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ABSTRACT

We present the quantum and classical mechanics formalisms for a particle with a position-dependent mass in the context of a deformed algebraic structure (named κ-algebra), motivated by the Kappa-statistics. From this structure, we obtain deformed versions of the position and momentum operators, which allow us to define a point canonical transformation that maps a particle with a constant mass in a deformed space into a particle with a position-dependent mass in the standard space. We illustrate the formalism with a particle confined in an infinite potential well and the Mathews–Lakshmanan oscillator, exhibiting uncertainty relations depending on the deformation.

I. INTRODUCTION

Minimum length scales are of crucial importance in several areas of physics, such as quantum gravity, string theory, and relativity, fundamentally due to the techniques developed for removing divergences in field theories maintaining the parameters lengths as universal constants of the theory in question (for a review, see, for instance, Ref. 1). In this sense, the seek for these minimum lengths in quantum mechanics has been translated into generalizations of the standard commutation relationship between the position and the momentum.[2] Further studies in noncommuting quantum spaces led to a Schrödinger equation with a position-dependent effective mass (PDM).[3] Along the last decades, the PDM systems have attracted attention because of their wide range of applicability in semiconductor theory,[4–7] nonlinear optics,[8] quantum liquids,[9,10] inversion potential for NH_3 in density functional theory,[11] particle physics,[12] many body theory,[13] molecular physics,[14] Wigner functions,[15] relativistic quantum mechanics,[16] superintegrable systems,[17] nuclear physics,[18] magnetic monopoles,[19,20] astrophysics,[21] nonlinear oscillations,[22–31] factorization methods and supersymmetry,[32–36] coherent states,[37–39] etc.

Complementarily, it has been found that the mathematical foundations of the PDM systems rely on the assumption of the noncommutativity between the mass operator $m(\hat{x})$ and the linear momentum operator $\hat{p}$, thus giving place to the ordering problem for the kinetic energy operator.[4,40–47] In addition, the development of generalized translation operators motivated the introduction of a position-dependent linear momentum for characterizing a particle with a PDM[7,48–56] that can be related to a generalized algebraic structure (called q-algebra[57]) inherited from the mathematical background of nonextensive statistics.[58] Concerning these formal structures, the κ-deformed statistics, originated from the κ-exponential and κ-logarithm functions, allows us to develop an algebraic structure, called κ-algebra,[59–74] with similar properties to those of the q-algebra. In particular, the κ-statistics has been employed in plasma physics,[75] astrophysics,[76] paramagnetic systems,[77] nonlinear diffusion,[78] social systems,[79] complex networks,[80] analysis of human DNA,[81] blackbody radiation,[82] quantum entanglement,[83] etc.

In this work, we employ the κ-algebra for generalizing classical and quantum mechanics with the aim of studying the properties of the resulting noncommuting space originated by the deformation. Between these properties, we found that the κ-deformed space, classical and quantum, allows us to characterize a PDM system with the mass being univocally determined by the κ-algebra. The work is organized as follows: In Sec. II, we review the properties of the κ-algebra that are used in the rest of the article. Next, we present in Sec. III the dynamics resulting from a generic PDM, and then, we specialize with the mass function $m(x)$ associated with the κ-algebra. Here, we obtain the Schrödinger equation associated with the κ-derivative, and we show that all the standard properties remain to be valid in the deformed structure such as the continuity equation, the wave-function normalization, and the classical limit. In Sec. IV, we illustrate our proposal with a particle in an infinite potential well. In Sec. V, we use the κ-deformed formalism to revisit the problem of the Mathews–Lakshmanan oscillator.[22–31] Finally, in Sec. VI, we draw some conclusions and outline future perspectives.

II. REVIEW OF THE κ-ALGEBRA

The κ-statistics emerges from a generalization of the Boltzmann–Gibbs entropy derived by means of a kinetic interaction principle that allows us to characterize nonlinear kinetics in particle systems (see, for instance, Ref. 59 for more details). In the last two decades, several theoretical developments have shown that the κ-formalism preserves features such as Legendre transform in thermodynamics,[62] H-theorem,[63] Lesche stability, and[64] composition law of the κ-entropy,[65] among others. The mathematical background of the κ-deformed formalism is based on generalizations of the standard exponential and logarithm functions, from which it is possible to introduce deformed versions of algebraic operators and calculus,[59–61] trigonometric and hyperbolic functions,[66,67] Fourier transform,[68] Gaussian law of error,[69] Stirling approximation and Gamma function,[70] Cantor set,[71] Lambert W-function,[72] information geometry,[73] and other possible exponential and logarithm functions.[74]

More specifically, the so-called κ-exponential is a deformation of the ordinary exponential function, defined by[59–61]

$$\exp_\kappa u \equiv \left(\kappa u + \sqrt{1+\kappa^2 u^2}\right)^{1/\kappa} = \exp\left(\frac{1}{\kappa}\operatorname{arcsinh}(\kappa u)\right), \quad (\kappa \in \mathbb{R}). \tag{1}$$

The inverse function of the κ-exponential is the κ-logarithm, given by

$$\ln_\kappa u \equiv \frac{u^\kappa - u^{-\kappa}}{2\kappa} = \frac{1}{\kappa}\sinh(\kappa \ln u), \quad (u > 0). \tag{2}$$

In the limit $\kappa \to 0$, the ordinary exponential and logarithmic functions are recovered, i.e., $\exp_0 x = \exp x$ and $\ln_0 x = \ln x$. These functions satisfy the properties $\exp_\kappa(a)\exp_\kappa(b) = \exp_\kappa(a \overset{\kappa}{\oplus} b)$, $\exp_\kappa(a)/\exp_\kappa(b) = \exp_\kappa(a \overset{\kappa}{\ominus} b)$, $\ln_\kappa(ab) = \ln_\kappa(a) \overset{\kappa}{\oplus} \ln_\kappa(b)$, and $\ln_\kappa(a/b) = \ln_\kappa(a) \overset{\kappa}{\ominus} \ln_\kappa(b)$, where the symbol $\overset{\kappa}{\oplus}$ represents the κ-addition operator defined by $a \overset{\kappa}{\oplus} b \equiv a\sqrt{1+\kappa^2 b^2} + b\sqrt{1+\kappa^2 a^2}$ and $\overset{\kappa}{\ominus}$ represents the κ-subtraction, $a \overset{\kappa}{\ominus} b \equiv a\sqrt{1+\kappa^2 b^2} - b\sqrt{1+\kappa^2 a^2}$.[59,66]

A κ-deformed calculus has been introduced in Ref. 59 from the deformed differential

$$d_\kappa u \equiv \lim_{u' \to u} u' \overset{\kappa}{\ominus} u = \frac{du}{\sqrt{1+\kappa^2 u^2}} + \mathcal{O}((du)^2). \tag{3}$$

The definition of a deformed variable u_κ (also named deformed κ-number) is

$$u_\kappa \equiv \frac{1}{\kappa}\operatorname{arcsinh}(\kappa u) = \ln[\exp_\kappa(u)], \tag{4}$$

which implies $d_\kappa u = du_\kappa$, i.e., the *deformed differential* of an ordinary variable u can be rewritten as with the ordinary differential of a *deformed variable* u_κ. In this way, one defines the κ-derivative operator by

$$D_\kappa f(u) \equiv \lim_{u' \to u} \frac{f(u') - f(u)}{u' \overset{\kappa}{\ominus} u} = \sqrt{1+\kappa^2 u^2}\frac{df(u)}{du}, \tag{5}$$

with the κ-exponential being an eigenfunction of D_κ, $D_\kappa \exp_\kappa u = \exp_\kappa u$. Similarly, the dual κ-derivative is defined by

$$\widetilde{D}_\kappa f(u) \equiv \lim_{u' \to u} \frac{f(u') \overset{\kappa}{\ominus} f(u)}{u' - u} = \frac{1}{\sqrt{1 + \kappa^2 [f(u)]^2}} \frac{df(u)}{du}, \tag{6}$$

which satisfies $\widetilde{D}_\kappa \ln_\kappa u = 1/u$. These operators obey $\widetilde{D}_\kappa x(y) = [D_\kappa y(x)]^{-1}$. In particular, we have $D_\kappa u = (\widetilde{D}_\kappa u)^{-1} = \sqrt{1 + \kappa^2 u^2}$. From Eqs. (5) and (6), we see that the standard derivative is recovered as $\kappa \to 0$. The deformed derivative operator (5) can be seen as the variation of the function $f(u)$ with respect to a nonlinear variation of the independent variable u, i.e., $D_\kappa f(u) = df(u)/du_\kappa$. On the other hand, the dual deformed derivative operator (6) is the rate of change of a nonlinear variation of the function $f(u)$ with respect to the standard variation of the independent variable u, $\widetilde{D}_\kappa f(u) = d_\kappa f(u)/du$. The deformed second derivatives satisfy

$$D_\kappa^2 f(u) = \sqrt{1 + \kappa^2 u^2} \frac{d}{du} \left[\sqrt{1 + \kappa^2 u^2} \frac{df}{du} \right] \tag{7}$$

and

$$\widetilde{D}_\kappa^2 f(u) = \frac{1}{\sqrt{1 + \kappa^2 [f(u)]^2}} \frac{d}{du} \left\{ \frac{1}{\sqrt{1 + \kappa^2 [f(u)]^2}} \frac{df}{du} \right\}. \tag{8}$$

These rules can be extended to deformed derivatives of higher order.

