

# $\kappa$ -Deformed Phase Space, Hopf Algebroid and Twisting\*

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**Abstract.** Hopf algebroid structures on the Weyl algebra (phase space) are presented. We define the coproduct for the Weyl generators from Leibniz rule. The codomain of the coproduct is modified in order to obtain an algebra structure. We use the dual base to construct the target map and antipode. The notion of twist is analyzed for  $\kappa$ -deformed phase space in Hopf algebroid setting. It is outlined how the twist in the Hopf algebroid setting reproduces the full Hopf algebra structure of  $\kappa$ -Poincaré algebra. Several examples of realizations are worked out in details.

*Key words:* noncommutative space;  $\kappa$ -Minkowski spacetime; Hopf algebroid;  $\kappa$ -Poincaré algebra; realizations; twist

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## 1 Introduction

Motivation for studying noncommutative (NC) spaces is related to the fact that general theory of relativity together with Heisenberg uncertainty principle leads to the uncertainty of position coordinates itself  $\Delta x_\mu \Delta x_\nu > l_{\text{Planck}}^2$  [22, 23]. This uncertainty in the position can be realized via NC coordinates. There are also arguments based on quantum gravity [22, 23, 36], and string theory models [20, 61], which suggest that the spacetime at the Planck length is quantum, i.e. noncommutative.

We will consider a particular example of NC space, the so called  $\kappa$ -Minkowski spacetime [13, 17, 37, 40, 41, 49, 50, 51, 53, 54, 56, 57, 58], which is a Lie algebraic deformation of the usual Minkowski spacetime. Here,  $\kappa$  is the deformation parameter usually interpreted as Planck mass or the quantum gravity scale. Investigations of physical theories on  $\kappa$ -Minkowski spacetime leads to many new properties, such as: modification of particle statistics [5, 18, 24, 27, 64, 65], deformed electrodynamics [28, 29], NC quantum mechanics [3, 4, 31, 46, 47], and quantum gravity effects [11, 21, 26, 30, 60].  $\kappa$ -Minkowski spacetime is also related to doubly-special and deformed relativity theories [1, 2, 10, 43, 44].

The symmetries of  $\kappa$ -Minkowski spacetime are described via Hopf algebra setting and they are encoded in the  $\kappa$ -Poincaré–Hopf algebra (in the same sense as are the symmetries of Minkowski spacetime encoded in the Poincaré–Hopf algebra). A Hopf algebra is a bialgebra equipped with an antipode map satisfying the Hopf axiom. The bialgebra is an (unital, associative) algebra which is also a (conunitary, coassociative) coalgebra such that certain compatibility conditions are satisfied. The antipode is an antihomomorphism of the algebra structure (an antialgebra

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homomorphism). Hopf algebras are used in various areas of mathematics and physics for fifty years. See [8, 52] for some examples.

It turns out that the notion of the Hopf algebra is too restrictive and it has to be generalized. For example, it is shown that the Weyl algebra (quantum phase space) can not have a structure of a Hopf algebra. Namely, the whole phase space (Weyl algebra) generated by  $p_\mu$  and  $x_\mu$  (or  $\hat{x}_\mu$ ) can not be equipped with the Hopf algebra structure, since one can not include  $\Delta x_\mu$  in a satisfactory way, i.e. the notion of Hopf algebra is too restrictive for the whole phase space (Weyl algebra). Several types of generalizations are possible: quasi-Hopf algebras, multiplier Hopf algebras and weak Hopf algebras. Our construction is very similar to the structure of the Hopf algebroid defined by Lu in [48].

Lu was inspired by the notion of the Poisson algebroid from the Poisson geometry. Namely, some Hopf algebras are quantization of the Poisson groups. Now, Hopf algebroids can be considered as the quantization of the Poisson groupoids. Lu introduces two algebras: the base algebra  $A$  and the total algebra  $H$ . One can consider the total algebra  $H$  as the algebra over the base algebra  $A$ . The left and right multiplications are given by the source and the target maps. Hence, the coproduct  $\Delta$  is defined on the total algebra  $H$  and the image lies in  $H \otimes_A H$  which is an  $(A, A)$ -bimodule but not an algebra. Namely,  $H \otimes_A H$  is the quotient of  $H \otimes H$  by the right ideal. G. Böhm and K. Szlachányi in [9] considered the same structure as Lu did, but they changed the definition of the antipode. For more comprehensive approach, see [8]. Let us mention that some ideas existed before the definition of Lu in which the base algebra or both the base algebra and the total algebra had to be commutative (see [8, 48] and references therein). Bialgebroid is equivalent to the notion of  $\times_A$ -bialgebra introduced much earlier by Takeuchi in [62].

One can analyze the structure of the Hopf algebra by twists. See [6, 7] for more details. P. Xu in [63] applies the twist to the bialgebroid (which he calls Hopf algebroid although he does not have the antipode). It is important to mention that Xu uses the definition of the bialgebroid which is equivalent to the definition from [48].

In [40]  $\kappa$ -Minkowski spacetime and Lorentz algebra are unified in a unique Lie algebra. Realizations and star products are defined and analyzed in general and specially, their relation to coproduct of the momenta is pointed out.

The deformation of Heisenberg algebra and the corresponding coalgebra by twist is performed in [57]. Here, the so called tensor exchange identities are introduced and coalgebras for the generalized Poincaré algebras are constructed. The exact universal  $R$ -matrix for the deformed Heisenberg (co)algebra is found.

The quantum phase space (Weyl algebra) and its Hopf algebroid structure is analyzed in [33]. Unification of  $\kappa$ -Poincaré algebra and  $\kappa$ -Minkowski spacetime is done via embedding into quantum phase space. The construction of  $\kappa$ -Poincaré-Hopf algebra and  $\kappa$ -Minkowski spacetime using Abelian twist in the Hopf algebroid approach has been elaborated.

Twists, realizations and Hopf algebroid structure of  $\kappa$ -deformed phase space are discussed in [34]. It is shown that starting from a given deformed coalgebra of commuting coordinates and momenta one can construct the corresponding twist operator.

In the present paper, the total algebra is the Weyl algebra  $\hat{\mathcal{H}}$  and the base algebra is the subalgebra  $\hat{\mathcal{A}}$  generated by noncommutative coordinates  $\hat{x}_\mu$ . The construction of the target map is obtained via dual realizations. The codomain of the coproduct is changed. We take a quotient of the image of the coproduct instead of quotient of  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$ . As a consequence, the right ideal by which Lu [48] has taken the quotient is now two-sided and the codomain of the coproduct has the algebra structure. The notion of the counit is related to realizations. Furthermore, we manage to incorporate the twist in our construction, obtaining the Hopf algebroid structure from the twist.

This paper is structured as follows. In Section 2 we introduce the  $\kappa$ -Minkowski spacetime and  $\kappa$ -deformed phase space, and we establish the connection between Leibniz rule and coproduct

for the Weyl generators. Also, the dual basis is introduced and elaborated. The Hopf algebroid structure of  $\kappa$ -deformed phase space  $\hat{\mathcal{H}}$  and undeformed phase space  $\mathcal{H}$  is presented in Section 3. In Section 4 we first discuss the realizations and then we provide the twist operator in the Hopf algebroid approach. It is shown that the twisted Hopf algebroid structure of phase space  $\mathcal{H}$  is isomorphic to the Hopf algebroid structure of  $\hat{\mathcal{H}}$ . Finally, in Section 5 we consider the  $\kappa$ -Poincaré–Hopf algebra in the *natural* realization (classical basis). It is outlined how the twist in Hopf algebroid setting reproduces the full Hopf algebra structure of  $\kappa$ -Poincaré algebra. Also, we discuss the existence and properties of twist in all types of deformations (space-, time- and light-like).

## 2 $\kappa$ -deformed phase space

### 2.1 $\kappa$ -Minkowski spacetime

Let us denote coordinates of the  $\kappa$ -Minkowski spacetime by  $\hat{x}_\mu$ . Latin indices will be used for the set  $\{1, \dots, n-1\}$  and Greek indices will be used for the set  $\{0, \dots, n-1\}$ . The Lorentz signature of the  $\kappa$ -Minkowski spacetime is defined by  $[\eta_{\mu\nu}] = \text{diag}(-1, 1, \dots, 1)$ . Let  $\mathfrak{g}_\kappa$  be the Lie algebra generated by  $\hat{x}_\mu$  such that

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \quad (2.1)$$

where  $a \in \mathbb{M}^n$ . The relation to  $\kappa$  mass parameter is  $a_\mu = \frac{1}{\kappa} u_\mu$ ,  $u_\mu \in \mathbb{M}^n$  ( $u^2 = -1$  time-like,  $u^2 = 1$  space-like and  $u^2 = 0$  light-like). The enveloping algebra  $\mathcal{U}(\mathfrak{g}_\kappa)$  of  $\mathfrak{g}_\kappa$  will be denoted by  $\hat{\mathcal{A}}$ .

