Justifications, Ontology, and Conservativity

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Abstract

An ontologically transparent semantics for justifications that interprets justifications as sets of formulas they justify has been recently presented by Artemov. However, this semantics of modular models has only been studied for the case of the basic justification logic J, corresponding to the modal logic K. It has been left open how to extend and relate modular models to the already existing symbolic and epistemic semantics for justification logics with additional axioms, in particular, for logics of knowledge with factive justifications.

We introduce modular models for extensions of J with any combination of the axioms (jd), (jt), (j4), (j5), and (jb), which are the explicit counterparts of standard modal axioms. After establishing soundness and completeness results, we examine the relationship of modular models to more traditional symbolic and epistemic models. This comparison yields several new semantics, including symbolic models for logics of belief with negative introspection (j5) and models for logics with the axiom (jb). Besides pure justification logics, we also consider logics with both justifications and a belief/knowledge modal operator of the same strength. In particular, we use modular models to study the conditions under which the addition of such an operator to a justification logic yields a conservative extension.

1 Introduction

Justification logics are epistemic logics that feature explicit justifications to evidence the agent's knowledge and/or belief. Instead of formulas $\Box A$, for *A is known*, the language of justification logic includes formulas of the form $t{:}A$ that stand for *A is known for the reason t*, where t is a so-called *justification term*. Justification logics also include operations on these terms to reflect the agent's reasoning power. For instance, if $A \to B$ is known for a reason s and A is known for a reason t, then B is known for the reason $s \cdot t$, where the binary operation $\cdot$ models the agent's ability to apply modus ponens.

The first justification logic, the Logic of Proofs LP, was originally developed by Artemov [1,2] to provide a classical provability semantics for intuitionistic logic. To this end, he introduced an arithmetic semantics for LP in which

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justification terms are interpreted as proofs in Peano arithmetic and the operations on terms correspond to computable operations on proofs in PA. Artemov established arithmetical completeness of LP with respect to this provability semantics and developed an algorithm embedding the modal logic S4 into LP, which, together with the well-known embeddings of intuitionistic logic into S4, solved the long-standing problem of finding a classical provability semantics for intuitionistic logic and S4.

The first non-arithmetic semantics for justification logics was introduced by Mkrtychev [25] in order to obtain decidability results for LP. In these models, which are now called M-models, an evaluation function $*$ assigns to each justification term t a set of formulas $*(t)$. The underlying principle is that

$$\Vdash t{:}F \qquad \Longleftrightarrow \qquad F \text{ is evidenced by } t \text{ according to } * \ . \tag{1}$$

However, if justifications are assumed to be factive, i.e., they can only support true facts, the clarity of (1) in M-models is muddled by truth thrown into the mix. Historically, it was a pragmatic choice of efficiency over philosophical transparency.

Later, Fitting [14], working independently from Mkrtychev, presented an epistemic, i.e., Kripke-style, semantics for justification logics with essentially the same machinery for handling justification terms as in M-models. In this semantics, commonly referred to as F-models, the truth of formulas also invades the causal space of justifications: a formula $t{:}F$ holds at a world w if and only if

(i) F is evidenced by t at w and

(ii) F is true at all worlds that the agent considers possible at w.

It is, therefore, standard to speak of terms as being admissible evidence that is not, however, decisive. F-models can be easily extended to the multi-agent case. Hence, they provide a powerful tool for epistemic applications of justification logics. For instance, F-models have been used for new analyses of traditional epistemic puzzles [4,5] and for investigating the evidential dynamics of public announcements [10,12] and common knowledge [3,9,11].

Despite their many applications, F-models present the same compromise as factive M-models. For the sake of efficiency, justification and truth are intertwined in that t is only evidence for F if F is true (in the same M-model or at all accessible worlds in the F-model). The philosophical objections to such a paradigm also have practical roots. In court, evidence is used to determine the truth of the matter. However, if the acceptability of the evidence were to depend on this truth, it would create a vicious circle. A clear ontological separation between justification and truth is achieved in *modular models* recently introduced by Artemov [6] (although they are less practicable than M-models or F-models, in some cases).

Similar to F-models, modular models consist of a Kripke structure together with an evaluation function $*_w$ for each world w. However, unlike in F-models, no formula is required to be true for t to be evidence for F at a world w. Additionally, modular models satisfy the condition of *justification yields be-*

lief (JYB), which provides a connection between justifications and the traditional possible world semantics for knowledge and belief. This connection principle states that *having evidence for F yields a belief that F*.

Artemov has studied modular models only for the case of the basic justification logic J. Extending them to justification logics with additional axioms, in particular to logics of knowledge with factive justifications, has been left open. We introduce modular models for extensions of J with any combination of the axioms (jd), (jt), (j4), (j5), and (jb), which are the explicit counterparts of standard modal axioms. The connection principle *justification yields belief* makes it possible to give uniform proofs for soundness and completeness for all these logics, which cannot be done in the case of F-models, where this connection principle need not hold. In particular, to obtain the soundness of a justification logic with respect to F-models, the models have to fulfill additional properties that depend on the axioms included in the logic. For modular models, however, these properties, e.g., monotonicity (for (jt)) or the strong evidence property (for (j5)), naturally follow from the fact that justification yields belief. We illustrate this point by developing F-models for justification logics with the axiom (jb) via modular models for these logics. The definition of the latter is trivially read from the axioms, whereas F-models additionally require the strong evidence property, which is not directly related to the axiom system of the logic.

This semantics for justification logics with the axiom (jb) is newly developed in this paper. This axiom was introduced by Brünnler et al. [8] as a justification counterpart of the usual axiom (b) from modal logic. The exploration of the relationship between modular models and M-models also leads to new M-models for justification logics of belief that include the axiom (j5) (some models of this kind are studied in [27]). This relationship, however, is more difficult to study mainly because we are not aware of a conceptually clear way of defining what constitutes an M-model.

Besides pure justification logics, we also consider logics with both justifications and a modal operator for knowledge/belief of the same strength. In particular, we show that the addition of such an operator to a justification logic yields a conservative extension for logics the justifications in which either have to be factive or may not be consistent. For logics with consistent but not necessarily factive justifications, conservativity requires a sufficient store of evidence, i.e., it is necessary to possess evidence for all axioms of the logic. We show that this additional requirement is essential by providing a counterexample to conservativity for the case when no evidence is present for any axiom.

2 Syntax

Justification terms are built from *constants* c_i and *variables* x_i according to the following grammar:

$$t ::= c_i \mid x_i \mid (t \cdot t) \mid (t + t) \mid !t \mid ?t \mid \bar{?}t \ .$$

We denote the *set of terms* by Tm. A term is called *ground* if it does not contain variables. The operations of $\cdot$ and $+$ are assumed to be left-associative in order to omit unnecessary parentheses. *Formulas* are built from *atomic propositions* p_i according to the following grammar:

$$F ::= p_i \mid \neg F \mid (F \to F) \mid t{:}F \ .$$

Prop and $\mathcal{L}_j$ denote the *set of atomic propositions* and the *set of formulas* respectively. We define $(A \land B) := \neg(A \to \neg B)$ and $\bot := p \land \neg p$ for a fixed atomic proposition p.

We consider a family of justification logics that differ in their axioms and in the availability of justifications for these axioms. By an *axiom* we understand a set of formulas in the language $\mathcal{L}_j$, called *axiom instances*. We consider sets of axioms that have form $\mathsf{L}(X) = \mathsf{J} \cup X$, where $\mathsf{J} = \{\mathrm{A1}, \mathrm{A2}, \mathrm{A3}\}$ is the smallest set of axioms with

A1 finite complete axiomatization of classical propositional logic,

A2 $t{:}(A \to B) \to (s{:}A \to (t \cdot s){:}B)$,

A3 $t{:}A \to (t + s){:}A$ and $s{:}A \to (t + s){:}A$,

and additional axioms $X \subseteq \{(\mathrm{jd}), (\mathrm{jt}), (\mathrm{j4}), (\mathrm{j5}), (\mathrm{jb})\}$ with

(jd) $t{:}\bot \to \bot$,

(jt) $t{:}A \to A$,

(j4) $t{:}A \to\, !t{:}t{:}A$,

(j5) $\neg t{:}A \to\, ?t{:}\neg t{:}A$,

(jb) $\neg A \to \bar{?}t{:}\neg t{:}A$.

We often write L instead of $\mathsf{L}(X)$ if the set of axioms X is not important. For a formula F and an axiom A, we write $F \in \mathrm{A}$ to mean that F is an instance of A.

The axiom (jb) was recently introduced in [8] (it was independently proposed by Ghari in an unpublished manuscript [18]). Note that the formulation of (jb) in [8] is slightly different, namely: $A \to \bar{?}t{:}\neg t{:}\neg A$.