III. κ-DEFORMED DYNAMICS OF A SYSTEM WITH POSITION-DEPENDENT MASS

A. κ-Deformed classical formalism

Let us first consider the problem of a particle with a position-dependent mass (PDM) $m(x)$ in 1D for the classical formalism. The Hamiltonian of the system is

$$\mathcal{H}(x, p) = \frac{p^2}{2m(x)} + V(x), \tag{9}$$

whose the linear momentum is $p = m(x)\dot{x}$, which leads to the equation of motion

$$m(x)\ddot{x} + \frac{1}{2} m'(x)\dot{x}^2 = F(x), \tag{10}$$

with $F(x) = -dV/dx$ being the force acting on the particle, where $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, and $m'(x) = dm/dx$ give velocity, acceleration, and mass gradient, respectively. The point canonical transformation (PCT)

$$\eta = \int^x \sqrt{\frac{m(y)}{m_0}} dy \quad \text{and} \quad \Pi = \sqrt{\frac{m_0}{m(x)}} p \tag{11}$$

maps the Hamiltonian (9) of a particle with the PDM $m(x)$ in the usual phase space (x, p) into another Hamiltonian of a particle with a constant mass m_0 represented in the deformed phase space (η, Π),

$$\mathcal{K}(\eta, \Pi) = \frac{1}{2m_0} \Pi^2 + U(\eta), \tag{12}$$

where $U(\eta) = V(x(\eta))$ is the potential expressed in the deformed space-coordinate η. When $m(x) = m_0$, both representations coincide.

Let us consider, in particular, the mass function

$$m(x) = \frac{m_0}{1 + \kappa^2 x^2}, \tag{13}$$

where the parameter κ has units of inverse length and controls the dependence of the mass with position, where $\kappa = 0$ corresponds to the standard case. Thus, the equation of motion (10) becomes

$$m_0 \left[\frac{\ddot{x}}{(1 + \kappa^2 x^2)} - \frac{\kappa^2 x \dot{x}^2}{(1 + \kappa^2 x^2)^2} \right] = F(x). \tag{14}$$

This equation can be compactly rewritten in the form of a deformed Newton's second law,

$$m_0 \widetilde{D}_\kappa^2 x(t) = F(x). \tag{15}$$

Moreover, for the mass function (13), the κ-deformed spatial coordinate and its conjugated linear momentum are

$$\eta = \frac{1}{\kappa}\text{arcsinh}(\kappa x) \equiv x_\kappa \tag{16a}$$

and

$$\Pi = \sqrt{1+\kappa^2 x^2}\, p \equiv \Pi_\kappa, \tag{16b}$$

with Poisson brackets $\{x_\kappa, \Pi_\kappa\}_{x,p} = 1$. The deformed displacement $d_\kappa x$ of a particle with the non-constant mass $m(x)$, given in Eq. (13), is mapped into the usual displacement dx_κ in a deformed space x_κ, provided with a constant mass m_0: $d_\kappa x \equiv (x+dx) \overset{\kappa}{\ominus} x = dx/\sqrt{1+\kappa^2 x^2}$, up to first order. The time evolution of the system is governed by the dual derivative, i.e. $\widetilde{D}_\kappa x(t) = \dot{x}/\sqrt{1+\kappa^2 x^2}$.

B. κ-Deformed quantum formalism

In the quantization of a PDM system, an ordering ambiguity arises for defining the kinetic energy operator in terms of the mass operator $m(\hat{x})$ and the linear momentum $\hat{p}$. There are several ways to define a Hermitian kinetic energy operator, and a general two-parameter form is given by

$$\hat{T} = \frac{1}{4}\Big\{[m(\hat{x})]^{-\alpha}\hat{p}[m(\hat{x})]^{-1+\alpha+\beta}\hat{p}[m(\hat{x})]^{-\beta} + [m(\hat{x})]^{-\beta}\hat{p}[m(\hat{x})]^{-1+\alpha+\beta}\hat{p}[m(\hat{x})]^{-\alpha}\Big\}. \tag{17}$$

For more details, see the discussions, for instance, of von Roos,[4] Lévy-Leblond,[40] and others. Among many particular cases in the literature, we point out the proposals by Ben Daniel and Duke ($\alpha = \beta = 0$),[41] Gora and Williams ($\alpha = 1, \beta = 0$),[42] Zhu and Kroemer ($\alpha = \beta = \frac{1}{2}$),[43] and Li and Kuhn ($\alpha = \frac{1}{2}, \beta = 0$).[44] Morrow and Brownstein[45] showed that only the case $\alpha = \beta$ satisfies the conditions of continuity of the wave-function at the boundaries of a heterojunction in crystals. In particular, Mustafa and Mazharimousavi[46] showed that the case $\alpha = \beta = \frac{1}{4}$ allows the mapping of a quantum Hamiltonian with a PDM into a Hamiltonian with a constant mass by means a PCT. More precisely, considering the quantum Hamiltonian

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2}[m(\hat{x})]^{-\frac{1}{4}}\hat{p}[m(\hat{x})]^{-\frac{1}{2}}\hat{p}[m(\hat{x})]^{-\frac{1}{4}} + V(\hat{x}), \tag{18}$$

the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$ in the position representation $\{|\hat{x}\rangle\}$ reads

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m_0}\sqrt[4]{\frac{m_0}{m(x)}}\frac{\partial}{\partial x}\sqrt{\frac{m_0}{m(x)}}\frac{\partial}{\partial x}\sqrt[4]{\frac{m_0}{m(x)}} + V(x)\right)\Psi(x,t), \tag{19}$$

with $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$ and E being the eigenvalue corresponding to the eigenfunction $\psi(x)$ of $\hat{H}$. It is straightforwardly verified that the probability density $\rho(x,t) \equiv |\Psi(x,t)|^2$ satisfies the continuity equation

$$\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}, \tag{20}$$

where the probability current is

$$J(x,t) \equiv \text{Re}\left\{\Psi^*(x,t)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left[\frac{1}{m(x)}\Psi(x,t)\right]\right\}. \tag{21}$$

Equation (19) can be conveniently rewritten by means of the transformation $\Psi(x,t) = \sqrt[4]{m(x)/m_0}\,\Phi(x,t)$ as

$$i\hbar\frac{\partial\Phi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m_0}\left(\sqrt{\frac{m_0}{m(x)}}\frac{\partial}{\partial x}\right)^2 + V(x)\right]\Phi(x,t). \tag{22}$$

Let us consider, in particular, the mass function (13). The modified wave-function $\Phi(x,t) = \sqrt[4]{1+\kappa^2x^2}\Psi(x,t)$ obeys a *κ-deformed Schrödinger wave-equation*

$$i\hbar\frac{\partial\Phi(x,t)}{\partial t} = -\frac{\hbar^2}{2m_0}\mathcal{D}_\kappa^2\Phi(x,t) + V(x)\Phi(x,t), \tag{23}$$

with $\mathcal{D}_\kappa = \sqrt{1+\kappa^2x^2}\partial_x$, which is the analog of the κ-derivative operator (5). Using Eq. (8), we obtain

$$i\hbar\frac{\partial\Phi(x,t)}{\partial t} = -\frac{\hbar^2(1+\kappa^2x^2)}{2m_0}\frac{\partial^2\Phi(x,t)}{\partial x^2} - \frac{\hbar^2\kappa^2 x}{2m_0}\frac{\partial\Phi(x,t)}{\partial x} + V(x)\Phi(x,t). \tag{24}$$

Equation (23) is indeed equivalent to a Schrödinger-like equation for $\Phi(x,t)$ with the non-Hermitian Hamiltonian operator

$$\hat{H}_\kappa = \frac{1}{2m_0}\hat{p}_\kappa^2 + V(\hat{x}), \tag{25}$$

where $\hat{p}_\kappa \equiv -i\hbar\mathcal{D}_\kappa = \sqrt{1+\kappa^2\hat{x}^2}\hat{p}$ stands for a κ-deformed non-Hermitian momentum operator and obeys the commutation relation

$$[\hat{x},\hat{p}_\kappa] = i\hbar\sqrt{\hat{1}+\kappa^2\hat{x}^2}. \tag{26}$$

This leads to the generalized uncertainty principle $\Delta x\Delta p_\kappa \geq \frac{\hbar}{2}\langle\sqrt{1+\kappa\hat{x}^2}\rangle$. We notice that if the standard wave-function $\Psi(x,t)$ is normalized, then $\Phi(x,t)$ is normalized under a κ-deformed integral. Indeed, we have

$$\int_{x_i}^{x_f}\Psi^*(x,t)\Psi(x,t)dx = \int_{x_i}^{x_f}\Phi^*(x,t)\Phi(x,t)d_\kappa x = 1. \tag{27}$$