### 2.2 Phase space

The momentum space  $\mathcal{T} = \mathbb{C}[[p_\mu]]$  is the commutative space generated by  $p_\mu$  such that

$$[p_\mu, \hat{x}_\nu] = -i\varphi_{\mu\nu}(p) \quad (2.2)$$

is satisfied for some set of real functions  $\varphi_{\mu\nu}$  (see [33, 34, 40] for details). Let us recall that  $\lim_{a \rightarrow 0} \varphi_{\mu\nu} = \eta_{\mu\nu} 1$  and  $\det \varphi \neq 0$ . We also require that generators  $\hat{x}_\mu$  and  $p_\mu$  satisfy Jacobi identities. This gives the set of restrictions on functions  $\varphi_{\mu\nu}$  (see equation (11) in [40] or equation (4) in [34]). The existence of such space  $\mathcal{T}$  is analyzed in several papers [40, 54]. One particularly interesting solution is the set  $\{p_\mu^L\}$  which is related to the so called *left covariant* realization [40, 54] where  $\varphi_{\mu\nu} = \eta_{\mu\nu} Z^{-1}$ , i.e. (2.2) leads to

$$[p_\mu^L, \hat{x}_\nu] = -i\eta_{\mu\nu} Z^{-1}. \quad (2.3)$$

Here  $Z$  denotes the shift operator defined by

$$[Z, \hat{x}_\mu] = ia_\mu Z, \quad [Z, p_\mu] = 0,$$

and for the *left covariant* realization is given by

$$Z^{-1} = 1 + (ap^L),$$

where we used  $(ap^L) \equiv a^\alpha p_\alpha^L$ . The phase space  $\hat{\mathcal{H}}$  is generated as an algebra by  $\hat{\mathcal{A}}$  and  $\mathcal{T}$  such that (2.1) and (2.2) are satisfied.

Let  $\blacktriangleright$  be the unique action of  $\hat{\mathcal{H}}$  on  $\hat{\mathcal{A}}$ , such that  $\hat{\mathcal{A}}$  acts on itself by left multiplication and  $t \blacktriangleright \hat{f} = [t, \hat{f}] \blacktriangleright 1$  for all  $t \in \mathcal{T}$  and  $\hat{f} \in \hat{\mathcal{A}}$ .  $\hat{\mathcal{A}}$  can be considered as an  $\hat{\mathcal{H}}$ -module.

### 2.3 Leibniz rule

We have already mentioned that  $\hat{\mathcal{H}}$  does not have the structure of the Hopf algebra, but it is possible to construct the structure of the Hopf algebroid. In this subsection we do the preparation for the coproduct which will be completely defined in Section 3. The formula for the coproduct can be built from the action  $\blacktriangleright$  and the Leibniz rule (see [40, Section 2.3] and [34]). In  $\kappa$ -Poincaré–Hopf algebra  $\mathcal{U}_\kappa(\mathcal{P})$  (where  $\mathcal{P}$  is generated by momenta  $p_\mu$  and Lorentz generators  $M_{\mu\nu}$ ) the coproducts of momenta and Lorentz generators are unique and  $\Delta|_{\mathcal{U}_\kappa(\mathcal{P})} : \mathcal{U}_\kappa(\mathcal{P}) \rightarrow \mathcal{U}_\kappa(\mathcal{P}) \otimes \mathcal{U}_\kappa(\mathcal{P})$ . However in the Hopf algebroid structure the coproduct of generators  $p_\mu$  and  $\hat{x}_\mu$  are not unique, modulo the right ideal  $\hat{\mathcal{R}}$  in (2.10).

Let  $\Delta(\hat{h}) = \hat{h}_{(1)} \otimes \hat{h}_{(2)}$  for  $\hat{h}_{(1)}, \hat{h}_{(2)} \in \hat{\mathcal{H}}$  (using Sweedler notation). Then

$$\hat{h} \blacktriangleright (\hat{f} \hat{g}) = m(\Delta(\hat{h}) \blacktriangleright (\hat{f} \otimes \hat{g})) = (\hat{h}_{(1)} \blacktriangleright \hat{f})(\hat{h}_{(2)} \blacktriangleright \hat{g}) \quad (2.4)$$

for  $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$ .

Now we recall the formula for the coproduct of  $p_\mu$  defined by  $\Delta|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}$ . Then

$$p_\mu \blacktriangleright (\hat{f} \hat{g}) = [p_\mu, \hat{f} \hat{g}] \blacktriangleright 1 = ([p_\mu, \hat{f}] \hat{g} + \hat{f} [p_\mu, \hat{g}]) \blacktriangleright 1 = [p_\mu, \hat{f}] \blacktriangleright \hat{g} + \hat{f} p_\mu \blacktriangleright \hat{g}. \quad (2.5)$$

For example let us write the coproduct of  $p_\mu^L$ . One finds by induction, starting with (2.3) that  $[p_\mu^L, \hat{f}] = (p_\mu^L \blacktriangleright \hat{f}) Z^{-1}$ ,  $\forall \hat{f} \in \hat{\mathcal{A}}$ . Inserting this result in the r.h.s. of (2.5) and comparison with r.h.s. of (2.4) for  $\hat{h} = p_\mu^L$  gives

$$\Delta(p_\mu^L) = p_\mu^L \otimes Z^{-1} + 1 \otimes p_\mu^L.$$

Now, let us consider elements  $\hat{x}_\mu$ . It is clear that

$$\Delta(\hat{x}_\mu) = \hat{x}_\mu \otimes 1 \quad (2.6)$$

since  $\hat{x}_\mu \blacktriangleright (\hat{f} \hat{g}) = (\hat{x}_\mu \hat{f}) \hat{g}$ . Formula (33) from [40] shows that

$$\hat{x}_\mu \blacktriangleright (\hat{f} \hat{g}) = (Z^{-1} \blacktriangleright \hat{f})(\hat{x}_\mu \blacktriangleright \hat{g}) - a_\mu (p_\alpha^L \blacktriangleright \hat{f})(\hat{x}_\alpha \blacktriangleright \hat{g})$$

and<sup>1</sup>

$$\Delta'(\hat{x}_\mu) = Z^{-1} \otimes \hat{x}_\mu - a_\mu p_\alpha^L \otimes \hat{x}^\alpha. \quad (2.7)$$

It is convenient to write (2.7) in the form

$$\Delta'(\hat{x}_\mu) = O_{\mu\alpha} \otimes \hat{x}^\alpha,$$

where

$$O_{\mu\alpha} = Z^{-1} \eta_{\mu\alpha} - a_\mu p_\alpha^L. \quad (2.8)$$

Hence, elements

$$\hat{R}_\mu = \hat{x}_\mu \otimes 1 - O_{\mu\alpha} \otimes \hat{x}^\alpha \quad (2.9)$$

satisfy  $m(\hat{R}_\mu \blacktriangleright (\hat{f} \otimes \hat{g})) = 0$  for all  $\hat{f}, \hat{g} \in \hat{\mathcal{A}}$  where  $m$  denotes the multiplication ( $m(\hat{f} \otimes \hat{g}) = \hat{f} \hat{g}$ ) and  $(a \otimes b) \blacktriangleright (\hat{f} \otimes \hat{g}) = (a \blacktriangleright \hat{f}) \otimes (b \blacktriangleright \hat{g})$ . Then

$$\hat{\mathcal{R}} = \mathcal{U}_+(\hat{R}_\mu) \hat{\mathcal{H}} \otimes \hat{\mathcal{H}} \quad (2.10)$$

<sup>1</sup>In Section 3.1, the coproduct will be defined and (2.6) and (2.7) will be equal, since both choices of coproducts of  $\hat{x}_\mu$  belong to the same congruence class.

is the right ideal in  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$ . Here we used that  $\mathcal{U}_+(\hat{R}_\mu)$  is the universal enveloping algebra generated by  $\hat{R}_\mu$  but without the unit element.

It is important to emphasize that such derived coproduct is an algebra homomorphism

$$\Delta(\hat{h}_1 \hat{h}_2) = \Delta(\hat{h}_1) \Delta(\hat{h}_2)$$

for any  $\hat{h}_1, \hat{h}_2 \in \hat{\mathcal{H}}$  which enables us to define the formula for the coproduct for all elements of  $\hat{\mathcal{H}}$ .