A *constant specification* CS for a set of axioms L is any subset

$$\mathsf{CS} \subseteq \{c{:}F \mid c \text{ is a constant and there is an axiom } \mathrm{A} \in \mathsf{L} \text{ such that } F \in \mathrm{A}\}.$$

Constant specifications determine axiom instances for which the logic provides justifications. A constant specification CS for a set of axioms L is called *axiomatically appropriate* (for L) if for each axiom $\mathrm{A} \in \mathsf{L}$ and for each axiom instance $F \in \mathrm{A}$, there is a constant c such that $c{:}F \in \mathsf{CS}$.

A *justification logic* L_{CS} is determined by its set of axioms L and its constant specification CS (for L). Whenever L_{CS} is used, it is assumed that CS is a constant specification for L. The *deductive system* L_{CS} is the Hilbert system given by the axioms L and by the rules modus ponens

$$\frac{A \quad A \to B}{B}\ (\mathrm{MP})$$

and axiom necessitation

$$\frac{}{!\cdots!c{:}\cdots{:}!!c{:}!c{:}c{:}F}\ (\text{AN!}),\quad \text{where } c{:}F \in \mathsf{CS}\ .$$

When (j4) $\in \mathsf{L}$, a simplified axiom necessitation rule can be used:

$$\frac{}{c{:}F}\ (\text{AN}),\quad \text{where } c{:}F \in \mathsf{CS}\ .$$

For instance, the deductive system $\mathsf{JD4}_{\mathsf{CS}}$ consists of the axioms A1–A3, (jd), and (j4) and of the rules (MP) and (AN). There are $2^5 = 32$ combinations of the five axioms (jd), (jt), (j4), (j5), and (jb), but they yield only 24 series of logics L_{CS}, because each instance of (jd) is also an instance of (jt).

For any justification logic L_{CS}, we write $\mathsf{L}_{\mathsf{CS}} \vdash A$ to mean that the *formula* A *is derivable in* L_{CS} and $\Delta \vdash_{\mathsf{L}_{\mathsf{CS}}} A$ to mean that the *formula* A *is derivable in* L_{CS} *from the set of formulas* Δ. When the logic L_{CS} is clear from the context, the subscript L_{CS} is omitted. We write Δ, A for $\Delta \cup \{A\}$.

The *deduction theorem* is standard for justification logics. Therefore, we omit its proof here.

Theorem 2.1 (Deduction Theorem, [4]) *For any justification logic* L_{CS}*, any* $\Delta \subseteq \mathcal{L}_j$*, and arbitrary* $A, B \in \mathcal{L}_j$*,* $\Delta, A \vdash_{\mathsf{L}_{\mathsf{CS}}} B \iff \Delta \vdash_{\mathsf{L}_{\mathsf{CS}}} A \to B$.

An important property of justification logics is their ability to internalize their own notion of proof, as stated in the following lemma, which can be easily proved by induction on the derivation.

Lemma 2.2 (Internalization for Variables, [4]) *Let* L_{CS} *be a justification logic with the axiomatically appropriate* CS*. For arbitrary formulas* $A, B_1, \ldots, B_n \in \mathcal{L}_j$*, if* $B_1, \ldots, B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} A$*, then there is a term* $t(x_1, \ldots, x_n) \in Tm$ *such that* $x_1{:}B_1, \ldots, x_n{:}B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} t(x_1, \ldots, x_n){:}A$ *for fresh variables* $x_1, \ldots, x_n$.

Corollary 2.3 (Constructive Necessitation, [4]) *Let* L_{CS} *be a justification logic with the axiomatically appropriate* CS*. For any formula* $A \in \mathcal{L}_j$*, if* $\mathsf{L}_{\mathsf{CS}} \vdash A$*, then* $\mathsf{L}_{\mathsf{CS}} \vdash t{:}A$ *for some ground term* $t \in Tm$.

Combining the previous results, we also obtain internalization for the case when the assumptions are justified by arbitrary terms.

Corollary 2.4 (Internalization for Arbitrary Terms) *Let* L_{CS} *be a justification logic with the axiomatically appropriate* CS*. For arbitrary formulas* $A, B_1, \ldots, B_n \in \mathcal{L}_j$ *and arbitrary terms* $s_1, \ldots, s_n \in Tm$*, if* $B_1, \ldots, B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} A$*, then there is a term* $t \in Tm$ *such that* $s_1{:}B_1, \ldots, s_n{:}B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} t{:}A$.

Proof. See Appendix A. □

Remark 2.5 There is another method for proving Corollary 2.4: use Internalization Lemma 2.2 to obtain $x_1{:}B_1, \ldots, x_n{:}B_n \vdash s'(x_1, \ldots, x_n){:}A$ and then replace $x_1, \ldots, x_n$ with $s_1, \ldots, s_n$ respectively. However, one has to be careful using this approach since substituting terms for variables need not preserve derivability. Consider $x{:}p \to p \in \text{(jt)}$ and let CS be an axiomatically appropriate constant specification for a logic with axioms $\mathsf{L} \ni \text{(jt)}$. Then, there is a constant c such that $\mathsf{L}_{\mathsf{CS}} \vdash c{:}(x{:}p \to p)$. Substituting a term t for x in $x{:}p \to p$ yields $t{:}p \to p \in \text{(jt)}$. Again, there is a constant d such that $\mathsf{L}_{\mathsf{CS}} \vdash d{:}(t{:}p \to p)$. However, without additional constraints on CS, there is no guarantee that $c = d$.

3 Basic Modular Models

Definition 3.1 *Let $X, Y \subseteq \mathcal{L}_j$ and $t \in Tm$. We define*

(i) $X \cdot Y := \{F \in \mathcal{L}_j \mid G \to F \in X \text{ and } G \in Y \text{ for some formula } G \in \mathcal{L}_j\}$;

(ii) $t{:}X := \{t{:}F \mid F \in X\}$.

Definition 3.2 (Basic evaluation) *A* basic evaluation for a logic L_{CS}*, or a basic L_{CS}-evaluation, is a function $*$ that maps atomic propositions to truth values $0, 1$ and maps justification terms to sets of formulas, $*\colon Prop \to \{0,1\}$ and $*\colon Tm \to \mathcal{P}(\mathcal{L}_j)$, such that for arbitrary $s, t \in Tm$ and any $F \in \mathcal{L}_j$,*

(i) $s^* \cdot t^* \subseteq (s \cdot t)^*$;

(ii) $s^* \cup t^* \subseteq (s + t)^*$;

(iii) $F \in t^*$ *for any conclusion* $t{:}F$ *of* (AN) *or* (AN!)*, whichever is a rule of* L_{CS};

(iv) $s{:}(s^*) \subseteq (!s)^*$ *for logics with* (j4) $\in \mathsf{L}$*; and*

(v) $F \notin t^*$ *implies* $\neg t{:}F \in (?t)^*$ *for logics with* (j5) $\in \mathsf{L}$.

Here p^* for $p \in Prop$ and t^* for $t \in Tm$ denote $*(p)$ and $*(t)$ respectively.

Definition 3.3 (Truth under a basic evaluation) *We define what it means for a formula to* hold under a basic evaluation $*$ *inductively as follows:*

$* \Vdash p$ *if and only if* $p^* = 1$ *for* $p \in Prop$;

$* \Vdash F \to G$ *if and only if* $* \nVdash F$ *or* $* \Vdash G$;

$* \Vdash \neg F$ *if and only if* $* \nVdash F$;

$* \Vdash t{:}F$ *if and only if* $F \in t^*$.

The definition does not depend on the logic for which $*$ is a basic evaluation. Thus, it is possible to talk about a basic evaluation without specifying its logic.

Definition 3.4 (Consistent, factive, and Brouwerian evaluation) *A basic L_{CS}-evaluation $*$ is called*

- consistent *if* $\bot \notin t^*$ *for any* $t \in Tm$;
- factive *if* $F \in t^*$ *implies* $* \Vdash F$ *for all* $t \in Tm$ *and* $F \in \mathcal{L}_j$;
- Brouwerian *if* $* \nVdash F$ *implies* $\neg t{:}F \in (\bar{?}t)^*$ *for all* $t \in Tm$ *and* $F \in \mathcal{L}_j$.

It is immediate from the last two definitions that a factive basic evaluation is always consistent.

Definition 3.5 (Basic modular model) *A* basic modular model for a logic L_{CS}*, or a basic modular L_{CS}-model, is a basic L_{CS}-evaluation $*$ such that*

$*$ *is*

(i) *consistent if* (jd) $\in \mathsf{L}$*;*

(ii) *factive if* (jt) $\in \mathsf{L}$*;*

(iii) *Brouwerian if* (jb) $\in \mathsf{L}$.

Theorem 3.6 (Soundness) *Let* L_{CS} *be a justification logic and* $F \in \mathcal{L}_j$*.*

$$\mathsf{L}_{\mathsf{CS}} \vdash F \qquad \Longrightarrow \qquad * \Vdash F \textit{ for all basic modular } \mathsf{L}_{\mathsf{CS}}\textit{-models } *.$$

Proof. See Appendix B. □

Completeness is established by a maximal consistent set construction.