Besides, we obtain the κ-deformed continuity equation

$$\frac{\partial\varrho(x,t)}{\partial t} + \mathcal{D}_\kappa\mathcal{J}(x,t) = 0, \tag{28}$$

with $\varrho(x,t) = |\Phi(x,t)|^2$ and

$$\mathcal{J}(x,t) \equiv \mathrm{Re}\left\{\Phi^*(x,t)\left(\frac{\hbar}{i}\mathcal{D}_\kappa\right)\left[\frac{\Phi(x,t)}{m_0}\right]\right\}. \tag{29}$$

It is worth noting that there is an equivalence between the Schrödinger equation for the Hermitian system (18) with the mass function $m(x)$ given by (13) and the non-Hermitian one (25) expressed in terms of a κ-deformed momentum operator, where $\Psi(x,t)$ must be replaced by $\Phi(x,t) = \sqrt[4]{1+\kappa^2x^2}\Psi(x,t)$. Moreover, we see that in the description of quantum systems with the mass function (13) in terms of the modified wave-function $\Phi(x,t)$, the usual derivative and integral with respect to the variable x are replaced by their corresponding κ-deformed versions. Analogous features apply in the classical formalism, with the motion equation expressed in terms of the dual κ-derivative [see Eq. (15)].

Using the change of variable $x \to x_\kappa = \ln[\exp_\kappa(x)]$ [see Eq. (4)], Eq. (23) can be rewritten in the κ-deformed space as

$$i\hbar\frac{\partial\Lambda(x_\kappa,t)}{\partial t} = -\frac{\hbar^2}{2m_0}\frac{\partial^2\Lambda(x_\kappa,t)}{\partial x_\kappa^2} + U(x_\kappa)\Lambda(x_\kappa,t), \tag{30}$$

with $\Lambda(x_\kappa,t) = \Phi(x(x_\kappa),t)$ and $U(x_\kappa) = V(x(x_\kappa))$ being a modified potential in terms of the original one V and the inverse transformation $x = x(x_\kappa)$, respectively. Therefore, the wave-equation for $\Psi(x,t)$ of a system with a PDM (13) with the potential $V(x)$ in the standard space $\{|\hat{x}\rangle\}$ is mapped into an equation for $\Lambda(x_\kappa,t)$ with the potential $U(x_\kappa) = V(x(x_\kappa))$ in the deformed space $\{|\hat{x}_\kappa\rangle\}$. The quantum Hamiltonian associated with the Schrödinger wave-equation (30) is $\hat{K}(\hat{x}_\kappa,\hat{\Pi}_\kappa) = \frac{1}{2m_0}\hat{\Pi}_\kappa^2 + U(\hat{x}_\kappa)$, which can be obtained by applying the point canonical transformation $(\hat{x},\hat{p}) \to (\hat{x}_\kappa,\hat{\Pi}_\kappa)$ on the quantum Hamiltonian (18), where

$$\hat{x}_\kappa = \frac{1}{\kappa}\mathrm{arcsinh}(\kappa\hat{x}), \tag{31a}$$

$$\hat{\Pi}_\kappa = \sqrt[4]{1+\kappa^2\hat{x}^2}\,\hat{p}\,\sqrt[4]{1+\kappa^2\hat{x}^2} = \frac{1}{2}(\hat{p}_\kappa^\dagger + \hat{p}_\kappa), \tag{31b}$$

with $[\hat{x}_\kappa, \hat{\Pi}_\kappa] = i\hbar\hat{1}$. In addition, we have that $\hat{\Pi}_\kappa$ is in accordance with the definition of a PDM pseudo-momentum operator introduced in Ref. 46. Thus, the dynamical variables (11) are the classical counterparts of the Hermitian operators (31).

From the eigenvalue equation $\hat{\Pi}_\kappa|k\rangle = \hbar k|k\rangle$, the eigenfunctions in the representation $\{|\hat{x}\rangle\}$ result in

$$\psi_k(x) = \frac{C}{\sqrt[4]{1+\kappa^2x^2}}[\exp_\kappa(x)]^{ik} = \frac{C}{\sqrt[4]{1+\kappa^2x^2}}\exp\left[\frac{ik}{\kappa}\operatorname{arcsinh}(\kappa x)\right], \tag{32}$$

where C is a constant. As in the non deformed case ($\kappa = 0$), the function $\psi_k(x)$ is not normalizable. Even though, a deformed wave-packet can be defined from the κ-deformed Fourier transform[68]

$$\psi(x) = \frac{1}{\sqrt[4]{1+\kappa^2x^2}}\int_{-\infty}^{+\infty} g(k)e^{\frac{ik}{\kappa}\operatorname{arcsinh}(\kappa x)}dk, \tag{33}$$

where $g(k)$ is the distribution function of the wave-vectors k. It is verified straightforwardly that the corresponding wave-packet of the operator $\hat{p}_\kappa$ is $\varphi(x) = \int_{-\infty}^{+\infty} g(k)[\exp_\kappa(x)]^{ik}dk$. The wave-packet in the representation of the deformed space is $\phi(x_\kappa) = \varphi(x(x_\kappa)) = \sqrt[4]{1+\kappa^2x^2}\,\psi(x(x_\kappa)) = \int_{-\infty}^{+\infty} g(k)e^{ikx_\kappa}dk$. From the Plancherel theorem, we have

$$\begin{aligned} g(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\phi(x_\kappa)e^{-ikx_\kappa}dx_\kappa \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi(x)[\exp_\kappa(x)]^{-ik}d_\kappa x \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\psi(x)}{\sqrt[4]{1+\kappa^2x^2}}e^{-\frac{ik}{\kappa}\operatorname{arcsinh}(\kappa x)}dx. \end{aligned} \tag{34}$$

IV. PARTICLE IN AN INFINITE POTENTIAL WELL

In Secs. IV and V, we illustrate the quantum and classical κ-deformed formalism with two paradigmatic examples.

A. Classical case

First, we consider the problem of a particle confined in an infinite square potential well between $x = 0$ and $x = L$. If $\mathcal{H}(x,p) = E$ is the energy of the classical particle, then the linear momentum is $p(x) = \pm\sqrt{2m_0E/(1+\kappa^2x^2)}$ and the velocity is $v(x) = \pm v_0\sqrt{1+\kappa^2x^2}$, with $v_0 = \sqrt{2E/m_0}$. For $v(0) = v_0$ and $0 < x < L$, the position as a function of time is $x(t) = \ln_\kappa[\exp(v_0 t)]$. Hence, the classical probability density $\rho_{\text{classic}}(x)dx \propto dx/v$ to find the particle within the interval $[x, x+dx]$ is

$$\rho_{\text{classic}}(x)dx = \frac{\kappa}{\ln\left(\kappa L + \sqrt{1+\kappa^2L^2}\right)}\frac{dx}{\sqrt{1+\kappa^2x^2}}, \tag{35}$$

from which the uniform distribution $\rho_{\text{classic}}(x) = 1/L$ is recovered when $\kappa \to 0$. The first and the second moments of the position and the linear momentum for the classical distribution (35) are

$$\frac{\overline{x}}{L} = \frac{\sqrt{1+\kappa^2L^2}-1}{\kappa L\,\ln\left(\kappa L + \sqrt{1+\kappa^2L^2}\right)}, \tag{36a}$$

$$\frac{\overline{x^2}}{L^2} = \frac{1}{2\kappa^2L^2}\left[\frac{\kappa L\sqrt{1+\kappa^2L^2}}{\ln\left(\kappa L + \sqrt{1+\kappa^2L^2}\right)} - 1\right], \tag{36b}$$

$$\overline{p} = 0, \tag{36c}$$

$$\overline{p^2} = 2m_0E\left[\frac{\kappa L}{\sqrt{1+\kappa^2L^2}\,\ln\left(\kappa L + \sqrt{1+\kappa^2L^2}\right)}\right]. \tag{36d}$$

We can verify that $\lim_{\kappa\to 0}\overline{x} = L/2$, $\lim_{\kappa\to 0}\overline{x^2} = L^2/3$, and $\lim_{\kappa\to 0}\overline{p^2} = 2m_0 E$. From the change of variable $x \to x_\kappa$, the PDM particle confined in an interval $[0, L]$ is mapped into a particle with a constant mass in $[0, L_\kappa]$, where $L_\kappa = \operatorname{arcsinh}(\kappa L)/\kappa$ corresponds to the length of the box in the deformed space.