## 2.4 Dual basis

In [40] we have introduced the notion of the dual basis. Let us recall some basic facts since it will be used for the definition of the target map. We define elements

$$\hat{y}_\mu = \hat{x}^\alpha O_{\mu\alpha}^{-1}, \tag{2.11}$$

where

$$O_{\mu\alpha}^{-1} = (\eta_{\mu\alpha} + a_\mu p_\alpha^L) Z \tag{2.12}$$

(it would be more precise to write  $(O^{-1})_{\mu\alpha}$ ). They have some interesting properties. Since

$$\hat{x}_\mu = \hat{y}^\alpha O_{\mu\alpha}, \tag{2.13}$$

$\hat{y}_\mu$  and  $p_\mu$  form a basis of  $\hat{\mathcal{H}}$  (it would be more correct to say that power series in  $\hat{y}_\mu$  and  $p_\mu$  form a basis of  $\hat{\mathcal{H}}$ ). Elements  $\hat{y}_\mu$  satisfy commutation relations similar to (2.1):

$$[\hat{y}_\mu, \hat{y}_\nu] = -i(a_\mu \hat{y}_\nu - a_\nu \hat{y}_\mu).$$

We call this basis the dual basis.

It is easy to check that  $\hat{x}_\mu$  and  $\hat{y}_\nu$  commute, i.e.

$$[\hat{x}_\mu, \hat{y}_\nu] = 0. \tag{2.14}$$

Also, the straightforward calculation shows that  $O_{\mu\nu}$  and  $O_{\lambda\rho}$  commute. It remains to consider commutation relations among  $O_{\mu\nu}$ ,  $\hat{x}_\mu$  and  $\hat{y}_\mu$ . The definition of  $O_{\mu\nu}$  yields  $[O_{\mu\nu}, \hat{x}_\lambda] = i(a_\mu \eta_{\lambda\nu} - a_\lambda \eta_{\mu\nu}) Z^{-1} = i(a_\mu O_{\lambda\nu} - a_\lambda O_{\mu\nu})$  and it shows that

$$[O_{\mu\nu}, \hat{x}_\lambda] = i C_{\mu\lambda}^\alpha O_{\alpha\nu}, \tag{2.15}$$

where  $C_{\mu\lambda\alpha} = a_\mu \eta_{\lambda\alpha} - a_\lambda \eta_{\mu\alpha}$  stands for structure constants. One can easily obtain

$$[O_{\mu\nu}^{-1}, \hat{x}_\lambda] = i(-a_\mu \eta_{\lambda\nu} + a_\lambda O_{\mu\nu}^{-1}), \quad [O_{\mu\nu}, \hat{y}_\lambda] = i(a_\mu \eta_{\lambda\nu} - a_\lambda O_{\mu\nu})$$

and

$$[O_{\mu\nu}^{-1}, \hat{y}_\lambda] = i(-a_\mu O_{\lambda\nu}^{-1} + a_\lambda O_{\mu\nu}^{-1}) = -i C_{\mu\lambda\alpha} (O^{-1})^{\alpha\nu}.$$

The commutation relation  $[O_{\mu\nu}^{-1}, \hat{x}_\lambda]$  can be also obtained from (2.15) multiplying by  $O_{\mu\alpha}^{-1}$  and  $O_{\beta\nu}^{-1}$  and using  $a^\alpha O_{\mu\alpha}^{-1} = a_\mu$ . Let us mention that elements  $O_{\mu\nu}$  satisfy

$$O_{\mu\nu} = \eta_{\mu\nu} + C_{\mu\nu}^\alpha p_\alpha^L. \tag{2.16}$$

One can easily check that  $\hat{y}_\mu \blacktriangleright 1 = \hat{x}_\mu$ . Using (2.14) and (2.16), it is easy to obtain that  $\hat{y}_\mu \blacktriangleright \hat{x}_\nu = \hat{x}_\nu \hat{x}_\mu$  and

$$\hat{f}(\hat{y}) \blacktriangleright \hat{g}(\hat{x}) = \hat{g}(\hat{x}) \hat{f}^{\text{op}}(\hat{x}).$$

Here  $\hat{f}^{\text{op}}$  stands for the *opposite* polynomial  $((\hat{x}_\mu \hat{x}_\nu)^{\text{op}} = \hat{x}_\nu \hat{x}_\mu)$ . Hence, the action  $\blacktriangleright$  of  $\hat{f}(\hat{y})$  can be understood as a multiplication from the right with  $\hat{f}^{\text{op}}(\hat{x})$ . One can show that  $\Delta(\hat{y}_\mu) = 1 \otimes \hat{y}_\mu$ . Note that the same construction as for  $\kappa$ -Minkowski space (2.1) could be generalized to arbitrary Lie algebra defined by structure constants  $C_{\mu\nu\lambda}$ .

### 3 Hopf algebroid

#### 3.1 Hopf algebroid structure of $\hat{\mathcal{H}}$

We define the source map, target map, coproduct, counit and antipode such that  $\hat{\mathcal{H}}$  has the structure of the Hopf algebroid.

In Hopf algebroid, the unit map is replaced by the source and target maps. In our case  $\hat{\mathcal{H}}$  is the total algebra and  $\hat{\mathcal{A}}$  is the base algebra. The source map  $\hat{\alpha} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{H}}$  is defined by

$$\hat{\alpha}(\hat{f}(\hat{x})) = \hat{f}(\hat{x}).$$

The target map  $\hat{\beta} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{H}}$  is defined by

$$\hat{\beta}(\hat{f}(\hat{x})) = \hat{f}^{\text{op}}(\hat{y}).$$

Let us recall that the source map is the homomorphism while the target map is the antihomomorphism. Relation (2.14) shows that

$$\hat{\alpha}(\hat{f}(\hat{x}))\hat{\beta}(\hat{g}(\hat{x})) = \hat{\beta}(\hat{g}(\hat{x}))\hat{\alpha}(\hat{f}(\hat{x})).$$

In order to define the coproduct on  $\hat{\mathcal{H}}$ , we consider the subspace  $\hat{\mathcal{B}}$  of  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$ :

$$\hat{\mathcal{B}} = \mathcal{U}(\hat{R}_\mu)(\hat{\mathcal{A}} \otimes \mathbb{C}) \triangle \mathcal{T},$$

where  $\mathcal{U}(\hat{R}_\mu)$  denotes the universal enveloping algebra generated by  $\hat{R}_\mu$  (see (3.1)). Here,  $\triangle \mathcal{T}$  denotes the subalgebra of  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$  generated by  $1 \otimes 1$  and elements  $\triangle(p_\mu)$ . For example, we can consider  $p_\mu^L$  and then  $\triangle \mathcal{T}$  is generated by  $1 \otimes 1$  and  $p_\mu^L \otimes Z^{-1} + 1 \otimes p_\mu^L$ . Since

$$[\hat{R}_\mu, \hat{R}_\nu] = i(a_\mu \hat{R}_\nu - a_\nu \hat{R}_\mu) = iC_{\mu\nu\alpha} \hat{R}_\alpha, \quad (3.1)$$

$$[\hat{x}_\mu \otimes 1, \hat{R}_\nu] = i(a_\mu \hat{R}_\nu - a_\nu \hat{R}_\mu), \quad (3.2)$$

$$[O_{\mu\alpha} \otimes \hat{x}^\alpha, \hat{R}_\nu] = 0, \quad (3.3)$$

$$[\triangle p_\mu^L, \hat{R}_\nu] = 0$$

and

$$[\hat{x}_\mu \otimes 1, p_\nu^L \otimes Z^{-1} + 1 \otimes p_\nu^L] = i\eta_{\mu\nu} Z^{-1} \otimes Z^{-1} \in \triangle \mathcal{T}, \quad (3.4)$$

$\hat{\mathcal{B}}$  is a subalgebra of  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$ . It is obvious that (3.3) is a consequence of (3.1) and (3.2) but we write it for completeness. Now, let us consider the subspace  $\hat{\mathcal{J}}$  of  $\hat{\mathcal{B}}$  defined by

$$\hat{\mathcal{J}} = \mathcal{U}_+(\hat{R}_\mu)(\hat{\mathcal{A}} \otimes \mathbb{C}) \triangle \mathcal{T},$$

where  $\mathcal{U}_+(\hat{R}_\mu)$  is the universal enveloping algebra generated by  $\hat{R}_\mu$  but without the unit element. Using (3.1)–(3.4) one can check that  $\hat{\mathcal{J}} = \hat{\mathcal{R}} \cap \hat{\mathcal{B}}$  and  $\hat{\mathcal{J}}$  is the twosided ideal in  $\hat{\mathcal{B}}$ .

