating reading) that there is fluid under the object in these cases.

Now Archimedes turns to objects denser than the fluid.

"**Proposition 7.** A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced."

We immediately see that Archimedes has not failed to consider the possibility of the object resting on the bottom of a fluid. However, in contrast with Proposition 6, he does not explicitly mention anything like a buoyant force. Instead, he speaks of what today we would call the object's "apparent weight" being lighter than the "true weight" by the weight of the displaced fluid. Archimedes does not say just how we are to weigh the object in the fluid. Let us consider some possible methods.

In the standard lecture demonstration, we use a thread of negligible mass to hang an object of mass m and volume V from a spring balance (or some other more modern scale) in the air and record the object's weight. The scale reading (and the tension T in the thread) will be mg, where g is the acceleration due to gravity. We have of course neglected the effects of air buoyancy. Then we fully immerse the object in water, still hanging from the spring balance, and show the class that the scale reading is now less by the weight of the displaced water, provided of course, that the object does not touch the bottom. The reading is now

$$T = mg - \rho gV, \tag{1}$$

where ρ is the density of the water. No difficulty here; the apparent weight of the object is the "true" weight minus the weight of the displaced fluid, just as Archimedes says. We moderns would probably go on to say that the difference is caused by the buoyant force.

What if the object is resting on the bottom in such a way that there is no fluid beneath it? For this we assume an object with a flat bottom, say a rectangular solid of height h and bottom surface area A. We also assume that the bottom of the fluid container is flat. There are two general approaches we might use to measure the apparent weight of the object, which we could style dynamic and static, respectively.

The Dynamic Apparent Weight

In this method, we try lifting the object off the bottom still using, for example, a thread and a spring balance. This is the general approach used in Refs. 2–4. The measured apparent weight (i.e., the reading of the spring balance) using this method will depend critically on many factors, e.g., whether or not (or to what degree) the fluid can seep in under the object as a lifting force is applied, the viscosity of the fluid, how rapidly the lifting force increases, the shape of the object/bottom interface, the shape of the object, the presence or absence of adhesive forces at the interface (suppose the object is glued!), the depth of the fluid, and so on. Needless to say, under these circumstances, there can be no uniquely defined "apparent weight"; each different set of experimental conditions could yield a different result.

Refs. 2–4 consider the particular case of no fluid seepage being allowed (as, for example, with a compressed suction cup) until the seal fails completely. Under those circumstances, the "dynamic apparent weight" (again ignoring viscous, molecular, and other extraneous forces) will be the "true weight" of the object plus the weight of the column of fluid above the object, which of course will depend on the depth of the fluid.

Is the "dynamic apparent weight" method a valid test of Archimedes' principle? I would submit that in general, it is not. The proofs of Archimedes' propositions are all done for objects and fluids in *static* equilibrium and as such avoid all the complications involved with actually moving an object in a fluid, especially if that motion involves a sudden — decidedly *nonstatic* — inrush of fluid as when, for example, the seal of a suction cup breaks, or when water gushes down the drain as we pull the plug in a sink.

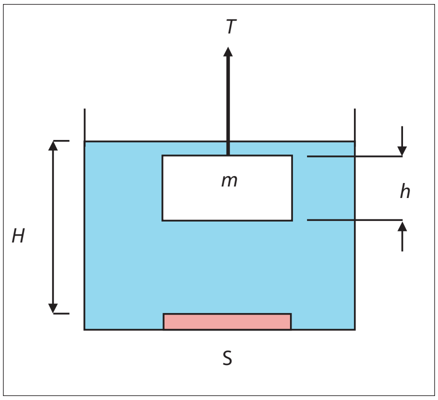

Fig. 1. The block is suspended in the fluid. The scale S reads $F_A = \rho gHA$. The magnitude of the suspending force T is ($mg - \rho ghA$), the apparent weight of the block.

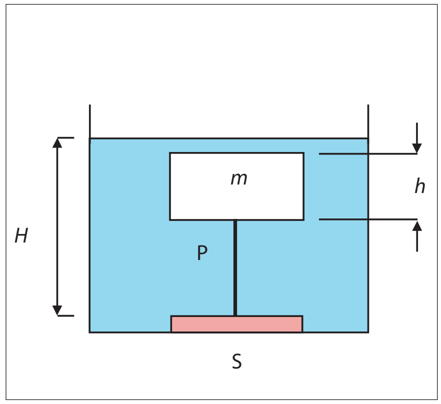

Fig. 2. The block is balanced on the pin P. Now scale S reads $F_A = \rho gHA + (mg - \rho ghA)$. The block's apparent weight is obtained by subtracting the quantity ρgHA from F_A.