B. Quantum case

Let us now analyze the problem in the κ-deformed quantum formalism. Considering $\Phi(x,t) = \varphi(x)e^{-iEt/\hbar}$, this leads to the time-independent Schrödinger-like equation $-\frac{\hbar^2}{2m_0}D_\kappa^2\varphi(x) = E\varphi(x)$, whose eigenfunctions are given by

$$\varphi_n(x) = \sqrt[4]{1+\kappa^2x^2}\,\psi_n(x) = C_\kappa \sin\left[\frac{k_{\kappa,n}}{\kappa}\operatorname{arcsinh}(\kappa x)\right] \tag{37}$$

for $0 \le x \le L$, and $\varphi_n(x) = 0$ elsewhere, with $C_\kappa^2 = 2/L_\kappa$ and $k_{\kappa,n} = n\pi/L_\kappa$, where n is an integer number and $L_\kappa = \kappa^{-1}\operatorname{arcsinh}(\kappa L)$. The energy levels corresponding to these eigenfunctions are

$$E_n = \frac{\hbar^2\pi^2n^2\kappa^2}{2m_0\operatorname{arcsinh}^2(\kappa L)} = \varepsilon_0\left[\frac{\kappa L}{\operatorname{arcsinh}(\kappa L)}\right]^2 n^2, \tag{38}$$

with $\varepsilon_0 = \hbar^2\pi^2/(2m_0L^2)$. The effect of the deformation parameter κ corresponds to a contraction of the space ($L_\kappa < L$ for $\kappa \ne 0$), and consequently, this leads to an increase in the energy levels of the particle. In Fig. 1, we illustrate the energy levels of the particle as a function of the quantum number for different values of κ.

The probability densities of the stationary states in the position space are

$$\rho_n(x) = |\psi_n(x)|^2 = \frac{2\kappa}{\operatorname{arcsinh}(\kappa L)}\frac{1}{\sqrt{1+\kappa^2x^2}}\sin^2\left[\frac{k_{\kappa,n}}{\kappa}\operatorname{arcsinh}(\kappa x)\right]. \tag{39}$$

Substituting Eq. (37) into the inverse Fourier transform (34), we obtain the eigenfunctions for the particle confined in a box in momentum space k,

$$g_n(k) = n\sqrt{\frac{L_\kappa}{2}}\left[\frac{1+(-1)^{n+1}e_\kappa^{-ikL}}{(kL_\kappa)^2-(n\pi)^2}\right]. \tag{40}$$

Consequently, its associated probability density results in

$$\gamma_n(k) = |g_n(k)|^2 = n^2L_\kappa\frac{1-\cos(n\pi)\cos(kL_\kappa)}{[(kL_\kappa)^2-(n\pi)^2]^2}. \tag{41}$$

Interestingly, the eigenfunctions (40) and the probability densities (41) have the same form as in the case of a particle with a constant mass but with L_κ instead of L. In Fig. 2, we plot the eigenfunctions $\psi_n(x)$ and their probability densities in the coordinate and momentum spaces,

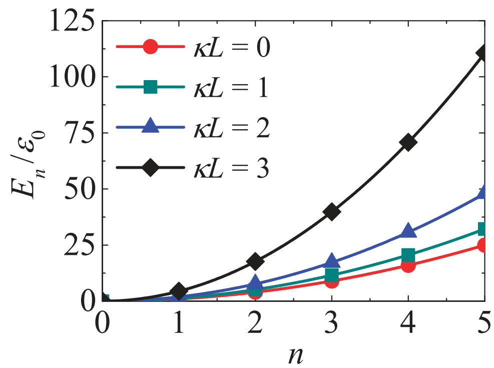

FIG. 1. Energy levels of a particle with the PDM $m(x) = m_0/(1+\kappa^2x^2)$ in an infinite square well of size L, for different quantum numbers n and values of κ, given in terms of the nondeformed fundamental energy $\varepsilon_0 = \frac{\hbar^2\pi^2}{2m_0L^2}$. The values of the energies are discrete, and the solid lines help in guiding the eyes.

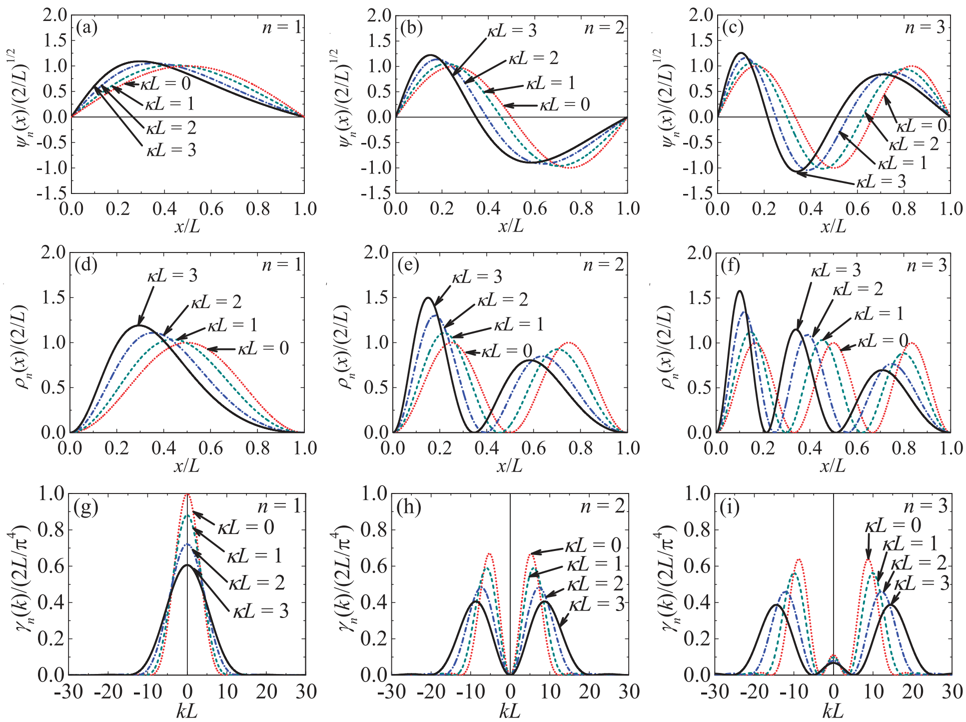

FIG. 2. Eigenfunctions $\psi_n(x)$ [(a)–(c)], probability densities $\rho_n(x) = |\psi_n(x)|^2$ [(d)–(f)], and $\gamma_n(k) = |g_n(k)|^2$ [(g)–(i)] for a particle with the PDM $m(x) = m_0/(1+\kappa^2x^2)$ and confined in an infinite square well for different parameters κL (the usual case, $\kappa L = 0$, is shown for comparison). [(a), (d), and (g)] $n = 1$ (ground state), [(b), (e), and (h)] $n = 2$ (first excited state), and [(c), (f), and (i)] $n = 3$ (second excited state).

$\rho_n(x)$ and $\gamma_n(k)$, for the three states of lower energy and for some values of the deformation parameter κ. We can see that as κ increases, $\rho_n(x)$ becomes more asymmetric and $\gamma_n(k)$ spread more along its domain. In Fig. 3, we show that the average value of the quantum probability density $\rho_n(x)$ approaches to the classical probability density $\rho_{\text{classic}}(x)$ (illustrated here for $n = 20$) in accordance with the correspondence principle. The distribution $\gamma_n(k)$ is also shown for the same state $n = 20$.

The eigenfunctions (37) constitute an orthonormal set of functions that obey the inner product $\int_0^L \varphi_n(x)\varphi_{n'}(x)d_\kappa x = \delta_{n,n'}$ so that any continuous function in the interval $[0, L]$ can be written as a linear combination

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left[n\pi \frac{\operatorname{arcsinh}(\kappa x)}{\operatorname{arcsinh}(\kappa L)}\right], \tag{42}$$

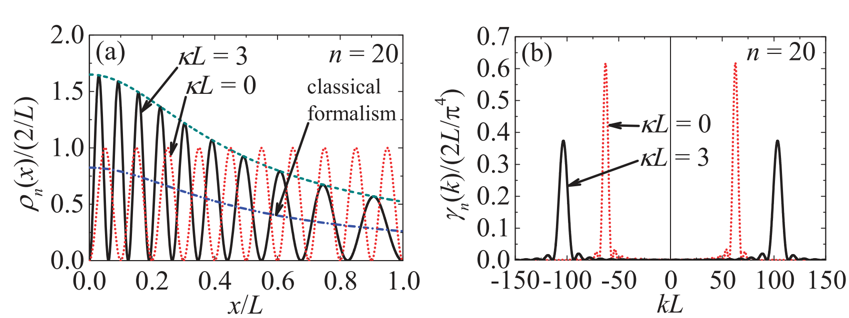

FIG. 3. Probability densities (a) $\rho_n(x) = |\psi_n(x)|^2$ and (b) $\gamma_n(k) = |g_n(k)|^2$ of a particle with the PDM confined in an infinite square well for $\kappa L = 3.0$ and for the eigenstate $n = 20$. In panel (a), the classical distribution [Eq. (35)] is shown for comparison, and the dotted upper line is $2\kappa L/[\operatorname{arcsinh}(\kappa x)\sqrt{1+\kappa^2x^2}]$.

with the coefficients c_n of the series given by

$$c_n = \frac{2\kappa}{\operatorname{arcsinh}(\kappa L)} \int_0^L f(x) \sin\left[n\pi \frac{\operatorname{arcsinh}(\kappa x)}{\operatorname{arcsinh}(\kappa L)}\right] d_\kappa x. \tag{43}$$

Concerning the Sturm–Liouville problem, Braga and Costa Filho[54] introduced a Fourier series in terms of deformed trigonometric functions that emerge from the formalism studied in Ref. 48. Likewise, we have that the κ-deformed Fourier series (42) has the same structure like the one proposed by Scarfone in Ref. 66, considering the κ-deformed mathematics. For the particular case $f(x) = 1$, we have $f(x) = \lim_{N\to\infty} f_N(x)$ with

$$f_N(x) = \frac{4}{\pi} \sum_{l=0}^{N} \frac{1}{2l+1} \sin\left[(2l+1)\pi \frac{\operatorname{arcsinh}(\kappa x)}{\operatorname{arcsinh}(\kappa L)}\right]. \tag{44}$$

Similarly, as was done in Ref. 54, we consider as a quantitative measure of the error the function defined by $R(N) = \int_0^L [f(x) - f_N(x)]^2 d_\kappa x$. In Fig. 4, we show that when N becomes large, the partial sum $f_N(x)$ converges to $f(x) = 1$, as well as $R(N)$ goes to zero.