Definition 3.7 (L_{CS}-consistent set) *Let* L_{CS} *be a justification logic. A set* $\Phi \subseteq \mathcal{L}_j$ *is called* L_{CS}-consistent *if there is* $F \in \mathcal{L}_j$ *such that* $\Phi \nvdash_{\mathsf{L}_{\mathsf{CS}}} F$*.* Φ *is called* maximal L_{CS}-consistent *if it is* L_{CS}*-consistent and has no* L_{CS}*-consistent proper extensions.*

As usual, a maximal L_{CS}-consistent set contains all instances of axioms from L and is closed under the inference rules of L_{CS}.

Theorem 3.8 (Completeness) *Let* L_{CS} *be a justification logic and* $F \in \mathcal{L}_j$*.*

$$* \Vdash F \textit{ for all basic modular } \mathsf{L}_{\mathsf{CS}}\textit{-models } * \qquad \Longrightarrow \qquad \mathsf{L}_{\mathsf{CS}} \vdash F \ .$$

Proof. See Appendix C. □

Remark 3.9 Basic modular models, introduced in [6], are closely related to M-models, introduced by Mkrtychev in [25] for the Logic of Proofs $\mathsf{LP}_{\mathsf{CS}}$, the logic with axioms $\mathsf{J} \cup \{(\mathrm{jt}), (\mathrm{j4})\}$, denoted $\mathcal{LP}_{AS}$ in [25]. Strictly speaking, Mkrtychev created two semantics, which he called *models* and *pre-models* and which he showed to be equivalent. M-models, which have been adapted to several other logics from the justification family in [20,21,26,27,28] and used to prove decidability of justification logics ([25]), to determine their complexity ([19,21,22]), and to study self-referentiality of modal logics ([23]), are the pre-models of [25]. However, the other semantics from [25], that of the models, is, in fact, exactly the semantics of basic modular models for $\mathsf{LP}_{\varnothing}$, i.e., of factive basic evaluations for $\mathsf{LP}_{\varnothing}$. The models of [25] are defined as *interpretations* with an additional condition, which is identical to our condition of factivity, while interpretations themselves are isomorphic modulo notation and terminology to basic evaluations for $\mathsf{LP}_{\varnothing}$. The machinery for handling the rule (AN) in [25] and in this paper is also essentially the same.

The difference between pre-models and models lies in the conditions under which $t{:}F$ holds. For models, it is the same as for basic evaluations, i.e., $F \in t^*$ is sufficient, whereas for pre-models, additionally $* \Vdash F$ is required. Clearly, this additional requirement replaces the requirement of factivity. Therefore, it should only be used when (jt) $\in \mathsf{L}$. Indeed, the M-models introduced in [21] for the logics with axioms J, $\mathsf{J} \cup \{(\mathrm{jd})\}$, $\mathsf{J} \cup \{(\mathrm{j4})\}$, or $\mathsf{J} \cup \{(\mathrm{jd}), (\mathrm{j4})\}$ are isomorphic to the basic modular models for these logics. By the same token, some of the basic modular models presented in this paper can and should be considered M-models.

Definition 3.10 (M-models for logics with (j5) but without (jt)) *An* M-model *for a logic* $\mathsf{L_{CS}}$ *with the axioms* $\mathsf{J} \cup \{(\text{j5})\}$*,* $\mathsf{J} \cup \{(\text{jd}), (\text{j5})\}$*,* $\mathsf{J} \cup \{(\text{j4}), (\text{j5})\}$*, or* $\mathsf{J} \cup \{(\text{jd}), (\text{j4}), (\text{j5})\}$ *is a basic modular model for* $\mathsf{L_{CS}}$*.*

4 Epistemic Models and Modularity

Justification logics also have an epistemic semantics, developed by Fitting in [14]. An *F-model* for a justification logic is a quadruple $\mathcal{M} = (W, R, \mathcal{E}, \mathcal{V})$, where $(W, R, \mathcal{V})$ is a Kripke model and where the admissible evidence function $\mathcal{E} \colon W \times \text{Tm} \to \mathcal{P}(\mathcal{L}_j)$ plays the role of a basic evaluation $*$ for each world. The crucial feature of F-models is that

$$\mathcal{M}, w \Vdash t{:}F \qquad \Longleftrightarrow \qquad F \in \mathcal{E}(w,t) \text{ and } \big(R(w,v) \Rightarrow \mathcal{M}, v \Vdash F\big) \ .$$

These epistemic models provide a way of comparing justification logics and their corresponding modal logics within the same semantics, as well as provide semantics for combinations of the two, which are sometimes called *logics of justifications and belief/knowledge*. However, F-models violate an important property of basic modular models: namely, the ontological separation of justification from truth. This separation is also violated in M-models for logics with the axiom (jt) (see Remark 3.9). Indeed, in such an M-model and in any F-model, to check whether a term t justifies a formula F, it must be observed whether F holds, in the M-model or in all accessible worlds of the F-model.

This prompted Artemov in [6] to introduce *modular models* for $\mathsf{J_{CS}}$ with a clear distinction between the truth and the justification of formulas. We now extend modular models to all the justification logics we are considering. Like F-models, these modular models can also be used for logics of justifications and belief/knowledge.

Definition 4.1 (Quasimodel) *A* quasimodel for $\mathsf{L_{CS}}$*, or an* $\mathsf{L_{CS}}$*-quasimodel, is a triple* $\mathcal{M} = (W, R, *)$*, where* $W \neq \varnothing$*,* $R \subseteq W \times W$*, and the* evaluation $*$ *maps each world* $w \in W$ *to a basic* $\mathsf{L_{CS}}$*-evaluation* $*_w$*. We will write* p^*_w *instead of* $*_w(p)$ *and* t^*_w *instead of* $*_w(t)$*.*

Definition 4.2 (Truth in quasimodels) *We define what it means for a formula to* hold at a world $w \in W$ of a quasimodel $\mathcal{M} = (W, R, *)$ *inductively as follows:*

$\mathcal{M}, w \Vdash p$ *if and only if* $p^*_w = 1$ *for* $p \in Prop$*;*

$\mathcal{M}, w \Vdash F \to G$ *if and only if* $\mathcal{M}, w \nVdash F$ *or* $\mathcal{M}, w \Vdash G$*;*

$\mathcal{M}, w \Vdash \neg F$ *if and only if* $\mathcal{M}, w \nVdash F$*;*

$\mathcal{M}, w \Vdash t{:}F$ *if and only if* $F \in t^*_w$*.*

As in the case of basic evaluations, this definition does not depend on the logic for which $\mathcal{M}$ *is a quasimodel. We write* $\mathcal{M} \Vdash F$ *if* $\mathcal{M}, w \Vdash F$ *for all* $w \in W$*.*

For a given quasimodel $\mathcal{M} = (W, R, *)$ and a world $w \in W$, we define

$$\Box_w := \{F \in \mathcal{L}_j \mid \mathcal{M}, v \Vdash F \text{ whenever } R(w, v)\} \ . \tag{2}$$

By analogy with basic modular models, we define the following notions:

Definition 4.3 (Consistent, factive, and Brouwerian quasimodel) *An* L_{CS}*-quasimodel* $\mathcal{M} = (W, R, *)$ *is called*

- consistent *if* $\bot \notin t^*_w$ *for any* $w \in W$ *and any* $t \in Tm$*;*
- factive *if* $F \in t^*_w$ *implies* $\mathcal{M}, w \Vdash F$ *for all* $w \in W$*,* $t \in Tm$*, and* $F \in \mathcal{L}_j$*;*
- Brouwerian *if* $\mathcal{M}, w \nVdash F$ *implies* $\neg t{:}F \in (\bar{?}t)^*_w$ *for all* $w \in W$*,* $t \in Tm$*, and* $F \in \mathcal{L}_j$*.*

Definition 4.4 (Modular model) *A* modular model $\mathcal{M} = (W, R, *)$ for L_{CS}*, or a modular* L_{CS}*-model, is an* L_{CS}*-quasimodel that meets the following conditions:*

(i) $t^*_w \subseteq \Box_w$ *for all* $t \in Tm$ *and* $w \in W$*;* *(JYB)*

(ii) *R is serial if* (jd) $\in \mathsf{L}$*;*

(iii) *R is reflexive if* (jt) $\in \mathsf{L}$*;*

(iv) *R is transitive if* (j4) $\in \mathsf{L}$*;*

(v) *R is Euclidean if* (j5) $\in \mathsf{L}$*;*

(vi) *R is symmetric if* (jb) $\in \mathsf{L}$*;*

(vii) $\mathcal{M}$ *is Brouwerian if* (jb) $\in \mathsf{L}$.

Conditions (i)–(vi) may seem superfluous since R plays no role in determining the truth of formulas. But Conditions (ii)–(vi) are well known for the corresponding modal axioms in modal logic and, hence, are needed, so to say, for backward compatibility: they ensure that the same semantics can be used for justification logics, logics of justifications and belief/knowledge, and modal logics. Condition (i) plays, in this respect, the role of a catalyzer allowing for a transition between these three formalisms. This condition essentially says that *justification yields belief*, abbreviated JYB. Indeed, whenever $F \in t^*_w$, we have $\mathcal{M}, w \Vdash t{:}F$ so that F has a justification at w. The requirement that F belong to $\Box_w$ says that F must be believed at w in the sense of Kripke models, i.e., hold at all worlds considered possible at w.