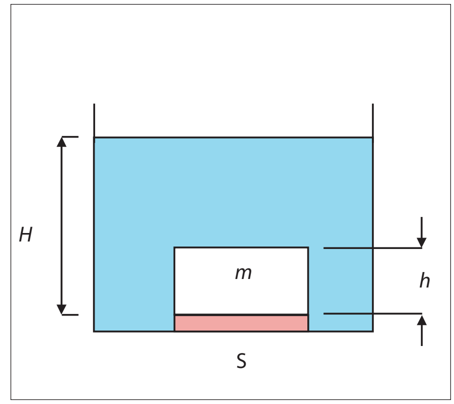

Fig. 3. The block now rests directly on S. Scale S still reads $F_A = \rho gHA + (mg - \rho ghA)$. The apparent weight of the block is the same as it is in Fig. 2.

The Static Apparent Weight

Let us now consider the following question: Is there a difference in a submerged object's apparent weight — measured *statically* — if that object is resting on the bottom, as opposed to being suspended in the fluid? In this method, we measure the apparent weight without moving the object or the fluid during the measurement, just as in the standard demonstration described above, except that now we propose to do the measurement with the object resting on the bottom.

How might we accomplish this? Referring to Fig. 1, let us suppose we have a force-sensing scale S located at the bottom of a container. The scale has a rectangular top of area A and is designed to measure the total force F_A on A (excluding the force due to atmospheric pressure). It could be, for example, a piezo or a capacitative force-transducer. When the container is empty, S reads zero. When covered by fluid to depth H, S reads ρgHA (i.e., the weight of the fluid column above it), where ρ is the fluid density and g is the acceleration due to gravity. Now let a rectangular solid of mass m, height h and bottom area A (shape matching the scale-top) be suspended directly above S. This requires a force T, pointing up, assuming that the block is denser than the fluid. The magnitude of T is ($mg - \rho ghA$), the apparent weight of the block (just as measured in the standard demo), and everyone will agree that there is a buoyant force exerted by the fluid on the block. The scale still reads $F_A = \rho gHA$.

In Fig. 2, the force T has been removed, and instead the block is balanced on a very thin pin P of negligible mass and cross-sectional area (or three such pins for stability), the pin bottom(s) resting on the scale. Since there is still undoubtedly a buoyant force acting on the block (it is still surrounded by fluid), the pin must exert a force on the scale equal to the apparent weight of the block. Scale S will therefore now read

$$F_A = \rho gHA + (mg - \rho ghA). \qquad (2)$$

To determine the apparent weight from the scale reading F_A, we simply subtract the quantity ρgHA from F_A.

In Fig. 3, the pin has been removed and the block rests on the scale. (It could even be glued there to make sure there is no fluid under the block). What does S read now? Simply the weight of the material piled upon it: The block's weight mg plus the weight of the fluid column above the block, $\rho g(H - h)A$. The scale therefore reads

$$F_A = mg + \rho g(H - h)A = \rho gHA + (mg - \rho ghA), \qquad (3)$$

exactly what we found in Eq. (2) above. Thus, the apparent weight of the block as measured by S (obtained, as established above, by subtracting ρgHA

from the scale reading) is still the same, *even though there is no fluid under the block*. If the experimenter isn't allowed to look inside the container, he or she would have no way of knowing from the scale reading whether the configuration is that of Fig. 2, or that of Fig. 3. This result, of course, is implied by Archimedes' Proposition 7.

Although we have determined the static apparent weight in a very detailed, step-by-step way, it really amounts to a very simple set of operations:

1. Place (or even glue) the submerged object on the scale and record the reading.
2. Remove the object and record the scale reading.*
3. The difference in readings is the apparent weight.

This procedure is not unlike that which we would use to find the object's "real" weight in air.

The above procedure works no matter what the submerged object's density may be. If its density is less than that of the fluid, the apparent weight would be negative and provision would have to be made to keep the object in force contact with the scale (e.g., by gluing it down or tethering it with a thread) during the first measurement. It wouldn't do to have the object float away!

Conclusion

Archimedes speaks of buoyant forces (or their equivalent) only when dealing with objects less dense than the fluid. In his statement of Proposition 7, which deals with objects denser than the fluid, he doesn't refer to buoyancy at all, only the "true" and (what we now call) the "apparent" weight of the submerged object, which if measured statically, is the same whether the submerged object is resting on the bottom or not. His proofs deal strictly with *hydrostatic* situations and thus avoid the many complications and ambiguities (interesting though they may be) associated with a submerged object moving in a fluid. As one of the most profound scientific and engineering geniuses of the ancient world, it should not be surprising that Archimedes enunciated his principle with remarkable precision and insight.

Postscript

It might be argued that Archimedes did not have access to a piezo or capacitative force-transducer. How then, would he have measured the apparent weight of an object at the bottom of a fluid with only, say, a pan balance? It turns out not to be too difficult to do just that by going through the steps described in Figs. 1–3 using a pan balance. The details are left as an exercise for the reader.

References

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* It is assumed in this step that H does not change.

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