Expected values of $\hat{x}$ and $\hat{\Pi}_\kappa$ for stationary states can be obtained from usual internal products of the eigenfunctions $\psi_n(x)$ or, equivalently, from the deformed internal products of the modified eigenfunctions $\varphi(x)$, i.e., $\langle \hat{x}^l \rangle = \int \psi_n^*(x)\hat{x}^l \psi_n(x)dx = \int \varphi_n^*(x)\hat{x}^l \varphi_n(x)d_\kappa x$ and $\langle \hat{\Pi}_\kappa^l \rangle = \int \psi_n^*(x)\hat{\Pi}_\kappa^l \psi_n(x)dx = \int \varphi_n^*(x)\hat{p}_\kappa^l \varphi_n(x)d_\kappa x$,, in which l is a positive integer. The expectation values $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p} \rangle$, and $\langle \hat{p}^2 \rangle$ for the eigenstates of the particle in a one dimensional infinite potential well are, respectively,

$$\frac{\langle \hat{x} \rangle}{L} = \frac{(\sqrt{1+\kappa^2L^2} - 1)(2\pi n)^2}{\kappa L \ln\left(\kappa L + \sqrt{1+\kappa^2L^2}\right)\left[\ln^2\left(\kappa L + \sqrt{1+\kappa^2L^2}\right) + (2\pi n)^2\right]}, \tag{45a}$$

$$\frac{\langle \hat{x}^2 \rangle}{L^2} = \frac{1}{2\kappa^2L^2}\left\{\frac{\kappa L\sqrt{1+\kappa^2L^2}(n\pi)^2}{\ln\left(\kappa L + \sqrt{1+\kappa^2L^2}\right)\left[\ln^2\left(\kappa L + \sqrt{1+\kappa^2L^2}\right) + (n\pi)^2\right]} - 1\right\}, \tag{45b}$$

$$\langle \hat{p} \rangle = 0, \tag{45c}$$

$$\langle \hat{p}^2 \rangle = \hbar^2\left[k_{\kappa,n}^2 \mathcal{I}_{1,0}(1) + \kappa^2\left(\frac{1}{2}\mathcal{I}_{1,0}(1) - \frac{5}{4}\mathcal{I}_{1,1}(1) - \mathcal{I}_{3,0}(1) + 5\mathcal{I}_{3,1}(1)\right)\right], \tag{45d}$$

with $\mathcal{I}_{j,l}(z) = 2\int_0^z \operatorname{sech}^{2j}(\lambda_\kappa u)\tanh^{2l}(\lambda_\kappa u)\sin^2(n\pi u)du$ and $\lambda_\kappa = \kappa L_\kappa$. The analytical form of the functions $\mathcal{I}_{j,l}(z)$ is expressed by means of the Appell hypergeometric function of two variables (http://functions.wolfram.com/ElementaryFunctions/Sech/21/01/14/01/10/01/0001/), and due to its complicated expression, it becomes convenient to write the expectation value (45d) in terms of $\mathcal{I}_{j,l}(z)$.

We can see that in the limit $n \to \infty$, Eq. (45) coincides with Eq. (36), which expresses the consistency of the classical limit. We can also verify that in the limit $\kappa \to 0$, we recover the usual results $\langle \hat{x} \rangle \to \frac{L}{2}$, $\langle \hat{x}^2 \rangle \to \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2}$, and $\langle \hat{p}^2 \rangle \to \hbar^2 k_n^2$ with $E_n = \hbar^2 k_n^2/2m_0$ ($k_n \equiv k_{0,n} = n\pi/L$). It is straightforward to verify that the expectation values of the pseudo-momentum satisfy

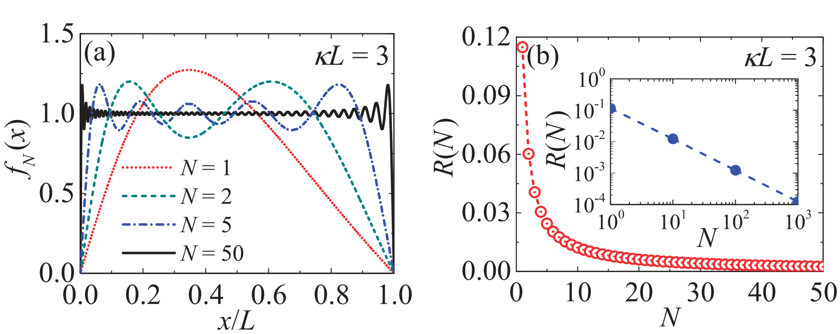

FIG. 4. (a) Partial sum $f_N(x)$ (κ-deformed Fourier series) [Eq. (44)] for $N = 1, 2, 5$, and 50. (b) Mean square error of the approximation $R(N)$ for the range from $N = 1$–50 (in the log–log graph inset: $N = 1$–10^3).

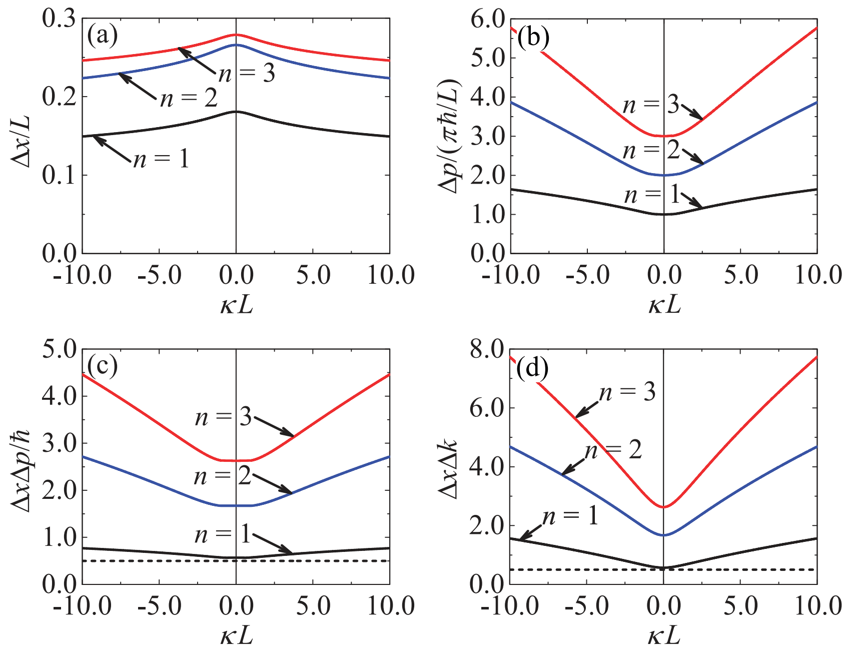

FIG. 5. Uncertainty in the function of κL of (a) the position Δx and (b) the momentum Δp along with the uncertainty relations [(c) and (d)] $\Delta x\Delta p$ and $\Delta x\Delta k$ for a particle with a PDM confined in a box for the ground state and the first two excited ones.

$$\langle\hat{\Pi}_\kappa\rangle = \hbar\langle k\rangle = 0, \tag{46a}$$

$$\langle\hat{\Pi}_\kappa^2\rangle = \hbar\langle k^2\rangle = \left(\frac{n\pi\hbar}{L_\kappa}\right)^2, \tag{46b}$$

with $\langle\hat{\Pi}_\kappa^2\rangle$ and $\langle\hat{p}^2\rangle$ different for $\kappa \neq 0$. In Fig. 5, we plot the uncertainty relation for different values of κ. Once the operators $\hat{x}$ and $\hat{p}$ are Hermitian and canonically conjugated, the uncertainty relation is satisfied for different values of κ, i.e., $\Delta x\Delta p \geq \frac{\hbar}{2}$. We can also see that the position and wave-vector satisfy the uncertainty relation $\Delta x\Delta k \geq \frac{1}{2}$. In Figs. 5(c) and 5(d), the minimum of the uncertainty relation is attained for $\kappa = 0$. Similar features have been observed in other system provided with a PDM. In Ref. 56, the Cramér–Rao, Fisher–Shannon, and LópezRuiz–Mancini–Calbet (LMC) complexities have been investigated for the problem of a particle with a PDM and confined in an infinite potential well within the framework of the q-algebra. In the context of these complexities, the conjugated variables exhibit a behavior similar to the standard Heisenberg uncertainty principle. For different states, the uncertainty relation associated with the Cramér–Rao, Fisher–Shannon, and LMC complexities exhibits a minimum lower bound when the mass of the particle is constant (i.e., with a null space deformation). This result is expectedly reasonable since the q-exponential[58] and the κ-exponential functions present a similar behavior when their deformation parameters recover the standard exponential ($q \to 1$ and $\kappa \to 0$).