**Remark.** We could also define the subalgebra  $\hat{\mathcal{B}}_3$  in  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$  by

$$\hat{\mathcal{B}}_3 = \mathcal{U}[(\hat{R}_\mu)_{1,2}, (\hat{R}_\mu)_{2,3}](\hat{\mathcal{A}} \otimes \mathbb{C} \otimes \mathbb{C})(\triangle \otimes 1)(\triangle \mathcal{T}),$$

where  $\mathcal{U}[(\hat{R}_\mu)_{1,2}, (\hat{R}_\mu)_{2,3}]$  denotes the universal enveloping algebra generated by  $1 \otimes 1 \otimes 1$ ,  $(\hat{R}_\mu)_{1,2} = \hat{R}_\mu \otimes 1$  and  $(\hat{R}_\mu)_{2,3} = 1 \otimes \hat{R}_\mu$  and we have that  $(\triangle \otimes 1)(\triangle \mathcal{T}) = (1 \otimes \triangle)(\triangle \mathcal{T})$  since  $\mathcal{T}$  is a Hopf algebra. Similarly, we can define  $\hat{\mathcal{B}}_n$  and then  $\hat{\mathcal{B}}$  would correspond to  $\hat{\mathcal{B}}_2$ . Also,  $\hat{\mathcal{R}}_n$  and  $\hat{\mathcal{J}}_n = \hat{\mathcal{R}}_n \cap \hat{\mathcal{B}}_n$  can be defined. See [48] for the similar discussion.

Now, we define the coproduct  $\Delta : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{B}}/\hat{\mathcal{J}} = \Delta\hat{\mathcal{H}}$  by

$$\begin{aligned}\Delta(\hat{x}_\mu) &= \hat{x}_\mu \otimes 1 + \hat{\mathcal{J}} = Z^{-1} \otimes \hat{x}_\mu - a_\mu p_\alpha^L \otimes \hat{x}_\alpha + \hat{\mathcal{J}} = O_{\mu\alpha} \otimes \hat{x}_\alpha + \hat{\mathcal{J}}, \\ \Delta(p_\mu^L) &= p_\mu^L \otimes Z^{-1} + 1 \otimes p_\mu^L + \hat{\mathcal{J}}.\end{aligned}\tag{3.5}$$

Notice that  $\hat{\mathcal{B}}/\hat{\mathcal{J}}$  is the ‘‘restriction’’ of Lu’s  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}/\hat{\mathcal{K}}$ , or in other words an  $(\hat{\mathcal{A}}, \hat{\mathcal{A}})$ -submodule of  $\hat{\mathcal{H}} \otimes \hat{\mathcal{H}}/\hat{\mathcal{K}}$  that turns out to be an algebra, which, in turn, allows us to define  $\Delta$  as an algebra homomorphism

$$\Delta(\hat{f}\hat{g}) = \Delta(\hat{f}) \Delta(\hat{g}).$$

The coproduct of  $\hat{y}_\mu$  is given by  $\Delta(\hat{y}_\mu) = 1 \otimes \hat{y}_\mu + \hat{\mathcal{J}} = \hat{y}_\alpha \otimes O_{\mu\alpha}^{-1} + \hat{\mathcal{J}}$ . One can check that such defined coproduct is coassociative.

The counit  $\hat{\epsilon} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{A}}$  is defined by

$$\hat{\epsilon}(\hat{h}) = \hat{h} \blacktriangleright 1.$$

This map is not a homomorphism. It is easy to check that  $m(\hat{\alpha}\hat{\epsilon} \otimes 1)\Delta = 1$  and  $m(1 \otimes \hat{\beta}\hat{\epsilon})\Delta = 1$ . In order to check the first identity, we write elements of  $\hat{\mathcal{H}}$  in the form  $\hat{f}(\hat{x})g(p)$  and for the second identity in the form  $\hat{f}(\hat{y})g(p)$ .

The antipode  $S : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$  is defined by

$$S(\hat{y}_\mu) = \hat{x}_\mu \quad \text{and} \quad S(p_\mu^L) = -p_\mu^L Z.$$

The antipode  $S(\hat{x}_\mu)$  can be calculated from (2.13). One obtains that

$$S(\hat{x}_\mu) = \hat{y}_\mu + ia_\mu(1 - n).\tag{3.6}$$

It follows that  $S^2(\hat{y}_\mu) = \hat{y}_\mu + ia_\mu(1 - n)$  (and  $S^2(\hat{x}_\mu) = \hat{x}_\mu + ia_\mu(1 - n)$ ) and  $S^2(p_\mu) = p_\mu$ . Previous two formulas can be written also as  $S^2(\hat{h}) = Z^{1-n}\hat{h}Z^{n-1}$ . It is enough to check it for the elements  $\hat{x}_\mu$  and  $p_\mu$  since  $S^2$  is a homomorphism. The expression of  $S^2(\hat{x}_\mu)$  can be written in terms of structure constants:

$$S^2(\hat{x}_\mu) = \hat{x}_\mu + iC_{\alpha\mu}{}^\alpha.\tag{3.7}$$

A nice way to check the consistency of the antipode is to start with (2.13) and apply the antipode  $S$  (note that  $S(O_{\mu\alpha}) = O_{\mu\alpha}^{-1}$ ):

$$S(\hat{x}_\mu) = O_{\mu\alpha}^{-1}\hat{x}^\alpha = O_{\mu\alpha}^{-1}\hat{y}_\beta O^{\alpha\beta}.$$

It produces

$$S^2(\hat{x}_\mu) = (O^{-1})^{\alpha\beta}\hat{x}_\beta O_{\mu\alpha}.$$

It remains to apply expressions for  $(O^{-1})^{\alpha\beta}$  and  $O_{\mu\alpha}$  (see (2.12) and (2.8)), use the abbreviation  $A^L = -a^\alpha p_\alpha^L = -(ap^L)$  and recall the identity  $Z = (1 - A^L)^{-1}$  (see [40]).

Let  $\mathcal{P} \subset \hat{\mathcal{H}}$  be the enveloping algebra of the Poincaré algebra  $\mathfrak{p}$ . It is possible to define the Hopf algebra structure on the subalgebra  $\mathcal{P}$  [40]. It is interesting to note that the coproduct and the antipode map defined above on  $\hat{\mathcal{H}}$  and restricted to  $\mathcal{P}$  coincides with the coproduct and the antipode map on the Hopf algebra  $\mathcal{P}$  [33]. For more details see Section 5.

It is easy to check that

$$S\hat{\beta} = \hat{\alpha}, \quad m(1 \otimes S)\Delta = \hat{\alpha}\hat{\epsilon}, \quad m(S \otimes 1)\Delta = \hat{\beta}\hat{\epsilon}S.\tag{3.8}$$

The first identity is obvious, the second one can be easily checked for the base elements and the third identity can be easily checked using the dual basis.

In [48], Lu analyzes the right ideal  $\hat{\mathfrak{K}}$  generated by  $\hat{Q}_\mu = \hat{y}_\mu \otimes 1 - 1 \otimes \hat{x}_\mu$  (right ideal  $\hat{\mathfrak{K}}$  is denoted by  $I_2$  in [48]). These elements are equal to  $\hat{R}_\alpha((O^{-1})^{\mu\alpha} \otimes 1)$ . It is important to mention that the identity  $m(1 \otimes S)\Delta = \hat{\alpha}\hat{\epsilon}$  is not satisfied in [48], because  $m(1 \otimes S)\hat{\mathfrak{K}} \neq 0$  and this is why the section  $\gamma$  is needed. In our approach, since we have  $\Delta : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{B}}/\hat{\mathcal{J}} = \Delta\hat{\mathcal{H}}$  and

$$m(1 \otimes S)\hat{\mathcal{J}} = 0,$$

it is easy to see that (3.8) holds  $\forall h \in \hat{\mathcal{H}}$ .

Let us point out that  $[\hat{R}_\mu, \hat{Q}_\nu] = 0$  and  $[\hat{Q}_\mu, \hat{Q}_\nu] = i(-a_\mu \hat{Q}_\nu + a_\nu \hat{Q}_\mu)$ . Also, it is easy to check that  $[\hat{Q}_\mu, \Delta p_\nu^L] = 0$  and  $[\hat{Q}_\mu, \hat{x}_\nu \otimes 1] = 0$ .

### 3.2 Hopf algebroid structure of $\mathcal{H}$

Now, let us consider the case when the deformation vector  $a_\mu$  is equal to 0. Then (2.1) transforms to

$$[\hat{x}_\mu, \hat{x}_\nu] = 0,$$

the algebra  $\hat{\mathcal{H}}$  becomes the Weyl algebra which we denote by  $\mathcal{H}$  and write  $x_\mu$  instead of  $\hat{x}_\mu$ . We have already mentioned that it is not possible to construct the Hopf algebra structure on  $\mathcal{H}$ . Let us repeat the Hopf algebroid structure on  $\mathcal{H}$  and set the terminology.