Note that, unlike for the case of basic modular models, we do not require that modular models for logics with (jd) be consistent or those for logics with (jt) be factive. Instead, these properties are derived from JYB and the corresponding restrictions on R.

Lemma 4.5 (Reflexive modular models are factive) *Let* (jt) $\in \mathsf{L}$ *and let* $\mathcal{M} = (W, R, *)$ *be a modular* L_{CS}*-model. Then* $\mathcal{M}$ *is factive.*

Proof. Suppose $F \in t^*_w$. Then $F \in \Box_w$ by JYB. Since $R(w, w)$ by reflexivity of R, we obtain $\mathcal{M}, w \Vdash F$ from (2). □

Lemma 4.6 (Serial modular models are consistent) *Let* (jd) $\in$ L *and let* $\mathcal{M} = (W, R, *)$ *be a modular* L_{CS}*-model. Then* $\mathcal{M}$ *is consistent.*

Proof. Assume towards a contradiction that $\bot \in t^*_w$. Then $\bot \in \Box_w$ by JYB. By seriality of R, there is $v \in W$ such that $R(w, v)$ and, by (2), we would have $\mathcal{M}, v \Vdash \bot$, which is impossible. $\Box$

There is an additional property that follows from JYB but is peculiar to the possible-worlds scenario.

Lemma 4.7 (Monotonicity) *Let* (j4) $\in \mathsf{L}$ *and* $\mathcal{M} = (W, R, *)$ *be a modular* L_{CS}*-model. Then for any* $t \in Tm$ *and for arbitrary* $a, b \in W$*,* $R(a, b)$ *implies* $t^*_a \subseteq t^*_b$.

Proof. Assume $R(a, b)$ and $F \in t^*_a$. Then $t{:}F \in (!t)^*_a$ because $*_a$ is a basic evaluation for L_{CS}. So $t{:}F \in \Box_a$ by JYB and $\mathcal{M}, b \Vdash t{:}F$ by (2), which means that $F \in t^*_b$. $\Box$

The soundness and completeness of justification logics with respect to modular models are almost obvious:

Theorem 4.8 (Soundness and Completeness, Modular Models I) *Let* L_{CS} *be a justification logic such that either* (jt) $\in \mathsf{L}$ *or* (jd) $\notin \mathsf{L}$ *and let* $F \in \mathcal{L}_j$.

$$\mathsf{L}_{\mathsf{CS}} \vdash F \qquad \Longleftrightarrow \qquad \mathcal{M} \Vdash F \textit{ for all modular } \mathsf{L}_{\mathsf{CS}}\textit{-models } \mathcal{M}\ . \tag{3}$$

Proof. See Appendix D. $\Box$

Remark 4.9 Unfortunately, single-world modular models are insufficient for proving the completeness of logics that are consistent but not factive. Indeed, any serial single-world model is automatically reflexive. Thus, JYB for such a model would yield factivity by Lemma 4.5, making it impossible to distinguish between consistency and factivity. In this case, the simplest completeness proof is via the canonical model construction akin to that from the proof of Theorem 3.8.

Theorem 4.10 (Soundness and Completeness, Modular Models II) *Let* L_{CS} *be a logic with* (jt) $\notin \mathsf{L}$ *and* (jd) $\in \mathsf{L}$*. Then* L_{CS} *is sound with respect to modular* L_{CS}*-models* ($\Longrightarrow$*-direction of* (3))*. If* CS *is axiomatically appropriate for* L*, then* L_{CS} *is also complete* ($\Longleftarrow$*-direction of* (3))*.*

Proof. See Appendix E. $\Box$

Remark 4.11 In fact, the completeness proof in Theorem 4.10 can be easily applied to all the logics covered by Theorems 4.8 and 4.10. The only addition would be the necessity to show R is reflexive if (jt) $\in \mathsf{L}$.

The relationship between F-models, mentioned at the beginning of this section, and modular models is rather straightforward. While an independent definition of F-models for all logics L_{CS}, except for those with (jb) $\in \mathsf{L}$, can be found in [4], we can describe them via modular models:

Definition 4.12 (F-models) *The definition of an* F-model $\mathcal{M}_{\mathsf{F}} = (W, R, *)$ for a logic L_{CS} *is identical to that of a modular* L_{CS}*-model (see Def. 4.4) with a new condition: if* (j4) $\in \mathsf{L}$*, then for all* $w, v \in W$ *and all* $t \in Tm$*,*

$$R(w,v) \qquad \Longrightarrow \qquad t^*_w \subseteq t^*_v \ ; \tag{4}$$

and with a restricted JYB analog: if (j5) $\in \mathsf{L}$ *or* (jb) $\in \mathsf{L}$*, then for all* $t \in Tm$*,* $F \in \mathcal{L}_j$*, and* $w \in W$*,*

$$F \in t^*_w \qquad \Longrightarrow \qquad F \in \Box^{\mathsf{F}}_w \ , \tag{5}$$

where

$$\Box^{\mathsf{F}}_w := \{G \in \mathcal{L}_j \mid \mathcal{M}_{\mathsf{F}}, v \Vdash_{\mathsf{F}} G \text{ whenever } R(w,v)\} \ . \tag{6}$$

Condition (4) *for F-models is traditionally called* monotonicity*, whereas the most common equivalent form of* (5) *is called the* strong evidence property*.*

The absence of the JYB requirement in the case when neither (j5) $\in \mathsf{L}$ nor (jb) $\in \mathsf{L}$ means that in an F-model, $F \in t^*_w$ says that t is *admissible* as evidence for F at the world w, but it need not be *decisive* as it is in modular models. Accordingly, the notion of truth $\Vdash_{\mathsf{F}}$ in F-models differs from $\Vdash$ in quasimodels with respect to formulas of the form $t{:}F$.

Definition 4.13 (Truth in F-models) *The definition of* $\mathcal{M}_{\mathsf{F}}, w \Vdash_{\mathsf{F}} A$ *is identical to that of* $\mathcal{M}_{\mathsf{F}}, w \Vdash A$ *for the* L_{CS}*-quasimodel* $\mathcal{M}_{\mathsf{F}}$ *(see Def. 4.2) except that the last clause is replaced by* $\mathcal{M}_{\mathsf{F}}, w \Vdash_{\mathsf{F}} t{:}F$ *if and only if* $F \in t^*_w$ *and* $F \in \Box^{\mathsf{F}}_w$*. Note that* $\Vdash_{\mathsf{F}}$ *can be applied to any quasimodel irrespective of its logic.*

Lemma 4.14 (From Modular to F-Models) *Every modular* L_{CS}*-model* $\mathcal{M} = (W, R, *)$ *is also an F-model for* L_{CS} *such that for all* $w \in W$ *and* $F \in \mathcal{L}_j$*,*

$$\mathcal{M}, w \Vdash_{\mathsf{F}} F \qquad \Longleftrightarrow \qquad \mathcal{M}, w \Vdash F \ . \tag{7}$$

Proof. See Appendix F. □

Remark 4.15 It is clear from the proof of Lemma 4.14 that (5) holds for any F-model based on a modular L_{CS}-model, even if neither (j5) $\in \mathsf{L}$ nor (jb) $\in \mathsf{L}$.

The converse direction is more interesting. An F-model need not be a modular model itself, but it always induces an equivalent modular model.

Lemma 4.16 (From F- to Modular Models) *Let* $\mathcal{M}_{\mathsf{F}} = (W, R, *_{\mathsf{F}})$ *be an F-model for a logic* L_{CS}*. Then* $\mathcal{M} := (W, R, *)$ *with*

$$t^*_w := \{F \in \mathcal{L}_j \mid F \in t^{*_{\mathsf{F}}}_w \text{ and } F \in \Box^{\mathsf{F}}_w\} = \{F \in \mathcal{L}_j \mid \mathcal{M}_{\mathsf{F}}, w \Vdash_{\mathsf{F}} t{:}F\} \tag{8}$$

is a modular L_{CS}*-model such that for all* $w \in W$ *and all* $F \in \mathcal{L}_j$*,*

$$\mathcal{M}, w \Vdash F \qquad \Longleftrightarrow \qquad \mathcal{M}_{\mathsf{F}}, w \Vdash_{\mathsf{F}} F \ . \tag{9}$$

Proof. See Appendix G. □

Remark 4.17 Soundness and completeness with respect to F-models for L_{CS} follow from Lemmas 4.14 and 4.16 and soundness and completeness with respect to modular L_{CS}-models, with the same requirement of axiomatic appropriateness for CS when (jd) $\in \mathsf{L}$. Thus, we have created F-models for the justification logics with (jb) $\in \mathsf{L}$.