V. κ-DEFORMED OSCILLATOR WITH POSITION-DEPENDENT MASS

A. κ-Deformed classical oscillator

Now, we consider a particle with the position-dependent mass (13) subjected to the potential $V(x) = \frac{1}{2}m(x)\omega_0^2x^2$. This problem is known as the Mathews–Lakshmanan oscillator,[22] where the classical Hamiltonian is given by

$$\mathcal{H}(x,p) = \frac{(1+\kappa^2x^2)p^2}{2m_0} + \frac{m_0\omega_0^2x^2}{2(1+\kappa^2x^2)}. \tag{47}$$

The deformed second Newton's law (15) for this oscillator becomes

$$\widetilde{D}_\kappa^2x(t) = -\frac{\omega_0^2x}{(1+\kappa^2x^2)^2}, \tag{48}$$

or more explicitly,

$$(1 + \kappa^2 x^2)\ddot{x} + \omega_0^2 x - \kappa^2 \dot{x}^2 x = 0. \tag{49}$$

The solution of Eq. (48) [or equivalently (49)] is

$$x(t) = A_\kappa \cos(\Omega_\kappa t + \delta_0), \tag{50}$$

with $A_\kappa = A_0/\sqrt{1-\kappa^2 A_0^2}$ being the amplitude of the oscillation, $\Omega_\kappa = \omega_0\sqrt{1-\kappa^2 A_0^2}$ being the angular frequency, and $A_0^2 = 2E/m_0\omega_0^2$. The potential of this oscillator has a finite well depth $W_\kappa = m_0\omega_0^2/2\kappa^2$. Since $E/W_\kappa = \kappa^2 A_0^2$, the oscillator has a closed (open) path in the phase space for $0 < \kappa^2 A_0^2 < 1$ ($\kappa^2 A_0^2 > 1$), according to Ref. 22. The point canonical transformation given in Eqs. (16) maps the Hamiltonian (47) into that of an anharmonic oscillator of the form

$$\mathcal{K}(x_\kappa, \Pi_\kappa) = \frac{1}{2m_0}\Pi_\kappa^2 + W_\kappa \tanh^2(\kappa x_\kappa), \tag{51}$$

with κ being a continuous parameter that controls the anharmonicity of the potential. In Fig. 6, we plot the phase spaces (x, p) and (x_κ, Π_κ) for different values of κA_0. The bounded motion in the interval $-A_\kappa < x < A_\kappa$ of the standard space turns out into the interval $-x_{\kappa,\max} < x_\kappa < x_{\kappa,\max} = \kappa^{-1}\operatorname{atanh}(\kappa A_0)$ in the deformed space. Besides, the unbounded motion has the interval of the linear momentum $0 < |p| < m_0\omega_0 A_0$ turned into $m_0\omega_0 A_0\sqrt{1-\frac{1}{\kappa^2 A_0^2}} < |\Pi_\kappa| < m_0\omega_0 A_0$. As the dimensionless parameter κA_0 increases from 0 to 1.1 within the interval $[0.9, 1.1]$, it is observed that the horizontal axes of the ellipses become infinite, thus giving place to an unbounded motion.

By means of the WKB approximation, we can obtain the energy levels of the corresponding quantum system. Using this method, we have

$$\begin{aligned}\left(n+\frac{1}{2}\right)\frac{\hbar}{2} &= \frac{1}{2\pi}\int_{-A_\kappa}^{A_\kappa} p(x)dx = \frac{m_0\Omega_\kappa}{2\pi}\int_{-A_\kappa}^{A_\kappa}\frac{\sqrt{A_\kappa^2 - x^2}}{1+\kappa^2 x^2}dx\\ &= \frac{m_0\Omega_\kappa A_\kappa^2}{4\pi}\int_0^{2\pi}\frac{\sin^2\theta_\kappa}{1+\kappa^2 A_\kappa^2\cos^2\theta_\kappa}d\theta_\kappa\\ &= \frac{m_0\Omega_\kappa}{2\kappa^2}\left(\sqrt{1+\kappa^2 A_\kappa^2}-1\right)\end{aligned} \tag{52}$$

with $n = 0, 1, 2, \ldots$. Since $E = \frac{1}{2}m_0\Omega_\kappa^2 A_\kappa^2$, we obtain

$$E_n = \hbar\omega_0\left(n+\frac{1}{2}\right) - \frac{\hbar^2\kappa^2}{2m_0}\left(n+\frac{1}{2}\right)^2, \tag{53}$$

which corresponds to the energy levels of an anharmonic oscillator.

From Eq. (50), the classical density probability of finding the particle between x and $x + dx$ results $\rho_{\text{classic}}(x) = \frac{1}{\pi\sqrt{A_\kappa^2 - x^2}}$. The first and second moments of the position and the linear momentum in terms of the amplitude or the energy for the deformed oscillator are

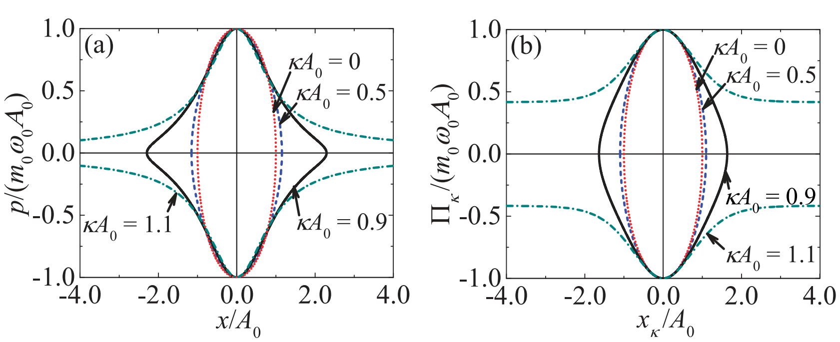

FIG. 6. Phase spaces of the κ-deformed oscillator in the (a) usual canonical coordinates (x, p) and the (b) deformed canonical ones (x_κ, Π_κ) for $\kappa A_0 = 0, 0.5, 0.9$, and 1.1.

$$\overline{x} = 0, \tag{54a}$$

$$\overline{x^2} = \frac{A_\kappa^2}{2} = \frac{E}{m_0\omega_0^2\left(1 - \frac{2E\kappa^2}{m_0\omega_0^2}\right)}, \tag{54b}$$

$$\overline{p} = 0, \tag{54c}$$

$$\overline{p^2} = \frac{m_0\omega_0^2 A_\kappa^2}{2(1+\kappa^2 A_\kappa^2)^{3/2}} = m_0 E\sqrt{1 - \frac{2E\kappa^2}{m_0\omega_0^2}}. \tag{54d}$$

The mean values of the kinetic and potential energies satisfy the relationship

$$\overline{T} = E - \overline{V} = \frac{m_0\omega_0^2}{2\kappa^2}\frac{1}{\sqrt{1+\kappa^2A_\kappa^2}}\left(1 - \frac{1}{\sqrt{1+\kappa^2A_\kappa^2}}\right), \tag{55}$$

with $\overline{V} = \int \rho_{\text{classic}}(x)V(x)dx$. Since $\overline{V} = \overline{T}/\sqrt{1-\kappa^2A_0^2}$, we have that the virial theorem ($\overline{V} = \overline{T}$) is satisfied only for $\kappa A_0 = 0$, which implies $\kappa = 0$.