Now,  $\varphi_{\mu\nu} = O_{\mu\nu} = \eta_{\mu\nu}$ ,  $Z = 1$  and  $\hat{y}_\mu = x_\mu$ . Let  $\mathcal{A}$  (the base algebra) be the subalgebra of  $\mathcal{H}$  generated by 1 and  $x_\mu$ . We define the action  $\triangleright$  of  $\mathcal{H}$  on  $\mathcal{A}$  in the same way as we did it in Section 2.2:  $f(x) \triangleright g(x) = f(x)g(x)$ ,  $p_\mu \triangleright 1 = 0$  and  $p_\mu \triangleright g(x) = [p_\mu, g(x)] \triangleright 1 = p_\mu g(x) \triangleright 1$ . Then  $\mathcal{A}$  can be considered as an  $\mathcal{H}$ -module. It is clear that the action  $\blacktriangleright$  transforms to the action  $\triangleright$  when the vector  $a$  is equal to 0.

The source and the target map are now equal  $\alpha_0 = \beta_0$  and  $\alpha_0; \beta_0 : \mathcal{A} \rightarrow \mathcal{H}$  reduces to the natural inclusion.

The counit  $\epsilon_0 : \mathcal{H} \rightarrow \mathcal{A}$  is defined by

$$\epsilon_0(h) = h \triangleright 1.$$

In order to define the coproduct, let us define relations  $(R_0)_\mu$  by

$$(R_0)_\mu = x_\mu \otimes 1 - 1 \otimes x_\mu.$$

Let  $\mathcal{U}[(R_0)_\mu]$  be the universal enveloping algebra generated by  $1 \otimes 1$  and  $(R_0)_\mu$ ,  $\mathcal{U}_+[(R_0)_\mu]$  be the universal enveloping algebra generated by  $(R_0)_\mu$  but without the unit element, and  $\Delta_0 \mathcal{T}$  be the algebra generated by  $1 \otimes 1$  and  $p_\mu \otimes 1 + 1 \otimes p_\mu$ . Note that  $\mathcal{T}$  is isomorphic to  $\Delta_0 \mathcal{T}$ . Now, we define  $\mathcal{B}_0$ , the subalgebra of  $\mathcal{H} \otimes \mathcal{H}$  of the form

$$\mathcal{B}_0 = \mathcal{U}[(R_0)_\mu](\mathcal{A} \otimes \mathbb{C}) \Delta_0 \mathcal{T}$$

and twosided ideal  $\mathfrak{J}_0$  of  $\mathcal{B}_0$  by

$$\mathfrak{J}_0 = \mathcal{U}_+[(R_0)_\mu](\mathcal{A} \otimes \mathbb{C}) \Delta_0 \mathcal{T}.$$

The coproduct  $\Delta_0 : \mathcal{H} \rightarrow \mathcal{B}_0/\mathfrak{J}_0 = \Delta_0 \mathcal{H}$  is a homomorphism defined by

$$\Delta_0(x_\mu) = x_\mu \otimes 1 + \mathfrak{J}_0, \quad \Delta_0(p_\mu) = p_\mu \otimes 1 + 1 \otimes p_\mu + \mathfrak{J}_0.$$



One checks that the coproduct  $\Delta_0$  and the counit  $\epsilon_0$  satisfy  $m(\alpha_0\epsilon_0 \otimes 1)\Delta_0 = 1$  and  $m(1 \otimes \beta_0\epsilon_0)\Delta_0 = 1$ .

The antipode  $S_0 : \mathcal{H} \rightarrow \mathcal{H}$  transforms to

$$S_0(x_\mu) = x_\mu, \quad S_0(p_\mu) = -p_\mu. \quad (3.9)$$

It is easy to check that

$$m(1 \otimes S_0)\Delta_0 = \alpha_0\epsilon_0, \quad m(S_0 \otimes 1)\Delta_0 = \beta_0\epsilon_0 S_0. \quad (3.10)$$

Similarly as in the deformed case, the expression  $m(1 \otimes S_0)\Delta_0$  is not well defined in [48], because  $m(1 \otimes S_0)\mathcal{K}_0 \neq 0$  and this is why the section  $\gamma$  is needed. In our approach, since

$$m(1 \otimes S_0)\mathfrak{I}_0 = 0$$

holds, one can check (3.10)  $\forall h \in \mathcal{H}$ .

## 4 Twisting Hopf algebroid structure

### 4.1 Realizations

The phase space satisfying (2.1) and (2.2) can be analyzed by realizations (see [40, 42, 56]). In Section 3.2, we have analyzed the Weyl algebra  $\mathcal{H}$  generated by  $p_\mu$  and commutative coordinates  $x_\mu$  satisfying

$$[p_\mu, x_\nu] = -i\eta_{\mu\nu}1.$$

Then, the noncommutative coordinates  $\hat{x}_\mu$  are expressed in the form

$$\hat{x}_\mu = x^\alpha \varphi_{\alpha\mu}(p) \quad (4.1)$$

such that (2.1) and (2.2) are satisfied. It is important to observe that the space  $\mathcal{H}$  is isomorphic to  $\hat{\mathcal{H}}$  as an algebra. Hence, we set  $\hat{\mathcal{H}} = \mathcal{H}$  and treat sets  $\{x_\mu, p_\nu\}$  and  $\{\hat{x}_\mu, p_\nu\}$  as different bases of the same algebra. However, we will use both symbols,  $\hat{\mathcal{H}}$  and  $\mathcal{H}$  in order to emphasize the basis. The action  $\triangleright$ , defined in Section 3.2 corresponds to  $\mathcal{H}$ . However,  $\mathcal{H}$  and  $\hat{\mathcal{H}}$ , considered as Hopf algebroids are different.

The restriction of the counit  $\epsilon_0|_{\hat{\mathcal{A}}}$ , introduced in Section 3.2, defines the bijection of vector spaces  $\hat{\mathcal{A}}$  and  $\mathcal{A}$ . By the abuse of notation, we denote it by  $\epsilon_0$  or  $\triangleright$ . Let us mention that the inverse map is simply  $\hat{\epsilon}|_{\mathcal{A}}$ . Then, the star product  $\star$  on  $\mathcal{A}$  is defined by  $(f \star g)(x) = \hat{f}(\hat{x})\hat{g}(\hat{x})\triangleright 1 = \hat{f}(\hat{x})\triangleright g(x)$  where  $f = \hat{f}\triangleright 1$  and  $g = \hat{g}\triangleright 1$ . The algebra  $\mathcal{A}$  equipped with the star product instead of pointwise multiplication will be denoted by  $\mathcal{A}_\star$  and the map  $\epsilon_0 : \hat{\mathcal{A}} \rightarrow \mathcal{A}_\star$  is an isomorphism of algebras.

It is possible to construct the dual realization  $\tilde{\varphi}_{\mu\nu}$  and the dual star product  $\star_{\tilde{\varphi}}$  such that

$$(f \star_{\tilde{\varphi}} g)(x) = (g \star_{\tilde{\varphi}} f)(x)$$

is satisfied (see [40, Section 5]). Now, elements  $\hat{y}_\mu$  are given by

$$\hat{y}_\mu = x^\alpha \tilde{\varphi}_{\alpha\mu}(p).$$

It is easy to check the following properties:

$$\hat{x}_\mu \triangleright f(x) = x_\mu \star_{\tilde{\varphi}} f(x) = f(x) \star_{\tilde{\varphi}} x_\mu$$

and

$$\hat{y}_\mu \triangleright f(x) = x_\mu \star_{\tilde{\varphi}} f(x) = f(x) \star_{\tilde{\varphi}} x_\mu.$$

### 4.1.1 Similarity transformations

The relation between realizations is given by the similarity transformations [34]. Let us consider two realizations. The first one is denoted by  $x_\mu$  and  $p_\mu$  and given by the set of functions  $\{\varphi_{\mu\nu}\}$  (and (2.2) or (4.1)). The second realization is denoted by  $X_\mu$ ,  $P_\mu$  and  $\Phi_{\mu\nu}$  ( $\hat{x}_\mu = X^\alpha \Phi_{\alpha\mu}(P)$ ). The similarity transformation  $\mathcal{E}$  is given by  $\mathcal{E} = \exp\{x^\alpha \Sigma_\alpha(p)\}$  such that  $\lim_{a \rightarrow 0} \Sigma_\alpha = 0$ . Now, the relation between realizations is given by

$$P_\mu = \mathcal{E} p_\mu \mathcal{E}^{-1}, \quad X_\mu = \mathcal{E} x_\mu \mathcal{E}^{-1}.$$

It is easy to see that  $P_\mu = P_\mu(p)$ . Since  $[P_\mu, \hat{x}_\nu] = -i\Phi_{\mu\nu}(P)$ ,

$$\frac{\partial P_\mu}{\partial p_\alpha} \varphi_{\alpha\nu} = \Phi_{\mu\nu}(P(p)) \quad \text{and} \quad \varphi_{\alpha\nu} = \left[ \frac{\partial P}{\partial p} \right]_{\alpha\mu}^{-1} \Phi_{\mu\nu}(P(p)).$$

It follows that the set of functions  $\varphi_{\mu\nu}$  can be obtained from the set of functions  $\Phi_{\mu\nu}$  and the expressions of  $P$  in terms of  $p$ . Since  $O_{\mu\nu} = O_{\mu\nu}(P(p))$ , it is easy to express  $O_{\mu\nu}$  in the realization determined by  $x_\mu$  and  $p_\mu$ .