5 Justifications and Belief

It should not be surprising that modular models can also be used for the joint language of justifications and belief. And while the condition JYB does not look out of place in justification logics, its real origins are, of course, modal, which is clearly seen in the following soundness proof. Many notions and conventions introduced in Sect. 2 are now generalized to the extended *language* $\mathcal{L}_\Box$ defined by the grammar:

$$F ::= p_i \mid \neg F \mid (F \to F) \mid t{:}F \mid \Box F \enspace .$$

For each set of axioms L considered earlier, we define the *set of axioms* $\mathsf{L}^\Box$ to consist of

- all the axioms of L in the extended language $\mathcal{L}_\Box$;
- axiom $\Box(A \to B) \to (\Box A \to \Box B)$;
- axiom $\Box\bot \to \bot$ if (jd) $\in \mathsf{L}$;
- axiom $\Box F \to F$ if (jt) $\in \mathsf{L}$;
- axiom $\Box F \to \Box\Box F$ if (j4) $\in \mathsf{L}$;
- axiom $\neg\Box F \to \Box\neg\Box F$ if (j5) $\in \mathsf{L}$;
- axiom $\neg F \to \Box\neg\Box F$ if (jb) $\in \mathsf{L}$; and
- axiom $t{:}F \to \Box F$.

The axiom $t{:}F \to \Box F$ is called the *connection axiom*. It formally states that justification yields belief. *Constant specifications for* $\mathsf{L}^\Box$ are defined in the obvious way. Given a constant specification CS for $\mathsf{L}^\Box$, the *deductive system* $\mathsf{L}^\Box_{\mathsf{CS}}$ is the Hilbert system given by the axioms $\mathsf{L}^\Box$ and by the rules (MP) and either (AN) or (AN!), as in L_{CS}, as well as by the usual necessitation rule from modal logic:

$$\frac{F}{\Box F}\,(\Box) \enspace .$$

A *basic evaluation for a logic* $\mathsf{L}^\Box_{\mathsf{CS}}$ and many other notions are defined in the same way as for its corresponding justification logic L_{CS} except that each instance of the language $\mathcal{L}_j$ should be replaced with $\mathcal{L}_\Box$ and those of the logic L_{CS} with $\mathsf{L}^\Box_{\mathsf{CS}}$. In particular, the definition of a *modular model for* $\mathsf{L}^\Box_{\mathsf{CS}}$ repeats that for L_{CS} with one extra clause for the truth of formulas $\Box F$:

$$\mathcal{M}, w \Vdash \Box F \qquad \Longleftrightarrow \qquad F \in \Box_w \enspace , \tag{10}$$

which is a standard definition recast in our notation.

We will not repeat definitions and proofs, unless there is a significant change due to the addition of modalities.

Theorem 5.1 (Soundness of Modular Models for $\mathsf{L}^{\Box}_{\mathsf{CS}}$) *Let* $\mathsf{L}^{\Box}_{\mathsf{CS}}$ *be a logic of justifications and belief and let* $F \in \mathcal{L}_{\Box}$*.*

$$\mathsf{L}^{\Box}_{\mathsf{CS}} \vdash F \qquad \Longrightarrow \qquad \mathcal{M} \Vdash F \text{ for all modular } \mathsf{L}^{\Box}_{\mathsf{CS}}\text{-models } \mathcal{M} \ .$$

Proof. Most of the proof repeats that of Theorem 3.6 or the standard argument for the modal axioms. The connection axiom $t{:}F \to \Box F$ is valid because of JYB and (10). □

Theorem 5.2 (Completeness of Modular Models for $\mathsf{L}^{\Box}_{\mathsf{CS}}$) *Let* $\mathsf{L}^{\Box}_{\mathsf{CS}}$ *be a logic of justifications and belief and let* $F \in \mathcal{L}_{\Box}$*.*

$$\mathcal{M} \Vdash F \text{ for all modular } \mathsf{L}^{\Box}_{\mathsf{CS}}\text{-models } \mathcal{M} \qquad \Longrightarrow \qquad \mathsf{L}^{\Box}_{\mathsf{CS}} \vdash F \ .$$

Proof. See Appendix H. □

6 Conservative Extensions of Modal Logics

Justification language and the closely related modal language provide a plethora of conservativity statements to be considered. Within justification logic itself, different sets of operations on justifications can be compared prompting the question whether larger sets are conservative over smaller ones, which is thoroughly studied in [15,24]. One can also ask whether logics of justifications and belief are conservative over the respective justification logics and/or the respective modal logics. While there are a few results of the latter type sprinkled through the literature so widely that listing all the papers here is impractical, the results of the former type are relatively rare. Artemov and Nogina in [7] introduced the logic $\mathsf{S4LP}$, which is $\mathsf{L}^{\Box}_{\mathsf{CS}}$ with $\mathsf{L} = \mathsf{J} \cup \{(\text{jt}), (\text{j4})\}$ in our notation. They also conjectured this logic to be conservative over L_{CS} (modulo a careful treatment of CS). In the introduction to [16], Fitting mentioned this conjecture as a fact without any proof. Most probably, Fitting's argument was semantic based on his semantics of F-models, which works both for these $\mathsf{L}^{\Box}_{\mathsf{CS}}$ and L_{CS} (see [13,14]). Finally, Ghari recently published a syntactic proof of the same fact in [17]. In this section, we use a semantic argument based on modular models to extend this conservativity result to other pairs of corresponding logics.

Theorem 6.1 (Conservativity)

(i) *Let* L_{CS} *be a justification logic with* (jd) $\notin \mathsf{L}$ *and let* $F \in \mathcal{L}_j$*. Then* $\mathsf{L}_{\mathsf{CS}} \vdash F \iff \mathsf{L}^{\Box}_{\mathsf{CS}} \vdash F$*.*

(ii) *Let* L_{CS} *be a justification logic with* (jd) $\in \mathsf{L}$*, let* CS *be an axiomatically appropriate constant specification for* L*, and let* $F \in \mathcal{L}_j$*. Then we have* $\mathsf{L}_{\mathsf{CS}} \vdash F \iff \mathsf{L}^{\Box}_{\mathsf{CS}} \vdash F$*.*

Proof. The statements in both cases follow from the fact that by Theorems 4.8, 4.10, 5.1, and 5.2, both logics are sound and complete with respect to the same class of modular models. □

The restriction on CS in the second part of this theorem originates from Theorem 4.10, whereas Theorem 5.2 does not feature any such restriction. The following example shows that this restriction is, in fact, essential for conservativity rather than being an artifact of the particular proof method.

Example 6.2 Consider $\mathsf{L}_\varnothing$ with $\mathsf{L} = \mathsf{J} \cup \{(\text{jd})\}$. We show that $\mathsf{L}_\varnothing^\Box$ is not conservative over $\mathsf{L}_\varnothing$. Indeed, $\neg y{:}x{:}\bot$ can be derived in $\mathsf{L}_\varnothing^\Box$ by taking the instance $x{:}\bot \to \bot$ of the axiom $(\text{jd}) \in \mathsf{L}^\Box$, applying normal modal reasoning to get $\Box x{:}\bot \to \Box\bot$, syllogizing with the modal seriality axiom $\Box\bot \to \bot \in \mathsf{L}^\Box$ to obtain $\Box x{:}\bot \to \bot$, and syllogizing once again with the instance $y{:}x{:}\bot \to \Box x{:}\bot$ of the connection principle $\in \mathsf{L}^\Box$. The final result is $\mathsf{L}_\varnothing^\Box \vdash y{:}x{:}\bot \to \bot$, or, equivalently, $\mathsf{L}_\varnothing^\Box \vdash \neg y{:}x{:}\bot$.

However, $y{:}x{:}\bot$ is shown to be satisfiable in an M-model for $\mathsf{L}_\varnothing$ in [22, Ex. 3.3.23]. Hence, $\mathsf{L}_\varnothing \nvdash \neg y{:}x{:}\bot$ due to the soundness of $\mathsf{L}_\varnothing$ with respect to its M-models or, equivalently, according to Remark 3.9, with respect to basic modular $\mathsf{L}_\varnothing$-models. Thus, $\mathsf{L}_\varnothing^\Box$ is not conservative over $\mathsf{L}_\varnothing$.

7 Fully Explanatory Models

Since in justification logics the modality of $\Box F$ is read existentially, i.e., as the existence of a justification for F, it is reasonable to ask whether the semantics we have presented supports this reading.

Definition 7.1 (Fully explanatory modular models) *A modular L_{CS}-model $\mathcal{M} = (W, R, *)$ is* fully explanatory *if for any $w \in W$ and any $F \in \mathcal{L}_j$, $F \in \Box_w$ implies $F \in t^*_w$ for some $t \in Tm$.*

This notion can be seen as the converse of JYB and, taking the latter into account, can be reformulated as $\Box_w = \bigcup_{t\in\mathrm{Tm}} t^*_w$. A similar notion was originally proposed by Fitting in [14] for F-models.