B. κ-Deformed quantum oscillator

The corresponding κ-deformed time-independent Schrödinger equation for the PDM oscillator is[23]

$$-\frac{\hbar^2}{2m_0}D_\kappa^2\varphi(x) + \frac{1}{2}\frac{m_0\omega_0^2x^2}{(1+\kappa^2x^2)}\varphi(x) = E\varphi(x). \tag{56}$$

Making the change of variable $x \to x_\kappa = \kappa^{-1}\operatorname{arcsinh}(\kappa x)$ [see Eq. (4)], we obtain a particle with the constant mass m_0 subjected to the Pöschl–Teller potential

$$-\frac{\hbar^2}{2m_0}\frac{d^2\phi(x_\kappa)}{dx_\kappa^2} - \frac{\hbar^2\kappa^2}{m_0}\frac{\nu(\nu+1)}{2}\operatorname{sech}^2(\kappa x_\kappa)\phi(x_\kappa) = \epsilon\phi(x_\kappa), \tag{57}$$

with $\epsilon = E - \hbar\omega_0/2\kappa^2a_0^2$, $\nu(\nu+1) = 1/\kappa^4a_0^4$, and $a_0^2 = \hbar/m_0\omega_0$. The solutions of Eq. (57) are

$$\phi(x_\kappa) = \sqrt{\frac{\kappa\mu(\nu-\mu)!}{(\nu+\mu)!}}P_\nu^\mu(\tanh(\kappa x_\kappa)), \tag{58}$$

where $\mu = \nu - n$, n is an integer, and P_ν^μ are the associated Legendre polynomials. Then, the eigenfunctions for the κ-deformed oscillator in the space representation x are

$$\psi_n(x) = \sqrt{\frac{\kappa(\nu-n)n!}{(2\nu-n)!}}\frac{1}{\sqrt[4]{1+\kappa^2x^2}}P_\nu^{\nu-n}\left(\frac{\kappa x}{\sqrt{1+\kappa^2x^2}}\right). \tag{59}$$

The energy levels are given by

$$E_n = \hbar\omega_\kappa\left(n+\frac{1}{2}\right) - \frac{\hbar^2\kappa^2}{2m_0}\left(n+\frac{1}{2}\right)^2 - \frac{\hbar^2\kappa^2}{8m_0}, \tag{60}$$

with $\omega_\kappa = \omega_0\sqrt{1+\frac{\hbar^2\kappa^4}{4m_0^2\omega_0^2}}$. It should be noted that the quantum energy levels differ from those obtained using the WKB approximation [Eq. (53)] by the constant term $-\frac{\hbar^2\kappa^2}{8m_0}$ and the frequency of small oscillations ω_0 replaced by ω_κ. This modification in the frequency is associated with the symmetrization problem of the classical Hamiltonian in order to construct its corresponding Hamiltonian operator in the quantum formalism (see Ref. 23 for more details). However, in the limit $\hbar \to 0$ with $n \gg 1$, Eq. (60) recovers the semi-classical approximation, Eq. (53). In Fig. 7, an illustration of the potential $V(x) = \frac{m_0\omega_0^2x^2}{2(1+\kappa^2x^2)}$ along with the energy levels for some values of κA_0 is shown. In Fig. 8, we show the wave-functions and the probability densities for the four lower energy states and for some values of κa_0. The values of κa_0 chosen are such that $\nu(\nu+1) = 1/\kappa^4a_0^4$ is satisfied with the ν integer. We consider $\nu = 4, 5, 10$, and ∞ in such a way that the corresponding values of κa_0 are $20^{-1/4}, 30^{-1/4}, 110^{-1/4}$, and 0.

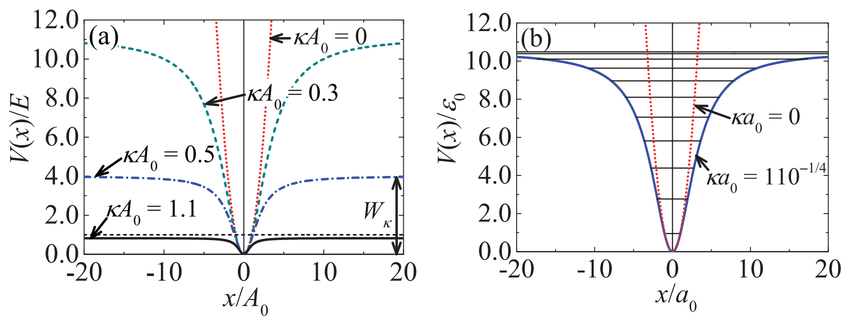

FIG. 7. (a) Potential $V(x) = \frac{m_0\omega_0^2 x^2}{2(1+\kappa^2x^2)}$ for $\kappa A_0 = 0, 0.3, 0.5$, and 1.1 with $A_0^2 = 2E/m_0\omega_0^2$. (b) Energy levels of the κ-deformed oscillator for $\kappa a_0 = 1/\sqrt[4]{110}$ with $a_0^2 = \hbar/m_0\omega_0$ and $\varepsilon_0 = \hbar\omega_0/2$.

From the Legendre differential equation

$$(1-u^2)\frac{d^2P_\nu^\mu(u)}{du^2} - 2u\frac{dP_\nu^\mu(u)}{du} + \left[\nu(\nu+1) - \frac{\mu^2}{1-u^2}\right]P_\nu^\mu(u) = 0, \tag{61}$$

the identities [see Eqs. (2) and (3) in page 965 of the Ref. 84], we obtain the expectation values of $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$, which are

$$\langle\hat{x}\rangle = 0, \tag{62a}$$

$$\begin{aligned}\langle\hat{x}^2\rangle &= \frac{E_n + \frac{\hbar^2\kappa^2}{2m_0}}{m_0\omega_0^2\left(1 - \frac{2E_n\kappa^2}{m_0\omega_0^2} - \frac{\hbar^2\kappa^4}{m_0^2\omega_0^2}\right)}\\ &= \frac{\hbar}{m_0\omega_0}\left\{\frac{\frac{\omega_\kappa}{\omega_0}\left(n+\frac{1}{2}\right) - \frac{\kappa^2a_0^2}{2}\left(n^2+n-\frac{1}{2}\right)}{1 - 2\kappa^2a_0^2\left[\frac{\omega_\kappa}{\omega_0}\left(n+\frac{1}{2}\right) - \frac{\kappa^2a_0^2}{2}\left(n^2+n-\frac{1}{2}\right)\right]}\right\},\end{aligned} \tag{62b}$$

$$\langle\hat{p}\rangle = 0, \tag{62c}$$

$$\begin{aligned}\langle\hat{p}^2\rangle &= m_0\left(E_n - \frac{\hbar^2\kappa^2}{4m_0}\right)\frac{z^2-(2n+1)z}{z^2-4}\\ &= m_0\hbar\omega_0\left[\frac{\omega_\kappa}{\omega_0}\left(n+\frac{1}{2}\right) - \frac{\kappa^2a_0^2}{2}(n^2+n+1)\right]\frac{z^2-(2n+1)z}{z^2-4},\end{aligned} \tag{62d}$$

with $z \equiv 2\nu + 1 = \sqrt{1 + \frac{4m_0^2\omega_0^2}{\kappa^4\hbar^2}}$.

In the limit $\kappa \to 0$, i.e., $z \to \infty$, the usual cases $\langle\hat{x}^2\rangle = \frac{\hbar}{m_0\omega_0}\left(n+\frac{1}{2}\right)$ and $\langle\hat{p}^2\rangle = m_0\hbar\omega_0\left(n+\frac{1}{2}\right)$ are recovered. According to the principle of correspondence, in the limit of large quantum numbers (or equivalently $\hbar \to 0$), we have $E_n \to E$ and $\omega_\kappa \to \omega_0$, and it is immediately verified that Eqs. (62b) and (62d) coincide with Eqs. (54b) and (54d), respectively. Indeed, when $\hbar \to 0$, one obtains that $z \approx 2m_0\omega_0/\hbar\kappa^2 \gg 1$, and we have

$$\begin{aligned}\lim_{\hbar\to0}\langle\hat{p}^2\rangle &= \lim_{\hbar\to0} m_0E_n\left(1 - \frac{2n+1}{z}\right)\\ &= \lim_{\hbar\to0} m_0E_n\sqrt{1 - \frac{2\kappa^2E_n}{m_0\omega_\kappa^2} - \frac{\hbar^2\kappa^4}{4m_0\omega_\kappa^2}}\\ &= m_0E_n\sqrt{1 - \frac{2\kappa^2E_n}{m_0\omega_\kappa^2}} + \mathcal{O}(\hbar^2).\end{aligned} \tag{63}$$