### 4.1.2 Examples

Let us consider three examples of realizations. The *noncovariant*  $\lambda$ -family of realizations is given by

$$\hat{x}_0 = x_0^{(\lambda)} - a_0(1 - \lambda)x_k^{(\lambda)}p_k^{(\lambda)}, \quad \hat{x}_k = x_k^{(\lambda)}Z^{-\lambda}, \quad (4.2)$$

and

$$\hat{y}_0 = \hat{x}_0 Z - ia_0 + a_0(\hat{x}p^L)Z, \quad \hat{y}_j = \hat{x}_j Z, \quad (4.3)$$

where  $Z = e^{A^{(\lambda)}}$  and  $\lambda \in \mathbb{R}$ . For this family we assume that  $a = (a_0, 0, \dots, 0)$ . Here,  $(\lambda)$  denotes the label. Generic realizations are denoted without the label. It is easy to obtain  $p_0^L = \frac{1}{a_0}(1 - Z^{-1})$  and  $p_k^L = p_k^{(\lambda)}Z^{\lambda-1}$ . Now, one calculates  $O_{\mu\nu}$  (see (2.8)) in terms of  $p_\mu^{(\lambda)}$ :  $O_{k\nu} = Z^{-1}\eta_{k\nu}$ ,  $O_{00} = -1$  and

$$O_{0k} = Z^{-1}\eta_{0k} - a_0 p_k^L = (\eta_{0k} - a_0 p_k^{(\lambda)} Z^\lambda) Z^{-1}.$$

The *left covariant* realization is defined by

$$\hat{x}_\mu = x_\mu^L(1 - A^L),$$

where  $Z = (1 - A^L)^{-1}$ . The element  $p^L$  that we have mentioned in Section 2.2 corresponds to the *left covariant* realization. It is easy to obtain that

$$\hat{y}_\mu = x_\mu^L + a_\mu(x^L p^L)$$

(see (2.11) for the definition of  $\hat{y}_\mu$ ).

The *right covariant* realization is defined by

$$\hat{x}_\mu = x_\mu^R - a_\mu(x^R p^R),$$

where  $Z = 1 + A^R$ . The relation between  $p_\mu^L$  and  $p_\mu^R$  is given by  $p_\mu^R = p_\mu^L Z$ . Now,

$$\hat{y}_\mu = x_\mu^R(1 + A^R).$$

Also, it is easy to calculate  $O_{\mu\nu}$  in terms of  $p_\mu^R$ :

$$O_{\mu\nu} = Z^{-1}\eta_{\mu\nu} - a_\mu p_\nu^L = (\eta_{\mu\nu} - a_\mu p_\nu^R) Z^{-1}.$$

One should notice the duality between the *left covariant* and the *right covariant* realizations.

## 4.2 Twist and Hopf algebroid

For each realization, there is the corresponding twist and vice versa [34]. The relation between the star product and twist is given by

$$f \star g = m(\mathcal{F}^{-1} \triangleright (f \otimes g))$$

for  $f, g \in \mathcal{A}$ . It follows that  $\mathcal{F}^{-1} \in \mathcal{H} \otimes \mathcal{H}/\mathfrak{K}_0$ . Now, we will use twists to reconstruct the Hopf algebroid structure described in Section 3.1, from the Hopf algebroid structure analyzed in Section 3.2. That is we will show that by twisting the Hopf algebroid structure of  $\mathcal{H}$  one can obtain the Hopf algebroid structure of  $\hat{\mathcal{H}}$ . Hence, we will consider twists  $\mathcal{F}$  such that  $\mathcal{F} : \Delta_0 \mathcal{H} \rightarrow \Delta \mathcal{H}$ . Here  $\mathfrak{J} \cong \hat{\mathfrak{J}}$  and  $\Delta \mathcal{H} \cong \Delta \hat{\mathcal{H}}$ . More precisely,  $\mathfrak{J}$  is the twosided ideal generated by elements  $R_\mu$  which are defined by

$$R_\mu = \mathcal{F}(R_0)_\mu \mathcal{F}^{-1}.$$

Let us mention that the relation between  $\hat{R}_\mu$  and  $R_\mu$  is given by

$$\hat{R}_\mu = R^\alpha \Delta (\varphi_{\alpha\mu}).$$

Also, it is easy to rebuild the realization from the twist. For the given twist  $\mathcal{F}$ , the corresponding realization is obtained by

$$\hat{x}_\mu = m(\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)).$$

Similarly,

$$\hat{y}_\mu = m(\tilde{\mathcal{F}}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)),$$

where  $\tilde{\mathcal{F}}^{-1}$  is given by  $\tilde{\mathcal{F}}^{-1} = \tau_0 \mathcal{F}^{-1} \tau_0$  ( $\tau_0$  stands for the flip operator with the property  $\tau_0(h_1 \otimes h_2) = h_2 \otimes h_1, \forall h_1, h_2 \in \mathcal{H}$ ).

The *noncovariant*  $\lambda$ -family of realizations have twists of the form

$$\mathcal{F}^{(\lambda)} = \exp(i(\lambda x_k^{(\lambda)} p_k^{(\lambda)} \otimes A^{(\lambda)} - (1 - \lambda)A^{(\lambda)} \otimes x_k^{(\lambda)} p_k^{(\lambda)})). \quad (4.4)$$

These twists belong to the family of Abelian twists (see [24]). The *left covariant* and the *right covariant* realizations, respectively, have twists of the form

$$\mathcal{F}^L = \exp(i(x^L p^L) \otimes \ln Z) \quad \text{and} \quad \mathcal{F}^R = \exp(-\ln Z \otimes i(x^R p^R)).$$

These two twists belong to the family of Jordanian twists (see [13]).

Let us reconstruct the source and the target maps from the twist. First, we define  $\alpha$  and  $\beta$ ,  $\alpha : \mathcal{A}_\star \rightarrow \mathcal{U}(\hat{x}_\mu) \subset \mathcal{H}$ ,  $\beta : \mathcal{A}_\star \rightarrow \mathcal{U}(\hat{y}_\mu) \subset \mathcal{H}$  by

$$\alpha(f(x)) = m(\mathcal{F}^{-1}(\triangleright \otimes 1)(\alpha_0(f(x)) \otimes 1)), \quad \alpha_0(f(x)) = f(x),$$

and

$$\beta(f(x)) = m(\tilde{\mathcal{F}}^{-1}(\triangleright \otimes 1)(\beta_0(f(x)) \otimes 1)), \quad \beta_0(f(x)) = f(x).$$

Now, the source and the target maps are given by

$$\hat{\alpha} = \alpha \epsilon_0|_{\hat{\mathcal{A}}} \quad \text{and} \quad \hat{\beta} = \beta \epsilon_0|_{\hat{\mathcal{A}}}.$$

The counit  $\hat{\epsilon} : \mathcal{H} \rightarrow \hat{\mathcal{A}}$  is given by

$$\hat{\epsilon}(h) = m(\mathcal{F}^{-1}(\triangleright \otimes 1)(\epsilon_0(h) \otimes 1)).$$

The coproduct can be calculated by the formula:

$$\Delta(h) = \mathcal{F}(\Delta_0(h))\mathcal{F}^{-1}.$$

For the *noncovariant*  $\lambda$ -family of realizations

$$\Delta(x_j^{(\lambda)}) = x_j^{(\lambda)} \otimes Z^\lambda = Z^{\lambda-1} \otimes x_j^{(\lambda)}, \quad (4.5)$$

$$\Delta(x_0^{(\lambda)}) = x_0^{(\lambda)} \otimes 1 + a_0(1-\lambda) \otimes x_k^{(\lambda)} p_k^{(\lambda)} = 1 \otimes x_0^{(\lambda)} - a_0 \lambda x_k^{(\lambda)} p_k^{(\lambda)} \otimes 1, \quad (4.6)$$

$$\Delta(p_j^{(\lambda)}) = p_j^{(\lambda)} \otimes Z^{-\lambda} + Z^{1-\lambda} \otimes p_j^{(\lambda)}, \quad (4.7)$$

and

$$\Delta(p_0^{(\lambda)}) = p_0^{(\lambda)} \otimes 1 + 1 \otimes p_0^{(\lambda)}. \quad (4.8)$$

It is a nice exercise to express  $\hat{x}_\mu$  in terms of  $x_\alpha^{(\lambda)}$  and  $p_\alpha^{(\lambda)}$  (see (4.2)), use (4.5)–(4.8) and obtain (3.5).