Theorem 7.2 *Let L_{CS} be a justification logic with the axiomatically appropriate CS. Then L_{CS} is sound and complete with respect to fully explanatory modular L_{CS}-models.*

Proof. Given Theorems 4.8 and 4.10 and Remark 4.11, it is sufficient to show that the canonical model $\mathcal{M}_c = (W, R, *)$ for L_{CS} constructed in the proof of Theorem 4.10 is fully explanatory. See Appendix I for details. □

8 Conclusion

Modular models provide an epistemic semantics for justification logics with a clear ontological separation of justification and truth. We have introduced modular models for the extensions of the basic justification logic J with any combination of the axioms (jd), (jt), (j4), (j5), and (jb).

One of the main properties of modular models is that justification yields belief, which has enabled us to study the relationship of modular models to more traditional epistemic semantics of F-models for justification logics and Kripke models for modal logics. We have also compared single-world variants of modular models to the existing symbolic semantics for justification logics. These comparisons have yielded several new semantics, including symbolic models for logics of belief with negative introspection (j5) and epistemic models for logics with the axiom (jb).

We have also extended the semantics of modular models to justification logics with an additional modal knowledge/belief operator and have exploited the common semantical framework to demonstrate that such extensions are typically conservative. All these conservativity results, with the exception of the conservativity of $\mathsf{S4LP}_{\mathsf{CS}}$ over $\mathsf{LP}_{\mathsf{CS}}$, are new.

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Appendix

A Proof of Corollary 2.4

Corollary 2.4 Let L_{CS} be a justification logic with the axiomatically appropriate CS. For arbitrary formulas $A, B_1, \ldots, B_n \in \mathcal{L}_j$ and arbitrary terms $s_1, \ldots, s_n \in \mathrm{Tm}$, if $B_1, \ldots, B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} A$, then there is a term $t \in \mathrm{Tm}$ such that $s_1{:}B_1, \ldots, s_n{:}B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} t{:}A$.

Proof. Assume $B_1, \ldots, B_n \vdash_{\mathsf{L}_{\mathsf{CS}}} A$. By Deduction Theorem 2.1,

$$\mathsf{L}_{\mathsf{CS}} \vdash B_1 \to (B_2 \to \cdots \to (B_n \to A) \cdots) \ .$$

By Constructive Necessitation 2.3, there is a ground term s' such that

$$\mathsf{L}_{\mathsf{CS}} \vdash s'{:}\Big(B_1 \to (B_2 \to \cdots \to (B_n \to A) \cdots)\Big) \ .$$

By repeated applications of A2 and modus ponens, for $t := s' \cdot s_1 \cdots s_n$,

$$s_1{:}B_1, \ldots, s_n{:}B_n \vdash_{\mathsf{L_{CS}}} t{:}A \ .$$

□

B Proof of Theorem 3.6

Theorem 3.6 Let $\mathsf{L_{CS}}$ be a justification logic and $F \in \mathcal{L}_j$. If $\mathsf{L_{CS}} \vdash F$, then $* \Vdash F$ for all basic modular $\mathsf{L_{CS}}$-models $*$.

Proof. As usual, the proof is by induction on the length of the derivation of F. Let $*$ be a basic modular $\mathsf{L_{CS}}$-model. It is obvious that all instances of propositional axioms hold under $*$ and the rule (MP) is respected by the semantics. Soundness of the axioms (A2), (A3), (j4), and (j5), as well as that of the rules (AN) and (AN!), immediately follows from the definition of a basic evaluation.

It is also easy to see that all instances of (jd) hold under all consistent basic evaluations, all instances of (jt) hold under all factive basic evaluations, and all instances of (jb) hold under all Brouwerian basic evaluations. The argument for (jt) is as follows: if $* \Vdash t{:}F$, then $F \in t^*$, so $* \Vdash F$ by factivity of $*$. □

C Proof of Theorem 3.8

Theorem 3.8 Let $\mathsf{L_{CS}}$ be a justification logic and $F \in \mathcal{L}_j$. If $* \Vdash F$ for all basic modular $\mathsf{L_{CS}}$-models $*$, then $\mathsf{L_{CS}} \vdash F$.

Proof. Assume that $\mathsf{L_{CS}} \nvdash F$. Then $\{\neg F\}$ is $\mathsf{L_{CS}}$-consistent and, hence, is contained in some maximal $\mathsf{L_{CS}}$-consistent set Φ. For this Φ, any $p \in \text{Prop}$, and any $t \in \text{Tm}$, we define

$$p^* := \begin{cases} 1 & p \in \Phi \\ 0 & p \notin \Phi \end{cases} \qquad \text{and} \qquad t^* := \{F \in \mathcal{L}_j \mid t{:}F \in \Phi\} \ . \tag{C.1}$$

It is easy to show that $*$ is a basic $\mathsf{L_{CS}}$-evaluation. By way of example, we will show Conditions (i) and (v); the rest is similar.

Condition (i) of Def. 3.2. Suppose $A \in s^* \cdot t^*$. Then there is $B \in \mathcal{L}_j$ such that $B \to A \in s^*$ and $B \in t^*$. By (C.1), $s{:}(B \to A) \in \Phi$ and $t{:}B \in \Phi$. By the maximal $\mathsf{L_{CS}}$-consistency of Φ, also $(s \cdot t){:}A \in \Phi$. Thus, $A \in (s \cdot t)^*$ by (C.1).

Condition (v) of Def. 3.2. Suppose (j5) $\in \mathsf{L}$ and $F \notin t^*$. By (C.1), we have $t{:}F \notin \Phi$. By the maximal $\mathsf{L_{CS}}$-consistency of Φ, then $\neg t{:}F \in \Phi$. Further, $?t{:}\neg t{:}F \in \Phi$ because (j5) $\in \mathsf{L}$. So $\neg t{:}F \in (?t)^*$ by (C.1).

We now show the so-called Truth Lemma: for all $D \in \mathcal{L}_j$,

$$D \in \Phi \qquad \Longleftrightarrow \qquad * \Vdash D \ . \tag{C.2}$$

We establish (C.2) by induction on the structure of D:

(i) $D = p \in \text{Prop}$. Then $p \in \Phi \quad \Leftrightarrow \quad p^* = 1 \quad \Leftrightarrow \quad * \Vdash p$.

(ii) The cases when $D = \neg A$ and $D = A \to B$ are standard.

(iii) $D = t{:}A$. Then $t{:}A \in \Phi \quad \Leftrightarrow \quad A \in t^* \quad \Leftrightarrow \quad * \Vdash t{:}A$.

To show that $*$ is a basic modular L_{CS}-model, we need to check Conditions (i)–(iii) of Def. 3.5:

(i) Assume towards a contradiction that $\bot \in t^*$. Then $t{:}\bot \in \Phi$ by (C.1). Since (jd) $\in \mathsf{L}$, we would have $\bot \in \Phi$, which contradicts the consistency of Φ.

(ii) Suppose $F \in t^*$. Then $t{:}F \in \Phi$ by (C.1). Since (jt) $\in \mathsf{L}$, we have $F \in \Phi$. Now $* \Vdash F$ follows by (C.2).

(iii) Suppose $* \nVdash F$. Then $F \notin \Phi$ by (C.2). By the maximal L_{CS}-consistency of Φ, we have $\neg F \in \Phi$. Since (jb) $\in \mathsf{L}$, also $\bar{?}t{:}\neg t{:}F \in \Phi$, and $\neg t{:}F \in (\bar{?}t)^*$ follows by (C.1).

Since $\neg F \in \Phi$ by the construction of Φ, we find $* \Vdash \neg F$ by (C.2). Thus, $* \nVdash F$ for the constructed basic modular L_{CS}-model $*$. Completeness of L_{CS} follows by contraposition. □

D Proof of Theorem 4.8

Theorem 4.8 Let L_{CS} be a justification logic such that either (jt) $\in \mathsf{L}$ or (jd) $\notin \mathsf{L}$ and let $F \in \mathcal{L}_j$. Then $\mathsf{L}_{\mathsf{CS}} \vdash F$ if and only if $\mathcal{M} \Vdash F$ for all modular L_{CS}-models $\mathcal{M}$.

Proof. It is sufficient to prove that any formula refutable by a basic modular model can be refuted at a world in a modular model and vice versa.
Soundness. Since R plays no role in the definition of truth in modular models, for any modular model $\mathcal{M} = (W, R, *)$ and for any world $w \in W$, the basic L_{CS}-evaluation $*_w$ satisfies exactly the same formulas as the world w of $\mathcal{M}$ does, i.e.,

$$\mathcal{M}, w \Vdash F \qquad \Longleftrightarrow \qquad *_w \Vdash F \ . \tag{D.1}$$

In particular, $*_w$ is factive if $\mathcal{M}$ is and Brouwerian if $\mathcal{M}$ is. Thus, it follows from Lemma 4.5 and from Condition (vii) for modular models that $*_w$ is a basic modular L_{CS}-model, which refutes all formulas refuted at the world w of $\mathcal{M}$.
Completeness. For the opposite direction, let $*$ be a basic modular L_{CS}-model. We define an L_{CS}-quasimodel $\mathcal{M} := (\{1\}, R, \star)$ with $\star(1, t) := *(t)$. Since $* = \star_1$, by (D.1) we have $* \Vdash F$ iff $\mathcal{M}, 1 \Vdash F$. Thus, $\mathcal{M}$ is Brouwerian if $*$ is. To show that $\mathcal{M}$ is a modular L_{CS}-model, which refutes all formulas refuted by $*$, it remains to make sure all the restrictions on R and the condition JYB are met. The choice of R depends on the logic.