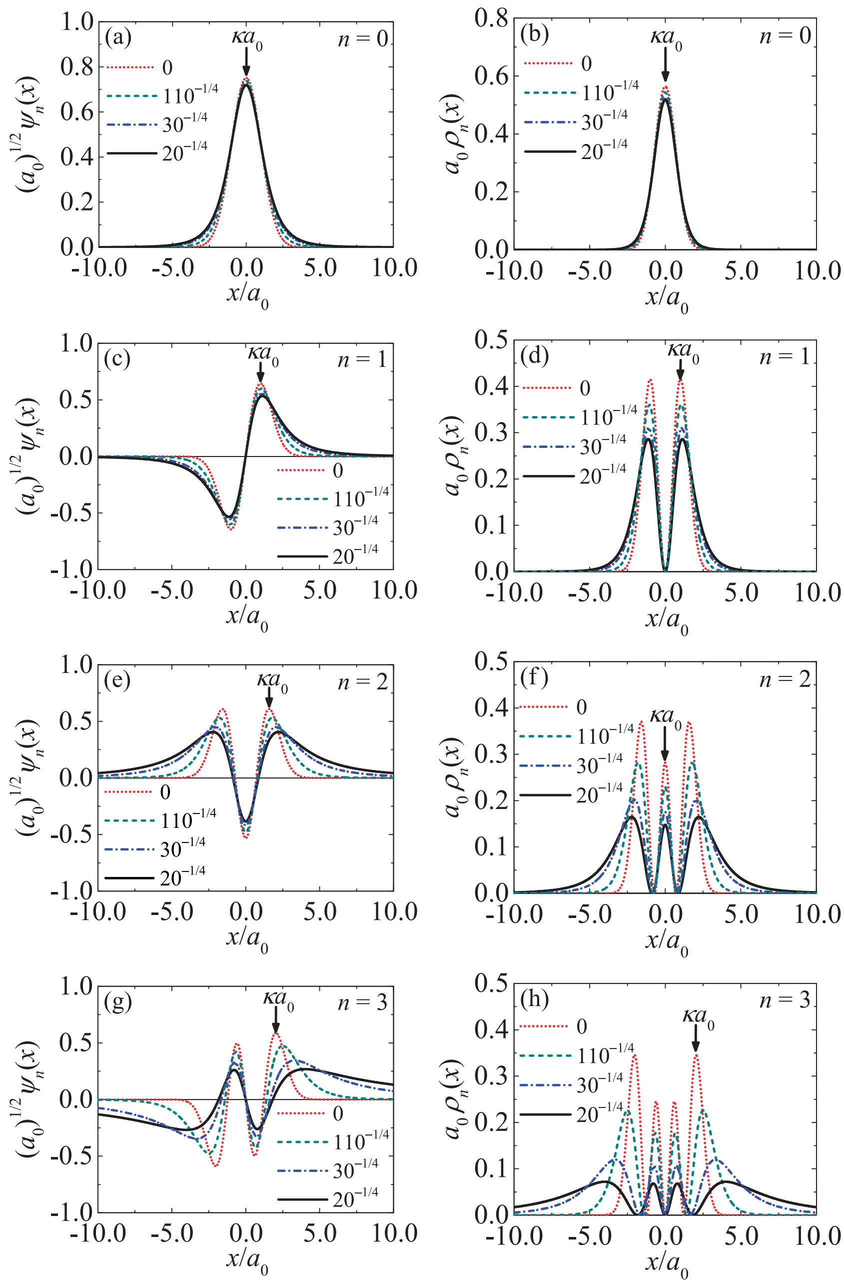

FIG. 8. Eigenfunctions $\psi_n(x)$ (upper line) and probability densities $\rho_n(x) = |\psi_n(x)|^2$ (bottom line) for a κ-deformed oscillator particle for the values of κa_0 such that $\nu(\nu + 1) = 1/(\kappa a_0)^4$ with $\nu = 4, 5, 10$, and ∞ in such a way that the corresponding values of κa_0 are $20^{-1/4}, 30^{-1/4}, 110^{-1/4}$, and 0. [(a) and (b)] $n = 0$ (ground state), [(c) and (d)] $n = 1$ (first excited state), [(e) and (f)] $n = 2$ (second excited state), and [(g) and (h)] $n = 3$ (third excited state).

The expectation values of the kinetic and potential energies satisfy

$$\langle \hat{T} \rangle = E_n - \langle \hat{V} \rangle = \frac{m_0\omega_0^2}{2\kappa^2}\frac{1}{\sqrt{1+\kappa^2 a_{n,\kappa}^2}}\left(\frac{\omega_0}{\omega_\kappa} - \frac{1}{\sqrt{1+\kappa^2 a_{n,\kappa}^2}}\right), \tag{64}$$

with $E_n = \frac{m_0\omega_0^2 a_{n,\kappa}^2}{2(1+\kappa^2 a_{n,\kappa}^2)}$ and the quantum amplitude

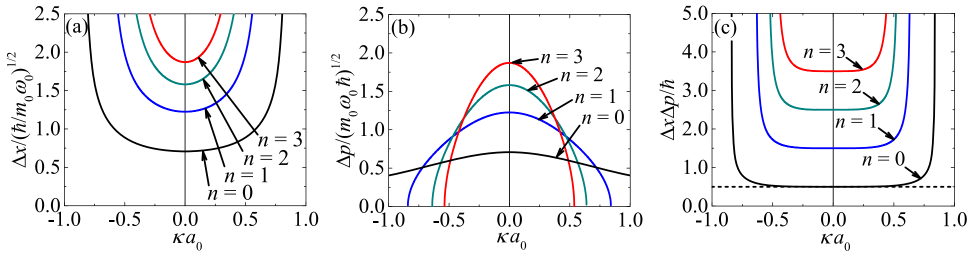

FIG. 9. Uncertainty of (a) the position Δx and (b) the momentum Δp of the product (c) $\Delta x\Delta p$, as a function of κa_0, for the quantum states with $n = 0$, 1, 2, and 3. The standard uncertainty relation $\Delta x\Delta p = (n + \frac{1}{2})\hbar$ is recovered for $\kappa a_0 \to 0$.

$$a_{n,\kappa} = a_0\left\{\frac{\frac{\omega_\kappa}{\omega_0}(2n+1) - \kappa^2 a_0^2\left(n^2+n+\frac{1}{2}\right)}{1-\kappa^2 a_0^2\left[\frac{\omega_\kappa}{\omega_0}(2n+1)-\kappa^2 a_0^2\left(n^2+n+\frac{1}{2}\right)\right]}\right\}^{1/2}. \tag{65}$$

In the classical limit, one has that $a_{n,\kappa} \to A_\kappa$ once $E_n \to E$, so the expectation value (64) recovers its classical average value (55). In Fig. 9, we show the uncertainty relation of the κ-deformed oscillator, along with the uncertainties Δx and Δp of x and p, for the ground state and the first three excited ones. As expected, while Δx increases as the dimensionless deformation parameter κa_0 varies within the interval $[-1, 1]$, Δp decreases and vice versa. In turn, this implies a generalized κ-uncertainty inequality [Fig. 9(c)], which is an increasing function of the quantum number n, and it also grows fast as κa_0 varies. The symmetry exhibited around the axis $\kappa a_0 = 0$ in the curves of Fig. 9 is a consequence of the invariance of the mass function (and then of the Hamiltonian too) given by Eq. (13) against the transformation $\kappa \to -\kappa$.

VI. CONCLUSIONS

We have presented the quantum and the classical mechanics that results from assuming a position-dependent mass related to the κ-algebra, which is the mathematical background underlying κ-statistics. Indeed, we have characterized both the quantum and classical formalism of a particle with a PDM determined univocally by the κ-algebra. The consistency of the κ-deformed formalism is manifested in the following arguments.

The κ-deformed Schrödinger equation turns out to be equivalent to a Schrödinger-like equation for a deformed wave-function provided with a κ-deformed non-Hermitian momentum operator. Within the κ-formalism, one can define deformed versions of the continuity equation, the Fourier transform, etc. In particular, a deformed Newton's second law in terms of the deformed dual κ-derivative [Eq. (15)] follows in the classical limit.

We have illustrated the approach with the problems of a particle confined in an infinite potential well and a κ-deformed oscillator, which is equivalent to the Mathews–Lakshmanan oscillator (in the standard space) or to the Pösch–Teller potential problem (in the κ-deformed space), provided with the change of variable $x \to x_\kappa$. We have obtained the distributions for the classical case as well as the eigenfunctions and eigenenergies for the quantum case. Although we have applied the mapping approach to a κ-deformed space in order to study the quantum Mathews–Lakshmanan oscillator, it is important to mention that other equivalent approaches can be found in the literature. For instance, factorization methods, supersymmetry, and coherent states have also been investigated for this nonlinear oscillator (see Refs. 34–38 and references therein).

Analogous to the quantum oscillator and the Hermite polynomials, the eigenvalues equation for the κ-deformed quantum oscillator is expressed in terms of the Legendre polynomials. Expectedly, in both examples, we have reported the localization and delocalization of the probability density functions corresponding to the conjugated variables x and p, from which the uncertainty relation follows (Figs. 5 and 9), with the particularity that the lower bound is an increasing function of the deformation parameter κ, satisfied by the ground state and the first three excited ones. This could be physically interpreted as if the quantum role of the deformation (or equivalently, of the non-constant mass) is to increase the intrinsic correlation between the conjugated operators $\hat{x}$ and $\hat{p}$. In addition, for the case of the κ-deformed oscillator, we have studied the effect of the deformation parameter κ on the phase space in the usual coordinates (x, p) and in the deformed ones (x_κ, Π_κ). It is verified that for a certain range of values of κ, the motion is unbounded (Fig. 6).

We consider that the techniques employed in this work could stimulate the seek of other generalizations of classical and quantum mechanical aspects, as has been reported in recent research studies by means of the q-algebra.[7,48–56]

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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