Similarly,

$$\Delta(x_\mu^L) = x_\mu^L \otimes Z = 1 \otimes (x_\mu^L + i a_\mu Z) \quad \text{and} \quad \Delta(p_\mu^L) = p_\mu^L \otimes Z^{-1} + 1 \otimes p_\mu^L$$

for the *left covariant* realization and

$$\Delta(x_\mu^R) = (x_\mu^R - i a_\mu Z) \otimes 1 = Z^{-1} \otimes x_\mu^R \quad \text{and} \quad \Delta(p_\mu^R) = p_\mu^R \otimes 1 + Z \otimes p_\mu^R$$

for the *right covariant* realization.

It remains to consider the antipode. Let

$$\chi^{-1} = m(S_0 \otimes 1)\mathcal{F}^{-1},$$

then

$$S(h) = \chi(S_0(h))\chi^{-1} \quad (4.9)$$

where  $S_0$  denotes the undeformed antipode map defined by (3.9) ( $S_0(x_\mu) = x_\mu$  and  $S_0(p_\mu) = -p_\mu$ ). For the similar approach regarding Hopf algebras, see [7, 6].

For the *noncovariant*  $\lambda$ -family of realizations,  $\chi$  has the form

$$\chi^{(\lambda)} = \exp(i(1-2\lambda)A^{(\lambda)}x_k^{(\lambda)}p_k^{(\lambda)} + \lambda(1-n)A^{(\lambda)}).$$

Then

$$S(p_j^{(\lambda)}) = -p_j^{(\lambda)}Z^{2\lambda-1}, \quad (4.10)$$

$$S(p_0^{(\lambda)}) = -p_0^{(\lambda)}, \quad (4.11)$$

$$S(x_j^{(\lambda)}) = x_j^{(\lambda)}Z^{1-2\lambda}, \quad (4.12)$$

$$S(x_0^{(\lambda)}) = x_0^{(\lambda)} - (1-2\lambda)a_0x_k^{(\lambda)}p_k^{(\lambda)} + \lambda i a_0(1-n). \quad (4.13)$$

Again, it is an exercise to express  $\hat{x}_\mu$  in terms of  $x_\alpha^{(\lambda)}$  and  $p_\alpha^{(\lambda)}$  (see (4.2)), use (4.10)–(4.13) and obtain (3.6). The antipode is given by

$$S(\hat{x}_j) = \hat{x}_j Z \quad (4.14)$$

and

$$S(\hat{x}_0) = \hat{x}_0 + a_0 x_k^{(\lambda)} p_k^{(\lambda)} + ia_0(1-n). \quad (4.15)$$

Let us recall that for the *noncovariant*  $\lambda$ -family of realizations we set  $a_\mu = (a_0, 0, \dots, 0)$ . Now, one can compare (4.14) and (4.15) with (3.6). The formula for the antipode of  $\hat{x}_\mu$  can be also obtained from the formula  $S(\hat{y}_\mu) = \hat{x}_\mu$ , formulas for the realization of  $\hat{x}_\mu$  and  $\hat{y}_\mu$ , (4.2) and (4.3) and formulas for  $S(p_\mu)$ . For all examples, it is easy to check that  $S(\hat{y}_\mu) = \chi(S_0(\hat{y}_\mu))\chi^{-1} = \hat{x}_\mu$ .

For the *left covariant* realization

$$(\chi^L)^{-1} = \exp(i(p^L x^L)A^L).$$

For the *right covariant* realization

$$(\chi^R)^{-1} = \exp(-iA^R(x^R p^R)).$$

There is a natural question if the antipode map on the Hopf algebroid  $\hat{\mathcal{H}}$  defined by (4.9) and the antipode map defined on the Hopf algebra  $\mathcal{U}(\mathfrak{igl}(n))$  coincide (see [38] for the formulas of the antipode). They coincide for  $h \in \hat{\mathcal{H}}$  for which  $\alpha\epsilon(h) = \beta\epsilon S_0(h)$ . For elements  $h$  for which  $\alpha\epsilon(h) \neq \beta\epsilon S_0(h)$ , the antipode maps do not coincide. For example,  $S_0(x_j p_j) = -x_j p_j + i$  in the Hopf algebroid, while  $S_0(x_j p_j) = -x_j p_j$  in the Hopf algebra (here no summation is assumed). See also [33].

Using (4.9), it is easy to obtain the expression for  $S^{-1}$ :

$$S^{-1}(h) = S_0(\chi)S_0(h)S_0(\chi^{-1}).$$

One can show that  $S_0(\chi) = Z^{n-1}\chi$ . Then  $S^{-1}(h) = Z^{n-1}S(h)Z^{1-n}$  and  $S^2(h) = Z^{1-n}hZ^{n-1}$ . For example,  $S^2(p_\mu) = p_\mu$ ,  $S^2(\hat{x}_\mu) = \hat{x}_\mu + ia_\mu(1-n)$  and  $S^2(\hat{y}_\mu) = \hat{y}_\mu + ia_\mu(1-n)$ . This coincides with results in Section 3 (see (3.6) and (3.7)).

## 5 $\kappa$ -Poincaré Hopf algebra from $\kappa$ -deformed phase space and twists

Let us consider the  $\kappa$ -Poincaré Hopf algebra in *natural* realization [54, 55, 56] (or classical basis [15, 44]). We start with the undeformed Poincaré algebra generated by Lorentz generators  $M_{\mu\nu}$  and translation generators (momentum)  $P_\mu$

$$\begin{aligned} [M_{\mu\nu}, M_{\lambda\rho}] &= \eta_{\nu\lambda}M_{\mu\rho} - \eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\lambda} + \eta_{\mu\rho}M_{\nu\lambda}, \\ [P_\mu, P_\nu] &= 0, \quad [M_{\mu\nu}, P_\lambda] = \eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu. \end{aligned}$$

The corresponding  $\kappa$ -deformed Poincaré–Hopf algebra can be written in a unified covariant way [25, 35, 40, 54, 56]. The coproduct  $\Delta$  is given by

$$\begin{aligned} \Delta P_\mu &= P_\mu \otimes Z^{-1} + 1 \otimes P_\mu - a_\mu p_\alpha^L Z \otimes P^\alpha, \\ \Delta M_{\mu\nu} &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - a_\mu (p^L)^\alpha Z \otimes M_{\alpha\nu} + a_\nu (p^L)^\alpha Z \otimes M_{\alpha\mu}, \end{aligned} \quad (5.1)$$

as well as the antipode  $S$  and counit  $\epsilon$

$$\begin{aligned} S(P_\mu) &= (-P_\mu - a_\mu p_\alpha^L P^\alpha)Z, \\ S(M_{\mu\nu}) &= -M_{\mu\nu} - a_\mu (p^L)^\alpha M_{\alpha\nu} + a_\nu (p^L)^\alpha M_{\alpha\mu}, \\ \epsilon(P_\mu) &= \epsilon(M_{\mu\nu}) = 0, \end{aligned} \quad (5.2)$$

where the momentum  $P_\mu$  is related to  $p_\mu^L$  via  $P_\mu = p_\mu^L - \frac{a_\mu}{2}(p^L)^2 Z$ . The above Hopf algebra structure unifies all three types of deformations  $a_\mu$ , i.e. time-like ( $a^2 < 0$ ), space-like ( $a^2 > 0$ ) and light-like ( $a^2 = 0$ ).

Using the action  $\blacktriangleright$  and coproduct  $\Delta$  we can get the whole algebra  $\{\hat{x}_\mu, M_{\mu\nu}, P_\mu\}$  (for details see [32, 33])

$$\begin{aligned} [M_{\mu\nu}, \hat{x}_\lambda] &= \eta_{\nu\lambda}\hat{x}_\mu - \eta_{\mu\lambda}\hat{x}_\nu - ia_\mu M_{\nu\lambda} + ia_\nu M_{\mu\lambda}, \\ [P_\mu, \hat{x}_\nu] &= -i(\eta_{\mu\nu}Z^{-1} - a_\mu P_\nu), \end{aligned} \quad (5.3)$$

where  $Z^{-1} = (aP) + \sqrt{1 + a^2 P^2}$  and from (5.3) it follows that the NC coordinates  $\hat{x}_\mu$  can be written in terms of canonical  $X_\alpha$  and  $P_\alpha$  ( $[X_\alpha, X_\beta] = 0$ ,  $[P_\mu, X_\nu] = -i\eta_{\mu\nu}1$ ) via  $\hat{x}_\mu = X_\mu Z^{-1} - (aX)P_\mu$  and satisfies (2.1).