If (jt) $\in \mathsf{L}$, set $R := \{(1,1)\}$. It is reflexive, symmetric, Euclidean, and transitive, so all restrictions on R are met. Further, if $F \in t^\star_1$, then $F \in t^*$ by the definition of $\star$. Since $*$ is factive, $* \Vdash F$. Thus, $\mathcal{M}, 1 \Vdash F$ by (D.1) and, consequently, $F \in \Box_1$. Thus, $\mathcal{M}$ meets JYB and is a modular L_{CS}-model.

If, on the contrary, (jt) $\notin \mathsf{L}$ and (jd) $\notin \mathsf{L}$, set $R := \varnothing$. It is symmetric, Euclidean, and transitive, and JYB in this case is met trivially, since $\Box_1 = \mathcal{L}_j$. □

E Proof of Theorem 4.10

Theorem 4.10 Let L_{CS} be a logic with (jt) $\notin \mathsf{L}$ and (jd) $\in \mathsf{L}$. Then L_{CS} is sound with respect to modular L_{CS}-models. If CS is axiomatically appropriate for L, then L_{CS} is also complete.

Proof. *Soundness.* Since $*_w$ from the soundness proof in Theorem 4.8 is consistent if $\mathcal{M}$ is and $\mathcal{M}$ is consistent by Lemma 4.6, the same soundness argument applies.
Completeness. We reuse the construction of a basic modular model $*_\Phi$ based on an L_{CS}-consistent set Φ from the proof of Theorem 3.8. From all such $*_\Phi$ we create the canonical modular L_{CS}-model and use (D.1) to transfer the properties of $*_\Phi$ proved for Theorem 3.8. Thus, we define $\mathcal{M}_c := (W, R, *)$, where $W := \{\Phi \subseteq \mathcal{L}_j \mid \Phi \text{ is a maximal } \mathsf{L}_{\mathsf{CS}}\text{-consistent set}\}$ and $*_\Phi$ for each $\Phi \in W$ is defined by (C.1). Finally, we set $R(\Phi, \Psi)$ iff $\Phi^\sharp \subseteq \Psi$, where $\Phi^\sharp := \{F \in \mathcal{L}_j \mid t{:}F \in \Phi\}$. If (jb) $\in \mathsf{L}$, then $\mathcal{M}_c$ is Brouwerian by (D.1) because each $*_\Phi$ is. To show that $\mathcal{M}_c$ is a modular L_{CS}-model, it remains to establish the appropriate properties of R and the condition JYB.

We start with the latter. Let $F \in t^*_\Phi$. Then $t{:}F \in \Phi$ by the definition of $*_\Phi$ and $F \in \Psi$ whenever $R(\Phi, \Psi)$ by the definition of R. By (C.2), $*_\Psi \Vdash F$, and $\mathcal{M}_c, \Psi \Vdash F$ by (D.1). Since Ψ is chosen arbitrarily, $F \in \Box_\Phi$.

In order to prove seriality of R, we have to use the axiomatical appropriateness of CS. It is sufficient to show that $\Phi^\sharp$ is consistent for any $\Phi \in W$. Then $\Phi^\sharp$ can be extended to a maximal consistent $\Psi \supseteq \Phi^\sharp$, which is accessible from Φ by the definition of R. Assume towards a contradiction that $\Phi^\sharp$ were not consistent. Then there would be $s_1{:}F_1, \dots s_n{:}F_n \in \Phi$ such that $F_1, \dots, F_n \vdash_{\mathsf{L}_{\mathsf{CS}}} \bot$. Since CS is axiomatically appropriate, by Corollary 2.4, there would be a term t such that $s_1{:}F_1, \dots s_n{:}F_n \vdash_{\mathsf{L}_{\mathsf{CS}}} t{:}\bot$. Hence, by (jd) and (MP), $s_1{:}F_1, \dots s_n{:}F_n \vdash_{\mathsf{L}_{\mathsf{CS}}} \bot$, which contradicts the consistency of Φ.

The argument for the other properties of R follows the same pattern. We only show the symmetry case. Let (jb) $\in \mathsf{L}$ and $R(\Phi, \Psi)$. To show that $R(\Psi, \Phi)$, assume towards a contradiction that $t{:}F \in \Psi$ but $F \notin \Phi$. Then $\neg F \in \Phi$ by the maximal L_{CS}-consistency of Φ and $\bar{?}t{:}\neg t{:}F \in \Phi$ for the same reason. Hence, $\neg t{:}F \in \Psi$ by the definition of R, which contradicts the consistency of Ψ. □

F Proof of Lemma 4.14

Lemma 4.14 Every modular L_{CS}-model $\mathcal{M} = (W, R, *)$ is also an F-model for L_{CS} such that for all $w \in W$ and $F \in \mathcal{L}_j$, we have $\mathcal{M}, w \Vdash_{\mathsf{F}} F$ if and only if $\mathcal{M}, w \Vdash F$.

Proof. We prove (7) for all $w \in W$ by induction on the structure of $F \in \mathcal{L}_j$. The only non-trivial case is when $F = t{:}G$, and the only non-trivial direction of this case is from right to left. If $\mathcal{M}, w \Vdash t{:}G$, then $G \in t^*_w$. Hence, $G \in \Box_w$ by JYB. By induction hypothesis, $G \in \Box^{\mathsf{F}}_w$. Thus, $\mathcal{M}, w \Vdash_{\mathsf{F}} t{:}G$.

It remains to show (4) for logics with (j4) $\in \mathsf{L}$ and (5) for those with (j5) $\in \mathsf{L}$ or (jb) $\in \mathsf{L}$. Since the former is exactly the statement of Lemma 4.7, we prove the latter. If $F \in t^*_w$, then $F \in \Box_w$ by JYB. By (7), $\Box_w = \Box^{\mathsf{F}}_w$ so that $F \in \Box^{\mathsf{F}}_w$.□

G Proof of Lemma 4.16

Lemma 4.16 Let $\mathcal{M}_\mathsf{F} = (W, R, *_\mathsf{F})$ be an F-model for a logic L_CS. Then $\mathcal{M} := (W, R, *)$ with

$$t^*_w := \{F \in \mathcal{L}_j \mid F \in t^{*_\mathsf{F}}_w \text{ and } F \in \Box^\mathsf{F}_w\} = \{F \in \mathcal{L}_j \mid \mathcal{M}_\mathsf{F}, w \Vdash_\mathsf{F} t{:}F\}$$

is a modular L_CS-model such that for all $w \in W$ and all $F \in \mathcal{L}_j$, we have $\mathcal{M}, w \Vdash F$ if and only if $\mathcal{M}_\mathsf{F}, w \Vdash_\mathsf{F} F$.

Proof. Although it is not yet proved that $\mathcal{M}$ is a quasimodel, we can still apply $\Vdash$ to it. Thus, we start by proving (9) for all $w \in W$ by induction on the structure of F. Again, the only non-trivial case is when $F = t{:}G$, and the statement for it follows immediately from (8).

We now use (9) to show that $\mathcal{M}$ is a modular L_CS-model. The conditions on R for F-models and modular models are identical, so we need to verify that $*_w$ is a basic L_CS-evaluation for each $w \in W$, that $\mathcal{M}$ is Brouwerian if (jb) $\in \mathsf{L}$, and that JYB holds. We check JYB first. Suppose $F \in t^*_w$. Then $F \in \Box^\mathsf{F}_w$ by (8). Hence, $F \in \Box_w$ by (9).

Suppose (jb) $\in \mathsf{L}$ and $\mathcal{M}, w \nVdash F$. Then $\mathcal{M}, w \nVdash_\mathsf{F} F$ by (9) so that $\neg t{:}F \in (\bar{?}t)^{*_\mathsf{F}}_w$ for any $t \in \mathrm{Tm}$ since $\mathcal{M}_\mathsf{F}$ is Brouwerian. Since (jb) $\in \mathsf{L}$, we can use JYB for $\mathcal{M}_\mathsf{F}$, which yields $\neg t{:}F \in \Box^\mathsf{F}_w$. Thus, $\neg t{:}F \in (\bar{?}t)^*_w$ by (8).

It remains to check that $*_w$ is a basic L_CS-evaluation.