Now we will discuss the realization of  $\kappa$ -Poincaré–Hopf algebra via phase space  $\hat{\mathcal{H}}$  and discuss the issue of the twist in the Hopf algebroid approach. Realization of  $M_{\mu\nu}$  in terms of canonical  $X_\alpha$  and  $P_\alpha$  is given by  $M_{\mu\nu} = i(X_\mu P_\nu - X_\nu P_\mu)$  which for  $\kappa$ -deformed phase space variables  $\hat{x}_\mu, P_\mu$  reads

$$M_{\mu\nu} = i(\hat{x}_\mu P_\nu - \hat{x}_\nu P_\mu)Z \in \hat{\mathcal{H}}.$$

This is a unique realization in  $\hat{\mathcal{H}}$  (see [54]). Using  $\Delta P_\mu$  (5.1),  $\Delta \hat{x}_\mu$  (3.5),  $\Delta Z = Z \otimes Z$  and relations  $\hat{R}_\mu$  (2.9) we obtain coproduct  $\Delta M_{\mu\nu}$  as in Hopf algebra (5.1). Note that the result for  $\Delta M_{\mu\nu}$  is unique in the  $\kappa$ -Poincaré–Hopf algebra  $\mathcal{U}_\kappa(\mathcal{P})$  since  $\mathcal{U}_\kappa(\mathcal{P}) \otimes \mathcal{U}_\kappa(\mathcal{P}) \cap \hat{\mathfrak{K}} = 0$  (which is obvious). Similarly we find  $S(M_{\mu\nu})$  within Hopf algebroid which coincides with  $S(M_{\mu\nu})$  in Hopf algebra (5.2) (for details see [33, 57]).

There is a question whether  $\Delta P_\mu$  and  $\Delta M_{\mu\nu}$  could be obtained from twist  $\mathcal{F}$  expressed in terms of Poincaré generators only.

1. For  $a_\mu$  light-like,  $a^2 = 0$ , such cocycle twist within Hopf algebra approach exists [35]

$$\mathcal{F} = \exp \left\{ a^\alpha P^\beta \frac{\ln[1 + (aP)]}{(aP)} \otimes M_{\alpha\beta} \right\}. \quad (5.4)$$

The cocycle condition for twist  $\mathcal{F}$  (5.4) can be checked using the results by Kulish et al. [45] in the Hopf algebra setting (see also [16]).

2. For  $a_\mu$  time- and space-like such twist does not exist within Hopf algebra. Namely, starting from  $\Delta P_\mu = \mathcal{F} \Delta_0 P_\mu \mathcal{F}^{-1}$  and  $\Delta M_{\mu\nu} = \mathcal{F} \Delta_0 M_{\mu\nu} \mathcal{F}^{-1}$  one can construct an operator  $\mathcal{F} = e^f$ , where  $f = f_1 + f_2 + \dots$  is expanded in  $a_\mu$  and expressed in terms of Poincaré generators and dilatation only. In the first order we found that the result is not unique, namely we have a one parameter family of solutions

$$f_1 = a^\alpha P^\beta \otimes M_{\alpha\beta} + u(M_{\alpha\beta} \otimes a^\alpha P^\beta - a^\alpha P^\beta \otimes M_{\alpha\beta} - D \otimes (aP) + (aP) \otimes D),$$

where  $u \in \mathbb{R}$  is a free parameter. However there is one solution ( $u = 0$ ) that can be expressed in terms of Poincaré generators only. Also up to first order in  $a_\mu$  cocycle condition is satisfied and one obtains the correct classical  $r$ -matrix (see equation (65) in [25]). In the second order for  $f_2$  we found a two parameter family of solutions. Here there is no solution without including dilatation, that is the operator  $\mathcal{F}$  can not be expressed in terms of Poincaré generators only. We have checked that the corresponding quantum  $R$ -matrix obtained using  $f_1$  and  $f_2$  is correct up to the second order. The cocycle condition is no longer satisfied in the Hopf algebra approach, that is  $\mathcal{F}$  is not a twist in the Drinfeld sense. However, after using tensor exchange identities [33, 34, 57] the cocycle condition is satisfied and  $\mathcal{F}$  is a twist in Hopf algebroid approach. It also reproduces the  $\kappa$ -Poincaré–Hopf

algebra (when applied to Poincaré generators) (see [34]). In [34], we have developed a general method for calculating operator  $\mathcal{F}$  for a given coproducts of  $x_\mu$  and  $p_\mu$ . In Section 3 of [34] the operator  $\mathcal{F}$  is constructed up to the third order for *natural* realization (classical basis) and it is shown that this operator  $\mathcal{F}$  gives the correct coproduct for  $M_{\mu\nu}$  (see equation (59) in [34]) and  $R$ -matrix (see equation (61) in [34]). We also stated that this operator  $\mathcal{F}$  can not be expressed in terms of  $\kappa$ -Poincaré generators only (see [34, p. 16]). From the results for  $f_1$ ,  $f_2$  and  $f_3$  (see equations (42), (46), (49) in [34]) one can show that they could be rewritten in terms of Poincaré generators and dilatation only (after using tensor exchange identities). For alternative arguments on nonexistence of cocycle twist for  $\kappa$ -Poincaré–Hopf algebra see [12, 14].

The main point that we want to emphasize is that the twist operator exists within Hopf algebroid approach, that the cocycle condition is satisfied [33, 34, 57] and that this twist gives the full  $\kappa$ -Poincaré–Hopf algebra (when applied to the generators of Poincaré algebra).

General statements on associativity of star product, twist and cocycle condition in Hopf algebroid approach are:

1. Lorentz generators  $M_{\mu\nu}$  can be written in terms of  $x^{(\lambda)}$  and  $p^{(\lambda)}$  (4.5)–(4.8). This defines the family of basis labeled by  $\lambda$ . The momenta  $p^{(\lambda)}$  do not transform as vectors under  $M_{\mu\nu}$ . The star product is associative for all  $\lambda \in \mathbb{R}$ . The corresponding twist  $\mathcal{F}^{(\lambda)}$  given in (4.4) is Abelian and satisfies the cocycle condition for all  $\lambda \in \mathbb{R}$ . Applying  $\mathcal{F}^{(\lambda)}$  to primitive coproduct  $\Delta_0 M_{\mu\nu}$  leads to  $\kappa$ -deformed  $\mathfrak{igl}(n)$  Hopf algebra (see [17, 25, 32, 38, 41]). However, if we apply conjugation by  $\mathcal{F}^{(\lambda)}$  to  $\Delta_0^{(\lambda)} M_{\mu\nu}^{(\lambda)}$  (which is not primitive coproduct) we obtain, in the Hopf algebroid approach [57], the correct coproduct  $\Delta^{(\lambda)} M_{\mu\nu}^{(\lambda)}$  corresponding to  $\lambda$  basis (for  $\lambda = 0$  see [33]). Similarly for the antipode  $S$ . Hence, the  $\kappa$ -Poincaré–Hopf algebra can be obtained by twist  $\mathcal{F}$  in the more generalized sense, i.e. in the Hopf algebroid approach.
2. If star product is associative in one base, then it is associative in any other base obtained by similarity transformations [34].
3. If star product is associative, then the corresponding twist  $\mathcal{F}$  satisfies cocycle condition in the Hopf algebroid approach, and vice versa. Note that, there exist star products which are associative but the corresponding twist operator  $\mathcal{F}$  does not satisfy the cocycle condition in the Hopf algebra approach.

## 6 Final remarks

It is important to note that the work presented in this paper is not genuinely different from Lu’s construction of Hopf algebroid [48] and that we use a particular choice of the algebra which makes it easier to construct the coproduct as an algebra homomorphism to the subalgebra  $\hat{\mathcal{B}}/\hat{\mathcal{J}}$ . By this particular choice of algebra we are able to satisfy

$$m(1 \otimes S)\Delta = \hat{\alpha}\hat{\epsilon}, \quad m(S \otimes 1)\Delta = \hat{\beta}\hat{\epsilon}S,$$

while in [48]  $m(1 \otimes S)\Delta$  is not well defined (for the version of coproduct in [48]) because  $m(1 \otimes S)\hat{\mathcal{K}} \neq 0$ , while in our case  $m(1 \otimes S)\hat{\mathcal{J}} = 0$ . Therefore we do not need the section  $\gamma$  in the first identity for the antipode. In our approach, since we have  $\Delta : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{B}}/\hat{\mathcal{J}} = \Delta\hat{\mathcal{H}}$  and

$$m(1 \otimes S)\hat{\mathcal{J}} = 0,$$

it is easy to see that (3.8) holds  $\forall h \in \hat{\mathcal{H}}$ . We are doing this in order to explain the structure of quantum phase space, i.e. Weyl algebra  $\hat{\mathcal{H}}$ .

An axiomatic treatment of the Hopf algebroid structure on general Lie algebra type non-commutative phase spaces, involving completed tensor products, has recently been proposed in [59].

The construction of QFT suitable for  $\kappa$ -Minkowski spacetime is still under active research [19, 39, 55]. We plan to apply  $\kappa$ -deformed phase space, Hopf algebroid approach and twisting to NCQFT and NC (quantum) gravity.

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