(i) Suppose $F \in s^*_w \cdot t^*_w$. Then there must exist a formula $G \in \mathcal{L}_j$ such that $G \to F \in s^*_w$ and $G \in t^*_w$. By (8), $G \to F \in s^{*_\mathsf{F}}_w$ and $G \in t^{*_\mathsf{F}}_w$. Thus, (a) $F \in s^{*_\mathsf{F}}_w \cdot t^{*_\mathsf{F}}_w \subseteq (s \cdot t)^{*_\mathsf{F}}_w$ since $*_\mathsf{F}(w)$ is a basic L_CS-evaluation. Also by (8), $G \to F \in \Box^\mathsf{F}_w$ and $G \in \Box^\mathsf{F}_w$. In other words, $\mathcal{M}_\mathsf{F}, v \Vdash_\mathsf{F} G \to F$ and $\mathcal{M}_\mathsf{F}, v \Vdash_\mathsf{F} G$ whenever $R(w, v)$. Clearly, $\mathcal{M}_\mathsf{F}, v \Vdash_\mathsf{F} F$ whenever $R(w, v)$ so that (b) $F \in \Box^\mathsf{F}_w$. From (a) and (b), $F \in (s \cdot t)^*_w$ follows by (8).

(ii) The proof that $s^*_w \cup t^*_w \subseteq (s + t)^*_w$ is similar.

(iii) Suppose $t{:}F$ is a conclusion of the (AN) or the (AN!) rule, whichever is present in L_CS. Then $F \in t^{*_\mathsf{F}}_w$ because $*_\mathsf{F}(w)$ is a basic L_CS-evaluation. It is now sufficient to show that $F \in \Box^\mathsf{F}_w$, which follows from the soundness of F-models. In [4], the soundness is established for most of the logics except for those with (jb) $\in \mathsf{L}$, for which F-models have not been defined. Thus, we need to show the soundness of (jb) in F-models for such L_CS. Suppose (jb) $\in \mathsf{L}$ and $\mathcal{M}'_\mathsf{F}, w' \nVdash_\mathsf{F} G$ for an arbitrary F-model $\mathcal{M}'_\mathsf{F} = (W', R', *_\mathsf{F}{}')$ for L_CS. Then $\neg s{:}G \in (\bar{?}s)^{*_\mathsf{F}{}'}_{w'}$ for any $s \in \mathrm{Tm}$ because $\mathcal{M}'_\mathsf{F}$ is Brouwerian. Also, $\neg s{:}G \in \Box^\mathsf{F}_{w'}$ by JYB. Thus, $\mathcal{M}'_\mathsf{F}, w' \Vdash_\mathsf{F} \bar{?}s{:}\neg s{:}G$.

(iv) Suppose (j4) $\in \mathsf{L}$ and $s{:}F \in s{:}(s^*_w)$, i.e., $F \in s^*_w$. By (8), this implies $\mathcal{M}_\mathsf{F}, w \Vdash_\mathsf{F} s{:}F$. By soundness of (j4) in monotone F-models that satisfy (iv), we get $\mathcal{M}_\mathsf{F}, w \Vdash_\mathsf{F} !s{:}s{:}F$. Now $s{:}F \in (!s)^*_w$ follows from (8).

(v) Suppose (j5) $\in \mathsf{L}$ and $F \notin t^*_w$. By (8), this implies $\mathcal{M}_\mathsf{F}, w \nVdash_\mathsf{F} t{:}F$. By soundness of (j5) in F-models that satisfy JYB and (v), we get that $\mathcal{M}_\mathsf{F}, w \Vdash_\mathsf{F} ?t{:}\neg t{:}F$. Now $\neg t{:}F \in (?t)^*_w$ follows from (8).

$\Box$

H Proof of Theorem 5.2

Theorem 5.2 Let $\mathsf{L}^{\Box}_{\mathsf{CS}}$ be a logic of justifications and belief. For all formulas $F \in \mathcal{L}_{\Box}$, if $\mathcal{M} \Vdash F$ for all modular $\mathsf{L}^{\Box}_{\mathsf{CS}}$-models $\mathcal{M}$, then $\mathsf{L}^{\Box}_{\mathsf{CS}} \vdash F$.

Proof. We define the canonical model $\mathcal{M}^{\Box}_{c} = (W, R, *)$ as follows:

$$W := \{\Phi \subseteq \mathcal{L}_{\Box} \mid \Phi \text{ is a maximal } \mathsf{L}^{\Box}_{\mathsf{CS}}\text{-consistent set}\}$$

and $*_{\Phi}$ for each $\Phi \in W$ is defined by (C.1) except that t^*_{Φ} consists of $\mathcal{L}_{\Box}$-formulas instead of $\mathcal{L}_j$-formulas. Finally, we set $R(\Phi, \Psi)$ iff $\Phi^{\Box} \subseteq \Psi$, where $\Phi^{\Box} := \{F \in \mathcal{L}_{\Box} \mid \Box F \in \Phi\}$.

As usual, the Truth Lemma is established by induction on D: for all formulas $D \in \mathcal{L}_{\Box}$ and all maximal $\mathsf{L}^{\Box}_{\mathsf{CS}}$-consistent sets Φ,

$$D \in \Phi \qquad \Longleftrightarrow \qquad \mathcal{M}^{\Box}_{c}, \Phi \Vdash D \ . \tag{H.1}$$

The cases for propositions and Boolean connectives are straightforward. The case for $D = t{:}F$ does not involve R and is essentially the same as in the proof of Theorem 3.8. The case for $D = \Box F$ is proved by the standard modal argument because R is defined as in the modal canonical model rather than as in Theorem 3.8.

It remains to show that $\mathcal{M}^{\Box}_{c}$ is a modular $\mathsf{L}^{\Box}_{\mathsf{CS}}$-model. The proof that $*_{\Phi}$ is a basic $\mathsf{L}^{\Box}_{\mathsf{CS}}$-evaluation is almost literally the same as in Theorem 3.8. The conditions on R are established by the standard modal argument. If (jb) $\in \mathsf{L}$, the proof that $\mathcal{M}^{\Box}_{c}$ is Brouwerian follows the relevant part of the proof of Theorem 3.8, only referring to the Truth Lemma (H.1) instead of (C.2).

Thus, the proof of JYB is the only thing that needs to be redone due to the change in the definition of R, compared to Theorem 4.10. Suppose $F \in t^*_{\Phi}$. Then $t{:}F \in \Phi$ by (C.1). Using the axiom instance $t{:}F \to \Box F$ and the maximal $\mathsf{L}^{\Box}_{\mathsf{CS}}$-consistency of Φ, we conclude that $\Box F \in \Phi$. Hence, $F \in \Psi$ whenever $R(\Phi, \Psi)$ by the definition of R. Thus, $\mathcal{M}, \Psi \Vdash F$ whenever $R(\Phi, \Psi)$ by (H.1), i.e., $F \in \Box_{\Phi}$. □

I Proof of Theorem 7.2

Theorem 7.2 Let L_{CS} be a justification logic with the axiomatically appropriate CS. Then L_{CS} is sound and complete with respect to fully explanatory modular L_{CS}-models.

Proof. Given Theorems 4.8 and 4.10 and Remark 4.11, it is sufficient to show that the canonical model $\mathcal{M}_c = (W, R, *)$ forL_{CS} constructed in the proof of Theorem 4.10 is fully explanatory.

Assume towards a contradiction that $F \in \Box_{\Phi}$ for some $F \in \mathcal{L}_j$ and $\Phi \in W$ but $F \notin t^*_{\Phi}$ for any $t \in \mathrm{Tm}$. Then $\Phi^{\sharp} \cup \{\neg F\}$ would be L_{CS}-consistent.

Indeed, if $\Phi^{\sharp} \cup \{\neg F\}$ were L_{CS}-inconsistent, then $G_1, \dots, G_n, \neg F \vdash_{\mathsf{L}_{\mathsf{CS}}} \bot$ for some $G_1, \dots, G_n \in \Phi^{\sharp}$. Equivalently, there would be terms $s_1, \dots, s_n$ such that $s_i{:}G_i \in \Phi$ for $i = 1, \dots, n$ and $G_1, \dots, G_n \vdash_{\mathsf{L}_{\mathsf{CS}}} F$. By Corollary 2.4, given the axiomatic appropriateness of CS, there would be a term t such that $s_1{:}G_1, \ldots, s_n{:}G_n \vdash_{\mathsf{L}_{\mathsf{CS}}} t{:}F$. By Deduction Theorem 2.1,

$$\mathsf{L}_{\mathsf{CS}} \vdash s_1{:}G_1 \to (s_2{:}G_2 \to \cdots \to (s_n{:}G_n \to t{:}F) \cdots)$$

so that $t{:}F \in \Phi$ by the maximal L_{CS}-consistency of Φ and $F \in t^*_\Phi$ by (C.1), contradicting our assumption.

Hence, the set $\Phi^\sharp \cup \{\neg F\}$ would be L_{CS}-consistent and could be extended to a maximal L_{CS}-consistent set Ψ. Clearly, $R(\Phi, \Psi)$ by the definition of R and $\mathcal{M}_c, \Psi \Vdash \neg F$ by (C.2) and (D.1). Thus, $\mathcal{M}_c, \Psi \nVdash F$, which contradicts our assumption that $F \in \Box_\Phi$. $\Box$